Generic Properties of Datatypes

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Generic Programming Summer School

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Outline

- Theorems For Free
- Commuting Datatypes (“Zips”)
- Relators, Fans and Membership
- Properties of Zips
- Conclusion

Parametric Polymorphism

Summary: parametric polymorphism is a verifiable form of (type) genericity.

Common Type = Common Properties

$$\mathsf{length} : \langle \forall \alpha :: \mathbb{N} \leftarrow \mathsf{List}.\alpha \rangle$$

For all types A and B and all functions f of type $A \leftarrow B$,

$$\mathsf{length}_A \circ \mathsf{List}.f \;=\; \mathsf{length}_B \quad .$$

Let sq denote the function that squares a number.

$$sq \circ \mathsf{length} : \langle \forall \alpha :: \mathbb{N} \leftarrow \mathsf{List}.\alpha \rangle$$

$$(sq \circ \mathsf{length}_A) \circ \mathsf{List}.f \;=\; sq \circ \mathsf{length}_B \quad .$$

Suppose $copycat$ appends a copy of a list to itself.

$$\mathsf{length} \circ copycat : \langle \forall \alpha :: \mathbb{N} \leftarrow \mathsf{List}.\alpha \rangle$$

$$(\mathsf{length}_A \circ copycat_A) \circ \mathsf{List}.f \;=\; \mathsf{length}_B \circ copycat_B \quad .$$

Polymorphism

Consider the type expressions defined by the following grammar:

$$\mathsf{Exp} \quad ::= \quad \mathsf{Exp} \times \mathsf{Exp} \;\mid\; \mathsf{Exp} \leftarrow \mathsf{Exp} \;\mid\; \mathsf{Const} \;\mid\; \mathsf{Var} \quad .$$

Here, Const denotes a set of constant types, like $\mathbb{N}$ (the natural numbers) and $\mathbb{Z}$ (the integers). Var denotes a set of type variables. We use Greek letters to denote type variables.

A term t is said to have *polymorphic* type $\langle\forall\alpha :: \mathsf{T}.\alpha\rangle$, where T is a type expression parameterised by type variables α, if t assigns to each type A a value t_A of type $\mathsf{T}.A$.

Mapping Relations to Relations

Type expressions are extended to denote functions from relations to relations.

$$R\times S : A\times B \sim C\times D \quad \Leftarrow \quad R : A\sim C \ \wedge \ S : B\sim D$$

$$((a,b)\ ,\ (c,d)) \in R\times S \quad \equiv \quad (a,c)\in R \ \wedge \ (b,d)\in S \quad .$$

$$R\leftarrow S : (A\leftarrow B) \sim (C\leftarrow D) \quad \Leftarrow \quad R : A\sim C \ \wedge \ S : B\sim D$$

$$(f,g) \in R\leftarrow S \quad \equiv \quad \langle \forall\, b,d :: (f.b\ ,\ g.d)\in R \Leftarrow (b,d)\in S \rangle \quad .$$

The constant type A is read as the identity relation id_A on A.

$$(x\ ,\ y)\in A \quad \equiv \quad x=y \quad .$$

Example

$$R{\times}R{\leftarrow}R \;:\; (A{\times}A \leftarrow A) \sim (B{\times}B \leftarrow B) \quad \Leftarrow \quad R : A \sim B$$

$$
\begin{array}{ll}
 & (f, g) \in R{\times}R{\leftarrow}R \\
= & \{ \quad \text{definition of} \leftarrow \text{on relations} \quad \} \\
 & \langle \forall\, a,b :: (f.a\,,\, g.b) \in R{\times}R \;\Leftarrow\; (a, b) \in R \rangle \\
= & \{ \quad \text{definition of} \times \text{on relations} \quad \} \\
 & \langle \forall\, a,b :: \\
 & \qquad (\mathsf{fst}.(f.a)\,,\, \mathsf{fst}.(g.b)) \in R \;\wedge\; (\mathsf{snd}.(f.a)\,,\, \mathsf{snd}.(g.b)) \in R \\
 & \qquad \Leftarrow (a, b) \in R \\
 & \rangle
\end{array}
$$

Example

$$\mathsf{id}_{\mathrm{Bool}} \leftarrow R\times R \;:\; (\mathrm{Bool} \leftarrow A\times A) \sim (\mathrm{Bool} \leftarrow B\times B) \quad \Leftarrow \quad R : A \sim B$$

$$
\begin{array}{ll}
 & (f,g) \;\in\; \mathsf{id}_{\mathrm{Bool}} \leftarrow R\times R \\
= & \qquad \{ \qquad \text{definition of } \leftarrow \text{ and } \times \text{ on relations} \quad \} \\
 & \langle \forall\, a,a',b,b' :: (f.(a,a')\,,\, g.(b,b')) \in \mathsf{id}_{\mathrm{Bool}} \;\Leftarrow\; (a,b) \in R \wedge (a',b') \in R \rangle \\
= & \qquad \{ \qquad \text{definition of } \mathsf{id}_{\mathrm{Bool}} \quad \} \\
 & \langle \forall\, a,a',b,b' :: f.(a,a') = g.(b,b') \;\Leftarrow\; (a,b) \in R \wedge (a',b') \in R \rangle
\end{array}
$$

Parametric

A term t of polymorphic type $\langle\forall\alpha :: T.\alpha\rangle$ is said to be *parametrically polymorphic* if, for each instantiation of relations R to type variables, $(t_A\ ,\ t_B) \in T.R$, where R has type $A \sim B$.

$$\mathsf{fst} : \langle\forall\, \alpha,\beta :: \alpha \leftarrow \alpha\times\beta\rangle$$

Suppose $R : A \sim B$ and $S : C \sim D$.

$$(\mathsf{fst}_{A,C}\ ,\ \mathsf{fst}_{B,D}) \ \in\ R \leftarrow R\times S$$

$=$ { definition of $\leftarrow$ and $\times$ on relations }

$$\langle\forall\, a,b,c,d :: (\mathsf{fst}_{A,C}.(a,c)\ ,\ \mathsf{fst}_{B,D}.(b,d)) \in R \ \Leftarrow\ (a,b) \in R \wedge (c,d) \in S\rangle$$

$=$ { definition of fst }

$$\langle\forall\, a,b,c,d :: (a,b) \in R \ \Leftarrow\ (a,b) \in R \wedge (c,d) \in S\rangle$$

$=$ { calculus }

$$\mathsf{true}$$

Ad Hoc Polymorphism

Suppose "==" denotes a polymorphic "equality" operator. That is,

$$== \;:\; \langle\forall\alpha :: \mathsf{Bool} \leftarrow \alpha\times\alpha\rangle$$

$$\begin{array}{ll}
& == \text{ is parametric} \\
= & \{ \quad \text{definition } (R \text{ ranges over relations of type } A{\leftarrow}B) \quad \} \\
& \langle\forall R :: (==_A\,,\,==_B) \;\in\; \mathsf{id}_{\mathtt{Bool}} \leftarrow R\times R\rangle \\
= & \{ \quad \text{definition of } \leftarrow \text{ and } \times \text{ on relations, and of } \mathsf{id}_{\mathtt{Bool}} \quad \} \\
& \langle\forall R :: \langle\forall u,v,x,y :: (u ==_A v) = (x ==_B y) \;\Leftarrow\; (u,x)\in R \wedge (v,y)\in R\rangle\rangle \\
\Rightarrow & \{ \quad \text{take } R \text{ to be an arbitrary function } f \\
& \quad\; (\text{so } (u,x)\in R \equiv u = f.x \text{ and } (v,y)\in R \equiv v = f.y \quad \} \\
& \langle\forall f :: \langle\forall x,y :: (f.x ==_A f.y) = (x == y)\rangle\rangle
\end{array}$$

Conclusion: all functions in the language of terms are injective, or "equality" is not both real equality and parametric.

