Cubical Type Theory

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Synthetic geometry

Euclid's postulates

1. **To draw a straight line from any point to any point.**
2. **To produce a finite straight line continuously in a straight line.**
3. **To describe a circle with any center and distance.**
4. **That all right angles are equal to one another.**
5. **Given a line and a point not on it, there is exactly one line through the point that does not intersect the line**

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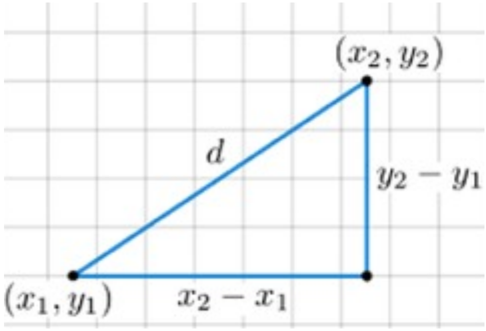
Cartesian



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models

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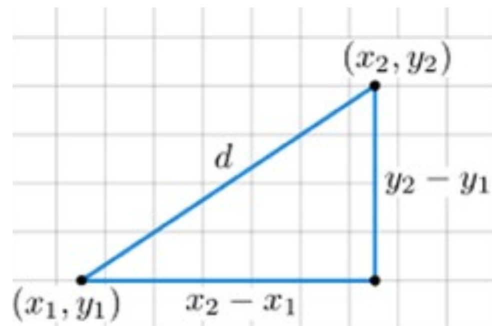

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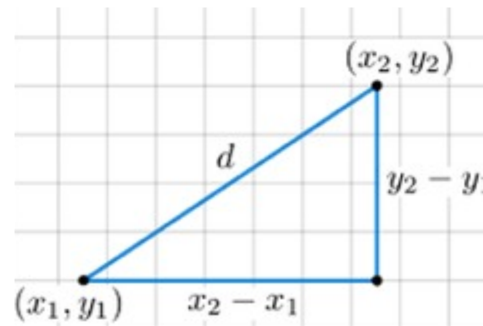

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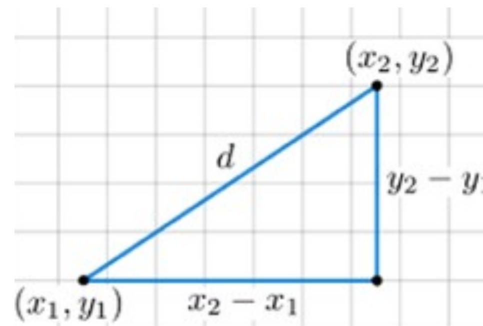

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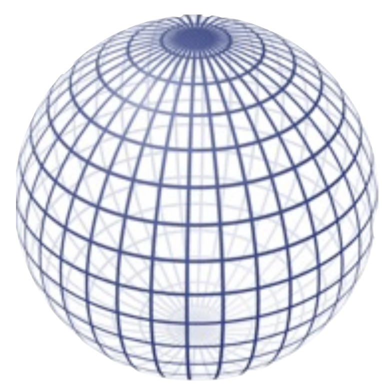

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2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. Two distinct lines meet at two antipodal points.

models

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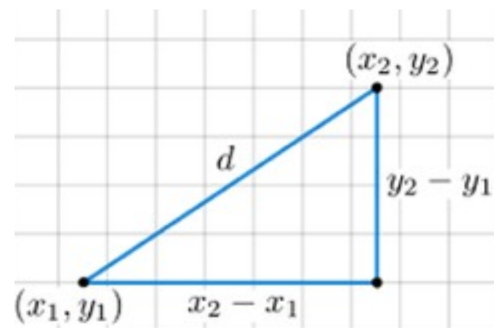

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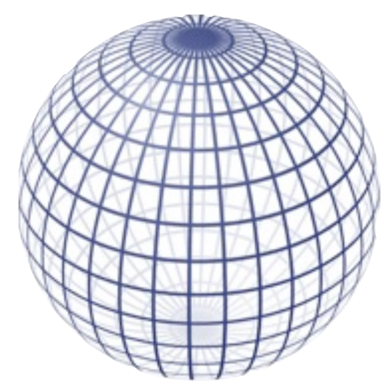
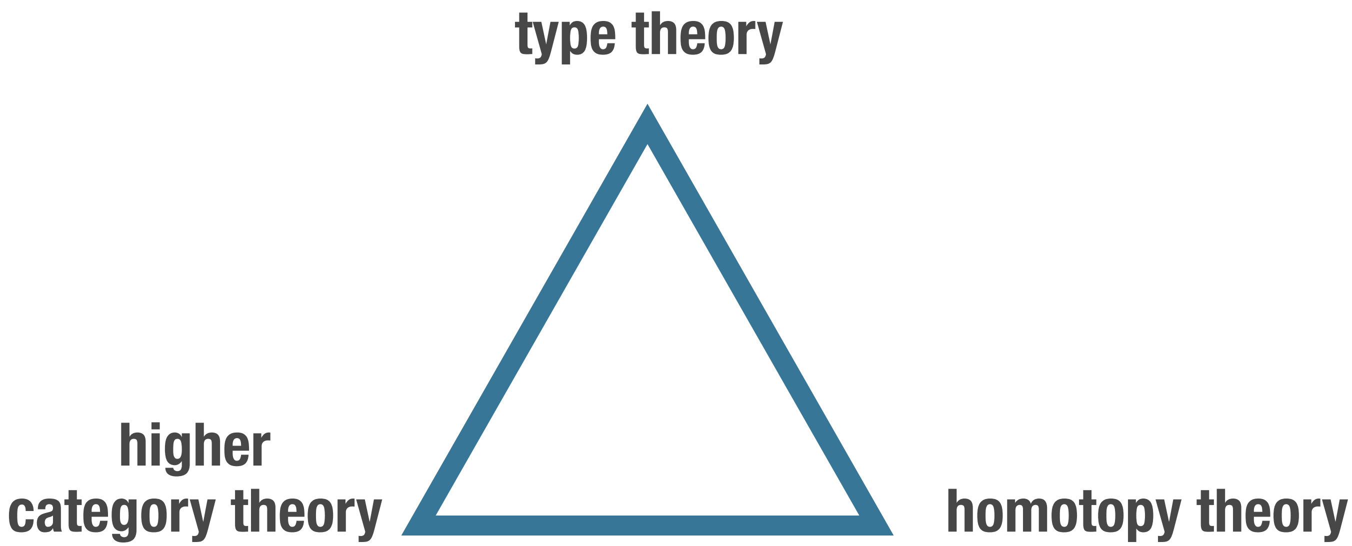


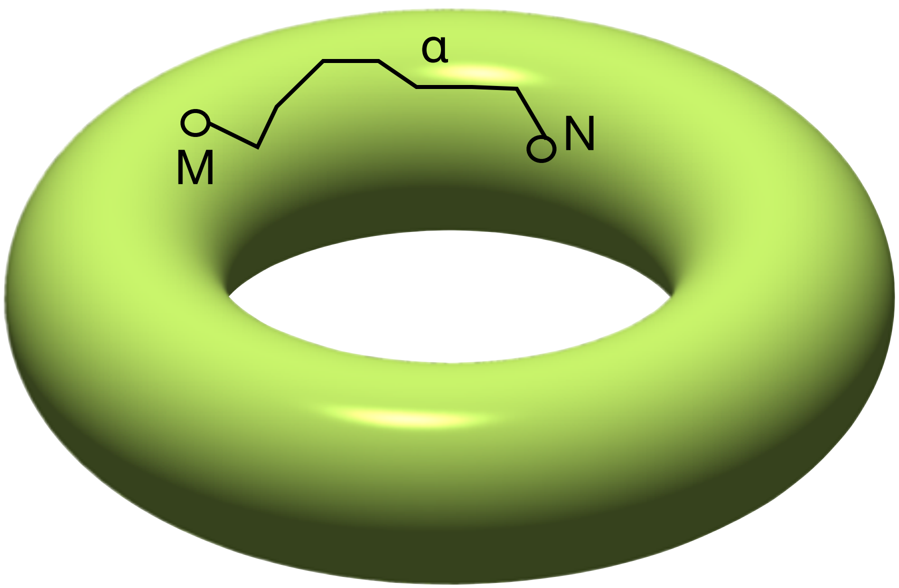
Homotopy type theory



[Awodey,Warren,Voevodsky,Streicher,Hofmann
Lumsdaine,Gambino,Garner,van den Berg]

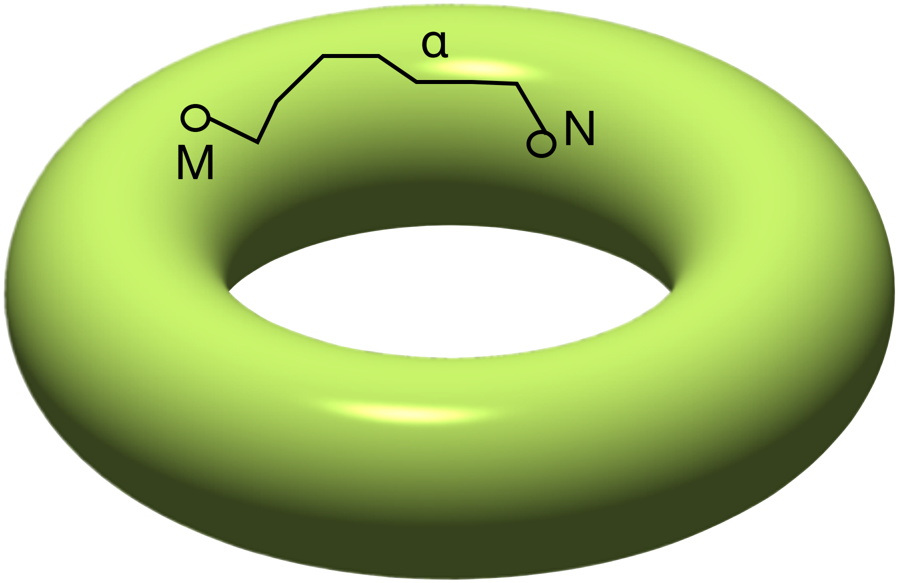
Homotopy type theory is a synthetic theory of spaces

Types as spaces



Types as spaces

type A is a space



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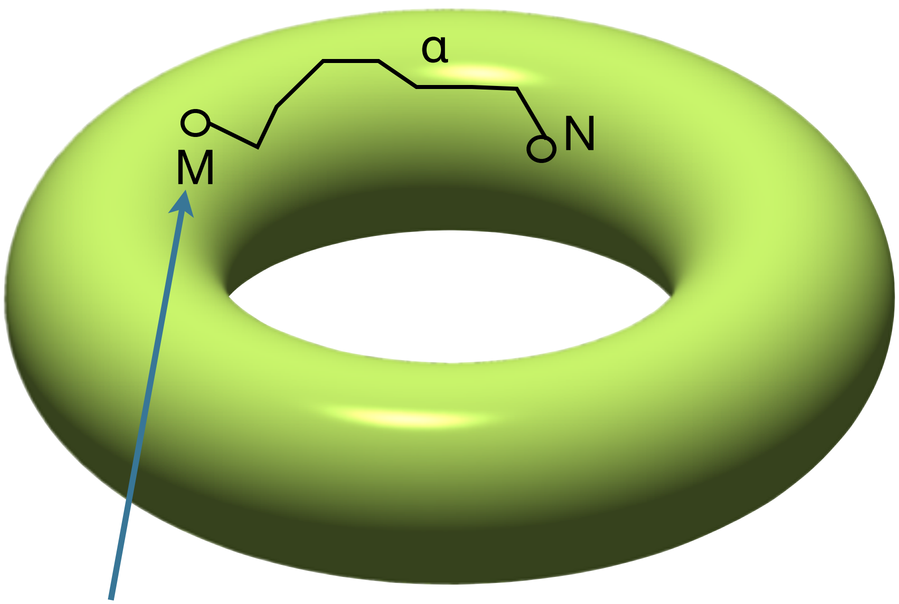

programs

M : A

are points

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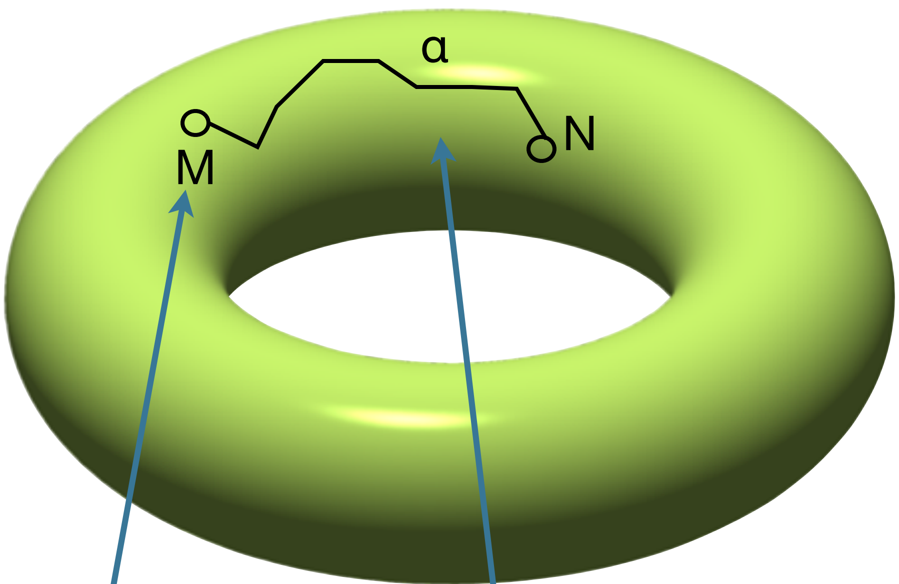

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$M : A$
are points

proofs of equality
$\alpha : M =_A N$
are paths

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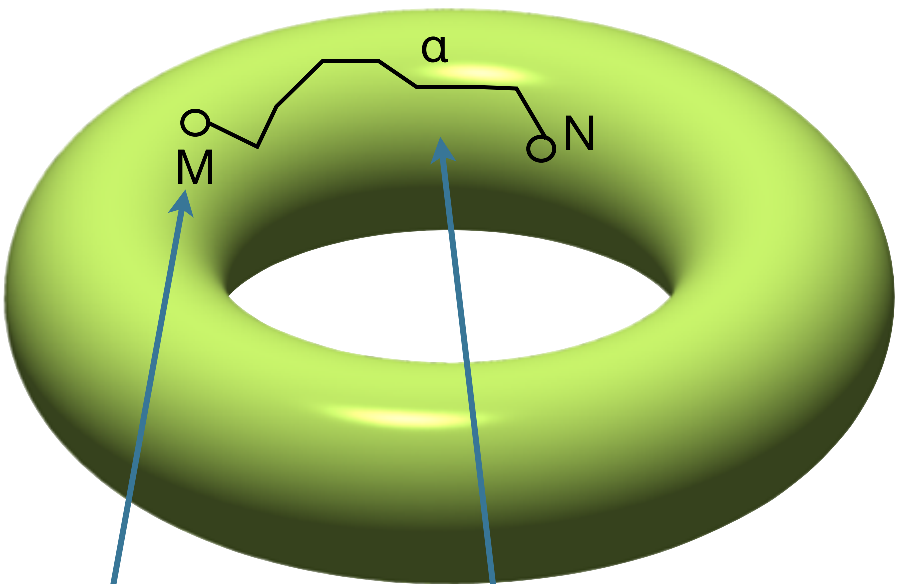

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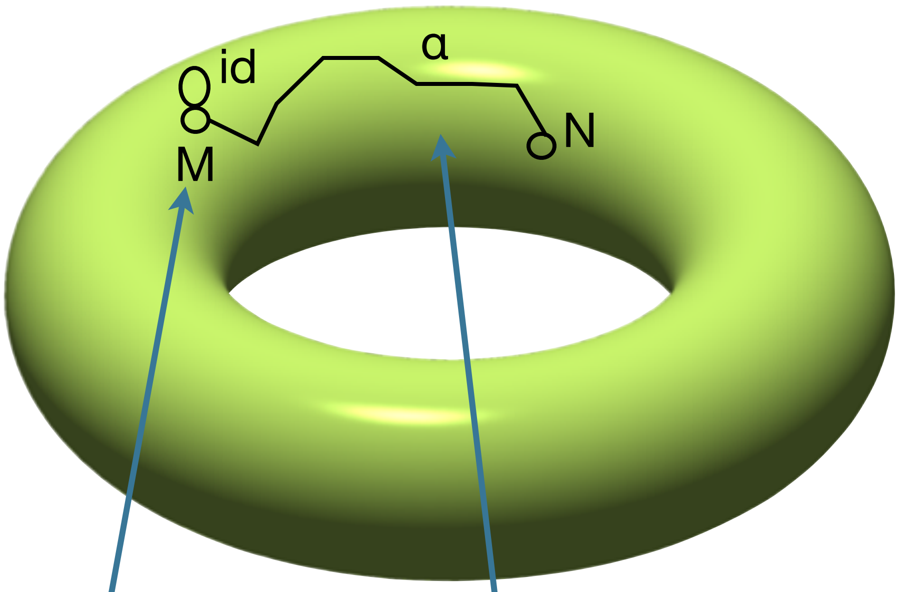

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path operations

```
id     : M = M (refl)
```

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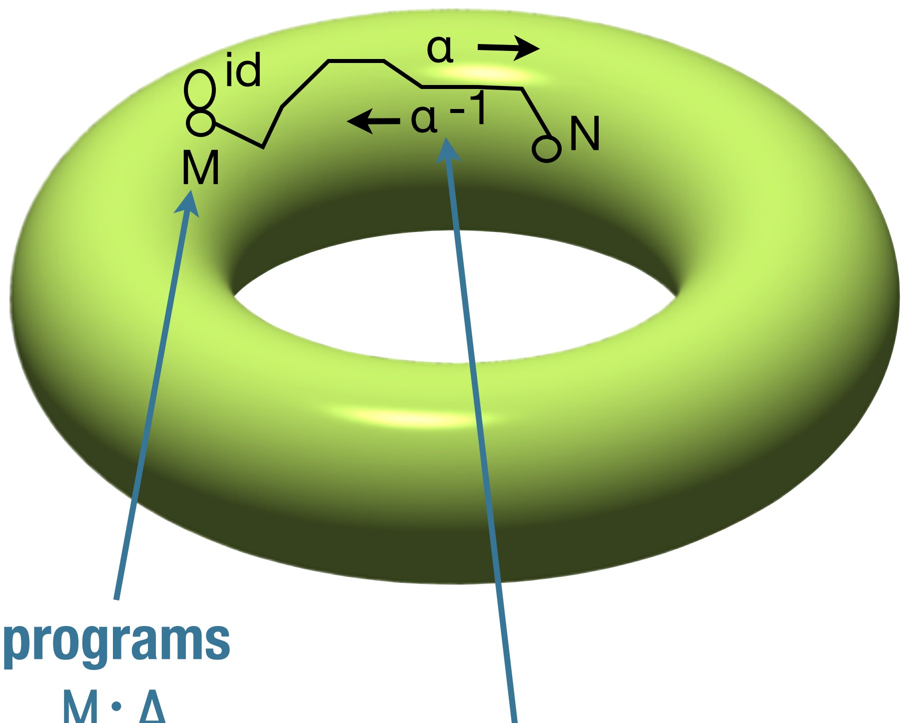

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α^{-1} : N = M (sym)

Types as spaces

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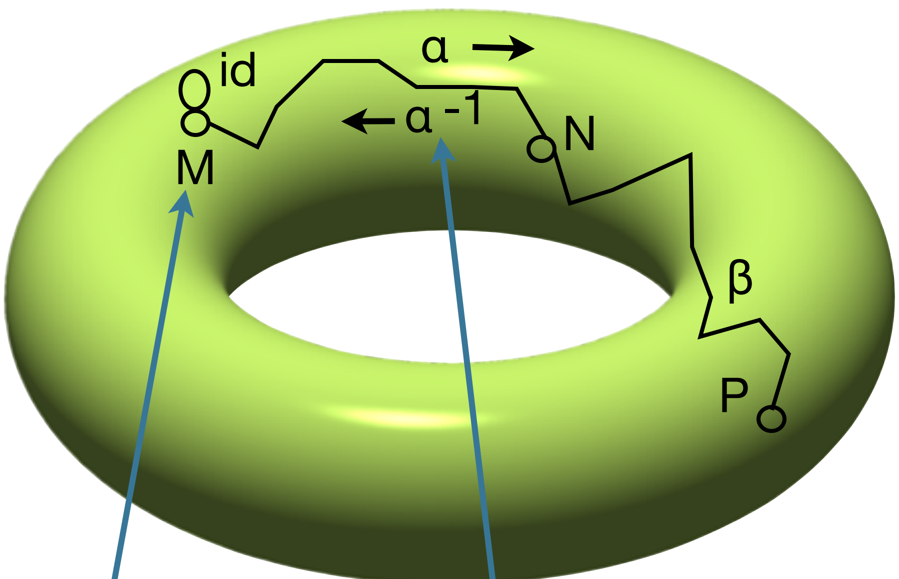

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β o α : M = P (trans)

Homotopy

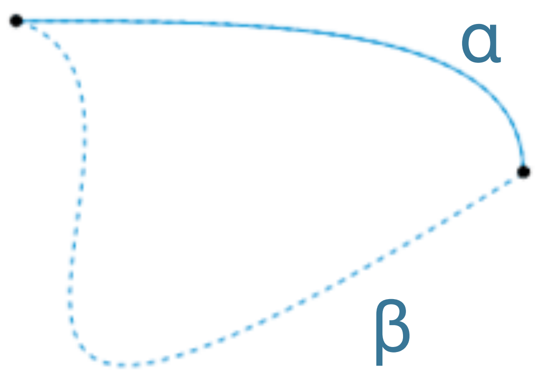
Deformation of one path into another

α

β

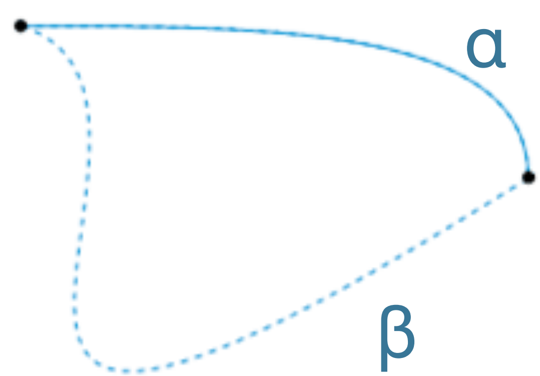
Homotopy

Deformation of one path into another



Homotopy

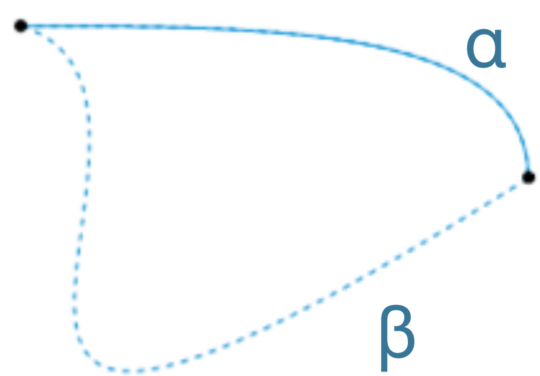
Deformation of one path into another



= 2-dimensional *path between paths*

Homotopy

Deformation of one path into another



$$\delta : \alpha =_{x=y} \beta$$

= 2-dimensional *path between paths*

Types as spaces

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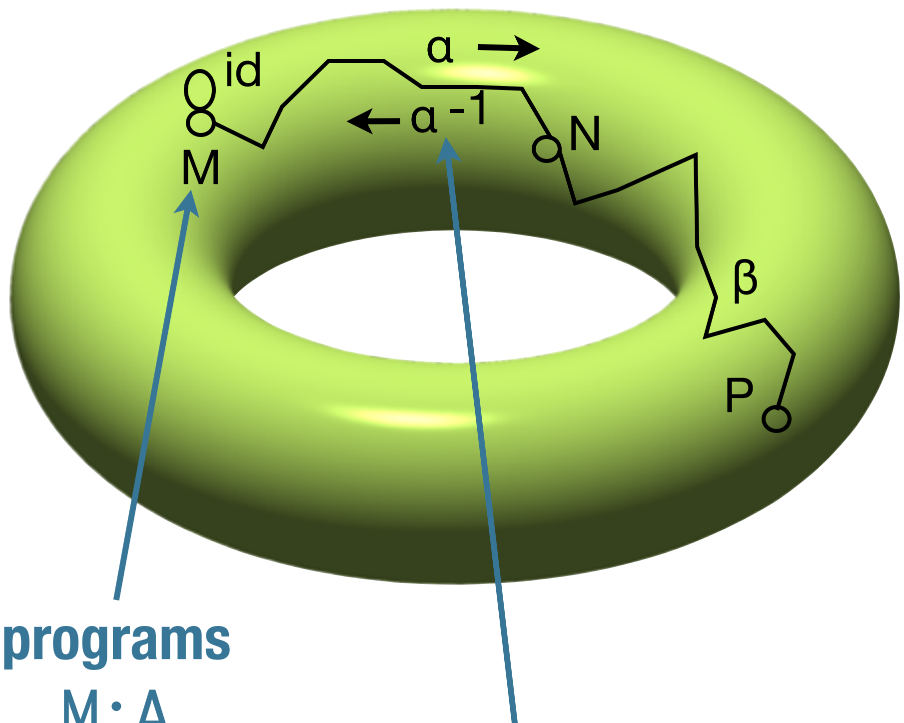

programs

$M : A$

are points

proofs of equality

$\alpha \ : \ M =_A N$

are paths

path operations

$id \quad : M = M$ (refl)

