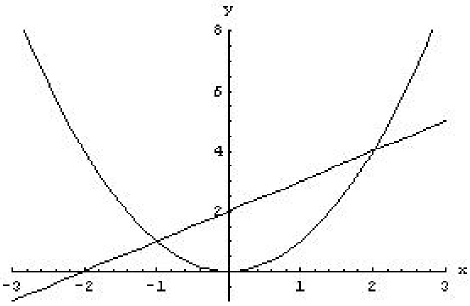
THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

Sample Solutions for *Specimen Test 1*

1. A.



The two curves $y = x^2$ and $y = x + 2$ meet when

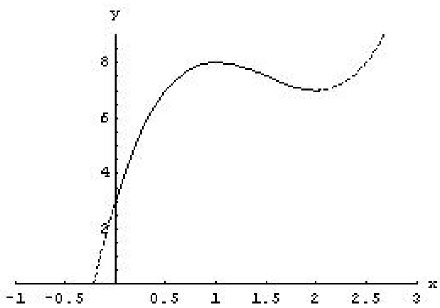
$$x^2 - (x+2) = (x+1)(x-2) = 0.$$

i.e. when $x = -1$ or $x = 2$. Within the region $-1 \leqslant x \leqslant 2$ then $x^2 \leqslant x + 2$ (see graphs) and so the area between the curves is

$$\begin{aligned}\int_{-1}^{2} \left(x + 2 - x^2\right) \mathrm{d}x &= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} \\ &= \frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) \\ &= \frac{9}{2}.\end{aligned}$$

The **answer is (c).**

B.



A function on the region $0 \leqslant x \leqslant 2$ takes it minimum (and likewise maximum) either at a point x_0 inside the region $0 < x < 2$ in which case $f'(x_0) = 0$ or at an endpoint $x_0 = 0$ or 2 (see graph). If

$$f(x) = 2x^3 - 9x^2 + 12x + 3$$

then

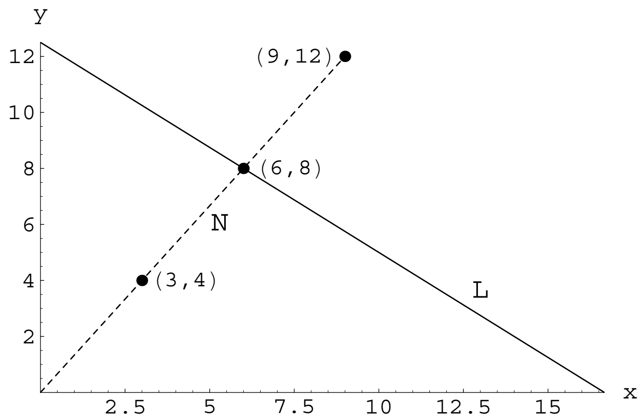
$$f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2).$$

The graph has two turning points at $x = 1$ and $x = 2$. From our knowledge of the shape of cubics or by looking at $f''(x)$ we know that $x = 1$ is a maximum and $x = 2$ is a minimum. So the minimum is either at 2, or possibly at the other endpoint $x = 0$ (even though this is not a turning point of f). Now

$$f(2) = 16 - 36 + 24 + 3 = 7, \qquad f(0) = 3.$$

So the minimum value of $f(x)$ for $0 \leqslant x \leqslant 2$ is 3 and the **answer is (b).**

C.



The gradient of the line L, with equation $3x+4y=50$, is $-3/4$ and so a normal to the line has gradient $4/3$. So the normal N to the line through $(3,4)$ has equation

$$y-4=\frac{4}{3}(x-3).$$

This normal N meets L when

$$\begin{aligned}3x+16+\frac{16}{3}(x-3) &= 50,\\ \Longrightarrow 25x &= 150,\\ \Longrightarrow (x,y) &= (6,8).\end{aligned}$$

The vector from $(3,4)$ along N to $(6,8)$ on L is $(3,4)$. Following this vector again to the mirror image gives $(9,12)$ and we see the **answer is (a)**.