Commuting Datatypes

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Introductory Examples

Zip (of lists)

$$([a_1, a_2, \ldots, a_n], [b_1, b_2, \ldots, b_n]) \mapsto [(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)]$$

$$\mathsf{Pair} \cdot \mathsf{List} \qquad \mapsto \qquad \mathsf{List} \cdot \mathsf{Pair}$$

Matrix Transposition

$$\mathsf{List} \cdot \mathsf{List} \qquad \mapsto \qquad \mathsf{List} \cdot \mathsf{List}$$

Broadcast

$$(a, [b_1, b_2, \ldots, b_n]) \mapsto [(a, b_1), (a, b_2), \ldots, (a, b_n)]$$

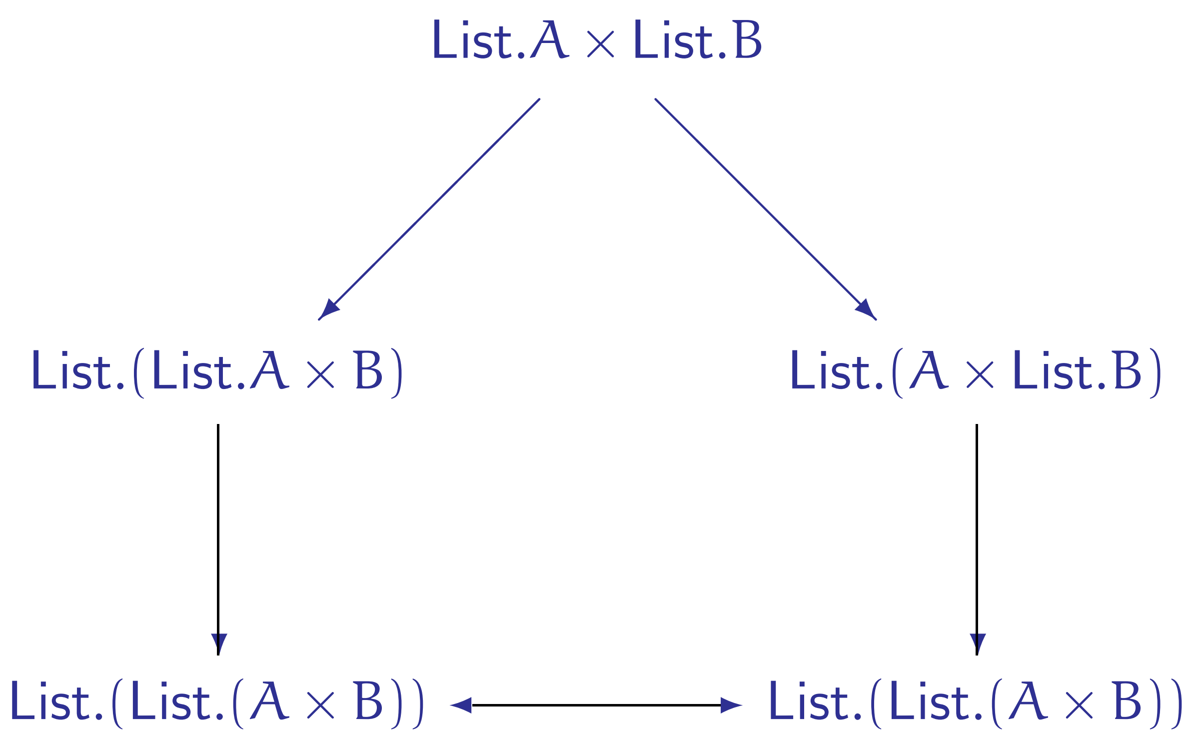
$$A\times \cdot \mathsf{List} \qquad \mapsto \qquad \mathsf{List} \cdot A\times$$

Primitive

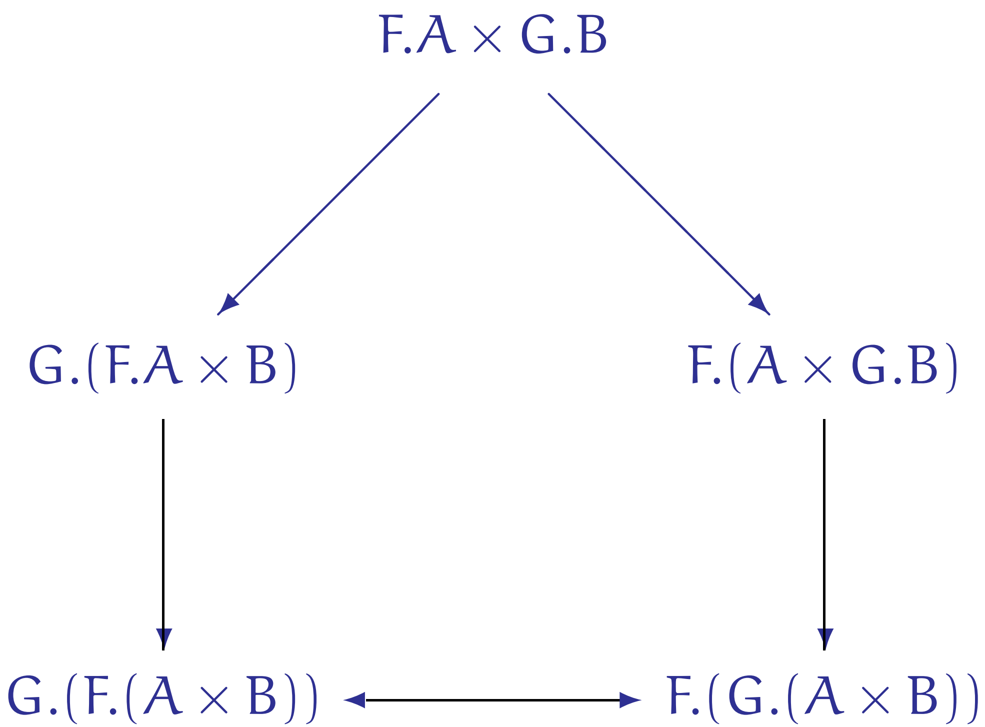
$$(A+B) \times (C+D) \mapsto (A\times C) + (B\times D)$$