$\alpha^{-1} \quad : N = M$ (sym)

$\beta \circ \alpha : M = P$ (trans)

homotopies

$ul : id \circ \alpha =_{M=N} \alpha$

$il : \alpha^{-1} \circ \alpha =_{M=M} id$

$asc : \gamma \circ (\beta \circ \alpha)$

$\qquad =_{M=P} (\gamma \circ \beta) \circ \alpha$

Equality type

$$x : A$$

$$p : x =_A y$$

$$? : p_1 =_{x=y} p_2$$

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Uniqueness of Identity Proofs (UIP)

```
Definition UIP_ :=
  forall (x y:U) (p1 p2:x = y), p1 = p2.
```

Equality type

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Uniqueness of Identity Proofs (UIP)

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Definition UIP_ :=
  forall (x y:U) (p1 p2:x = y), p1 = p2.
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Proof-relevant equality

$x : A$

$p : x =_A y$

$q : p_1 =_{x=y} p_2$

Proof-relevant equality

$x : A$

$p : x =_A y$

$q : p_1 =_{x=y} p_2$

$q_1 =_{p1=p2} q_2$

Proof-relevant equality

$x : A$

$p : x =_A y$

$q : p_1 =_{x=y} p_2$

$r : q_1 =_{p1=p2} q_2$

Proof-relevant equality

$$x : A$$

$$p : x =_A y$$

$$q : p_1 =_{x=y} p_2$$

$$r : q_1 =_{p1=p2} q_2$$

$$\vdots$$

Homotopy groups of spheres

k[th] homotopy group

n-dimensional sphere

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^3	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^4	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{12}$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{24}\times\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{120}\times\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^5$
S^5	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{30}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{72}\times\mathbf{Z}_2$
S^6	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{24}\times\mathbf{Z}_2$	$\mathbf{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{120}$	$\mathbf{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{120}$

[image from wikipedia]

Univalence [Voevodsky]

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- *Equivalence of types* is a generalization to spaces of bijection of sets

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- *Equivalence of types* is a generalization to spaces of bijection of sets
- Univalence axiom, roughly:
  all structures/properties respect equivalence

Univalence

- Transporting along an equality is a generic program that lifts equivalences
- Can do parametricity-like reasoning about modules
- Provides “right” equality for mathematical structures (groups, categories, …)

Higher inductive types

[Bauer,Lumsdaine,Shulman,Warren]

New way of forming types:

Inductive type specified by generators
not only for points (elements), but also for paths

Higher inductive types

- Subsume quotient types, which have been problematic in intensional type theory
- Direct constructive definitions of spaces and other mathematical concepts
- Some nascent programming applications

Homotopy in HoTT

$\pi_1(S^1) = \mathbb{Z}$

$\pi_{k<n}(S^n) = 0$

Hopf fibration

$\pi_2(S^2) = \mathbb{Z}$

$\pi_3(S^2) = \mathbb{Z}$

James Construction

$\pi_4(S^3) = \mathbb{Z}_?$

Freudenthal

$\pi_n(S^n) = \mathbb{Z}$

K(G,n)

Blakers-Massey

Cohomology axioms

Van Kampen

Covering spaces

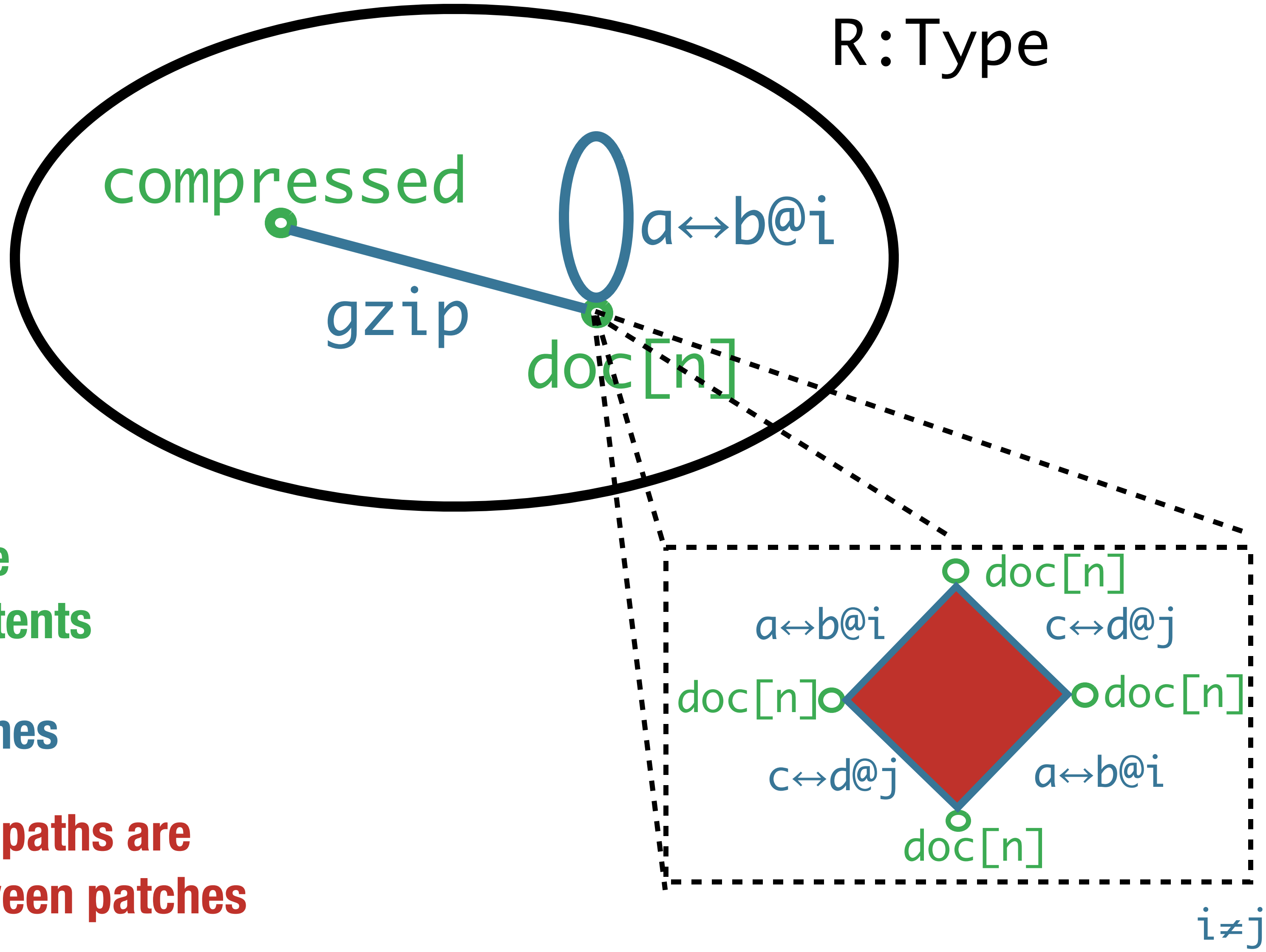
Whitehead for n-types

Mayer-Vietoris

$T^2 = S^1 \times S^1$

[Brunerie, Cavallo, Finster, Hou, Licata, Lumsdaine, Shulman]

A patch theory as a HIT



points describe repository contents

paths are patches

paths between paths are equations between patches

Computation

[Coquand,Huber,Bezem,Barras,
Licata,Harper,Brunerie,Shulman,
Altenkirch,Kaposi,Polansky…]

- Bezem,Coquand,Huber, 2013 gave a constructive model of type theory in Kan cubical sets; evaluator based on this
- This work: a syntactic type theory based on these ideas

Everything Respects Equivalence

Respect Equivalence

$$\frac{\alpha : A \simeq B \qquad \alpha' : A' \simeq B'}{\alpha \times \alpha' : A \times A' \simeq B \times B'}$$

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$$\alpha\times\alpha'(a{:}A, a'{:}A') = (\alpha\ a,\ \alpha'\ a')$$

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$$\alpha\times\alpha'(a{:}A, a'{:}A') = (\alpha\ a,\ \alpha'\ a')$$

$$(\alpha\times\alpha')^{-1}(b{:}B, b'{:}B') = (\alpha^{-1}\ b,\ \alpha'^{-1}\ b')$$

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$$(\alpha{\to}\alpha')^{-1}(g{:}B \to B') = \alpha'^{-1} \,.\, g \,.\, \alpha$$

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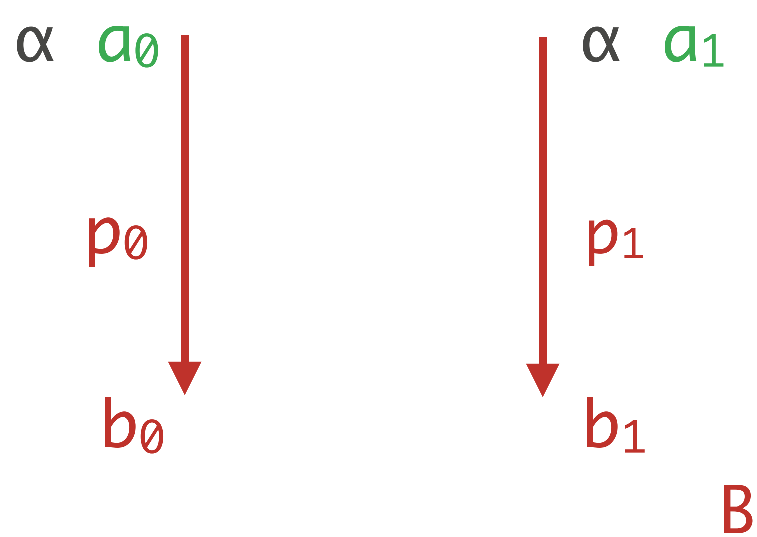

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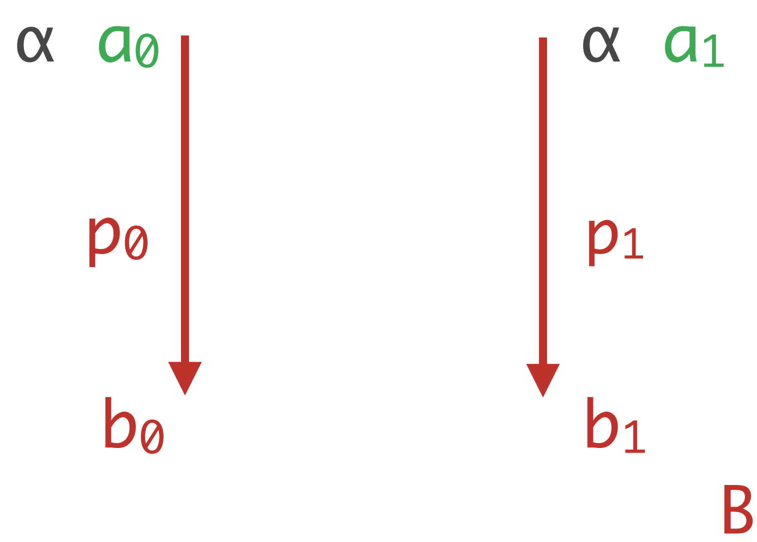

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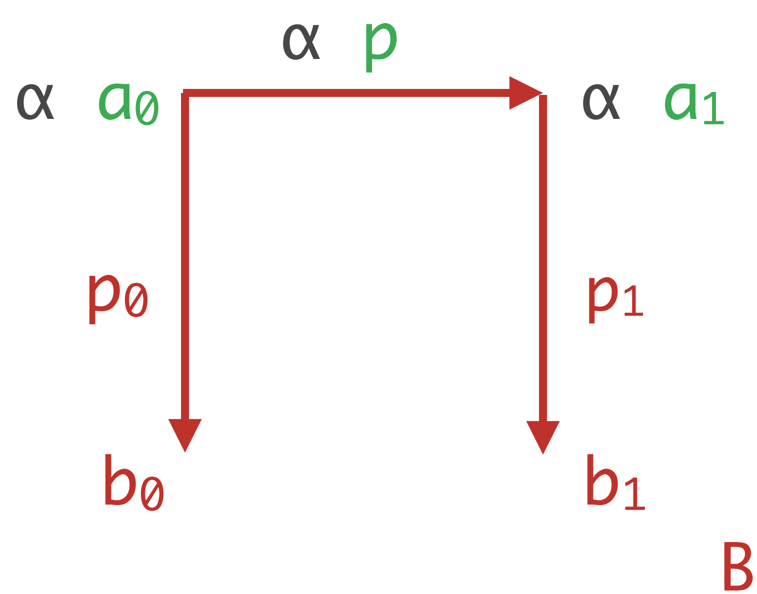

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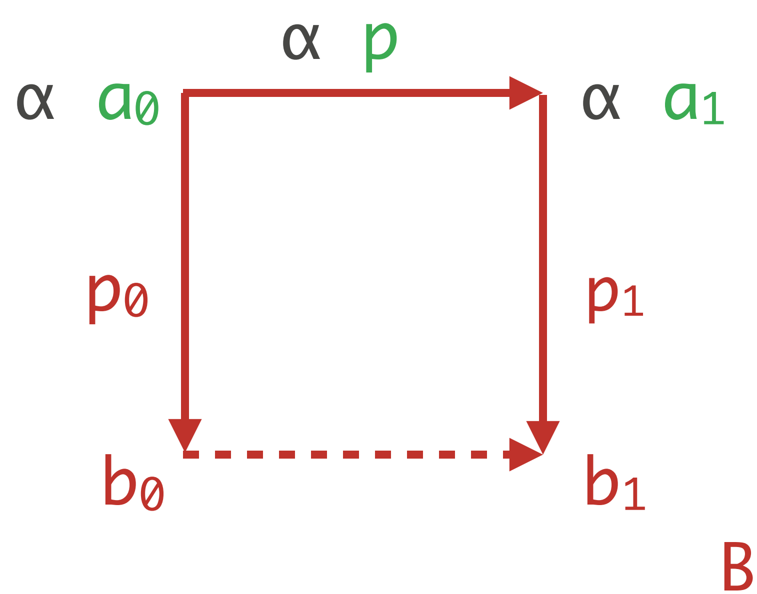

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$$p_0 : a_0 =_A \alpha^{-1}\ b_0$$

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$$p : a_0 =_A a_1 \qquad ▯$$

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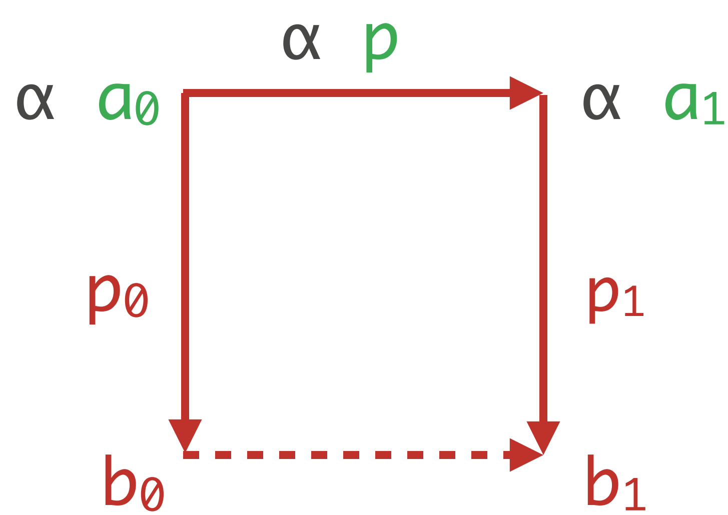

$! \ p_0 ; \alpha\ p ; p_1 \qquad B$

Missing Sides

$$\frac{p : x =_{A} y \qquad p' : x' =_{A'} y'}{(p,p') : (x,x') =_{A \times A'} (y,y')}$$

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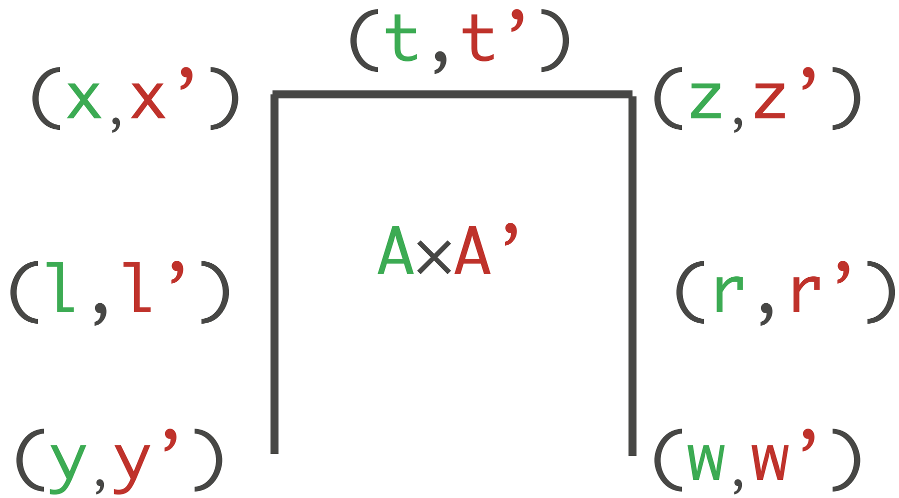
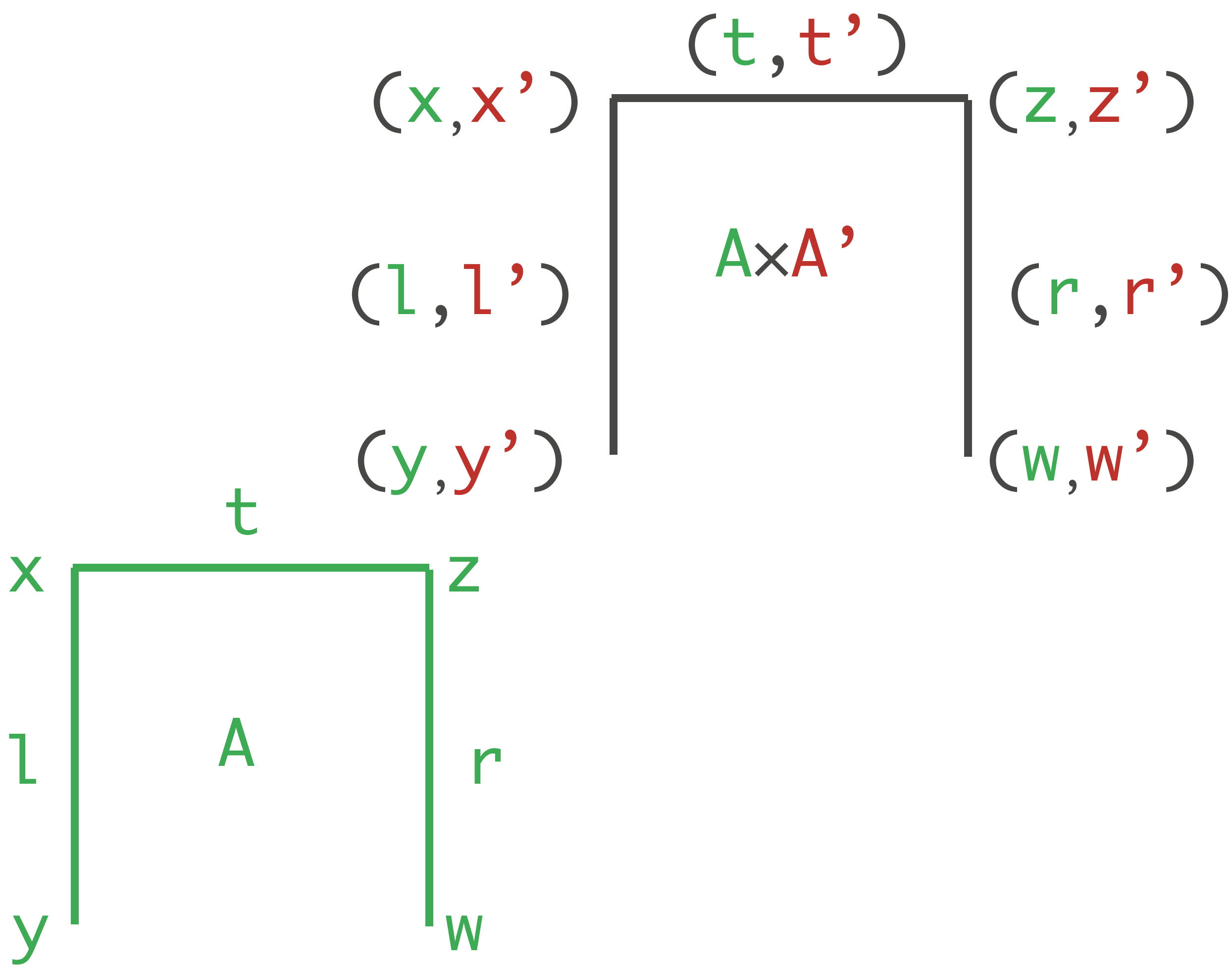
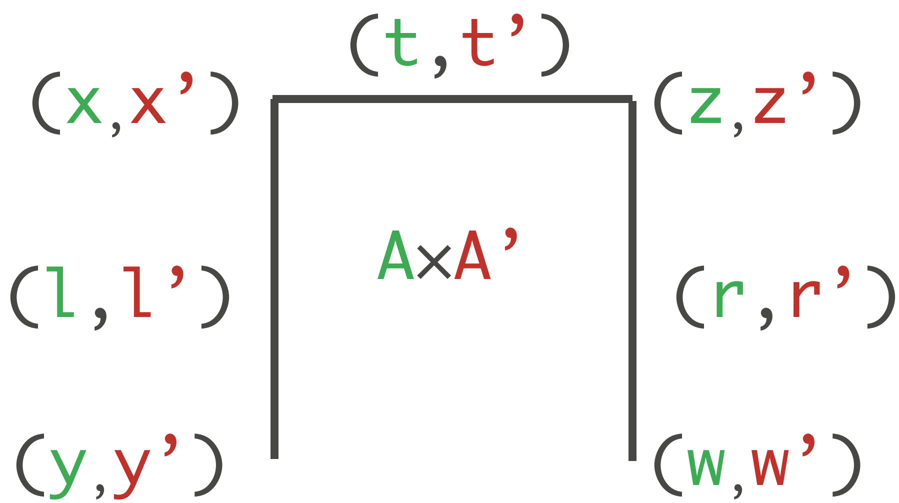
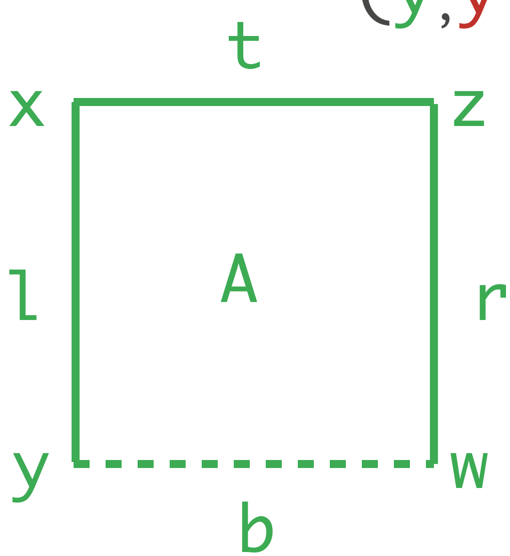


$$\frac{p : x =_{A} y \qquad p' : x' =_{A'} y'}{(p,p') : (x,x') =_{A \times A'} (y,y')}$$



$$\frac{p : x =_A y \qquad p' : x' =_{A'} y'}{(p,p') : (x,x') =_{A\times A'} (y,y')}$$