D. Let

$$f(x)=x^3-30x^2+108x-104.$$

By inspection we note that

$$f(2)=8-120+216-104=0,$$

so that $(x-2)$ is a factor of $f(x)$. We see

$$\begin{aligned}f(x) &= (x-2)\left(x^2-28x+52\right)\\ &= (x-2)(x-2)(x-26).\end{aligned}$$

Hence $x=2$ is a repeated root of the equation and the **answer is (d).**

E. It has been noted that $6\times 7=42$.

- This certainly isn't a counter-example to the product of two odd numbers being odd, as this statement is true and so there are no counter-examples.
- The product 42 is not a multiple of 4, but the numbers 6 and 7 are consecutive, so $6\times 7=42$ *is* a counter-example to (b).
- As 42 *isn't* a multiple of 4 then $6\times 7=42$ can have no bearing on the truth or not of statement (c).
- Statement (d) is false, but to provide a counter-example we would need to show that *all* the possible factorisations of some even integer included an odd number.

So the **answer is (b).**

F. We have

$$\begin{aligned}
2\cos^2 x + 5\sin x = &\quad 4, \\
\implies 2 - 2\sin^2 x + 5\sin x = &\quad 4, \\
\implies 2\sin^2 x - 5\sin x + 2 = &\quad 0, \\
\implies (2\sin x - 1)(\sin x - 2) = &\quad 0.
\end{aligned}$$

Hence $\sin x = 1/2$, which occurs at $\pi/6$ and $5\pi/6$ in the given range, or $\sin x = 2$, which is true for no values of x. Hence there are two solutions and the **answer is (a)**.

G. Note

$$x^2 + 3x + 2 > 0 \implies (x+1)(x+2) > 0 \implies x > -1 \quad \text{or} \quad x < -2. \tag{1}$$

Also

$$x^2 + x < 2 \implies (x-1)(x+2) < 0 \implies -2 < x < 1. \tag{2}$$

The only x for which (1) and (2) agree are in the range $-1 < x < 1$. Hence the **answer is (b).**

H. We are given that

$$\log_{10} 2 = 0.3010\ldots \text{ and that } 10^{0.2} < 2.$$

So

$$\log_{10} 2^{100} = 100 \log_{10} 2 = 30.10\ldots$$

Hence

$$2^{100} = 10^{30.10\ldots} = 10^{0.10\ldots} \times 10^{30}.$$

This means that 2^{100} is 31 digits long, with its first digit being determined by $10^{0.10\ldots}$. As

$$10^{0.10\ldots} < 10^{0.2} < 2$$

that first digit is less than 2 and so must be a 1. This shows the **answer is (c).**

I. This question could be approached by simply working out the 11 coefficients. More systematically we know that the coefficient of x^k in $(1 + x/2)^k$ is

$$c_k = \frac{10!}{k!\,(10-k)!}\left(\frac{1}{2}\right)^k.$$

Note that

$$\frac{c_{k+1}}{c_k} = \frac{10!}{(k+1)!\,(9-k)!} \times \frac{k!\,(10-k)!}{10!} \times \frac{2^k}{2^{k+1}} = \frac{10-k}{2\,(k+1)}.$$

From this we can see that $c_{k+1}/c_k > 1$, (i.e. that the c_k are growing) if

$$10 - k > 2k + 2 \implies 8 > 3k$$

which means $k \leqslant 2$. So the c_k grow up to c_3 and then decrease thereafter — that is the **answer is (b)**.

J. If $x^2y^2\,(x+y) = 1$ then it is clear that $x \neq 0$ and $y \neq 0$. This means that the curve never crosses the x- and y-axes. This eliminates (a) and (b) as options. From $x^2y^2\,(x+y) = 1$ it follows that

$$x + y = \frac{1}{x^2y^2} > 0$$

and so the curve lies entirely in the region $x + y > 0$, which eliminates (d) as a possibility. Hence the correct **answer is (c)**.

2. (i) Writing c for $\cos\theta$ and s for $\sin\theta$ we have

$$\begin{aligned}(x-1)\left(x^2-(c+s)\,x+cs\right) &= \left[x^3-(c+s)\,x^2+csx\right]-\left[x^2-(c+s)\,x+cs\right]\\ &= x^3-(1+c+s)\,x^2+(cs+c+s)\,x-cs,\end{aligned}$$

as required. Factorising the quadratic further we get

$$x^2-(c+s)\,x+cs=(x-c)\,(x-s)\,,$$

and hence the three roots of the cubic are $1, c, s$.