$$\times \cdot + \qquad \mapsto \qquad + \cdot \times$$

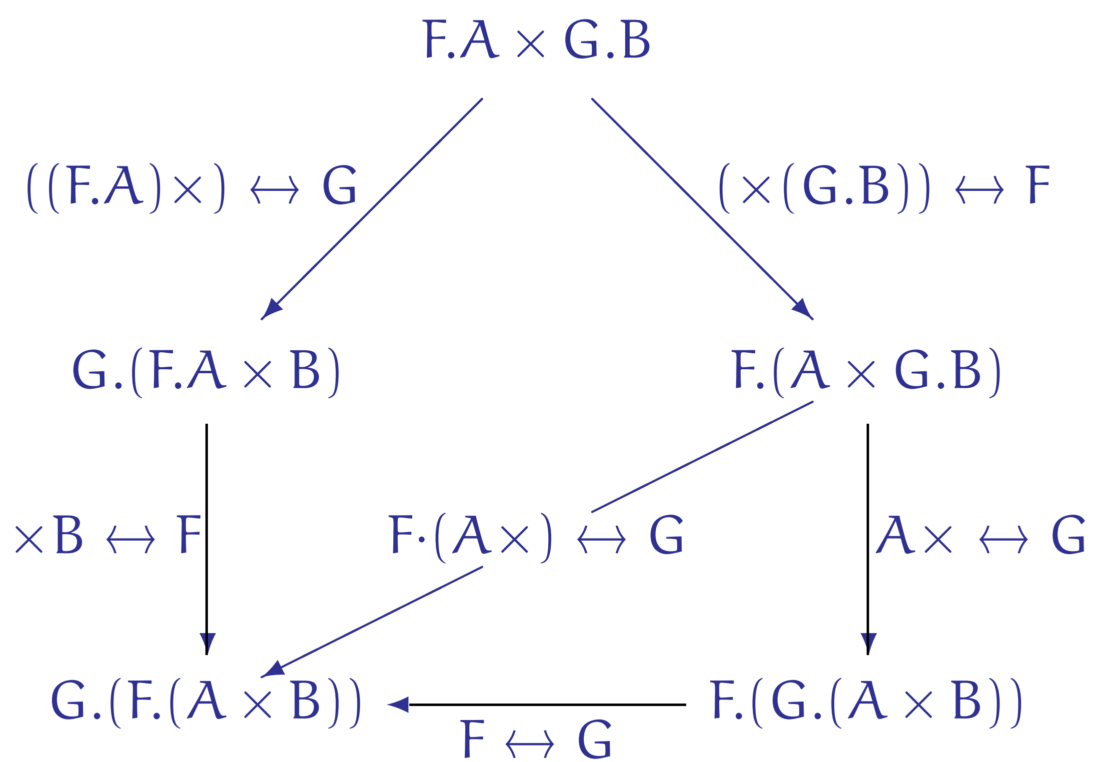
Structure Multiplication ...



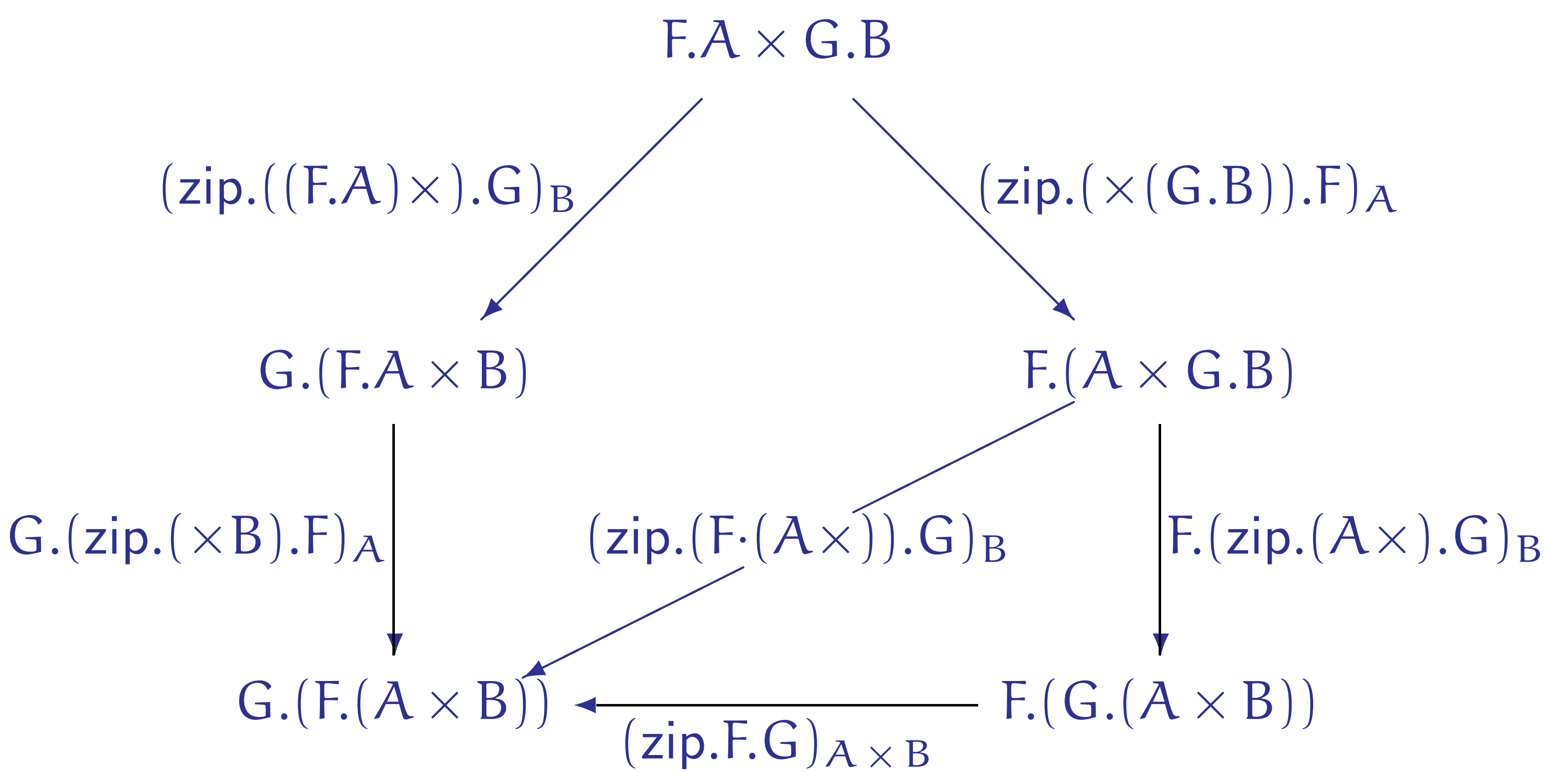
... Generalised ...