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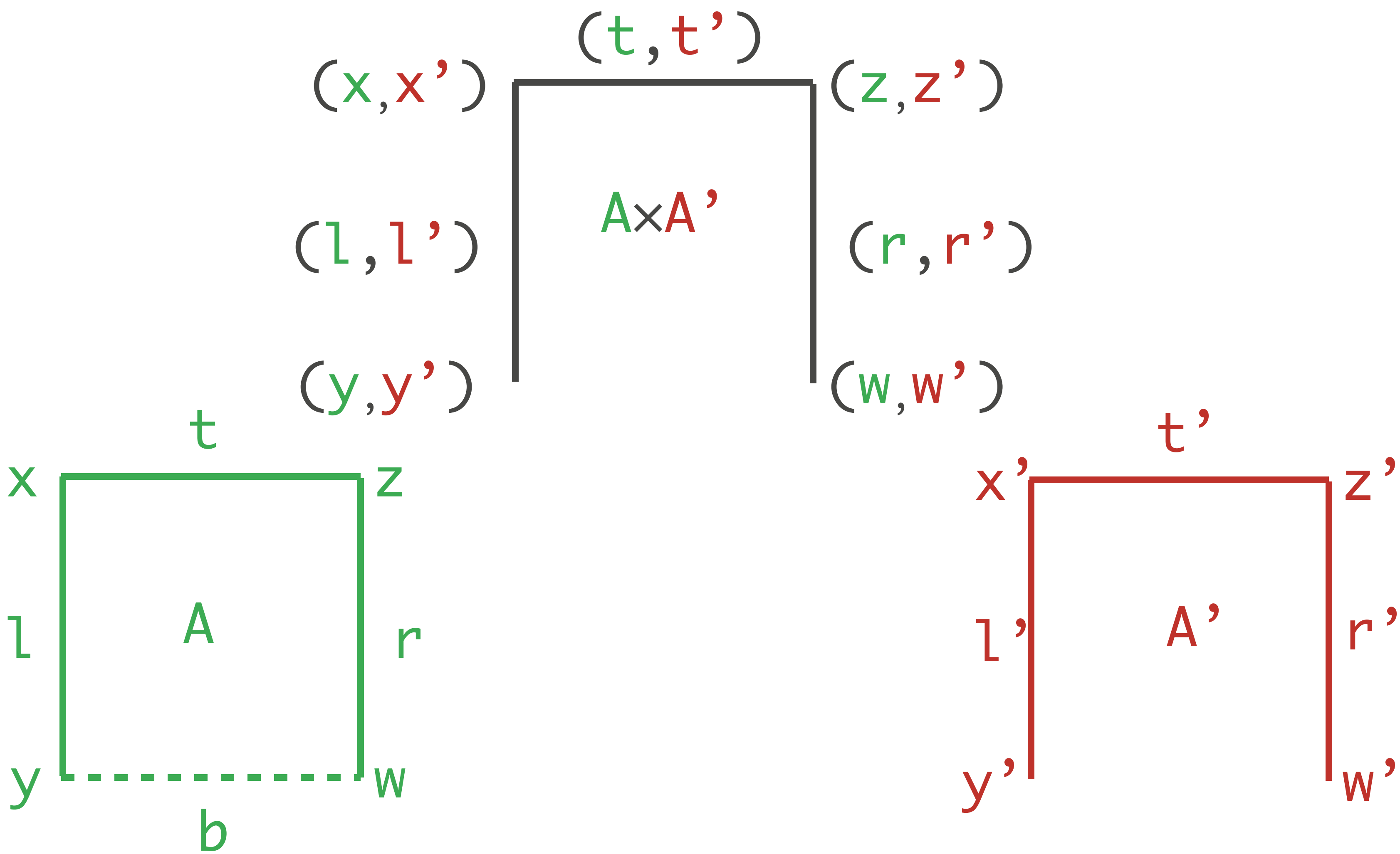
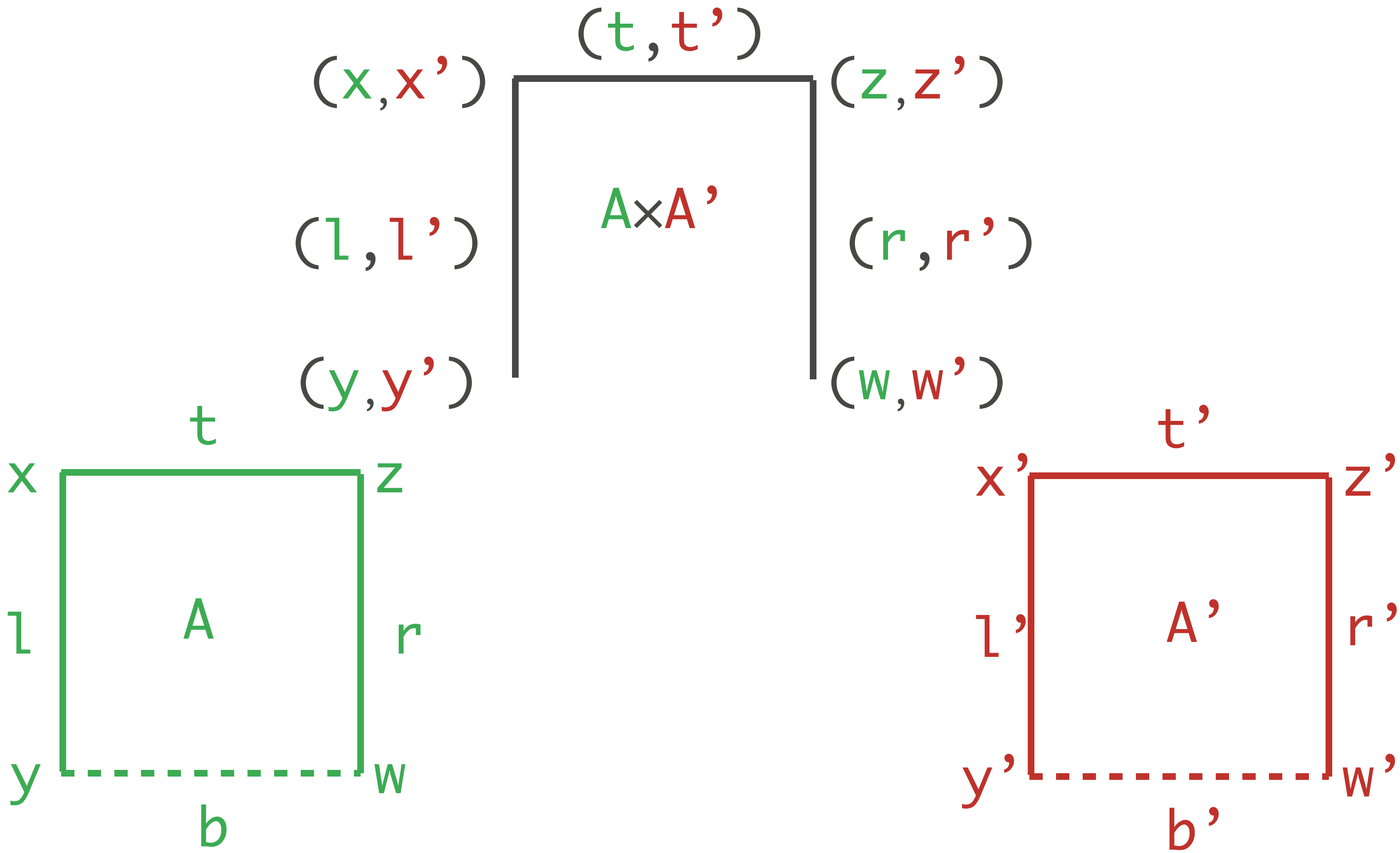
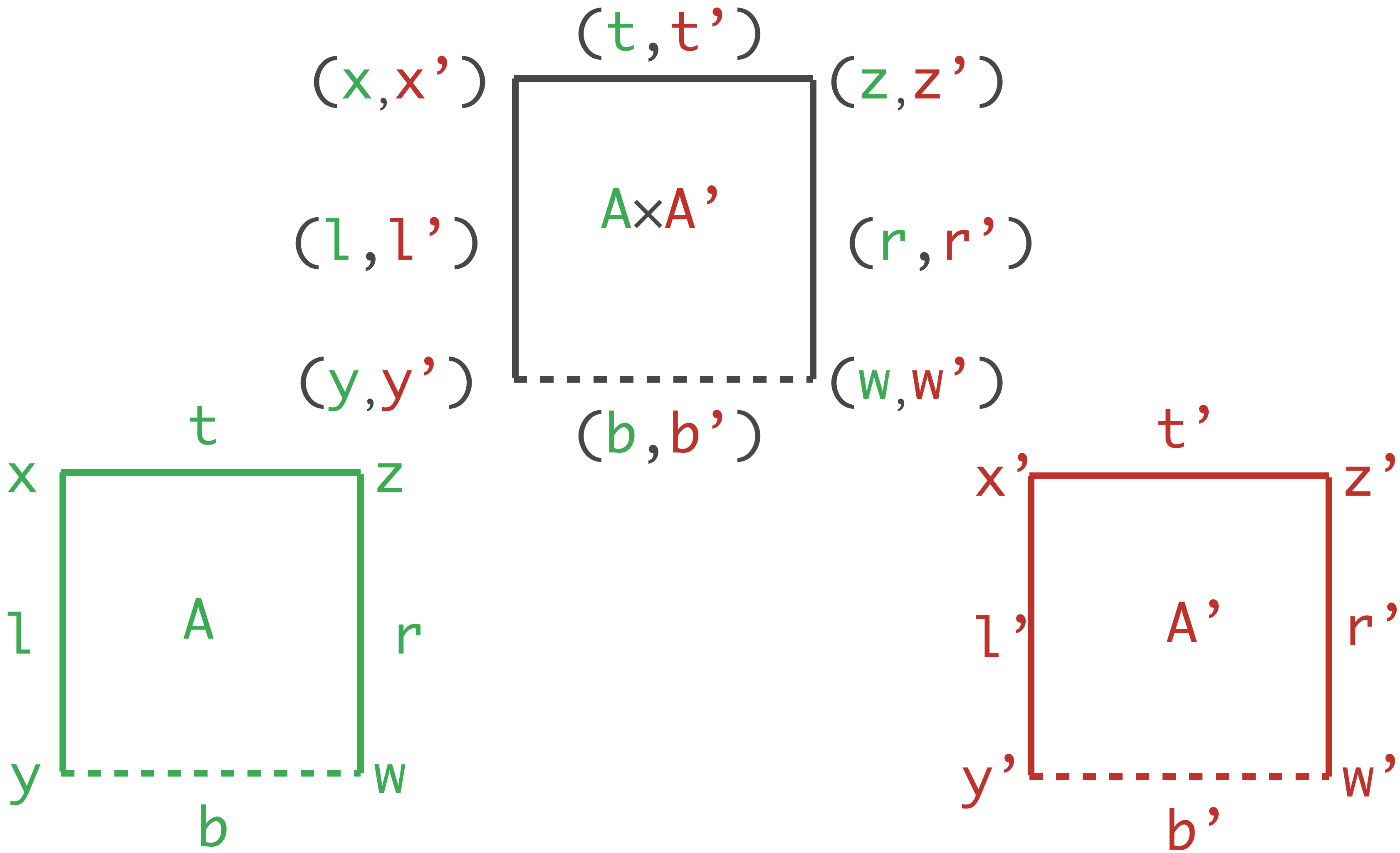


$$\frac{p : x =_A y \qquad p' : x' =_{A'} y'}{(p,p') : (x,x') =_{A\times A'} (y,y')}$$

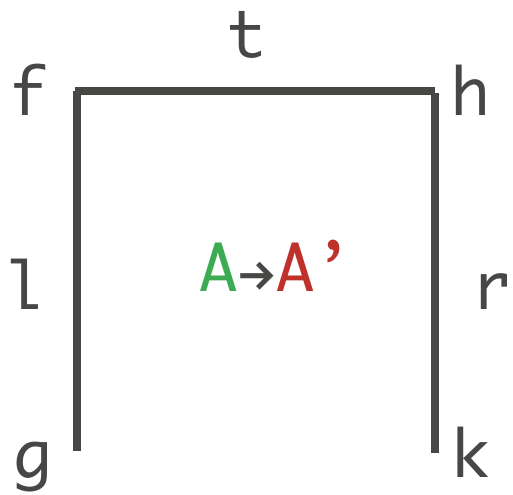


$$\frac{p : x =_{A} y \qquad p' : x' =_{A'} y'}{(p,p') : (x,x') =_{A\times A'} (y,y')}$$



$$\frac{x{:}A \vdash p : f\ x =_{A'} g\ x}{\lambda x.p : f =_{A \to A'} g}$$

$$\frac{x{:}A \vdash p : f\ x =_{A'} g\ x}{\lambda x.p : f =_{A\to A'} g}$$



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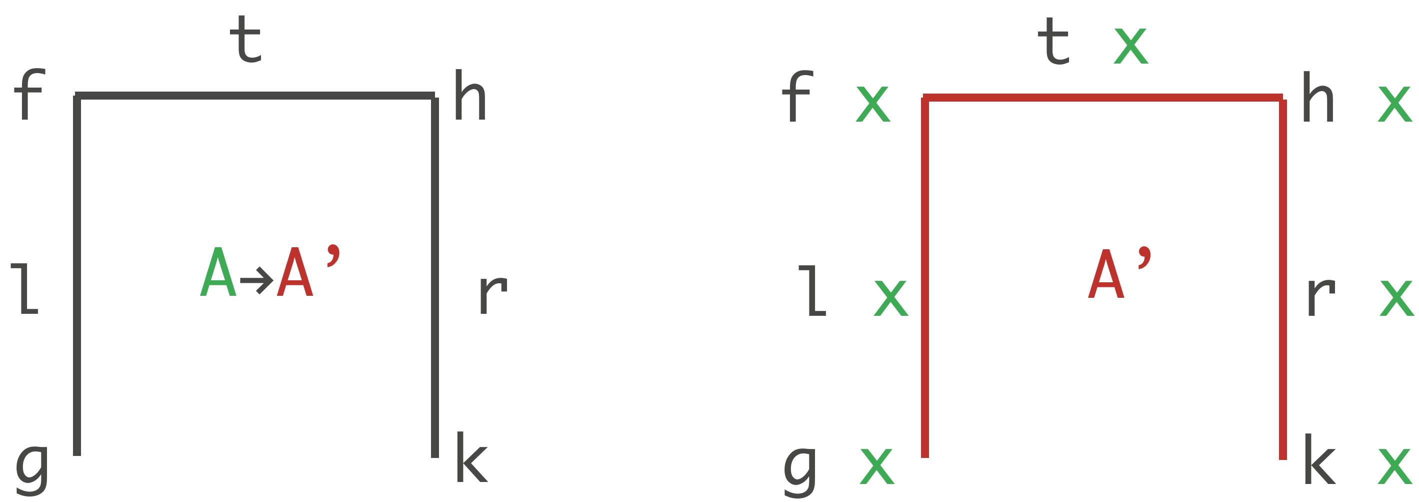

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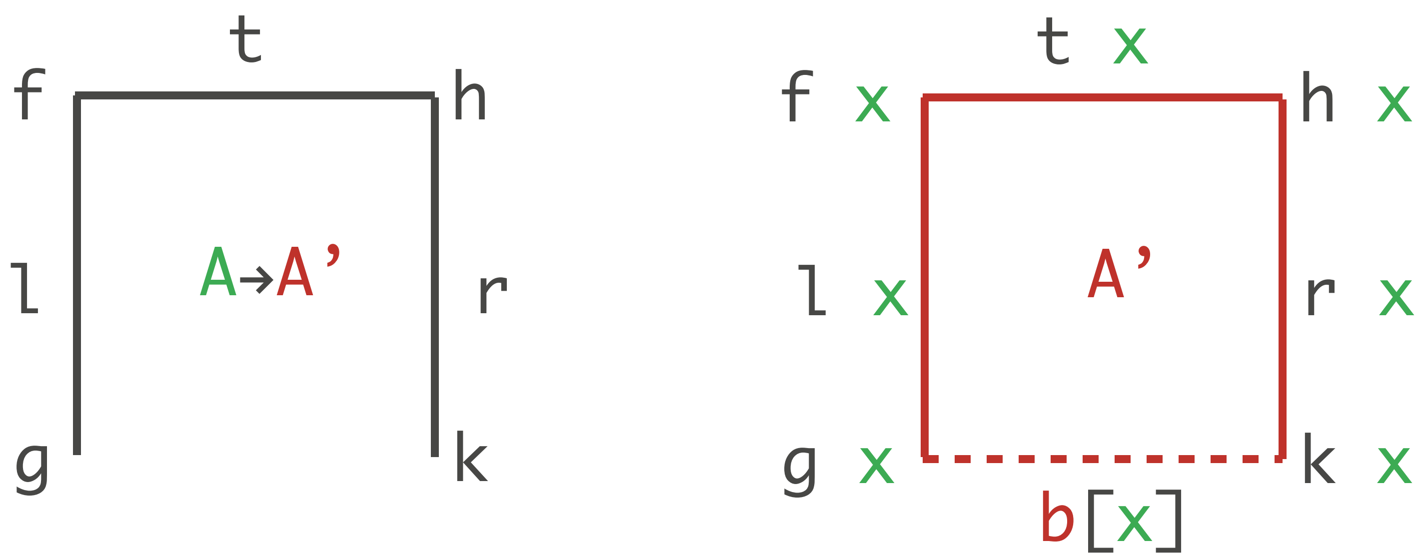

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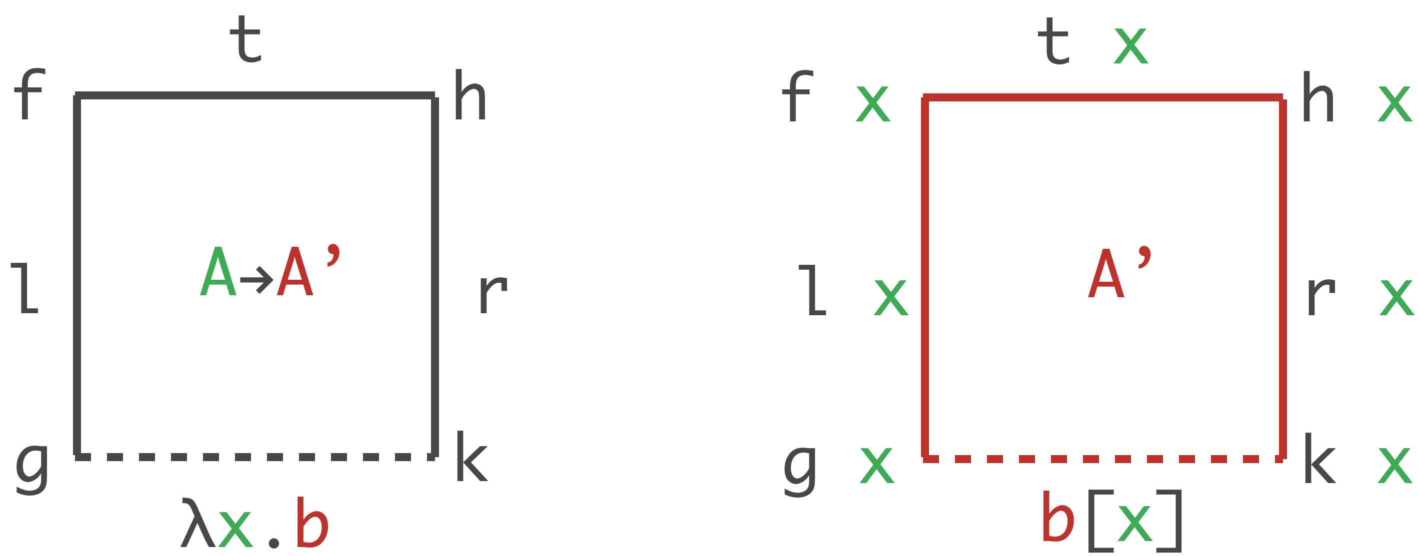
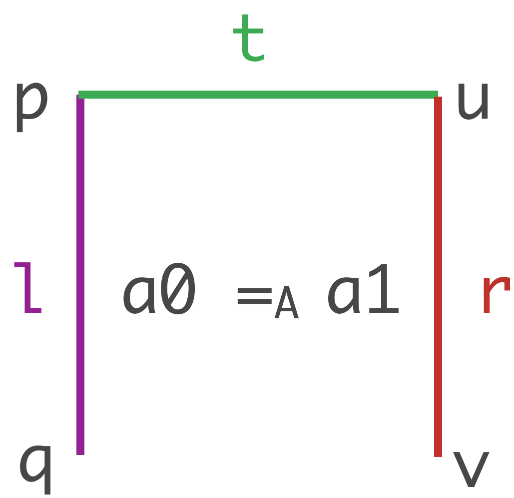
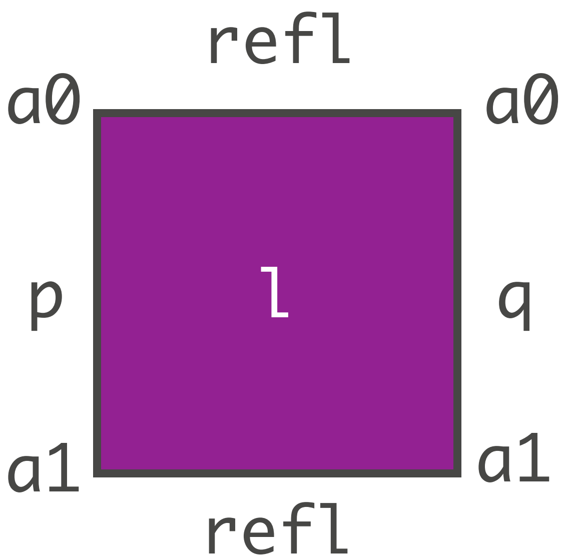
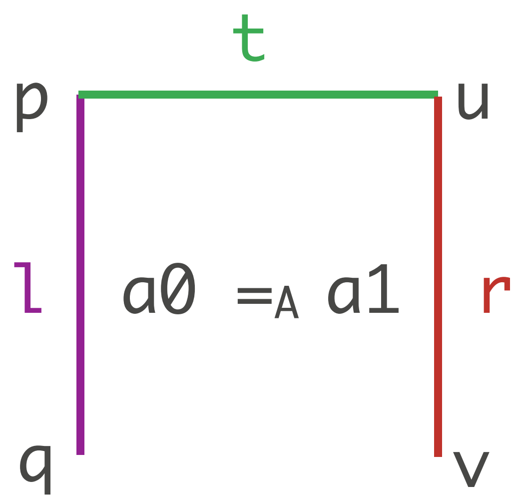




$$p\ q\ u\ v : a0 =_A a1$$

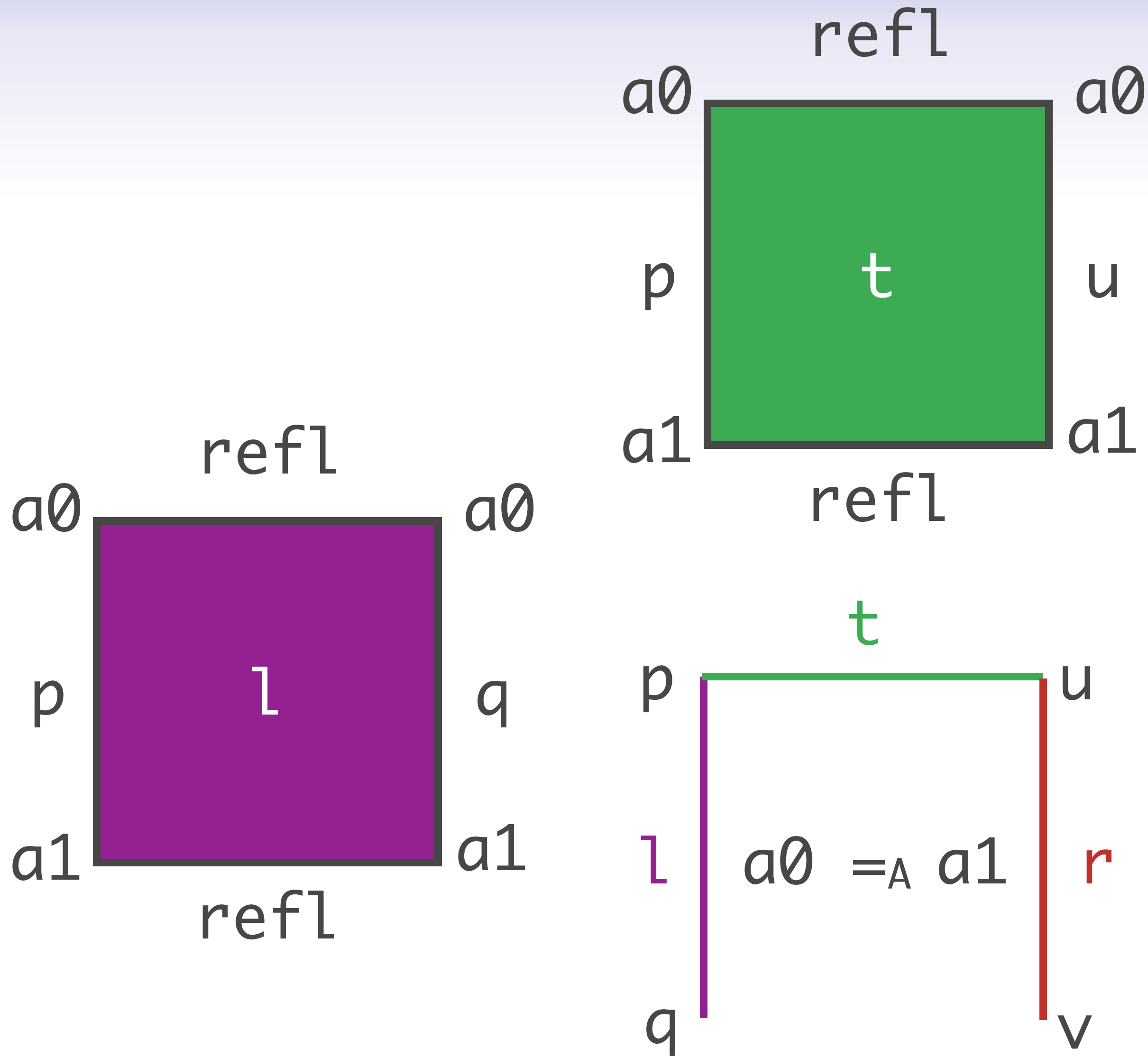
$l : p =_{a0=a1} q \qquad t : p =_{a0=a1} u \qquad r : u =_{a0=a1} v$