(ii) When $\theta=\pi/3$ then the three roots are 1, $c=\frac{1}{2}$ and $s=\frac{\sqrt{3}}{2}$.

(iii) Two of the three roots can be equal when

- $s=1$ which only occurs in the range $0\leqslant\theta<2\pi$ when $\theta=\pi/2$;
- $c=1$ in which case $\theta=0$;
- $s=c$ in which case $\tan\theta=1$ and $\theta=\pi/4$ or $5\pi/4$.

So the list of possibilities is $\theta=0,\ \pi/4,\ \pi/2,\ 5\pi/4$.

(iv) As s and c vary between -1 and 1, and the other root is 1, then the greatest difference possible is 2. However, $|s-c|$ cannot equal 2, so the difference is greatest when $s=-1$ or when $c=-1$. These cases occur at $\theta=3\pi/2$ and $\theta=\pi$ respectively

When $s=-1$ then $c=0$, and when $c=-1$ then $s=0$. As the cubic is symmetric in s and c then the cubic is the same in each case — or we might explicitly calculate it in each case to get

$$(x-1)\,(x-0)\,(x+1)=x^3-x.$$

3. (i) As

$$f(x) = x^2 - 2px + 3$$

then $f'(x) = 2x - 2p = 2(x-p) = 0$ at $x = p$. So the stationary point is in the range $0 < x < 1$ only if $0 < p < 1$, and is otherwise outside that x-range.

(ii) The minimum value $m(p)$ attained by $f(x)$ in the range $0 \leqslant x \leqslant 1$ will occur either at an endpoint ($x = 0$ or $x = 1$) or at a stationary point in between. If $p \geqslant 1$ then we have seen there is no stationary point in between; as $f(0) = 3$ and $f(1) = 4 - 2p < 3$ then $m(p) = 4 - 2p$.

(iii) If $p \leqslant 0$ then again there is no stationary point in the range $0 < x < 1$. However this time

$$f(1) = 4 - 2p > 3 = f(0)$$

and so $m(p) = 3$.

(iv) If $0 < p < 1$ then there is a stationary point at $x = p$, which is a minimum (because of the U-shape of the parabola, or by calculating $f''(p) = 2$). Now

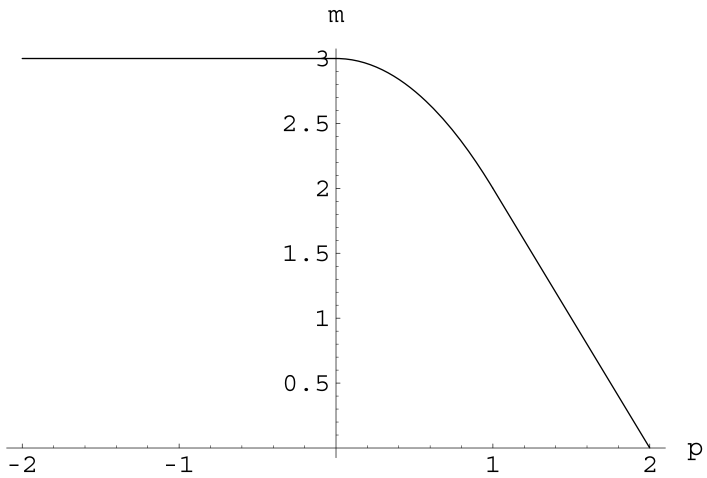
$$f(p) = p^2 - 2p^2 + 3 = 3 - p^2.$$

From our knowledge of the graph's shape, or by checking that:

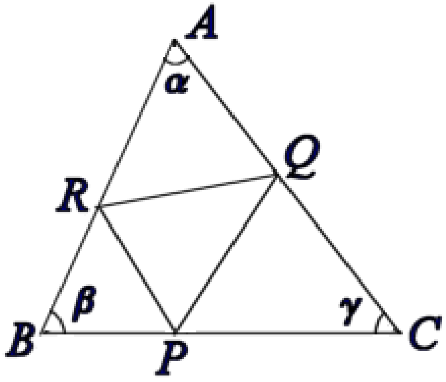
$$\begin{aligned} 3 - p^2 &< 3 = f(0); \\ 3 - p^2 < 4 - 2p = f(1) \quad \text{as} \quad 1 - 2p + p^2 &= (1-p)^2 > 0; \end{aligned}$$