... Illustrates Generic Requirements



Multi-Coloured Zips



Broadcasts ...

A broadcast copies a given value across all storage locations of a datatype.

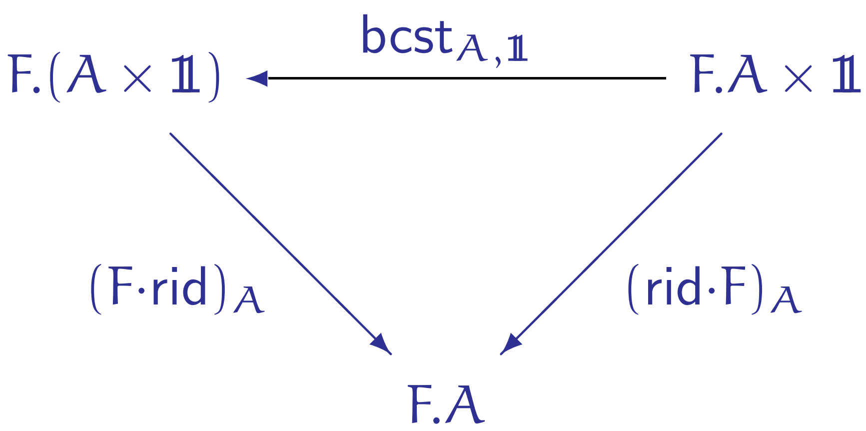
Formally, a family of functions bcst, where

$$\mathsf{bcst}_{A,B} \ : \ F.(A \times B) \leftarrow F.A \times B$$

is said to be a *broadcast* for datatype F iff it is parametrically polymorphic in the parameters A and B and $\mathsf{bcst}_{A,B}$ behaves coherently with respect to product in the following sense:

... Respect the Unit of Product ...

The following diagram



(where $\mathsf{rid}_A \ : \ A \leftarrow A \times \mathbb{1}$ is the obvious natural isomorphism)
commutes.

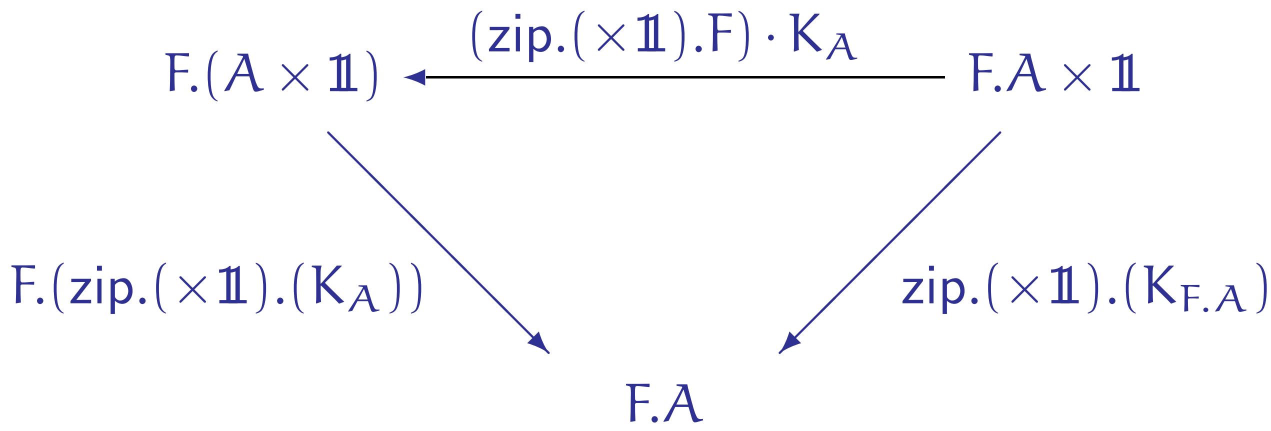
… and Associativity of Product

The following diagram

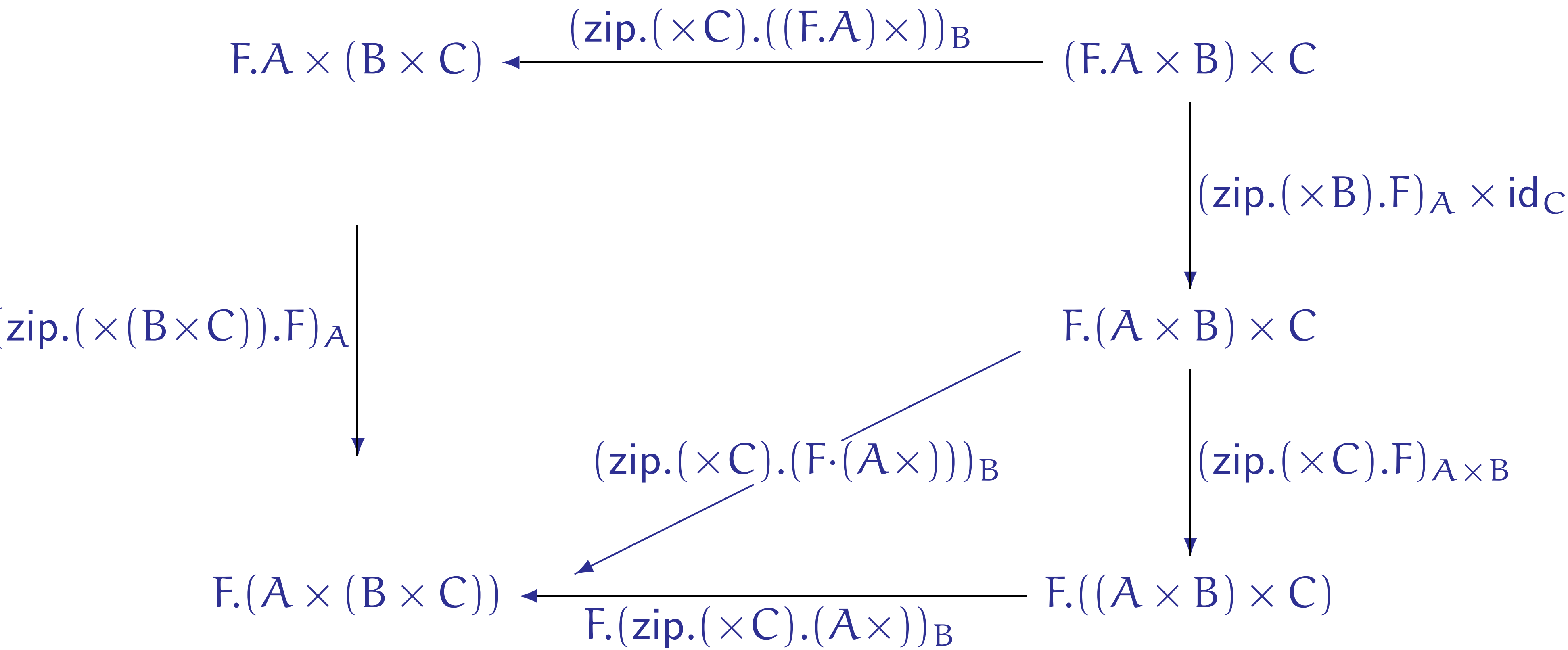
$$
\begin{array}{ccc}
F.A \times (B \times C) & \xleftarrow{\ \mathsf{ass}_{F.A,B,C}\ } & (F.A \times B) \times C \\
 & & \Big\downarrow \mathsf{bcst}_{A,B} \times \mathsf{id}_C \\
\mathsf{bcst}_{A\,,\,B\times C} \Big\downarrow & & F.(A \times B) \times C \\
 & & \Big\downarrow \mathsf{bcst}_{A\times B\,,\,C} \\
F.(A \times (B \times C)) & \xleftarrow[\ F\cdot\mathsf{ass}_{A,B,C}\ ]{} & F.((A \times B) \times C)
\end{array}
$$