$p\ q\ u\ v : a0 =_A a1$

$l : p =_{a0=a1} q$ $t : p =_{a0=a1} u$ $r : u =_{a0=a1} v$



$$p\ q\ u\ v : a0 =_A a1$$

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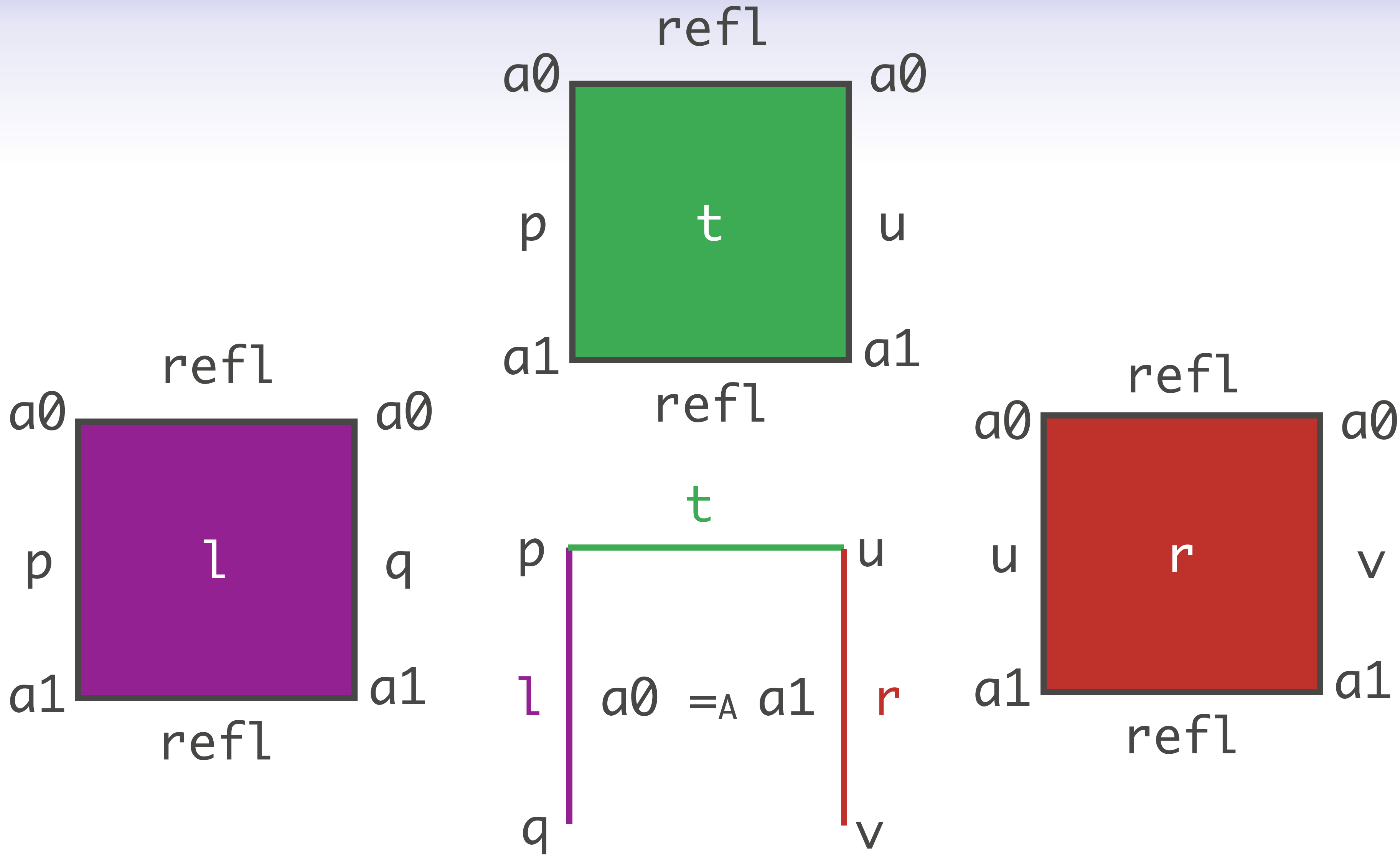

$p \; q \; u \; v : a0 =_A a1$

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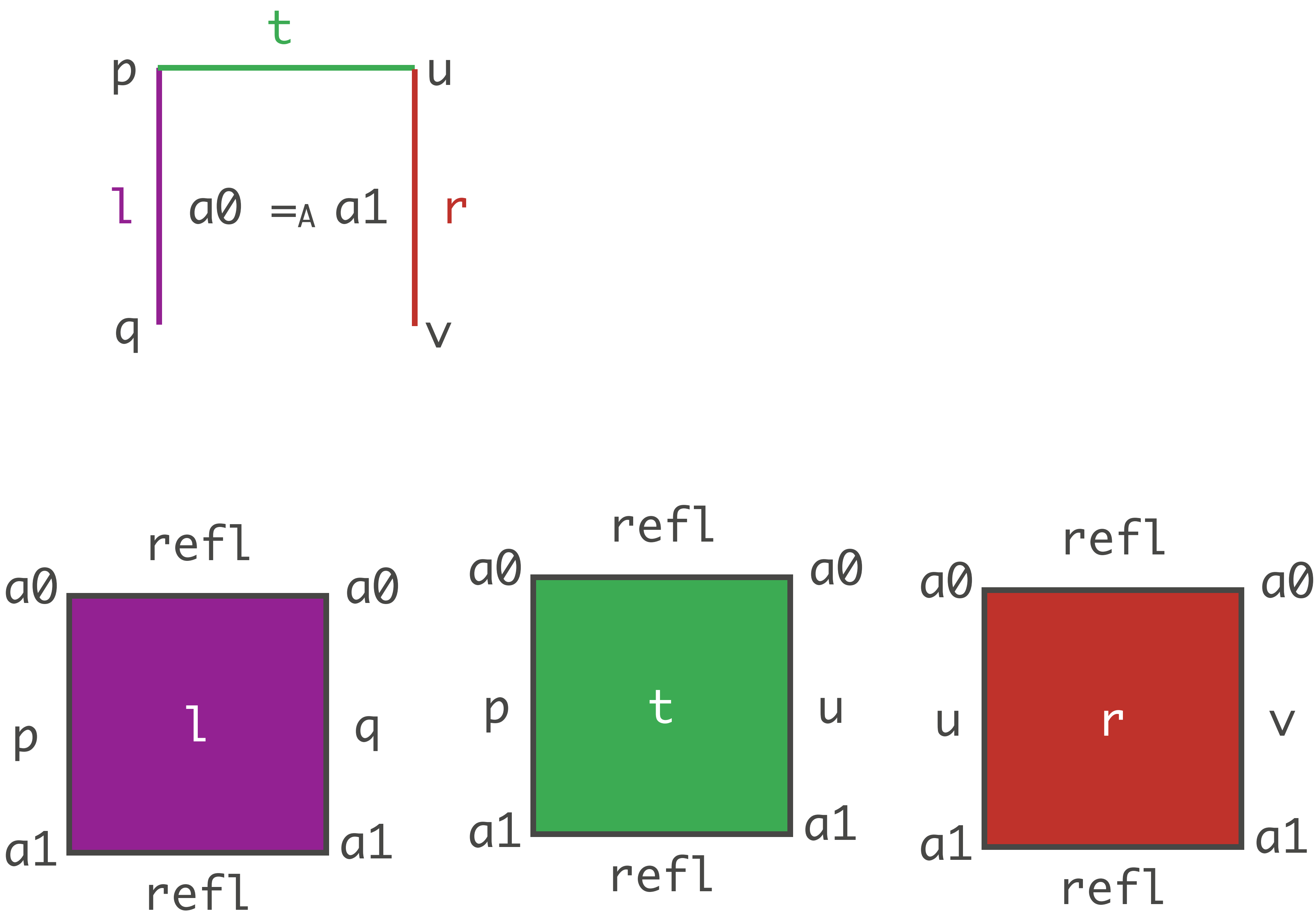

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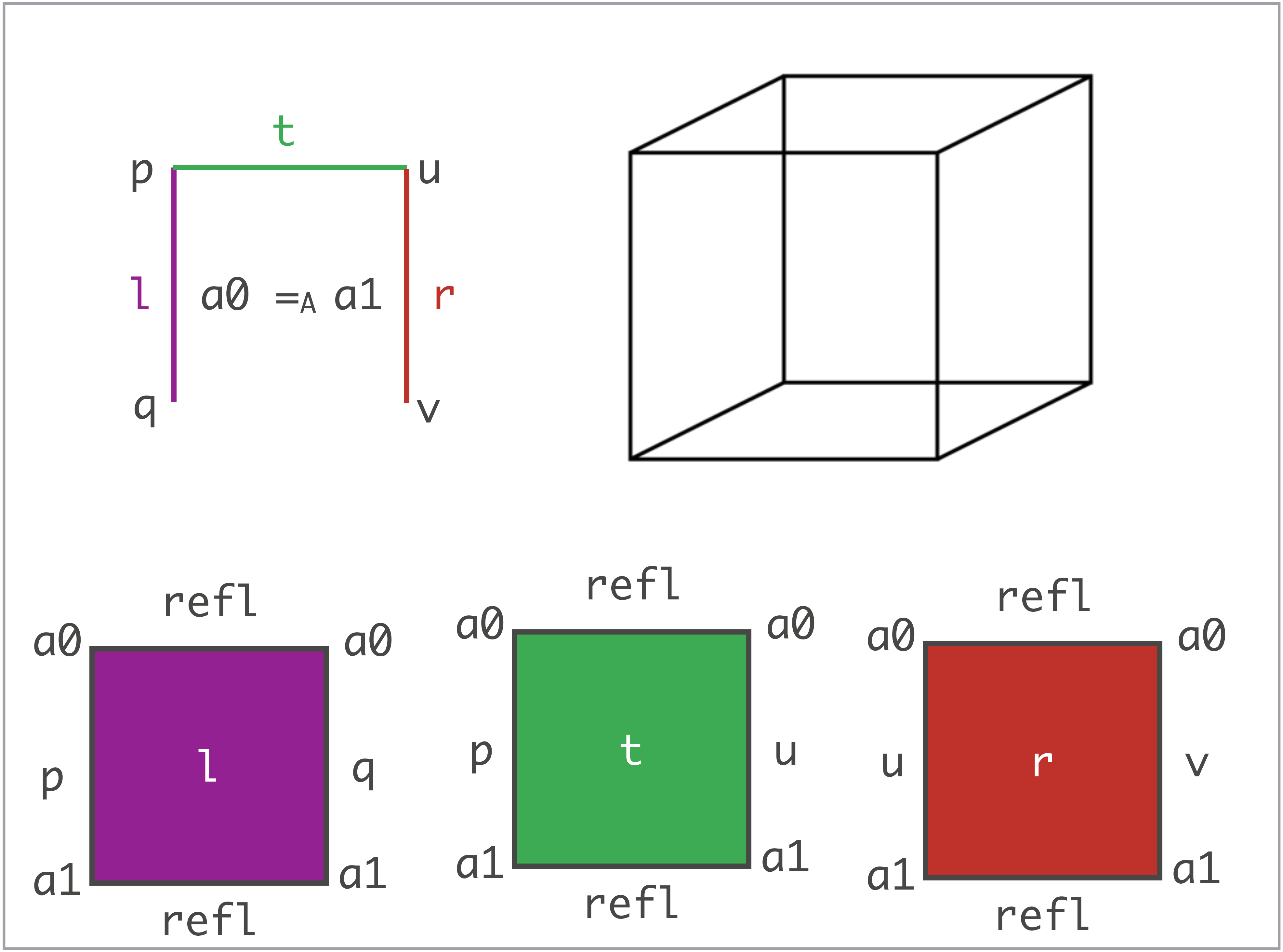
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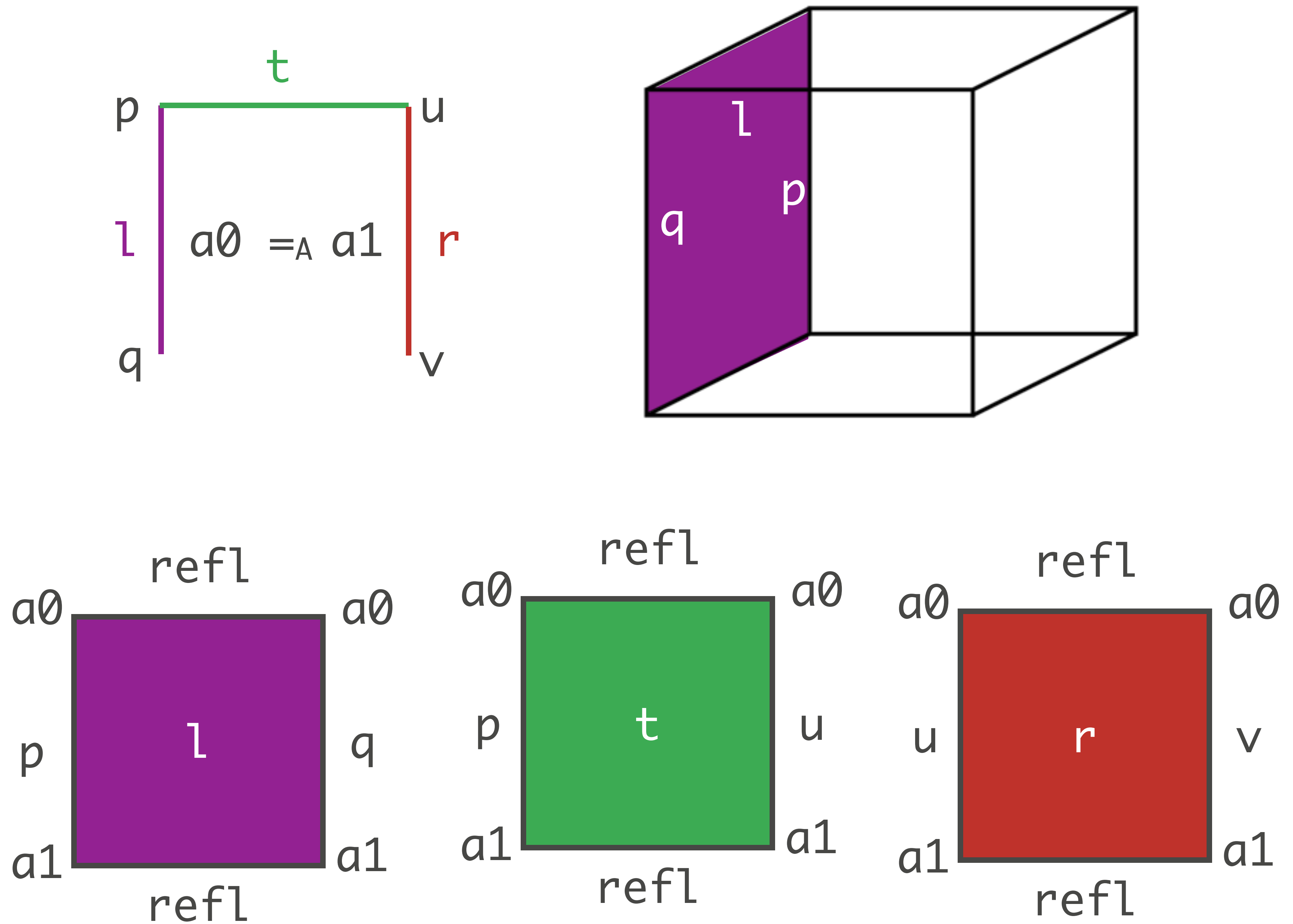
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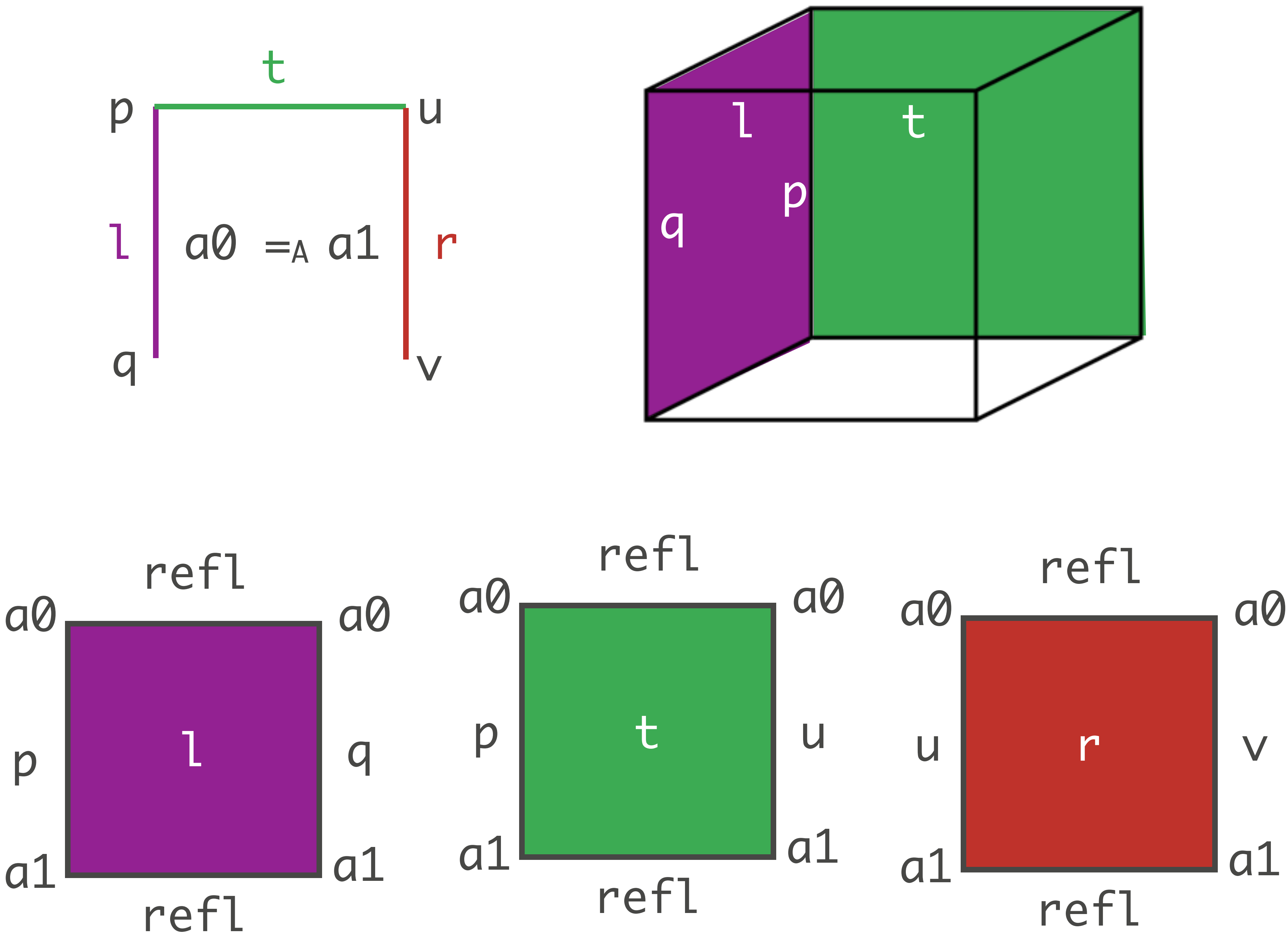
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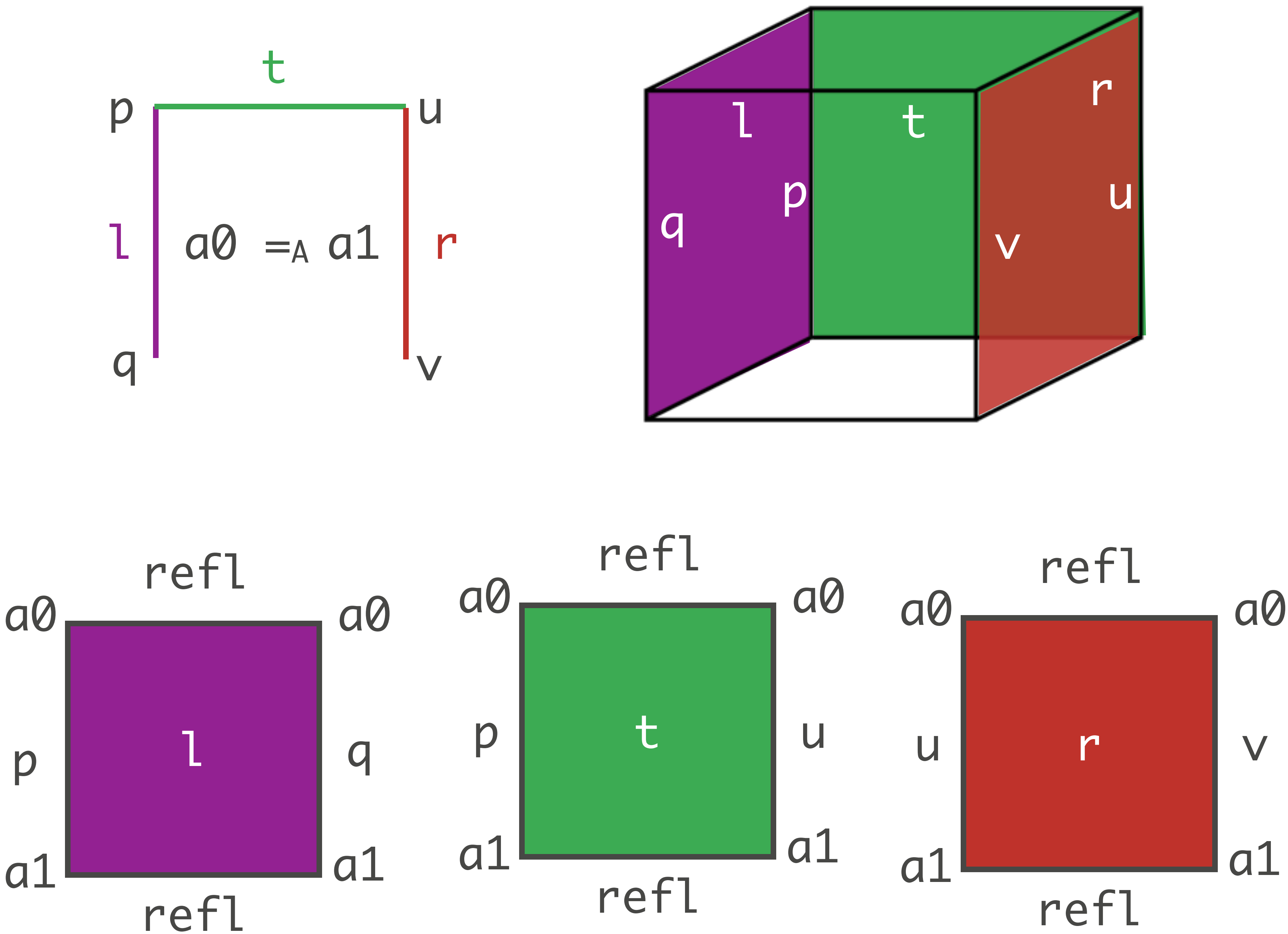
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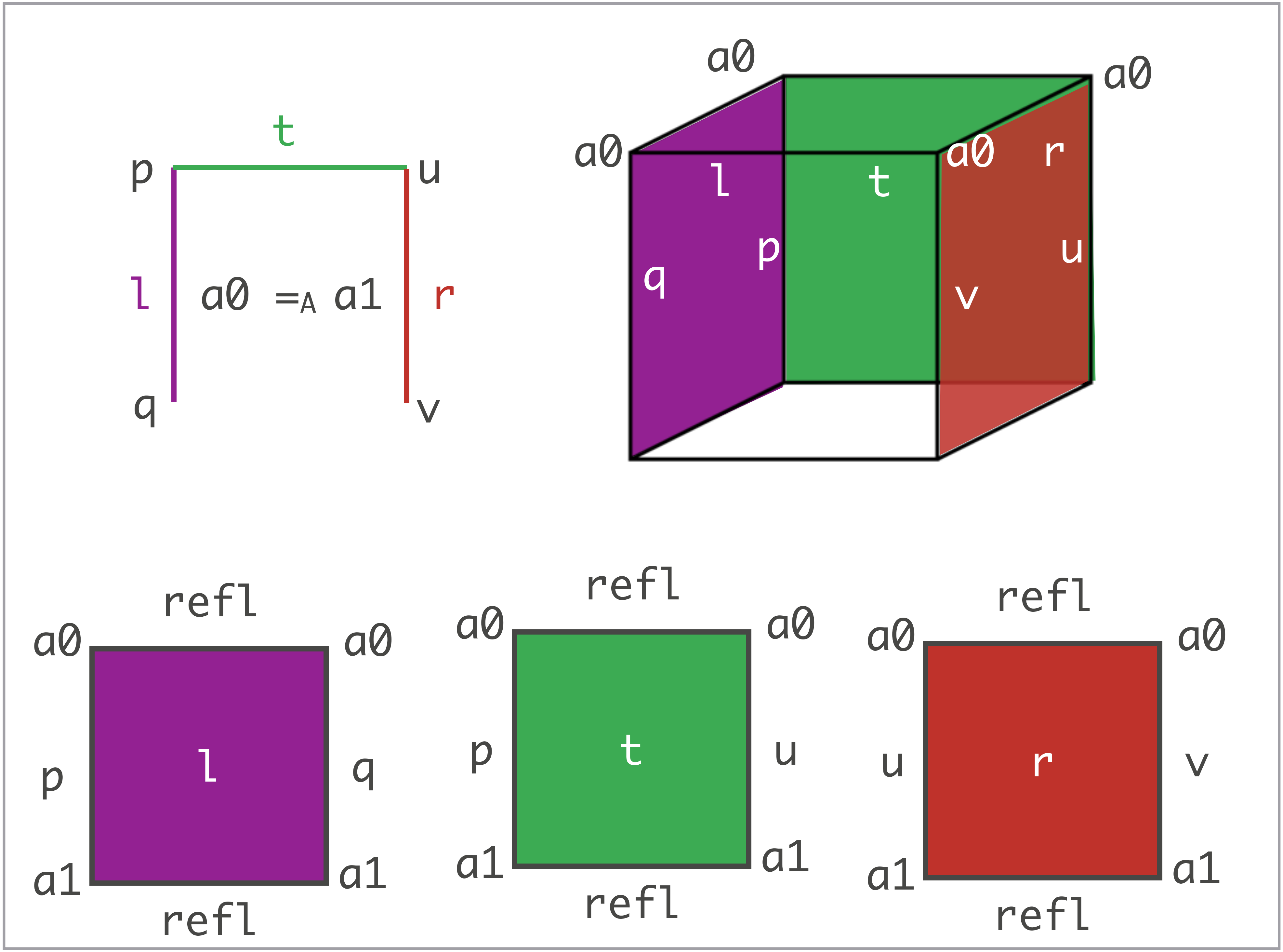
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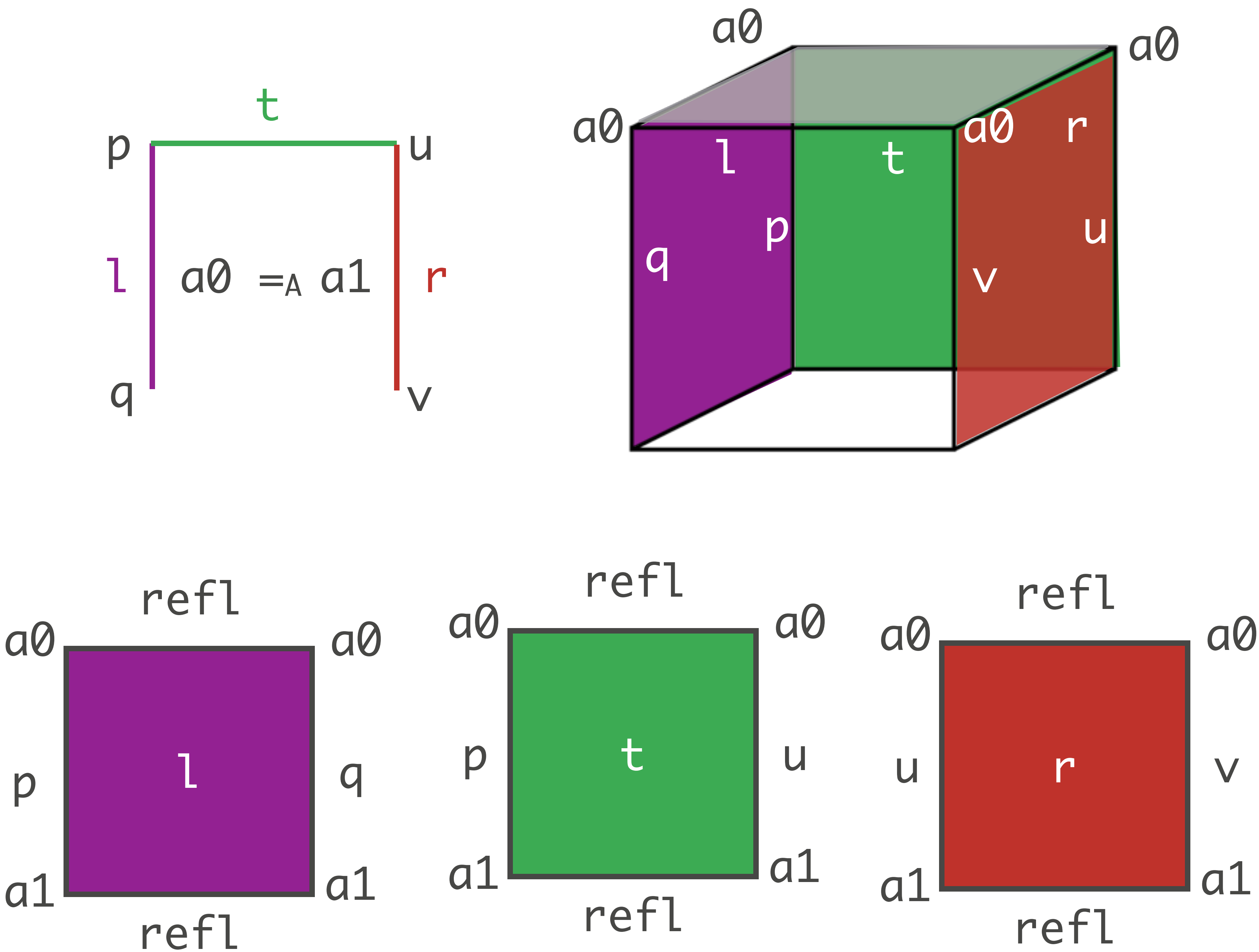
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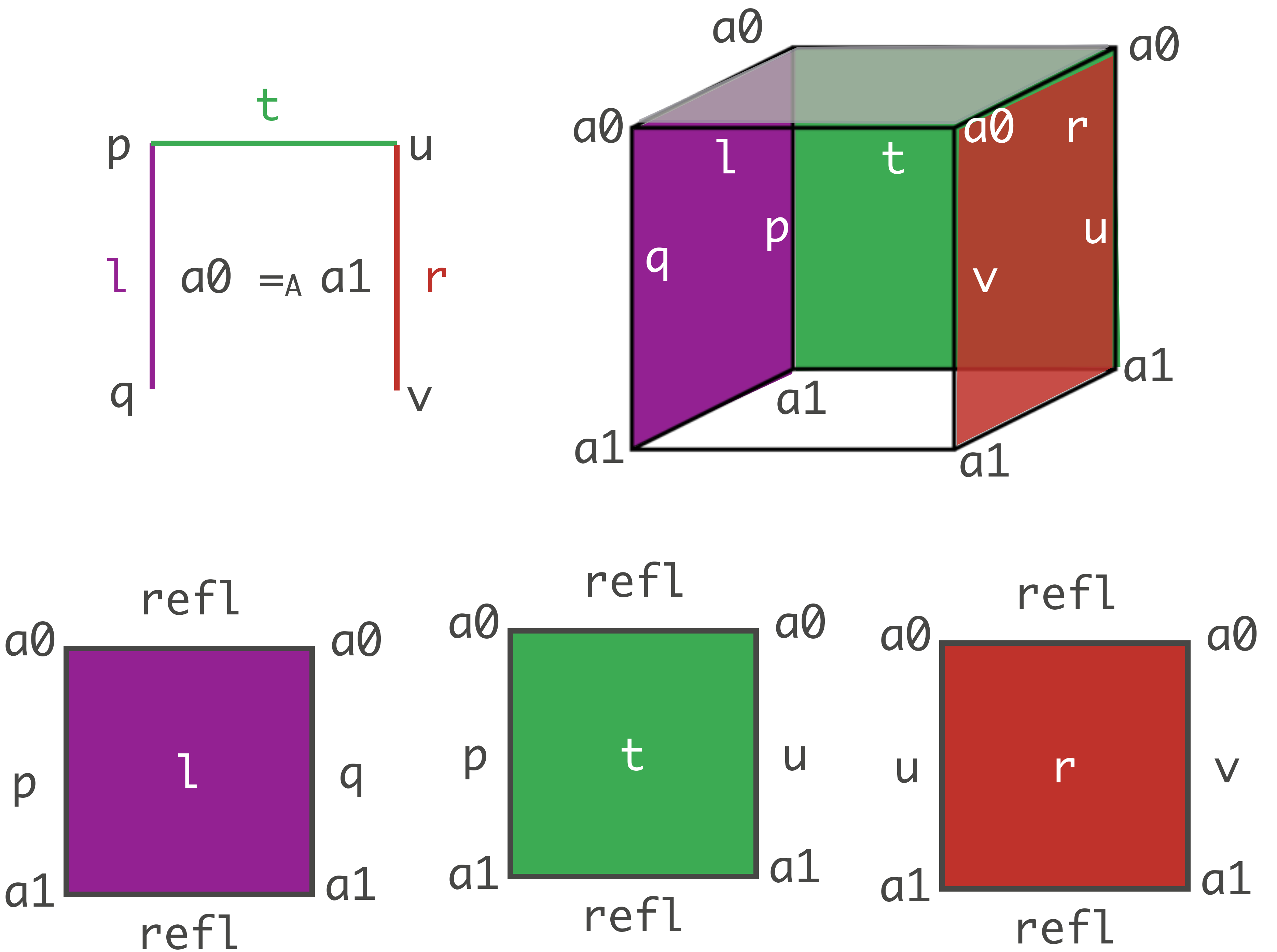
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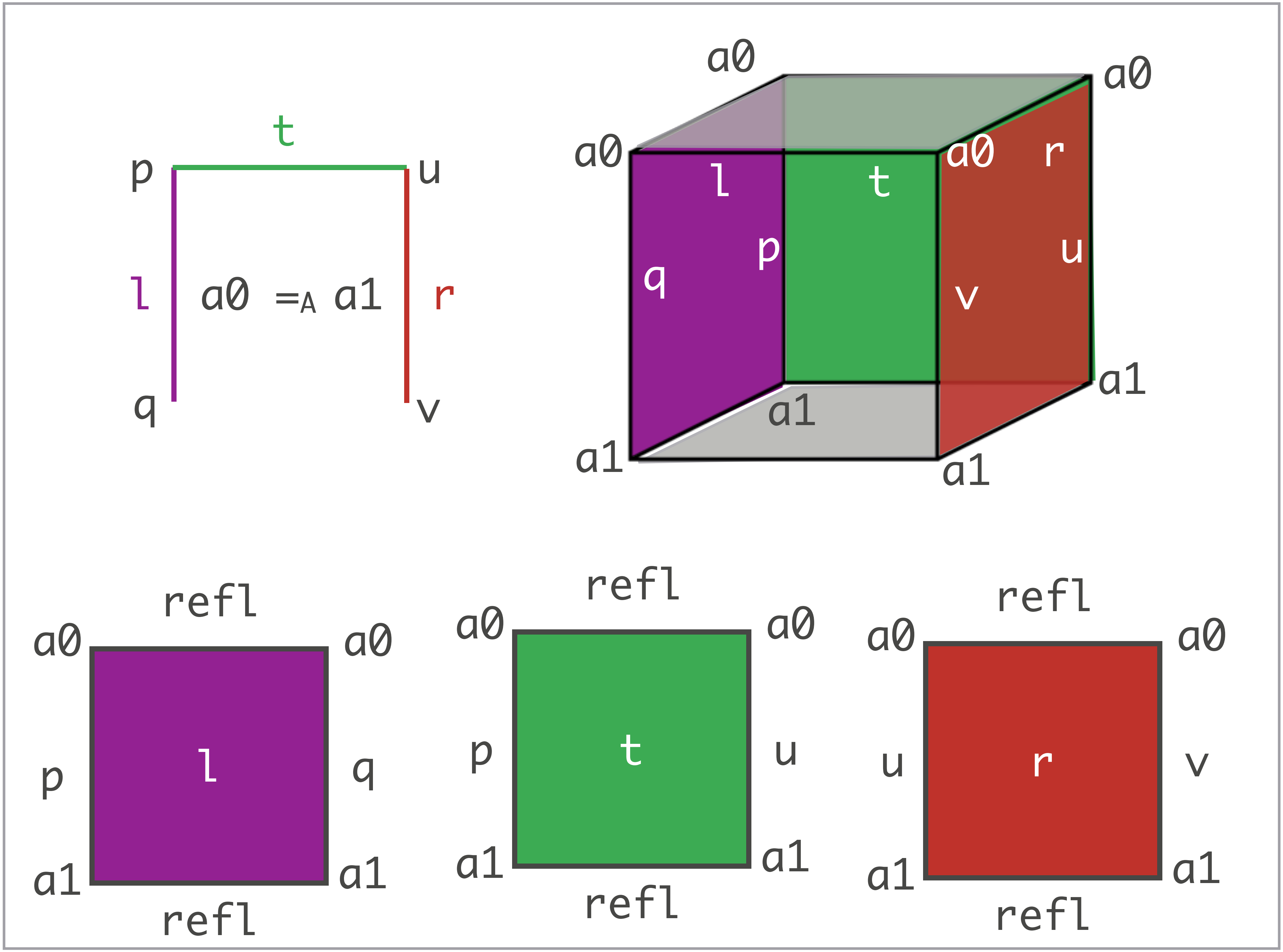