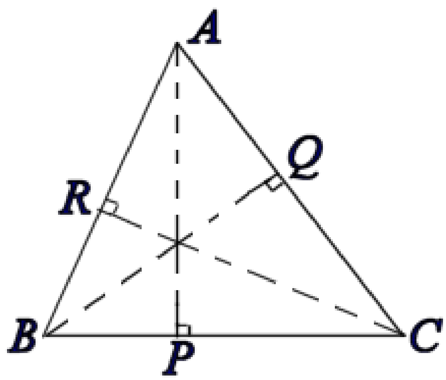
then $x = p$ is the minimum on the whole range $0 \leqslant x \leqslant 1$ and $m(p) = 3 - p^2$ when $0 < p < 1$.

(v)



4.





(i) Taking AB as the base, which is of length c, and CR as the height, which is of length

$$AC \sin(\angle CAR) = b \sin \alpha,$$

we see the triangle has area

$$\frac{1}{2} \times \text{ base } \times \text{ height } = \frac{1}{2} cb \sin \alpha.$$

By similar considerations, the area is also given by

$$\text{Area}(ABC) = \frac{1}{2} bc \sin \alpha = \frac{1}{2} ac \sin \beta = \frac{1}{2} ab \sin \gamma.$$

Dividing these equations by $abc/2$, and then inverting, we obtain the sine rule

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

(ii) From the first part of the question we see that

$$\begin{aligned} \text{Area}(AQR) &= \frac{1}{2} AQ \cdot AR \sin \alpha \\ &= \frac{AQ}{c} \times \frac{AR}{b} \times \text{Area}(ABC) \\ &= \cos^2 \alpha \times \text{Area}(ABC) \end{aligned}$$

since BAQ and RAC are right-angled triangles. Similarly

$$\begin{aligned} \text{Area}(BPR) &= \cos^2 \beta \times \text{Area}(ABC); \\ \text{Area}(CQP) &= \cos^2 \gamma \times \text{Area}(ABC). \end{aligned}$$

Hence

$$\begin{aligned} \text{Area}(PQR) &= \text{Area}(ABC) - \text{Area}(AQR) - \text{Area}(BPR) - \text{Area}(CQP) \\ &= (1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma) \times \text{Area}(ABC). \end{aligned}$$

(iii) If

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

in a triangle then the area of PQR is zero. This can only happen when two of P, Q and R coincide at what will be the right-angle of the triangle — so the equality holds only if the triangle ABC is right-angled.

5. (i) The only way to create longer songs, from previous ones, is by Rule II. xy is a song of length two (as it can be made from Rule II as x-no notes-no notes-y) and by Rule III yx is also a song. With these two songs and using Rule II again, we can produce two songs

$$xxyxyy \quad \text{and} \quad xyxyxy$$

of length six, and applying Rule III we also see that

$$yyxyxx \quad \text{and} \quad yxyxyx$$

are songs.

(ii) As was noted in (i) longer songs may only be produced from shorter ones using Rule II. Given any song s of length m then the song $xssy$ is also a song. By this means, k songs of length m create k new songs of length $2m+2$. The only way to produce further songs of length $2m+2$ is to swap the xs and ys in the newly made songs by Rule III. So $y\tilde{s}\tilde{s}x$ will also be a song of length $2m+2$ where $\tilde{s}$ denotes the song s with all xs and ys swapped. As the new batch of songs end in an x and the previous batch ended in a y then the second batch of length $2m+2$ songs contains none of the first batch. In total, then, we have $2k$ new songs of length $2m+2$.

Each time this process of generating new songs produces twice as many songs of the next allowable length as for the previous allowed length. As there is one song of the "noughth" allowed length there will be 2^n of the nth allowed length for $n \geqslant 0$. What is the nth allowed length? These allowed lengths follow the rule

$$\begin{aligned}
\text{length}_0 &= 0 \\
\text{length}_1 &= 2 \times 0 + 2 = 2 \\
\text{length}_2 &= 2 \times 2 + 2 = 2^2 + 2 \\
\text{length}_3 &= 2 \times \left(2^2 + 2\right) + 2 = 2^3 + 2^2 + 2 \\
&\vdots \\
\text{length}_n &= 2^n + 2^{n-1} + \cdots + 2.
\end{aligned}$$

This length_n is a geometric sum with n terms, common ratio 2, and first term 2, and so we have

$$\text{length}_n = \frac{2\left(2^n - 1\right)}{(2-1)} = 2^{n+1} - 2.$$

[For those with knowledge of *mathematical induction* then part (ii) could be attempted using this, for full marks, but no presumption about such knowledge has been made.]