(where $\mathsf{ass}_{A,B,C} : A \times (B \times C) \leftarrow (A \times B) \times C$ is the obvious natural isomorphism) commutes as well.

Unit of Product is a "zip"



Associativity of Product is a "Zip"



Conclusion

- Commuting Datatypes ("Zips") are everywhere!
- Generic specification and proof is (potentially) very effective.
- A relational framework is necessary.
- Challenge: give generic specification of "commuting datatypes" from which "zips" can be constructed calculationally.

Relators, Fans and Membership

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Allegories

Categorical formulation of (point-free) relation algebra.

Category (objects A, B, C, arrows —"relations"— R, S)

$$\mathsf{R}\circ\mathsf{S} : A \leftarrow \mathsf{B} \quad \Leftarrow \quad \mathsf{R} : A \leftarrow \mathsf{C} \wedge \mathsf{S} : \mathsf{C} \leftarrow \mathsf{B} \quad ,$$

$$\mathsf{id}_A : A \leftarrow A \quad .$$

Arrows of same type are partially ordered by $\subseteq$.

$$\mathsf{S}_1\circ\mathsf{T}_1 \subseteq \mathsf{S}_2\circ\mathsf{T}_2 \quad \Leftarrow \quad \mathsf{S}_1 \subseteq \mathsf{S}_2 \wedge \mathsf{T}_1 \subseteq \mathsf{T}_2 \quad .$$

$$\mathsf{X} \subseteq \mathsf{R} \wedge \mathsf{X} \subseteq \mathsf{S} \quad \equiv \quad \mathsf{X} \subseteq \mathsf{R}\cap\mathsf{S} \quad .$$

Converse

$$\mathsf{R}^{\cup} \subseteq \mathsf{S} \quad \equiv \quad \mathsf{R} \subseteq \mathsf{S}^{\cup} \quad ,$$

$$(\mathsf{R}\circ\mathsf{S})^{\cup} = \mathsf{S}^{\cup}\circ\mathsf{R}^{\cup} \quad ,$$

$$\mathsf{R}\circ\mathsf{S} \cap \mathsf{T} \subseteq (\mathsf{R} \cap \mathsf{T}\circ\mathsf{S}^{\cup})\circ\mathsf{S} \quad .$$

Relator

Relator: functor that is monotonic and respects converse.

Let $\mathcal{A}$ and $\mathcal{B}$ be allegories. A mapping F from objects of $\mathcal{A}$ to objects of $\mathcal{B}$ and arrows of $\mathcal{A}$ to arrows of $\mathcal{B}$ is a relator iff

$$\mathsf{F.R} : \mathsf{F.}A \leftarrow \mathsf{F.B} \quad \Leftarrow \quad \mathsf{R} : A \leftarrow \mathsf{B} \quad ,$$

$$\mathsf{F.R} \circ \mathsf{F.S} = \mathsf{F.(R \circ S)} \quad \text{for each } \mathsf{R} : A \leftarrow \mathsf{B} \text{ and } \mathsf{S} : \mathsf{B} \leftarrow \mathsf{C} \quad ,$$

$$\mathsf{F.id}_A = \mathsf{id}_{\mathsf{F.}A} \quad \text{for each object } A \quad ,$$

$$\mathsf{F.R} \subseteq \mathsf{F.S} \quad \Leftarrow \quad \mathsf{R} \subseteq \mathsf{S} \quad \text{for each } \mathsf{R} : A \leftarrow \mathsf{B} \text{ and } \mathsf{S} : A \leftarrow \mathsf{B} \quad ,$$

$$(\mathsf{F.R})^{\cup} = \mathsf{F.}(\mathsf{R}^{\cup}) \quad \text{for each } \mathsf{R} : A \leftarrow \mathsf{B} \quad .$$

Examples: List is an endorelator. $\times$ is a binary relator.

Functions

Relation $R : A \leftarrow B$ is *total* iff

$$\mathsf{id}_B \subseteq R^{\cup} \circ R \quad ,$$

and relation R is single-valued or *simple* iff

$$R \circ R^{\cup} \subseteq \mathsf{id}_A \quad .$$

A function is a relation that is total and simple.

Relators preserve totality

$$(F.R)^{\cup} \circ F.R$$
$$= \quad \{ \text{ relators respect converse } \}$$
$$F.(R^{\cup}) \circ F.R$$
$$= \quad \{ \text{ relators distribute through composition } \}$$
$$F.(R^{\cup} \circ R)$$
$$\supseteq \quad \{ \text{ assume } id_B \subseteq R^{\cup} \circ R \text{, relators are monotonic } \}$$
$$F.id_B$$
$$= \quad \{ \text{ relators preserve identities } \}$$
$$id_{F.B} \quad .$$

Similarly, relators preserve simplicity. Hence relators preserve functions.

Parametricity — point-free

Recall

$$(f,g) \in R \leftarrow S \quad \equiv \quad \langle \forall c,d :: (f.c\,,\,g.d) \in R \Leftarrow (c,d) \in S \rangle \quad .$$

Point-free:

$$(f,g) \in R \leftarrow S \;\equiv\; f^{\cup} \circ R \circ g \supseteq S \quad .$$

Equivalently, using *shunting* rule:

$$(f,g) \in R \leftarrow S \;\equiv\; R \circ g \supseteq f \circ S \quad .$$

Relators are Parametric

Type:

$$F.R : F.A \leftarrow F.B \quad \Leftarrow \quad R : A \leftarrow B \quad .$$

That is,

$$F \;:\; \langle \forall \alpha,\beta :: (F.\alpha \leftarrow F.\beta) \leftarrow (\alpha \leftarrow \beta) \rangle \quad .$$

F is parametric iff, for all relations R and S, and all functions f and g,

$$(F.f\,,\,F.g) \in F.R \leftarrow F.S \quad \Leftarrow \quad (f,g) \in R \leftarrow S \quad .$$

Exercise: verify that this is the case using point-free definition of $R \leftarrow S$.