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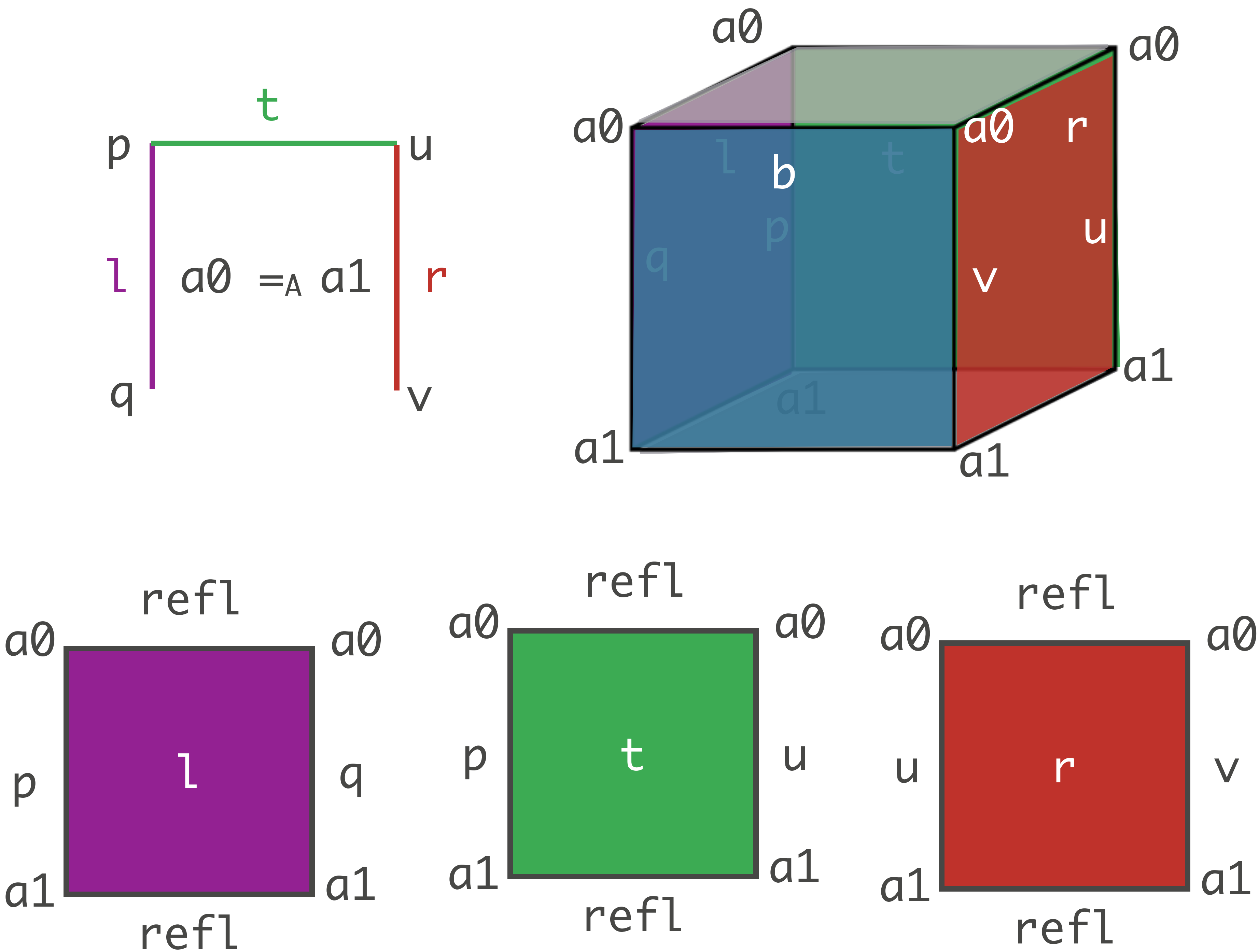
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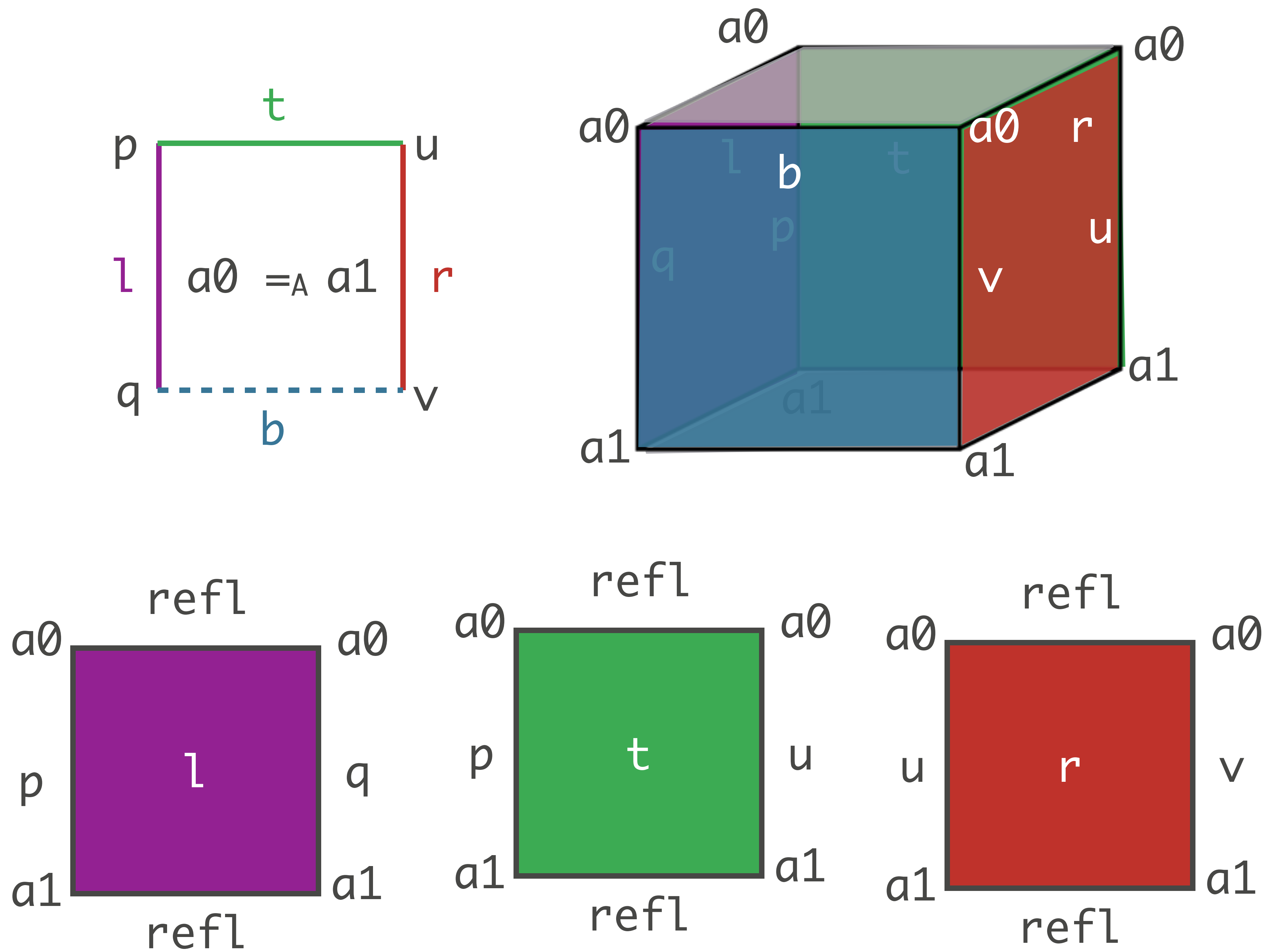
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Kan condition:
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Cubes

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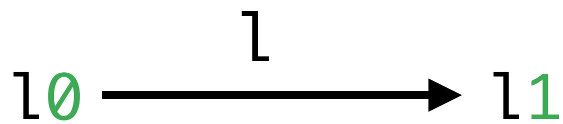

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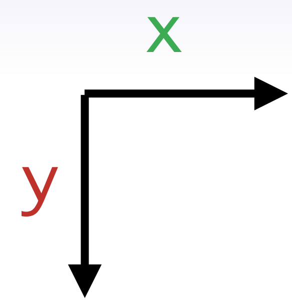

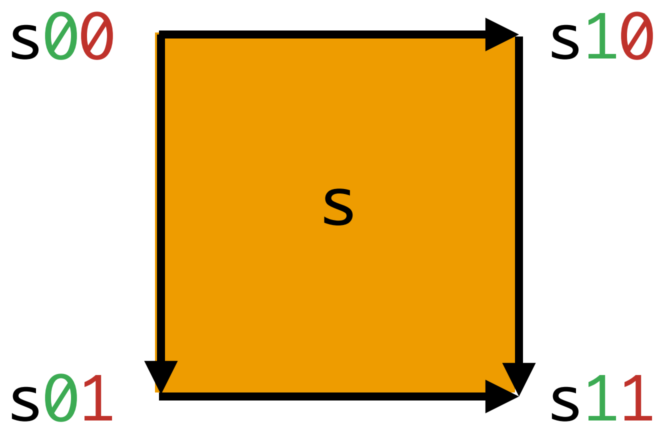

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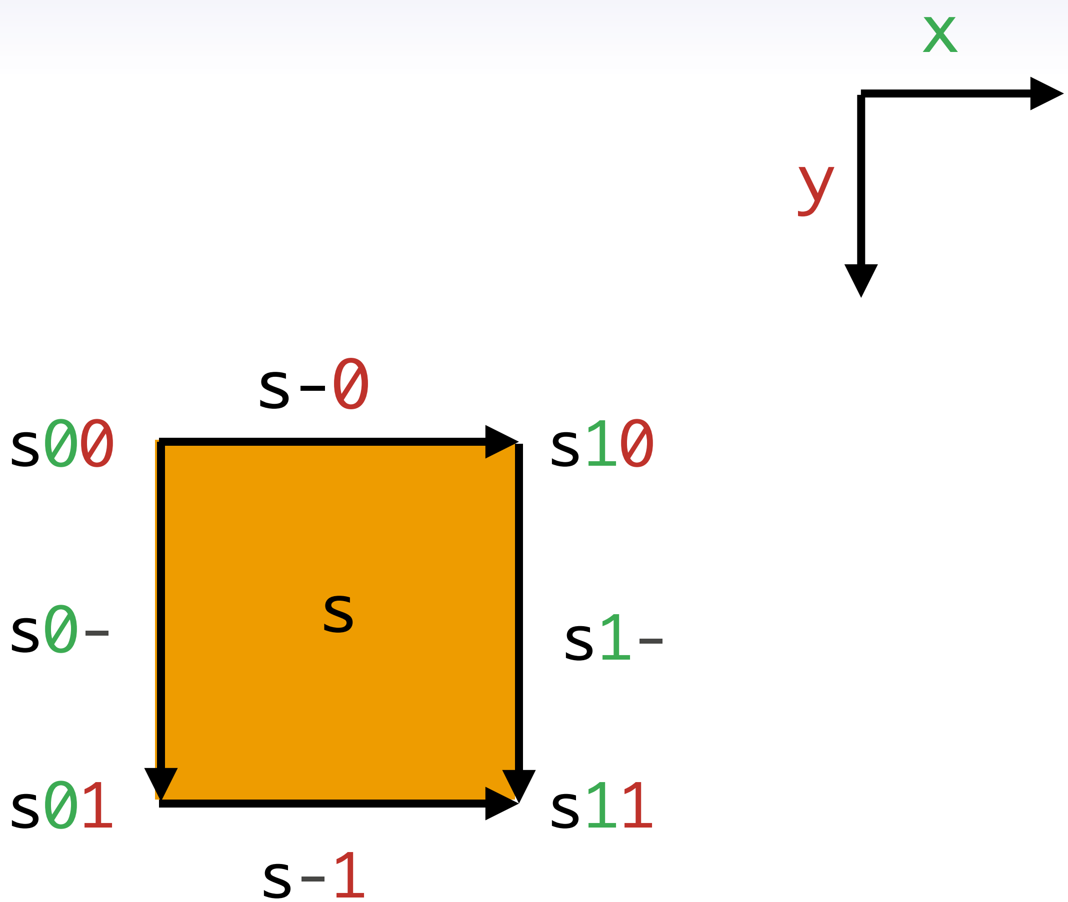
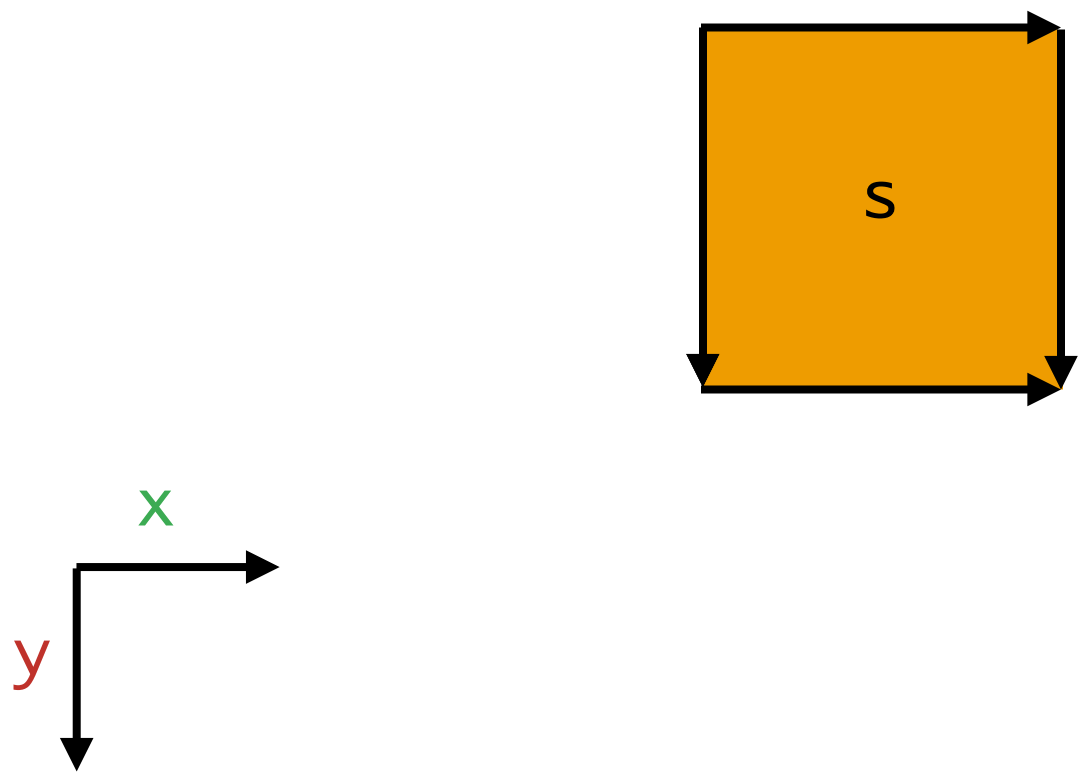