(iii) For any positive whole number n, it is possible to produce a song of greater length using Rules I, II, III, because we can find a k such that $2^{k+1} - 2 > n$. Set $N = 2^{k+1} - 2$. If this longer (length N) song ends in a y we may reduce its length using Rule IV to make another song one note shorter. On the other hand, if the song of length N ends in an x we can swap all the xs and ys by Rule III to produce a song, also of length N, which now ends in an y. Removing the final y by Rule IV we again have produced a song of length $N-1$.

Repeating this process, one note at a time, we eventually produce a song of length n, and so songs of all possible lengths exist in the Martian later period.

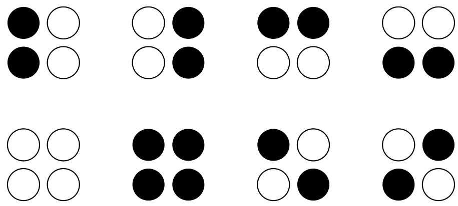
6. (i) The three hats are not all of the same colour. If Alice sees a black and white hat then she cannot conclude anything about her own — but in the event she see two hats of the same colour she can conclude that her own must be of the opposite colour. So Alice can see two black hats and herself wears a white one.

(ii) If Bob and Charlie have the same coloured hats then Alice can deduce what she is wearing. As she cannot then Bob and Charlie must be wearing different hats. As Bob can see Charlie's hat then he can deduce his own is of the opposite colour.

(iii) If there were two black hats and one white one then, as in part (i), one of the three would be able to deduce that they were wearing a white hat. Hence there must be one or no black hats.

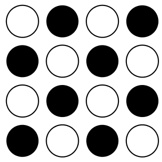
(iv) If there were three white hats then on the second time of questioning all three would again be unsure about the colour of their own hat. If there is one black hat then the two who can see it know that it is the only black hat and that they are wearing white. Given the responses we know that Alice is wearing black, and that Bob and Charlie are each wearing white.

7. (i) The possible patterns are



The possible numbers of white counters are $0, 2, 4$.

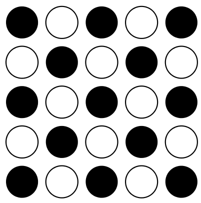
(ii)



Note that we started with an even number of whites. Each allowed move flips a row or column. If the number of whites in the flipped row/column is i then it becomes $4 - i$. In particular, if there was an odd (respectively even) number of whites before there will be an odd (respectively even number) afterwards. As the remaining counters are unchanged then each flip preserves the parity (oddness or evenness) of the total number of whites. As we started out with an even number of whites, there will always remain an even number of whites — in particular, there can never be just one white.

(An alternative proof is to note that any two-by-two subsquare is affected by each move either not at all or in the same ways as moves in the two-by-two game, and must start with a pattern identical or complementary to that of the previous part. So if it were possible to reach a four-by-four square with only one white then it would be possible to reach a two-by-two square with only one white, and the previous part demonstrates that this is not possible.)

(iii)



In the five-by-five case, the number of whites in each row or column changes from i to $5 - i$ under each flip. Arguing as in the previous part, the total number of whites changes from odd to even, or vice versa, with every move. Since there is an even number of whites to start with and an odd number to end up with (an all-white grid with 25 whites), there must be an odd number of moves.

It is fairly easy to see how to change the starting pattern to all white with five moves (flip rows 1, 3, and 5 and then columns 2 and 4). Can it be done with fewer moves? We already know that the number of moves must be odd so it suffices to show that 3 moves is not sufficient. The easiest way to see this is to note that one cannot touch all the black counters in three moves — at most three black counters lie in a row or column and there are thirteen counters that need turning in all.

(These sample solutions have been produced by Dr. Richard Earl, who is the Organising Secretary for admissions in Mathematics, Statistics and Computer Science in Oxford. Any questions regarding the test, applying to Oxford to read for these subjects, or comments on these solutions, would be welcome at earl@maths.ox.ac.uk)