Natural Transformations

Parametricity of reverse function, rev, on lists, and of fork:

$$\mathsf{List.R} \circ \mathsf{rev}_B \supseteq \mathsf{rev}_A \circ \mathsf{List.R}$$

$$\mathsf{R}{\times}\mathsf{R} \circ \mathsf{fork}_B \supseteq \mathsf{fork}_A \circ \mathsf{R}$$

In fact,

$$\mathsf{List.R} \circ \mathsf{rev}_B = \mathsf{rev}_A \circ \mathsf{List.R} \quad .$$

But, it is *not* the case that, for all R,

$$\mathsf{R}{\times}\mathsf{R} \circ \mathsf{fork}_B = \mathsf{fork}_A \circ \mathsf{R} \quad .$$

For example,

$$\{(0,0),(1,0)\} \times \{(0,0),(1,0)\} \circ \mathsf{fork}_B \neq \mathsf{fork}_A \circ \{(0,0),(1,0)\} \quad .$$

fork is a (lax) *natural transformation*, rev is a *proper* natural transformation.

Natural Transformations

$$\theta : F \hookleftarrow G \quad = \quad F.R \circ \theta_B \supseteq \theta_A \circ G.R \quad \text{for each } R : A \leftarrow B$$

$$\theta : F \hookrightarrow G \quad = \quad F.R \circ \theta_B \subseteq \theta_A \circ G.R \quad \text{for each } R : A \leftarrow B \quad .$$

Facts:

$$(F.f \circ \theta_B = \theta_A \circ G.f \quad \text{for each function } f : A \leftarrow B) \;\Leftarrow\; \theta : F \hookleftarrow G \quad .$$

In a “tabular allegory”,

$$\theta : F \hookleftarrow G \;\Leftarrow\; (F.f \circ \theta_B = \theta_A \circ G.f \quad \text{for each function } f : A \leftarrow B) \quad .$$

In words, $\theta : F \hookleftarrow G$ iff θ is a (categorical) natural transformation in the underlying category of maps.

Conclusion: we take $\theta : F \hookleftarrow G$ to be the definition of a *natural transformation* in an allegory.

Division

An allegory is *locally complete* if for each set $\mathcal{S}$ of relations of type $A \leftarrow B$, the union $\cup\mathcal{S} : A \leftarrow B$ exists and, furthermore, intersection and composition distribute over arbitrary unions.

$\perp\!\!\!\perp_{A,B}$ is the smallest relation of type $A \leftarrow B$ and $\top\!\!\!\top_{A,B}$ is the largest relation of the same type.

In a *division* allegory, composition distributes through union. That is, there are two *division* operators "\" and "/", such that, for all $R : A \leftarrow B$, $S : B \leftarrow C$ and $T : A \leftarrow C$,

$$R \circ S \subseteq T \quad \equiv \quad S \subseteq R \backslash T \quad ,$$

$$R \circ S \subseteq T \quad \equiv \quad R \subseteq T / S \quad ,$$

$$S \subseteq R \backslash T \quad \equiv \quad R \subseteq T / S \quad .$$

Domain and Range

The *range* of a relation R is the set of all x such that $(x,y) \in R$ for some y.

Formally, the range operator "$<$" is defined by, for all $R : A \leftarrow B$ and all $X \subseteq \mathsf{id}_A$,

$$R< \subseteq X \;\; \equiv \;\; R \subseteq X \circ \top\!\top_{A,B} \;\; .$$

The *domain* $R>$ is defined by

$$R> \;\; = \;\; (R\cup)< \;\; .$$

Membership

The membership relation of a relator F is a family of relations mem_A, indexed by objects A, such that

$$\mathsf{mem}_A \ : \ A \leftarrow \mathsf{F}.A \quad , \text{ and}$$

for all A, all $\mathsf{X} \subseteq \mathsf{id}_A$ and $\mathsf{Y} \subseteq \mathsf{id}_{\mathsf{F}.A}$,

$$\mathsf{F}.\mathsf{X} \supseteq \mathsf{Y} \ \equiv \ (\mathsf{mem}_A \circ \mathsf{Y})_< \subseteq \mathsf{X} \quad .$$

In words, $\mathsf{F}.\mathsf{X}$ is the largest subset Y of F-structures, each of type $\mathsf{F}.A$, such that the data stored in elements is in the set X.

Weakest Liberal Precondition

For all $X \subseteq id_A$ and $Y \subseteq id_{F.A}$,

$$
\begin{array}{ll}
 & (mem_A \circ Y)^{<} \ \subseteq \ X \\
= & \{ \quad \text{definition of range} \quad \} \\
 & mem_A \circ Y \subseteq X \circ \top\!\top \\
= & \{ \quad \text{division} \quad \} \\
 & Y \subseteq mem_A \backslash (X \circ \top\!\top) \\
= & \{ \quad Y \subseteq id_{F.A} \quad \} \\
 & Y \subseteq mem_A \backslash (X \circ \top\!\top) \cap id_{F.A} \quad .
\end{array}
$$

For those familiar with the wp calculus: $mem_A \backslash (X \circ \top\!\top) \cap id_{F.A}$ is the weakest liberal precondition guaranteeing a state satisfying X after "execution" of mem.

Properties of F structures

For all A, all $\mathsf{X} \subseteq \mathsf{id}_A$ and $\mathsf{Y} \subseteq \mathsf{id}_{\mathsf{F}.A}$,

$$\mathsf{F.X} \supseteq \mathsf{Y} \quad \equiv \quad \mathsf{mem}_A \backslash (\mathsf{X} \circ \top\!\!\top) \cap \mathsf{id}_{\mathsf{F}.A} \;\supseteq\; \mathsf{Y} \quad .$$

So,

$$\mathsf{F.X} \;=\; \mathsf{mem}_A \backslash (\mathsf{X} \circ \top\!\!\top) \cap \mathsf{id}_{\mathsf{F}.A} \quad .$$

Interpreting $\mathsf{X} \subseteq \mathsf{id}_A$ as a property of values of type A, $\mathsf{F.X}$ is a property of values of type $\mathsf{F}.A$. The identity says that a property of an F-structure is characterised by properties of the values stored in the structure (its "members").

Largest Natural Transformations

Recall: for each object A,

$$\mathsf{mem}_A \;:\; A \leftarrow \mathsf{F}.A \quad .$$

Membership is parametric: for all R,

$$\mathsf{R} \circ \mathsf{mem} \;\supseteq\; \mathsf{mem} \circ \mathsf{F.R} \quad .$$

Equivalently,

$$\mathsf{mem} : \mathrm{Id} \hookleftarrow \mathsf{F} \quad .$$

Also,

$$\mathsf{mem}\backslash\mathsf{id} : \mathsf{F} \hookleftarrow \mathrm{Id} \quad .$$

Theorem: The fan of relator F, $\mathsf{mem}\backslash\mathsf{id}$, is the largest natural transformation of type $\mathsf{F} \hookleftarrow \mathrm{Id}$. The membership of relator F is the largest natural transformation of type $\mathrm{Id} \hookleftarrow \mathsf{F}$.