Square with its boundary



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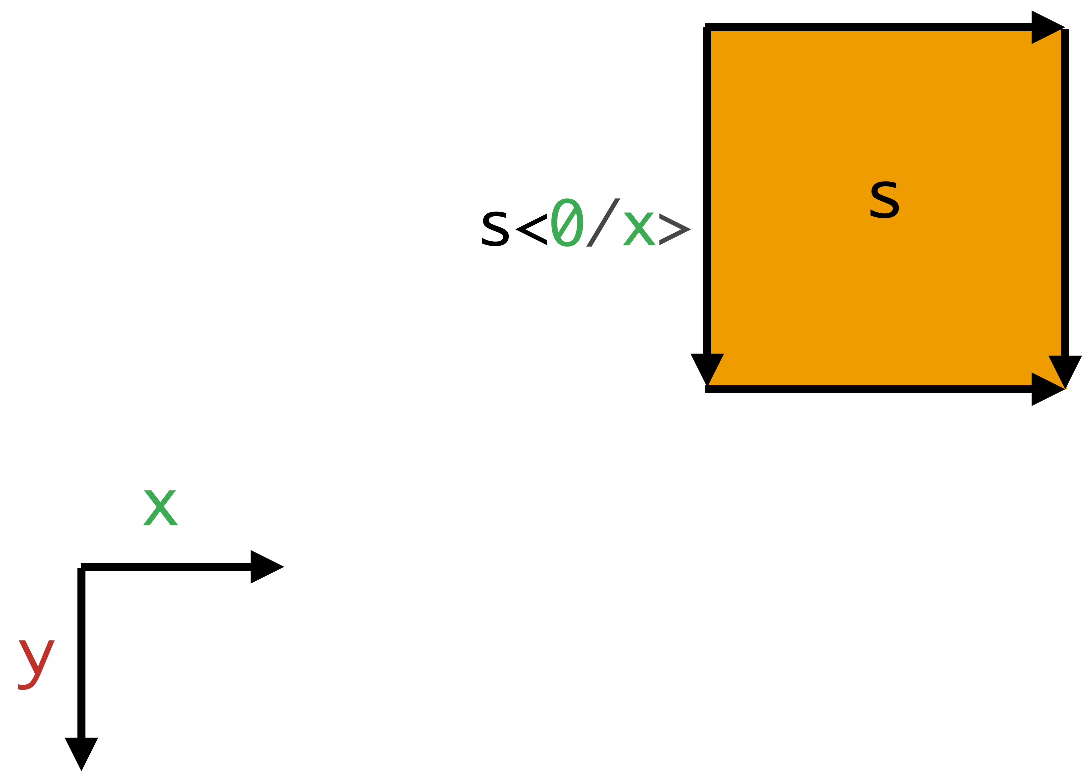

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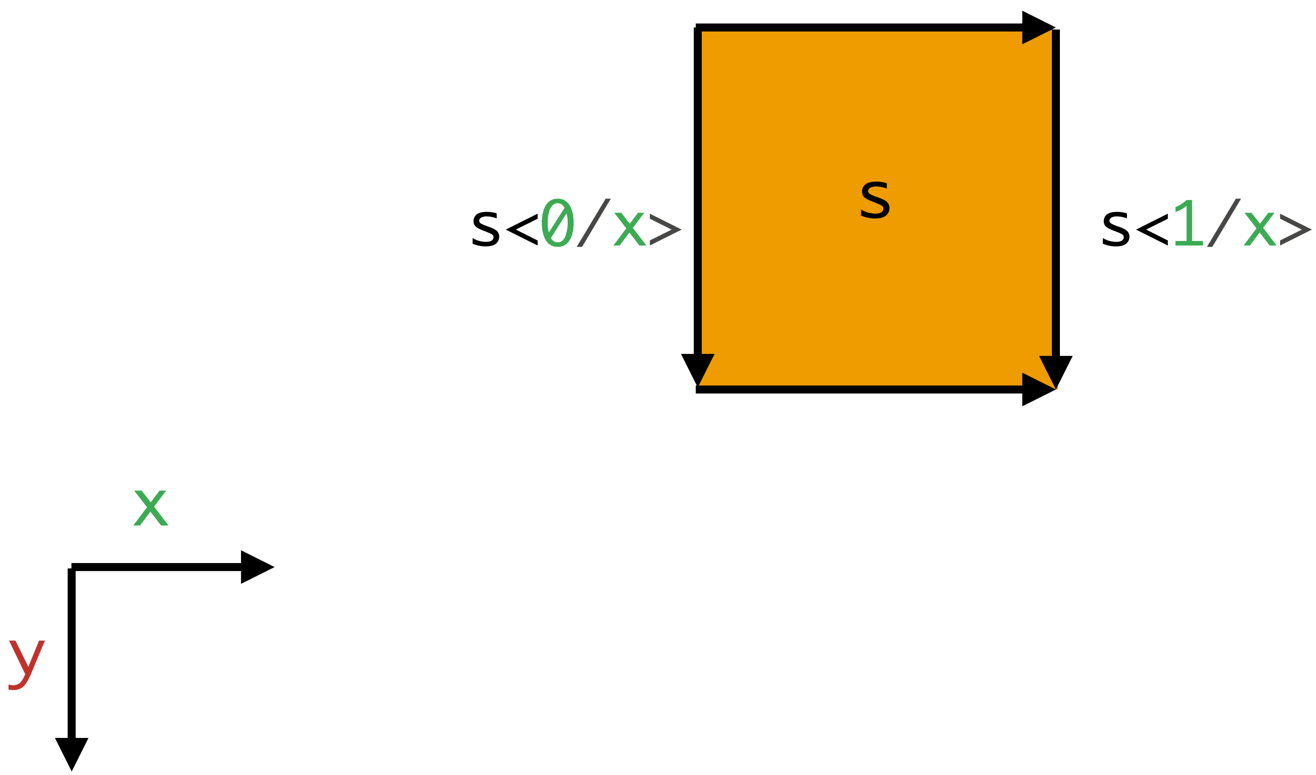

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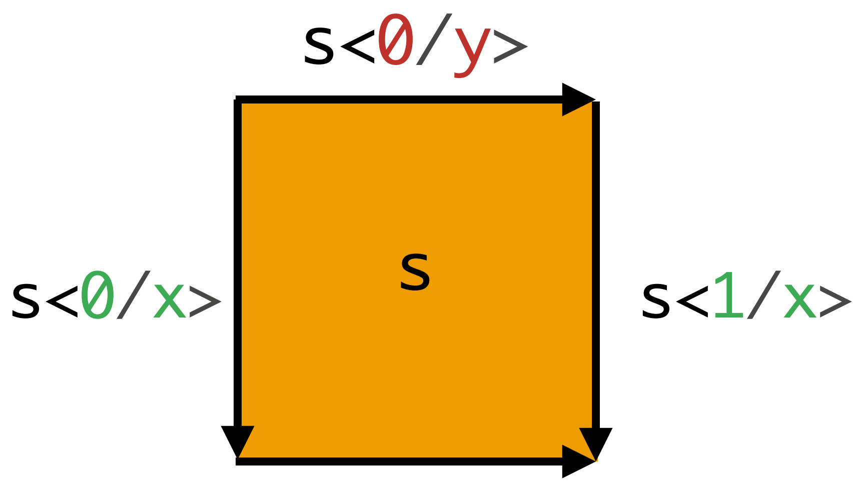

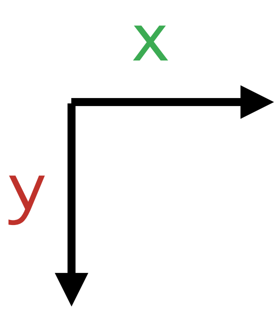

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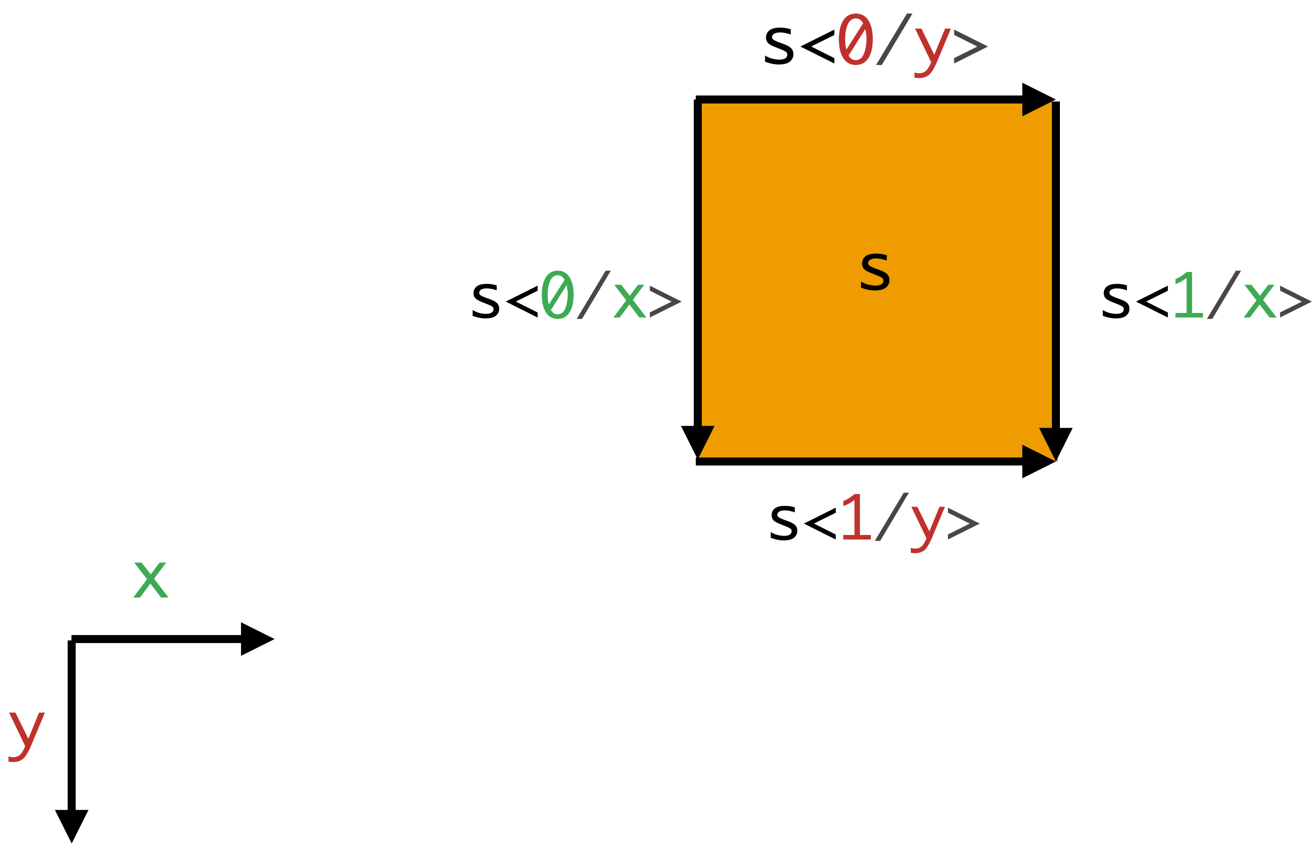

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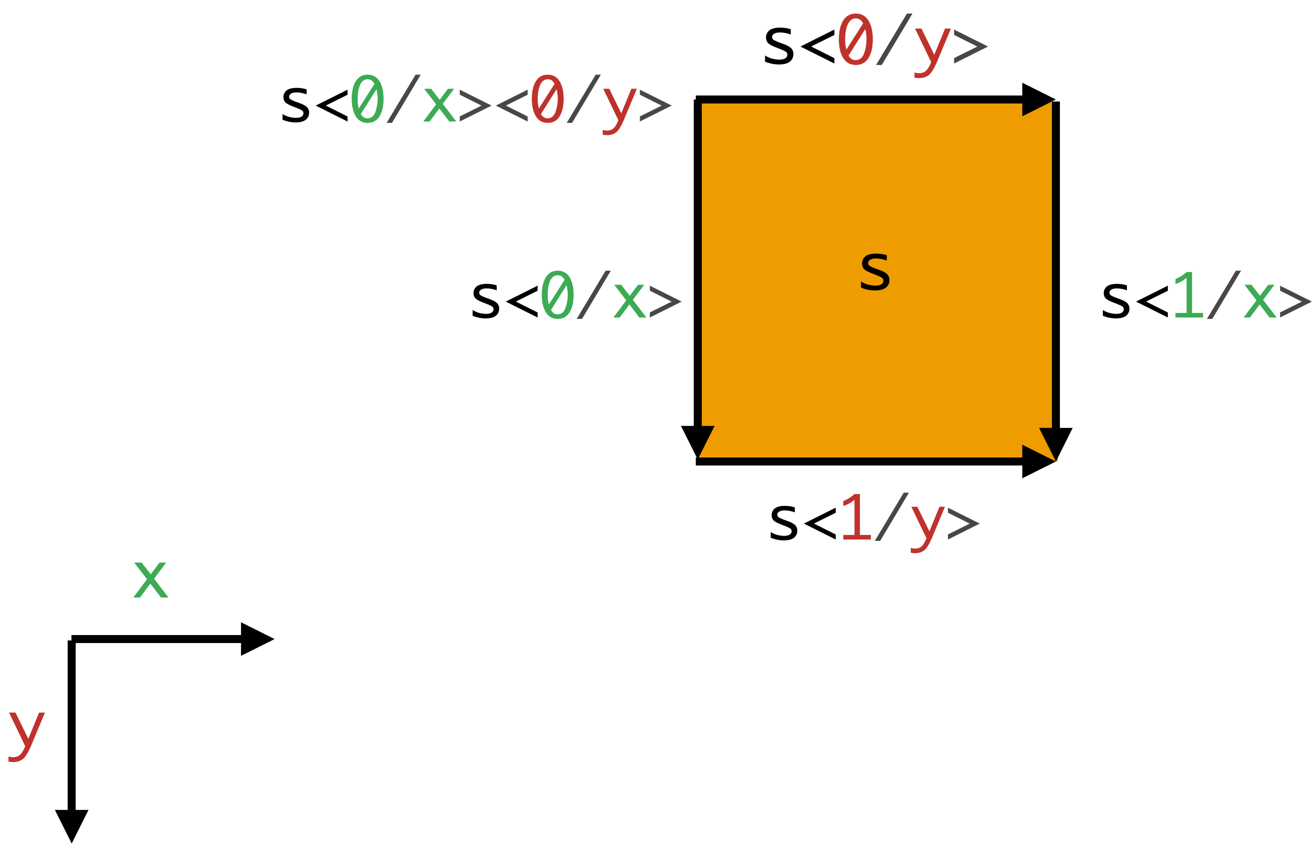

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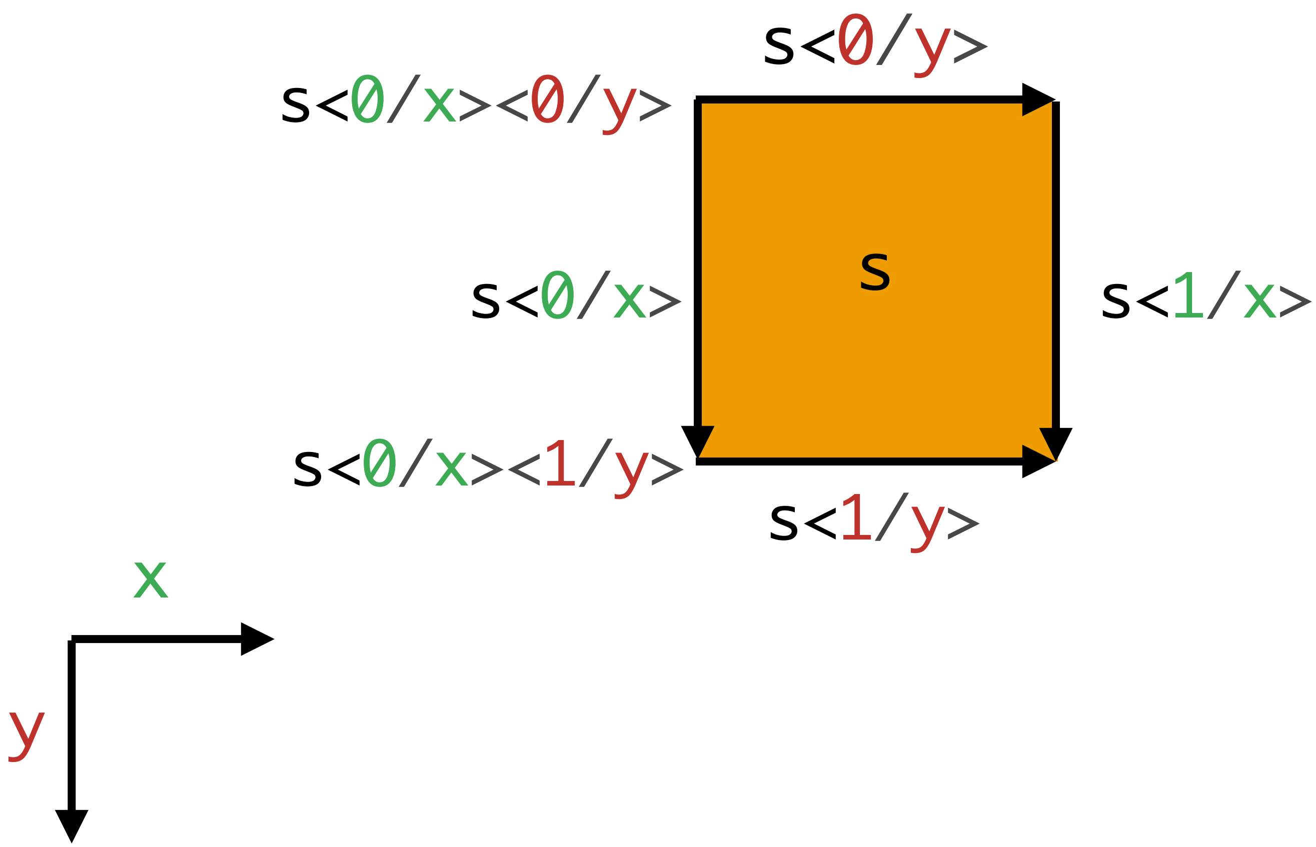

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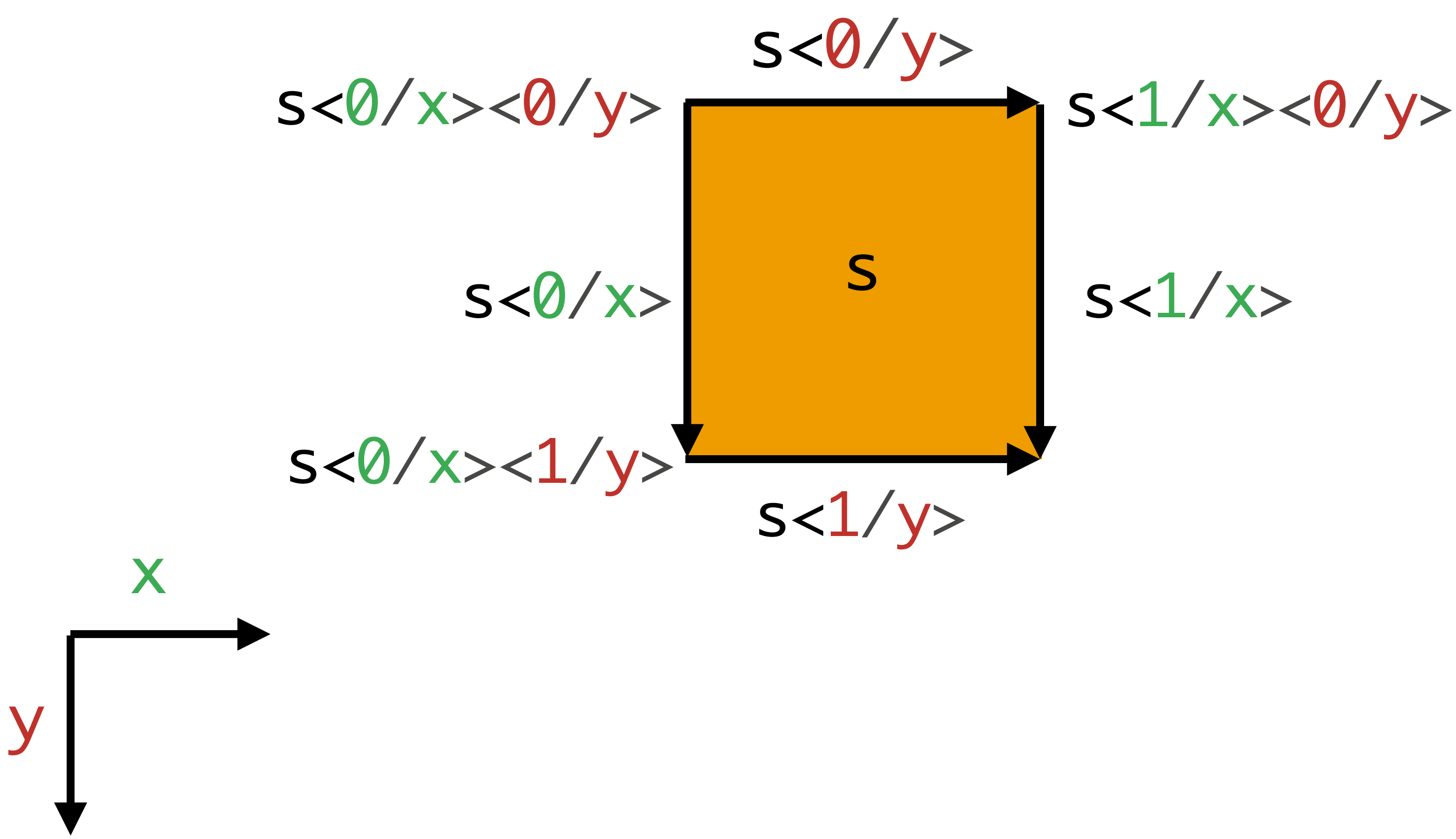

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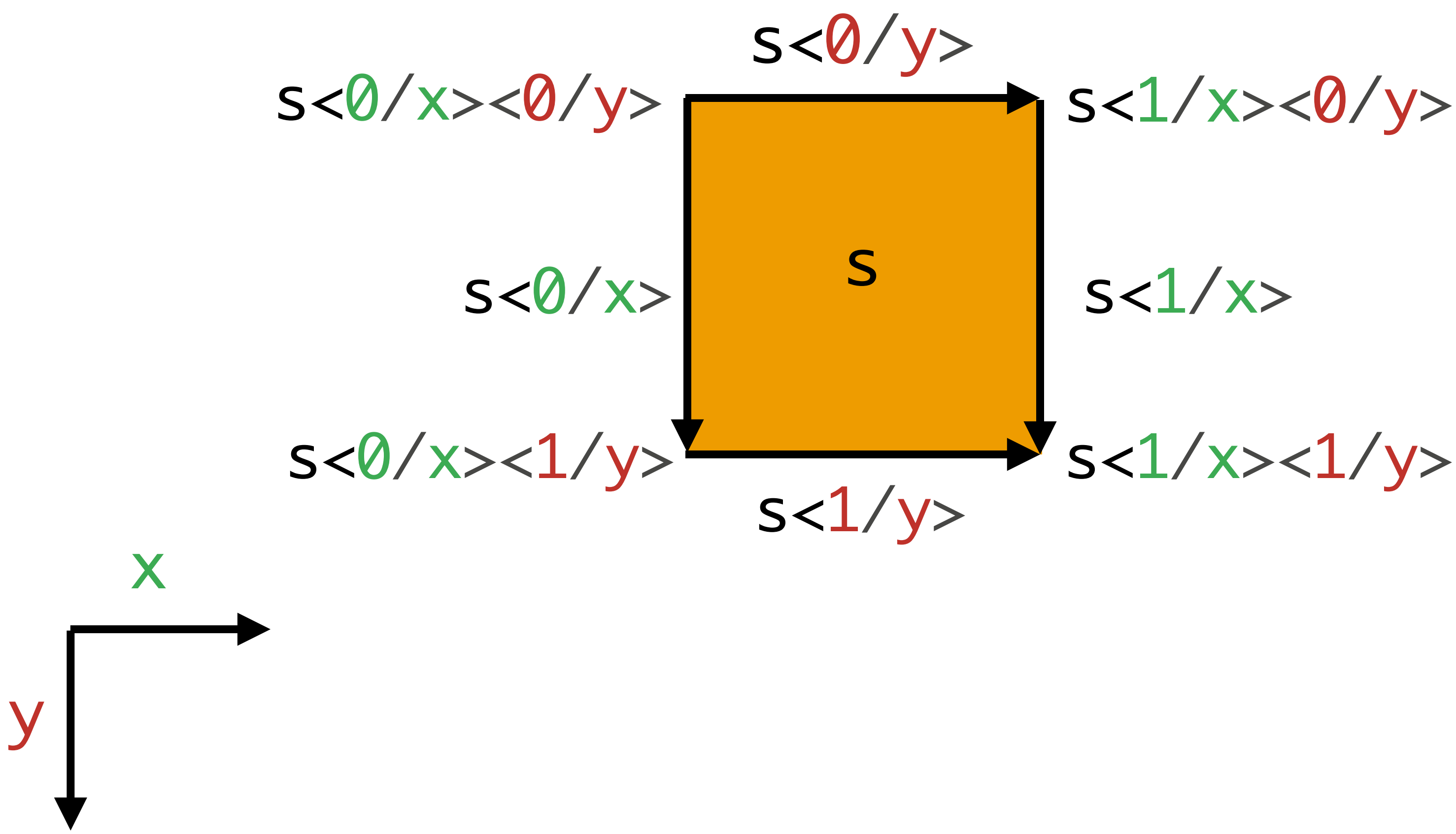