Understanding Natural Transformations

Theorem: Suppose F and G are relators with memberships $\mathsf{mem.F}$ and $\mathsf{mem.G}$ respectively. Then the largest natural transformation of type $\mathsf{F} \hookleftarrow \mathsf{G}$ is $\mathsf{mem.F} \backslash \mathsf{mem.G}$.

Interpretation: A natural transformation of type $\mathsf{F} \hookleftarrow \mathsf{G}$ changes structure only. Stored values may be lost or duplicated, but no computation is performed on them.

A *proper* natural transformation to F from G changes the structure without loss or duplication of stored values.

The Specification of a Generic Zip

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(Lower Order) Naturality

$$\mathsf{zip.F.G} \quad : \quad \mathsf{G{\bullet}F} \leftarrow \mathsf{F{\bullet}G} \quad .$$

A zip is a *proper* natural transformation.

A zip transforms one structure to another without loss or duplication of values.

(Higher Order) Naturality

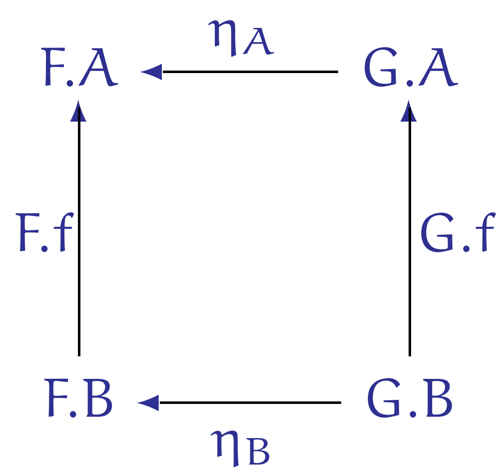
$$\mathsf{zip.F} \;:\; (\bullet\mathsf{F}) \longleftarrow (\mathsf{F}\bullet) \quad .$$

Categorical Nat Trans (Revision)

A natural transformation is an arrow in the functor category. I.e.,

$$\eta : F \leftarrow G$$

means that the following diagram commutes (for all A, B and $f : A \leftarrow B$)



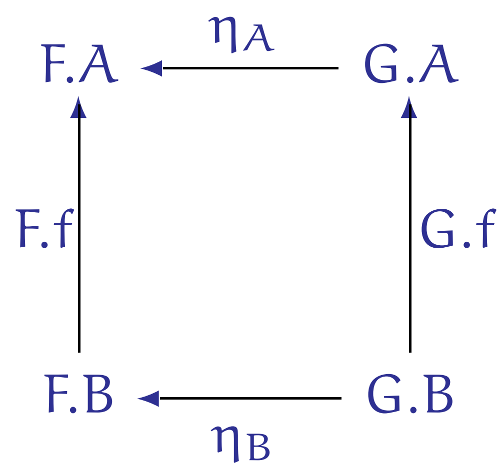
Now, if F is a functor, $(\bullet F)$ and $(F\bullet)$ are endofunctors on the functor category.

$(\bullet F)$ maps functor (object) G to $G\bullet F$ and natural transformation (arrow) η to $\eta\bullet F$, where $(\eta\bullet F)_A = \eta_{F.A}$.

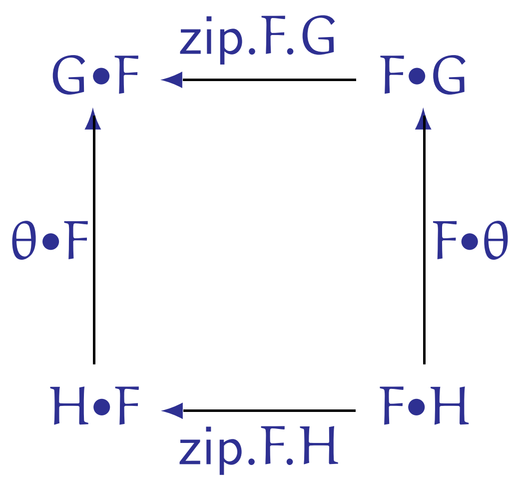
$(F\bullet)$ maps functor (object) G to $F\bullet G$ and natural transformation (arrow) η to $F\bullet\eta$, where $(F\bullet\eta)_A = F.(\eta_A)$.

Categorical NT Revision (Continued)

Diagram defining $\eta : F \leftarrow G$



instantiated for $\mathsf{zip}.F : (\bullet F) \leftarrow (F\bullet)$



where $\theta : G \leftarrow H$ is a natural transformation.

Allegorical Naturality

Recall that parametricity was defined in terms of *relations*.

Recall also that, in the particular case that t has type $\langle\forall\alpha :: F.\alpha \leftarrow G.\alpha\rangle$, t is parametric is equivalent to t is a natural transformation (in the underlying category of maps).

This is a stroke of luck for functional programmers, BUT their luck has run out!

The equality in

$$(\theta{\bullet}F) \circ zip.F.H \quad = \quad zip.F.G \circ (F{\bullet}\theta)$$

is too severe — because

- θ may be nondeterministic.
- Zips are partial.

Nondeterminism

Take $\mathsf{F} := \mathsf{List}$ and $\mathsf{G} = \mathsf{H} := \times$.

$\mathsf{zip.F.H}$ and $\mathsf{zip.F.G}$ are both the inverse of conventional zips. They unzip a list of pairs to a pair of lists.

Take $\theta := \mathsf{id} \cup \mathsf{swap}$.

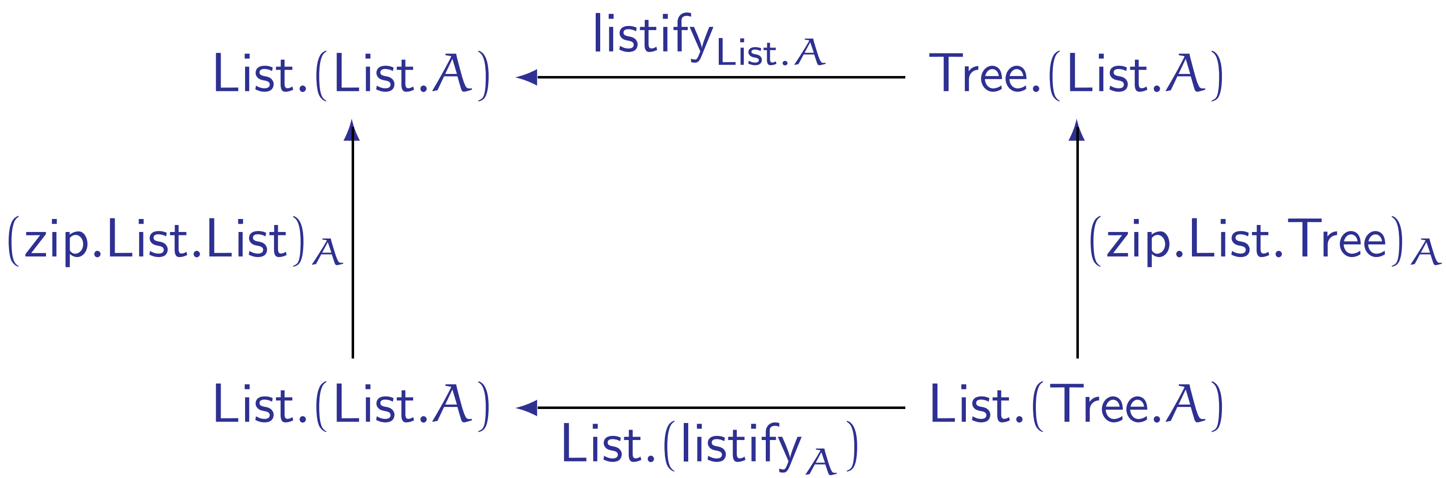
θ nondeterministically swaps the elements of a pair or not.

$(\theta \bullet \mathsf{F}) \circ \mathsf{zip.F.H}$ unzips a list of pairs into a pair of lists and swaps the lists or not.

$\mathsf{zip.F.G} \circ (\mathsf{F} \bullet \theta)$ first swaps some of the elements of a list of pairs and then unzips it into a pair of lists.

$$(\theta \bullet \mathsf{F}) \circ \mathsf{zip.F.H} \quad \subset \quad \mathsf{zip.F.G} \circ (\mathsf{F} \bullet \theta) \quad .$$

Partiality



View both paths through the diagram as partial relations of type List.(List.A) $\leftarrow$ List.(Tree.A).

The lower path (via List.(List.A)) includes the upper path (via Tree.(List.A)).

Reason: for the lower path, the sizes of the trees must be the same; for the upper path, the trees must have the same shape.

zip.F **is parametric.**

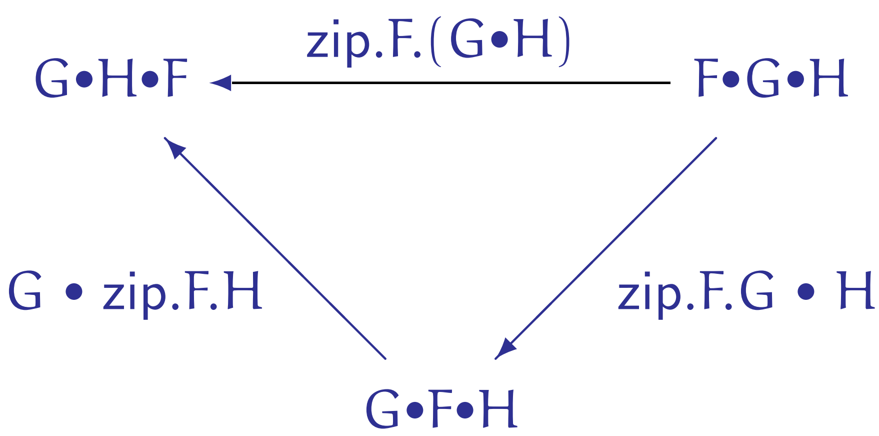
That is, for all $\theta : G \hookleftarrow H$,

$$(\theta \bullet F) \circ zip.F.H \quad \subseteq \quad zip.F.G \circ (F \bullet \theta) \quad .$$

Compositionality

Informally, zip.F is a monoid homomorphism.

(Note: more than this: zip.F should respect pointwise extension of relators. For full discussion see Hoogendijk's thesis.)



$$\mathsf{zip.F.(G{\bullet}H)} \quad = \quad (\mathsf{G} \bullet \mathsf{zip.F.H}) \circ (\mathsf{zip.F.G} \bullet \mathsf{H}) \quad .$$

$$\mathsf{zip.F.Id} \quad = \quad \mathsf{id{\bullet}F} \quad .$$

Zips

Definition 1 (Half Zip) Consider a fixed relator F and a pointwise closed class of relators $\mathcal{G}$. Then the members of the collection $\mathsf{zip.F.G}$, where G ranges over $\mathcal{G}$, are called *half-zips* iff
(a) $\mathsf{zip.F.G} : \mathsf{G} \bullet \mathsf{F} \leftarrow \mathsf{F} \bullet \mathsf{G}$, for each G in $\mathcal{G}$,
(b) $(\theta \bullet \mathsf{F}) \circ \mathsf{zip.F.H} \subseteq \mathsf{zip.F.G} \circ (\mathsf{F} \bullet \theta)$ for each $\theta : \mathsf{G} \hookleftarrow \mathsf{H}$,
(c) $\mathsf{zip.F.(G \bullet H)} = (\mathsf{G} \bullet \mathsf{zip.F.H}) \circ (\mathsf{zip.F.G} \bullet \mathsf{H})$ for all G and H,
(d) $\mathsf{zip.F.Id} = \mathsf{id} \bullet \mathsf{F}$.
□

Definition 2 (Commuting Relators) The half-zip $\mathsf{zip.F.G}$ is said to be a *zip* of (F, G) if there exists a half-zip $\mathsf{zip.G.F}$ such that

$$\mathsf{zip.F.G} = (\mathsf{zip.G.F})^{\cup}$$

We say that datatypes F and G *commute* if there exists a zip for (F, G).
□

Constructing Zips

See Hoogendijk's thesis for how these are calculated:

$$
\begin{aligned}
\mathsf{zip}.\mathsf{K}_A.\mathsf{G} &= \mathsf{fan}.\mathsf{G} \bullet \mathsf{K}_A \quad ,\\
\mathsf{zip}.{+}.\mathsf{G} &= \mathsf{G}.\mathsf{inl} \triangledown \mathsf{G}.\mathsf{inr} \quad ,\\
\mathsf{zip}.{\times}.\mathsf{G} &= (\mathsf{G}.\mathsf{outl} \vartriangle \mathsf{G}.\mathsf{outr})^{\cup} \quad ,\\
\mathsf{zip}.\mathsf{T}.\mathsf{G} &= (\!| \mathsf{id}_{\mathsf{G}} \otimes ;\ \mathsf{G}.\mathsf{in} \circ (\mathsf{zip}.\otimes.\mathsf{G} \bullet \mathsf{Id}\,\Delta\,\mathsf{T}) |\!) \quad .
\end{aligned}
$$

where T is the tree relator with pattern relator $\otimes$.

$$
\begin{aligned}
\mathsf{fan}.\mathsf{K}_A &= \top\!\!\top_{A,\_}\\
\mathsf{fan}.{+} &= (\mathsf{id} \triangledown \mathsf{id})^{\cup}\\
\mathsf{fan}.{\times} &= \mathsf{id} \vartriangle \mathsf{id}\\
\mathsf{fan}.\mathsf{T} &= (\!| \mathsf{id} \otimes ;\ (\mathsf{fan}.\otimes)^{\cup} |\!)^{\cup}
\end{aligned}
$$

where T is the tree relator with pattern relator $\otimes$.