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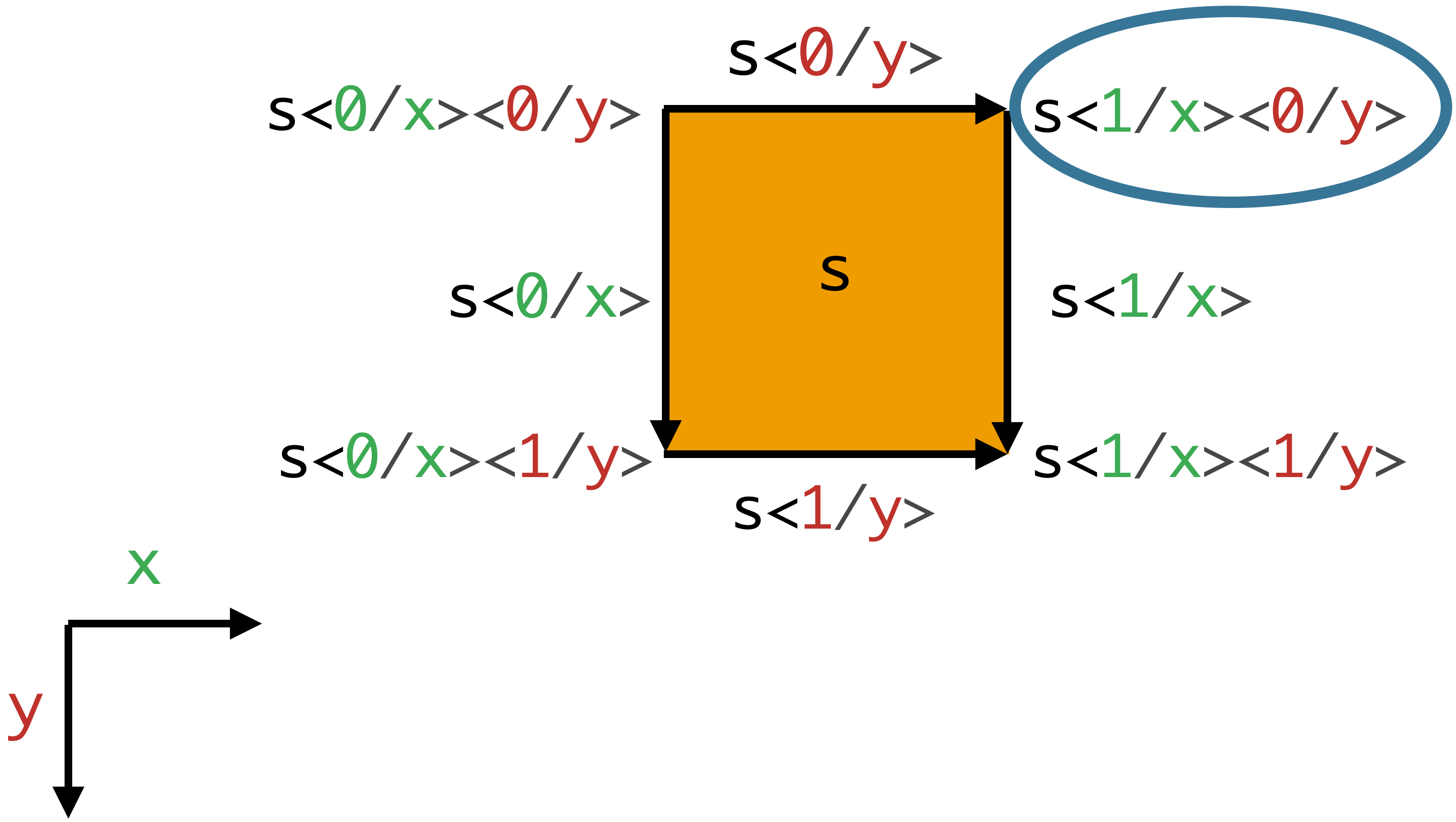

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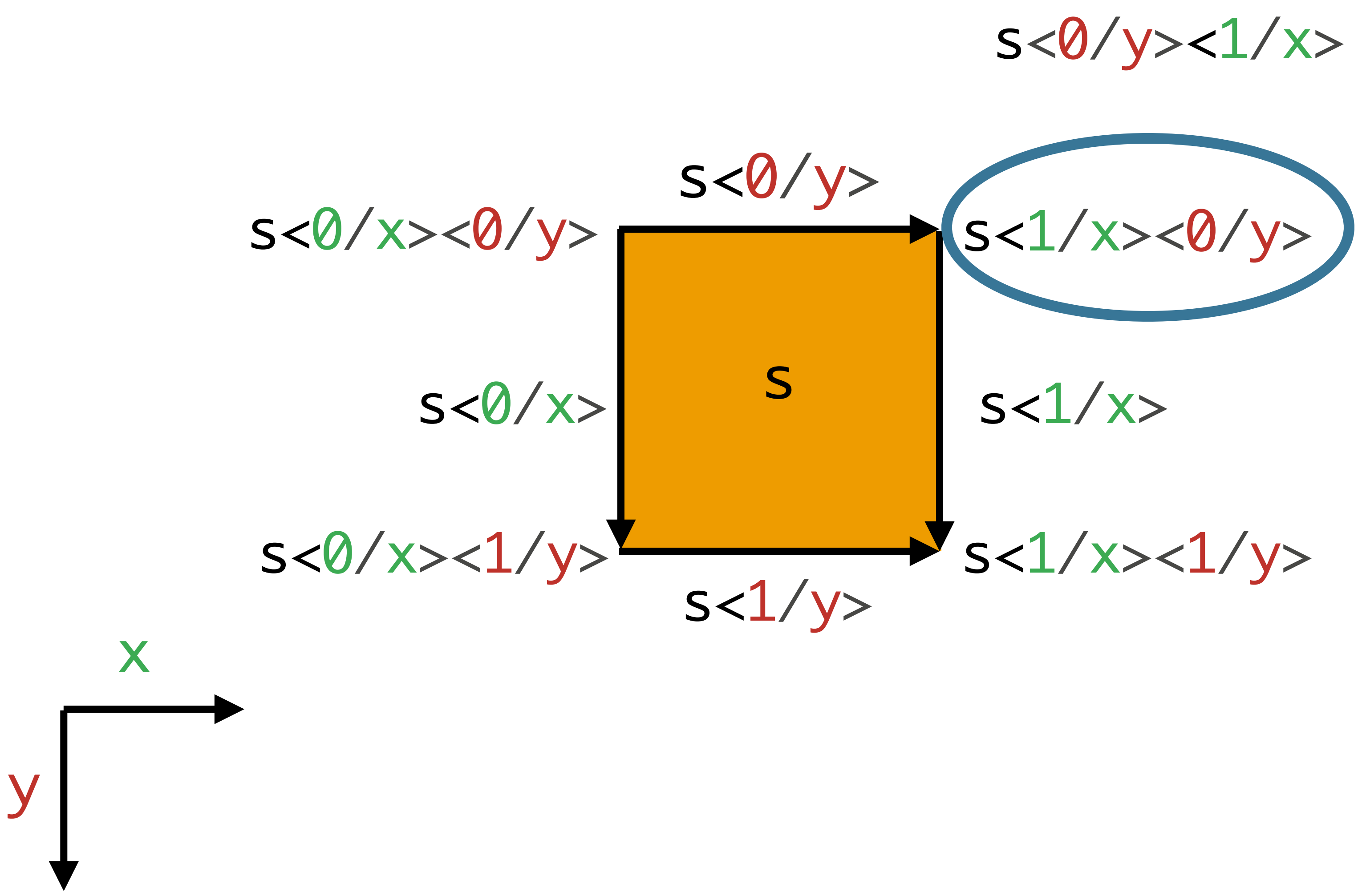

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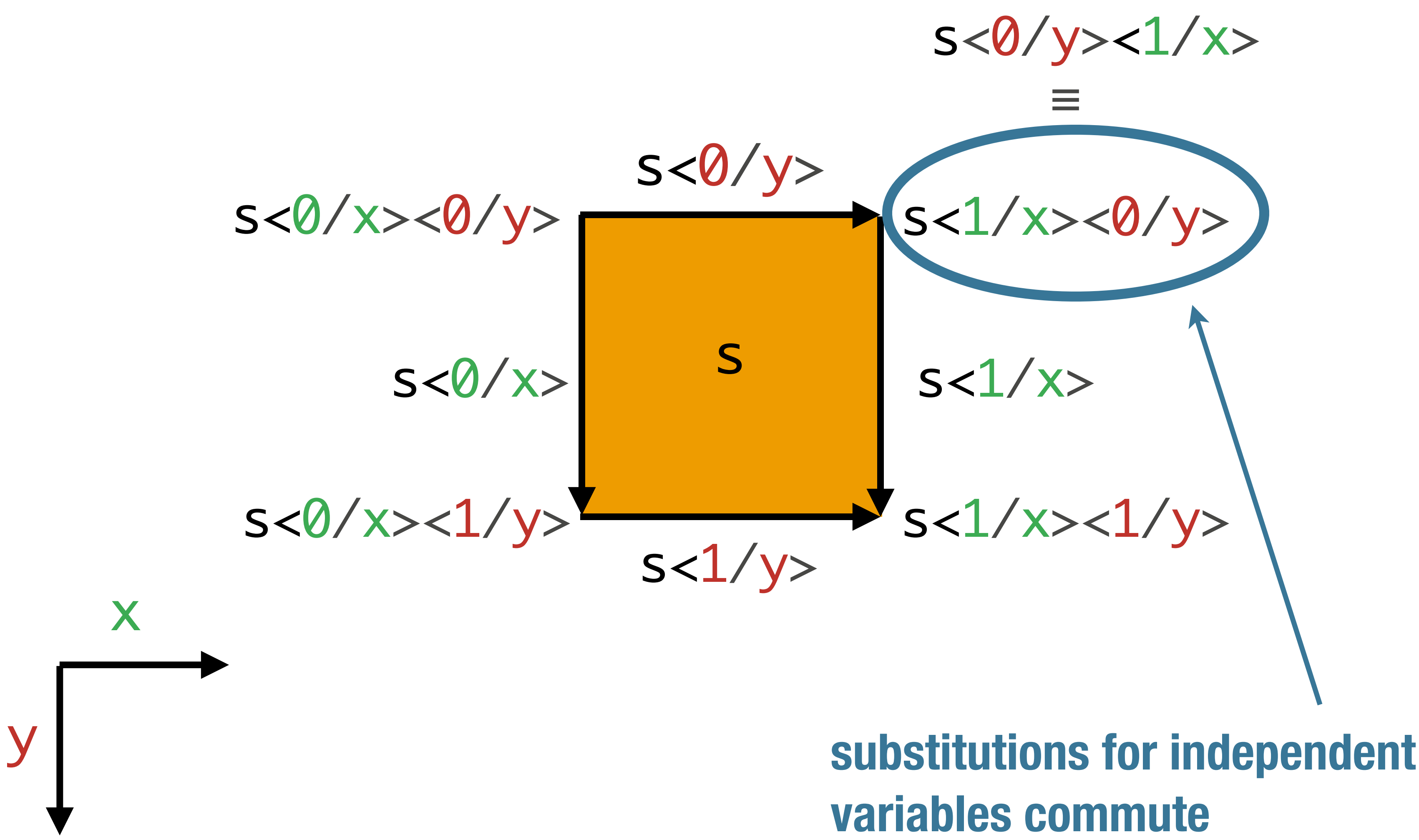

Degeneracies

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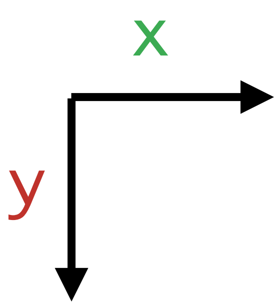

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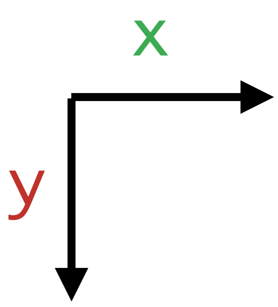

Degeneracies

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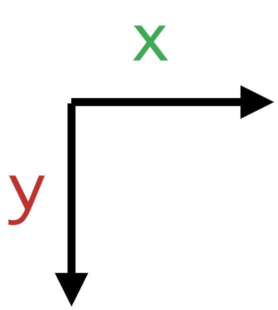
$$a{\uparrow}^{x} \langle 0/x \rangle = a$$



Degeneracies

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a↑x <0/x> = a

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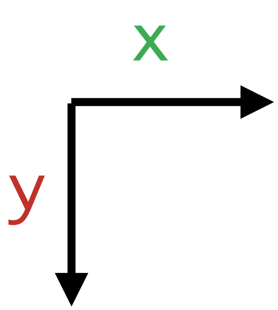

Degeneracies

$$a{\uparrow}^{x}\langle 0/x\rangle = a$$

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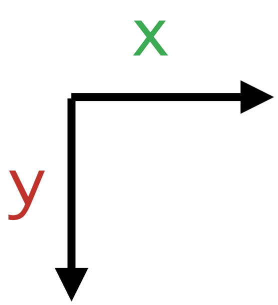

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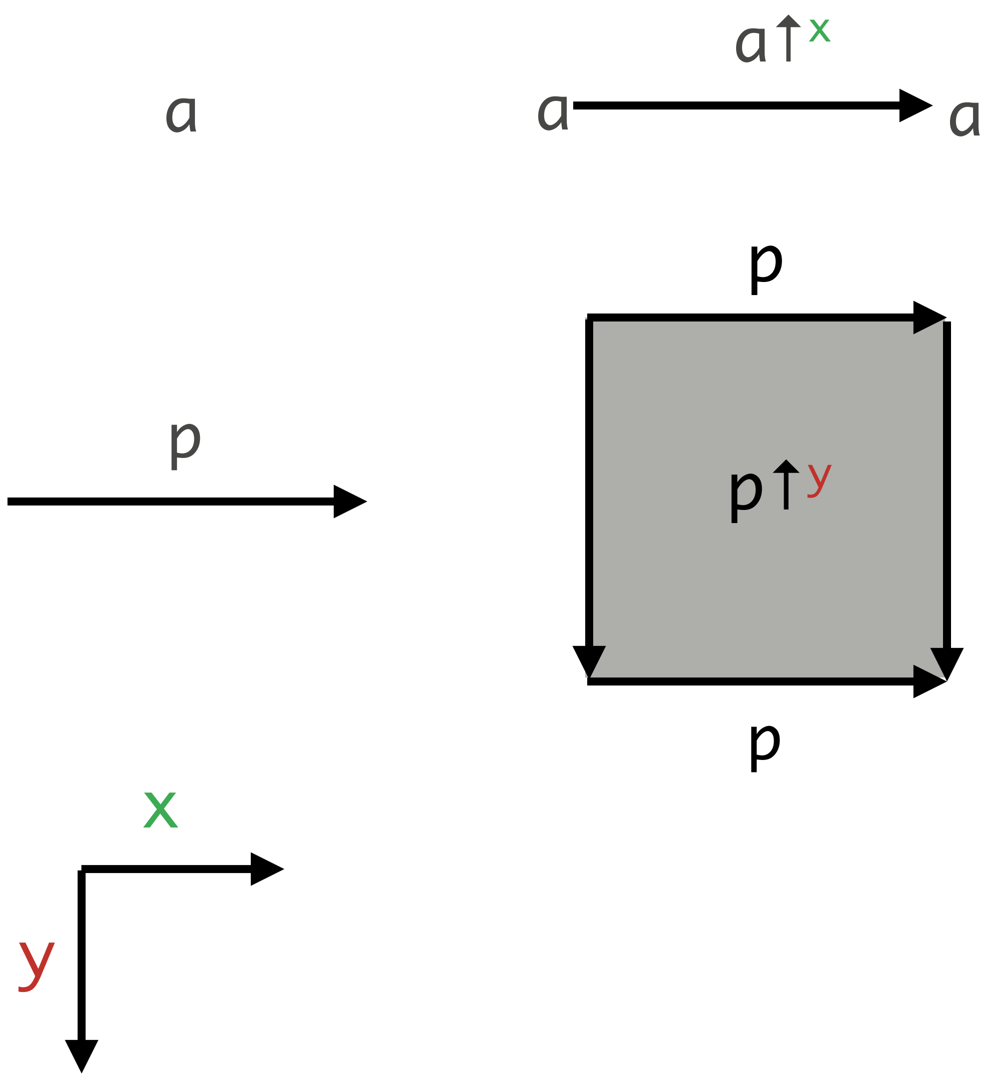

$a{\uparrow}^{x}\langle 0/x\rangle = a$

$a{\uparrow}^{x}\langle 1/x\rangle = a$

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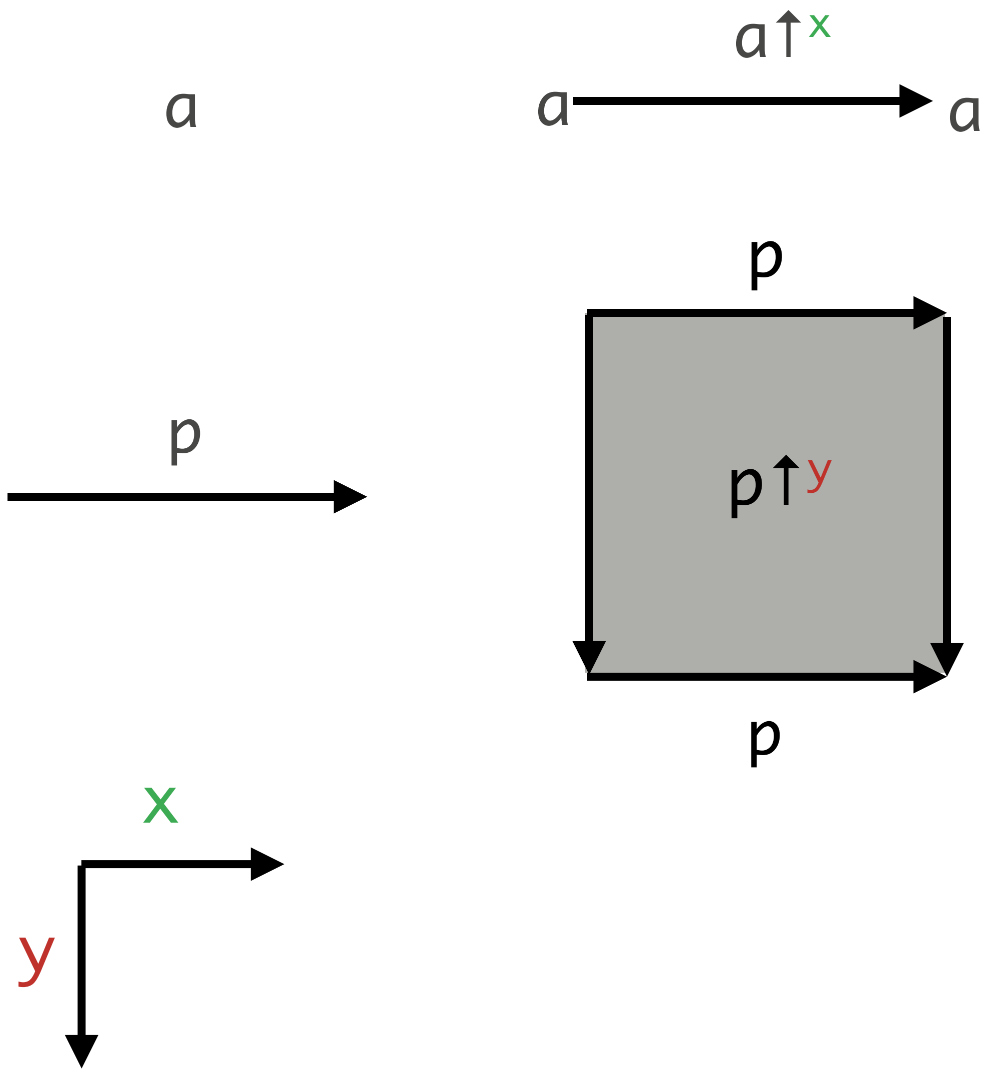


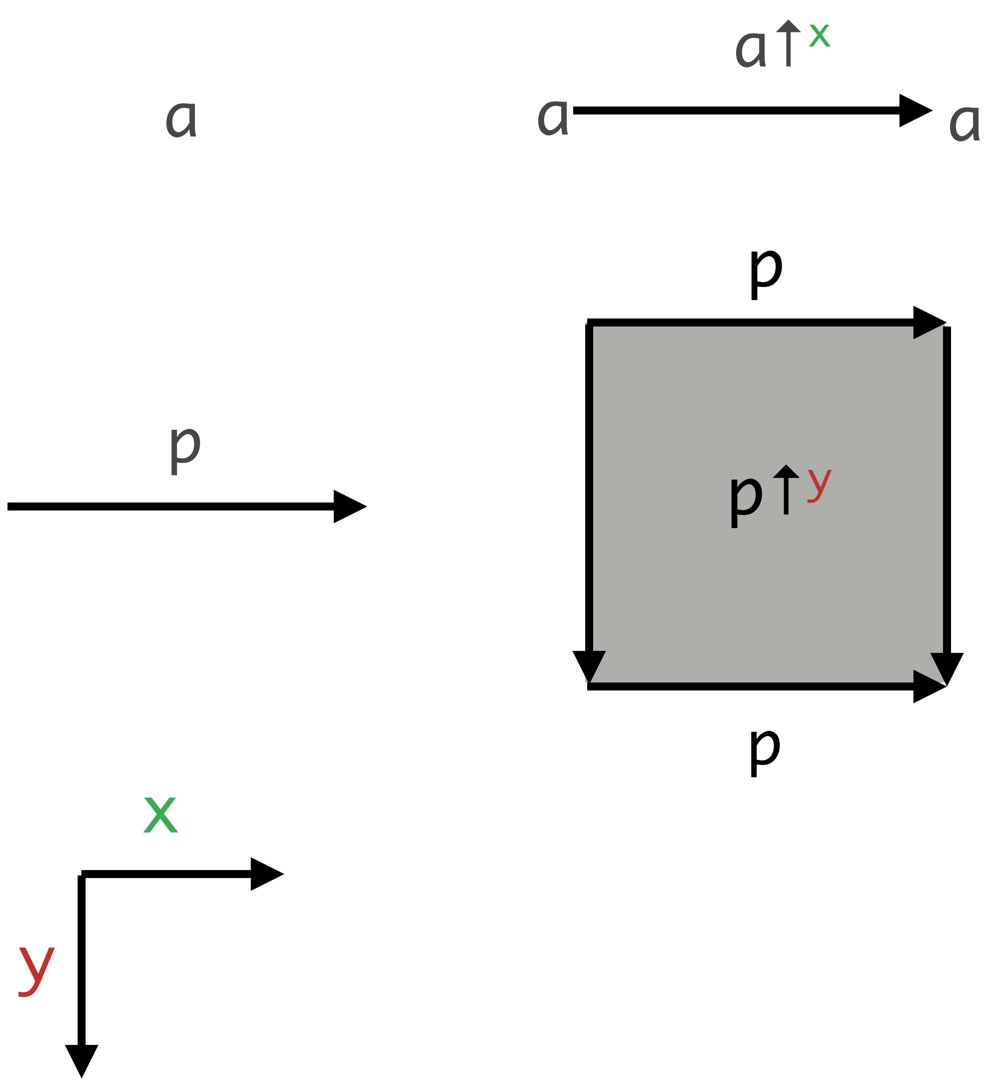
$a{\uparrow}^{x}\ \langle 0/x \rangle = a$

$a{\uparrow}^{x}\ \langle 1/x \rangle = a$

$p{\uparrow}^{y}\langle 0/y \rangle = p$

$p{\uparrow}^{y}\langle 1/y \rangle = p$

Degeneracies



$$a{\uparrow}^{x}\langle 0/x\rangle = a$$

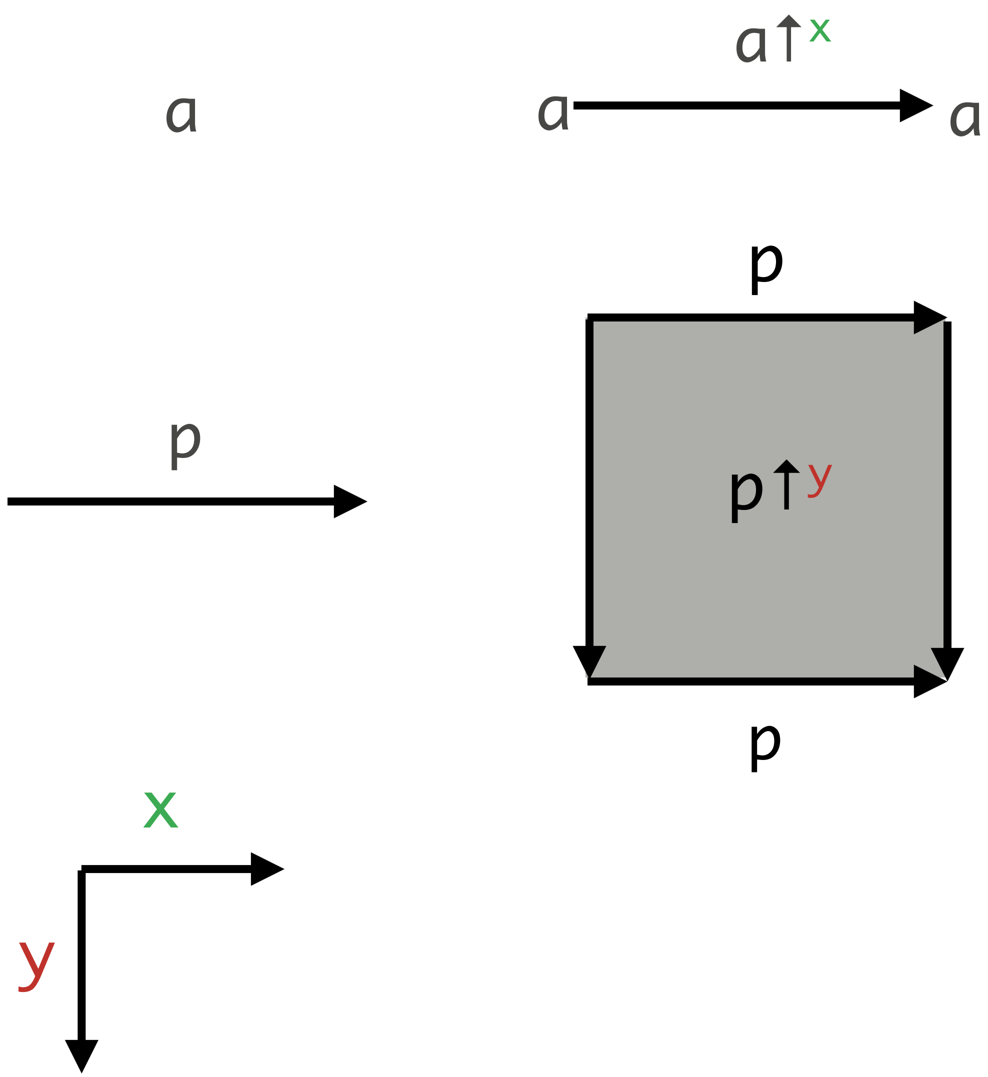
$$a{\uparrow}^{x}\langle 1/x\rangle = a$$

$$p{\uparrow}^{y}\langle 0/y\rangle = p$$

$$p{\uparrow}^{y}\langle 1/y\rangle = p$$

$$p{\uparrow}^{y}\langle 0/x\rangle = (p\langle 0/x\rangle){\uparrow}^{y}$$

Degeneracies



$$a{\uparrow}^{x}\langle 0/x\rangle = a$$

$$a{\uparrow}^{x}\langle 1/x\rangle = a$$

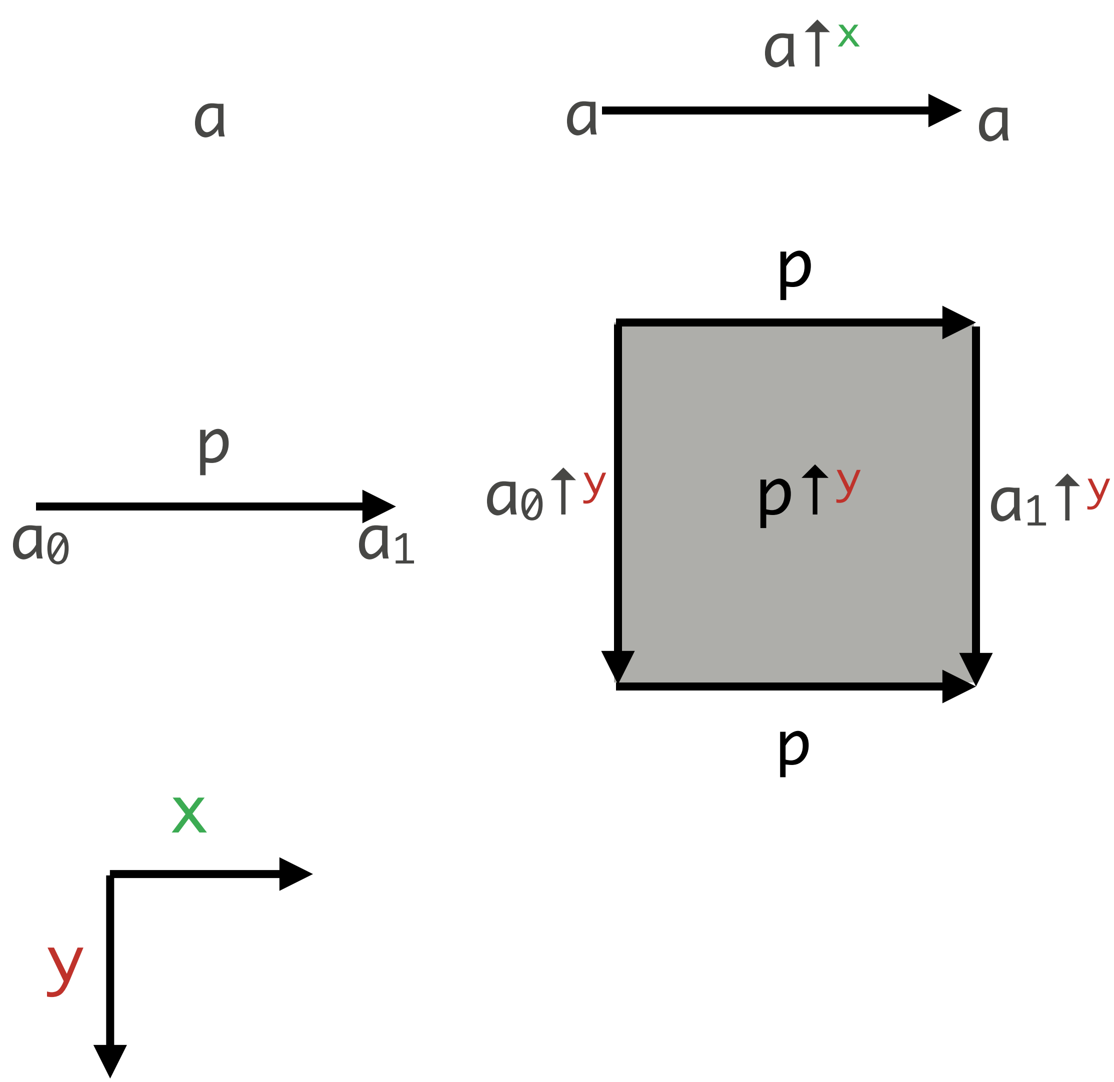
$$p{\uparrow}^{y}\langle 0/y\rangle = p$$

$$p{\uparrow}^{y}\langle 1/y\rangle = p$$

$$p{\uparrow}^{y}\langle 0/x\rangle = (p\langle 0/x\rangle){\uparrow}^{y}$$

$$p{\uparrow}^{y}\langle 1/x\rangle = (p\langle 1/x\rangle){\uparrow}^{y}$$

Degeneracies



$$a\uparrow^{x} \langle 0/x\rangle = a$$

$$a\uparrow^{x} \langle 1/x\rangle = a$$

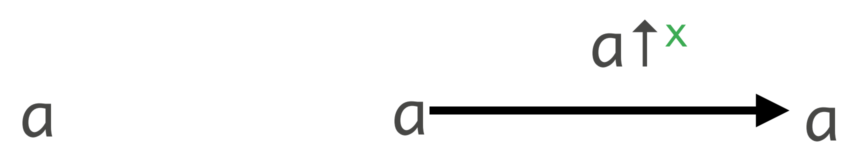
$$p\uparrow^{y}\langle 0/y\rangle = p$$

$$p\uparrow^{y}\langle 1/y\rangle = p$$

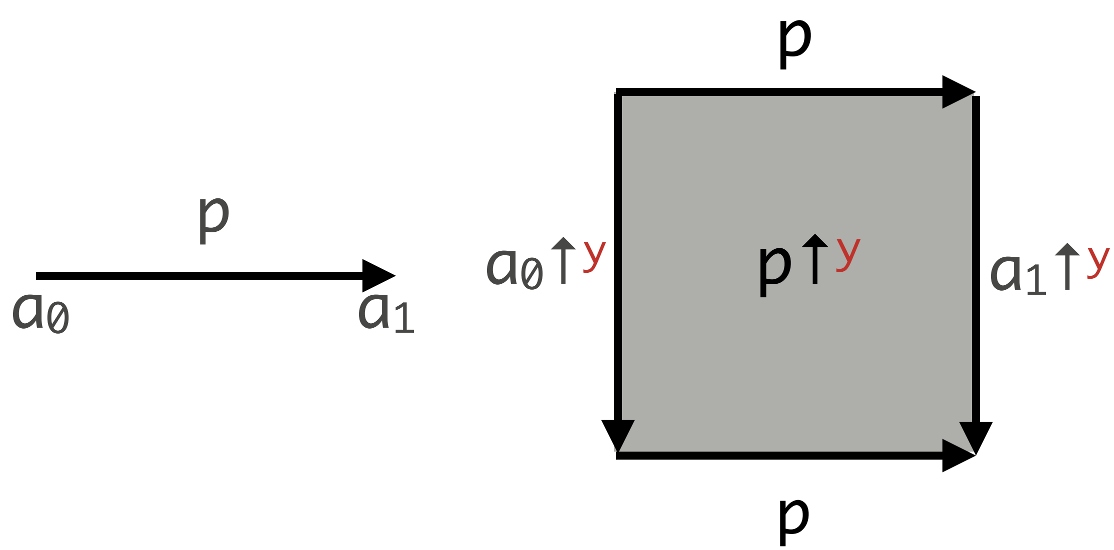
$$p\uparrow^{y}\langle 0/x\rangle = (p\langle 0/x\rangle)\uparrow^{y}$$

$$p\uparrow^{y}\langle 1/x\rangle = (p\langle 1/x\rangle)\uparrow^{y}$$

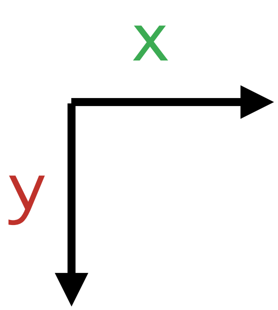
Degeneracies



$$a{\uparrow}^{x}\langle 0/x\rangle = a$$
$$a{\uparrow}^{x}\langle 1/x\rangle = a$$



$$p{\uparrow}^{y}\langle 0/y\rangle = p$$
$$p{\uparrow}^{y}\langle 1/y\rangle = p$$
$$p{\uparrow}^{y}\langle 0/x\rangle = (p\langle 0/x\rangle){\uparrow}^{y}$$
$$p{\uparrow}^{y}\langle 1/x\rangle = (p\langle 1/x\rangle){\uparrow}^{y}$$

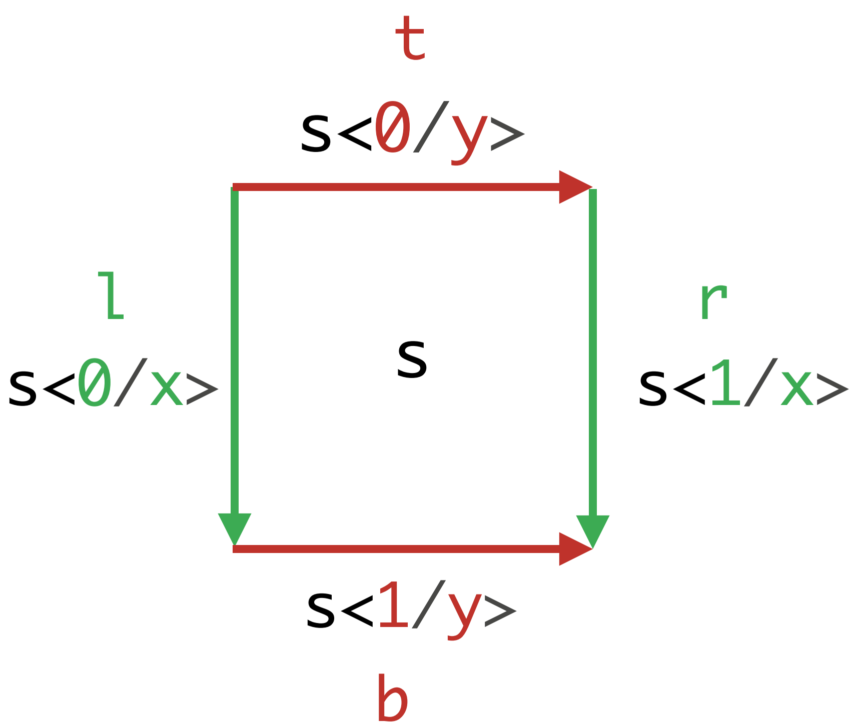


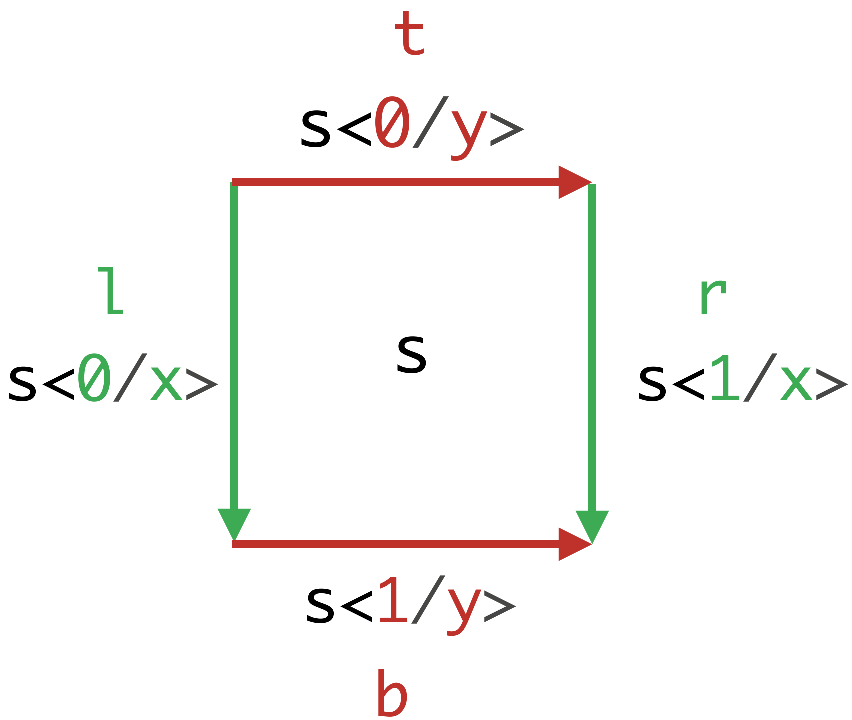
substitution after weakening is identity, otherwise pushes inside

Dimensions [Coquand,Pitts]

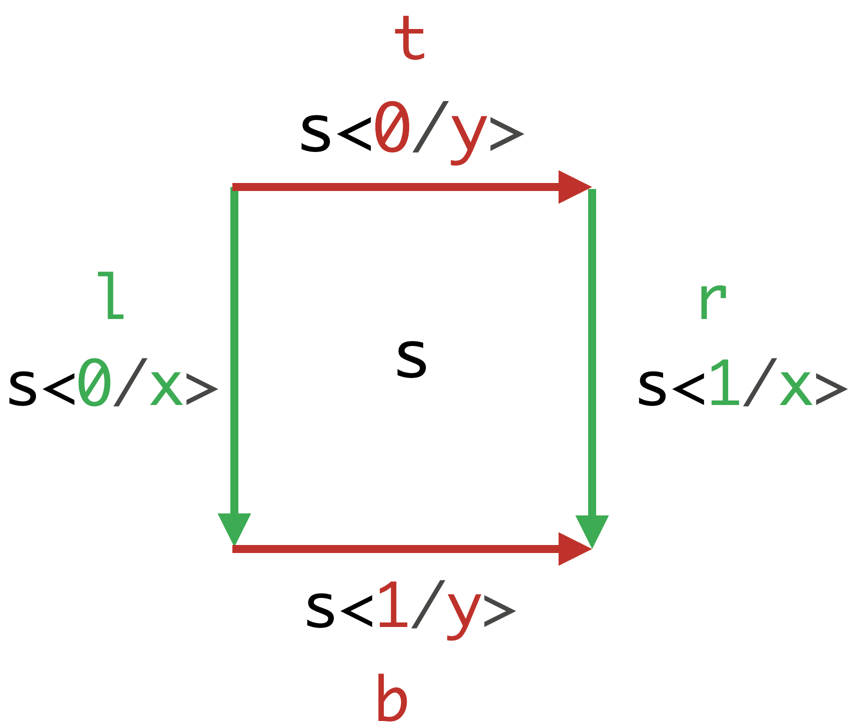
- n-dimensional cube has n dimension names free
- α-equivalence: make {x,…}-cube into {x’,...}-cube
- Substitution of 0 or 1: faces
- Weakening:degeneracy/reflexivity

Properties are *cubical identities*


t
s<0/y>
l
s<0/x>
s
r
s<1/x>
s<1/y>
b

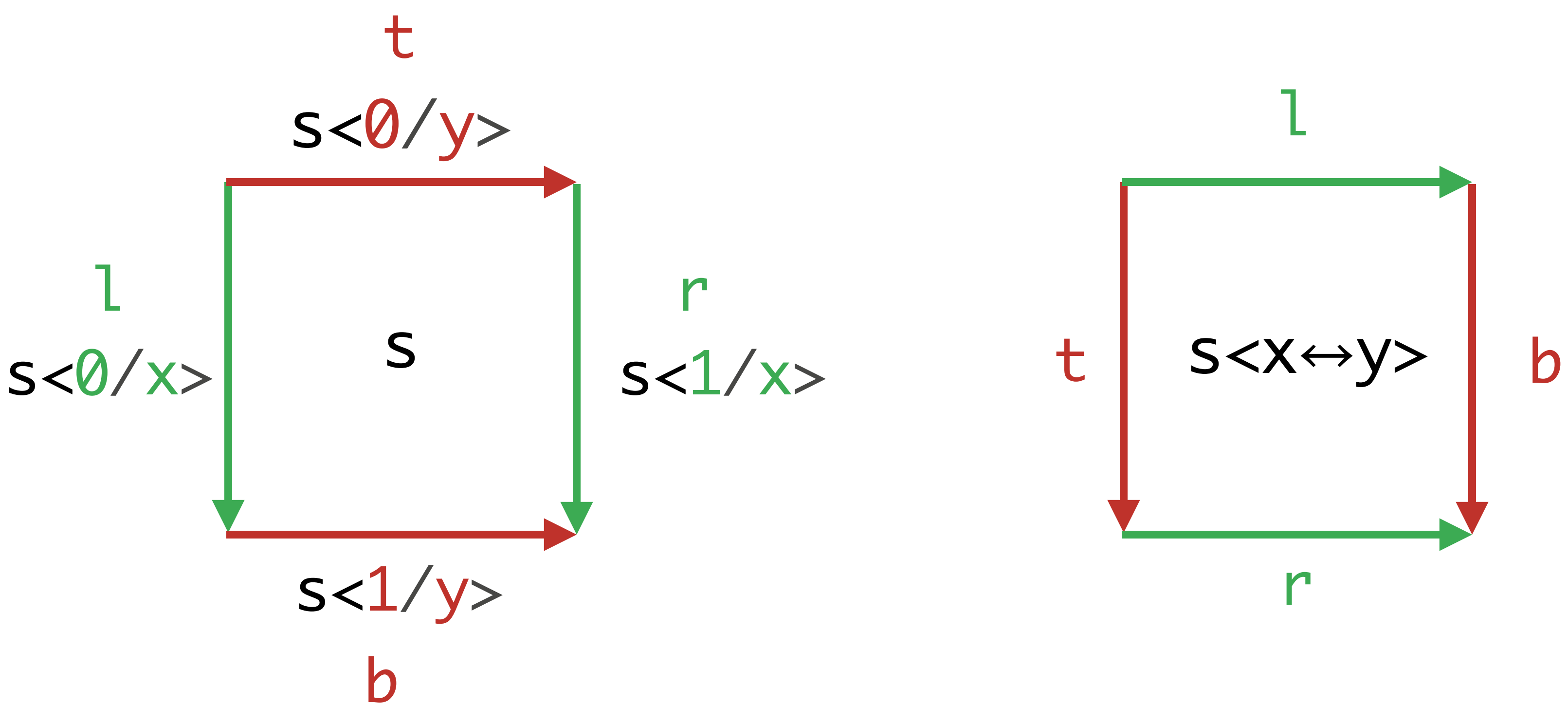


s<x↔y><0/x> = s<0/y>



s<x↔y><0/x> = s<0/y>

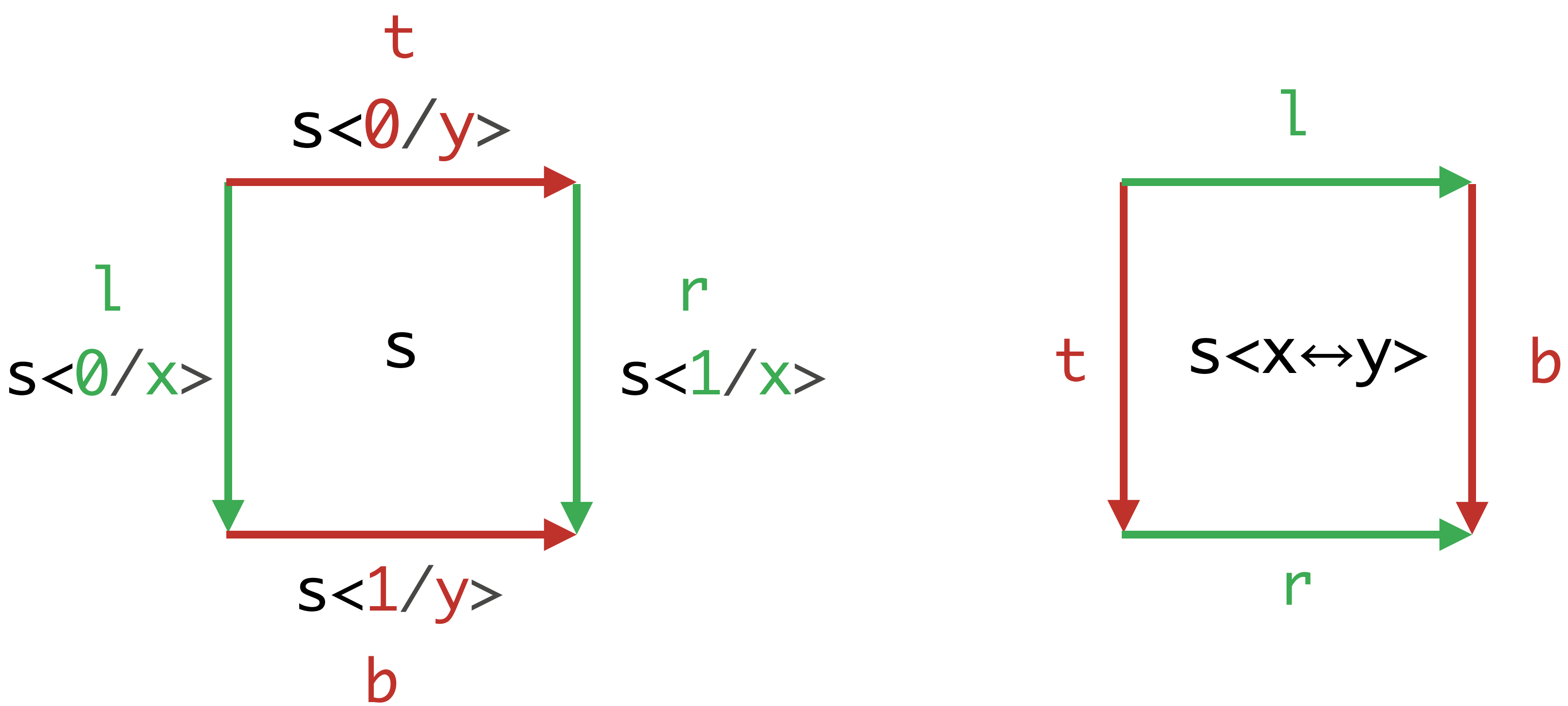
s<x↔y><0/y> = s<0/x>



$$s\langle x \leftrightarrow y\rangle\langle 0/x\rangle = s\langle 0/y\rangle$$

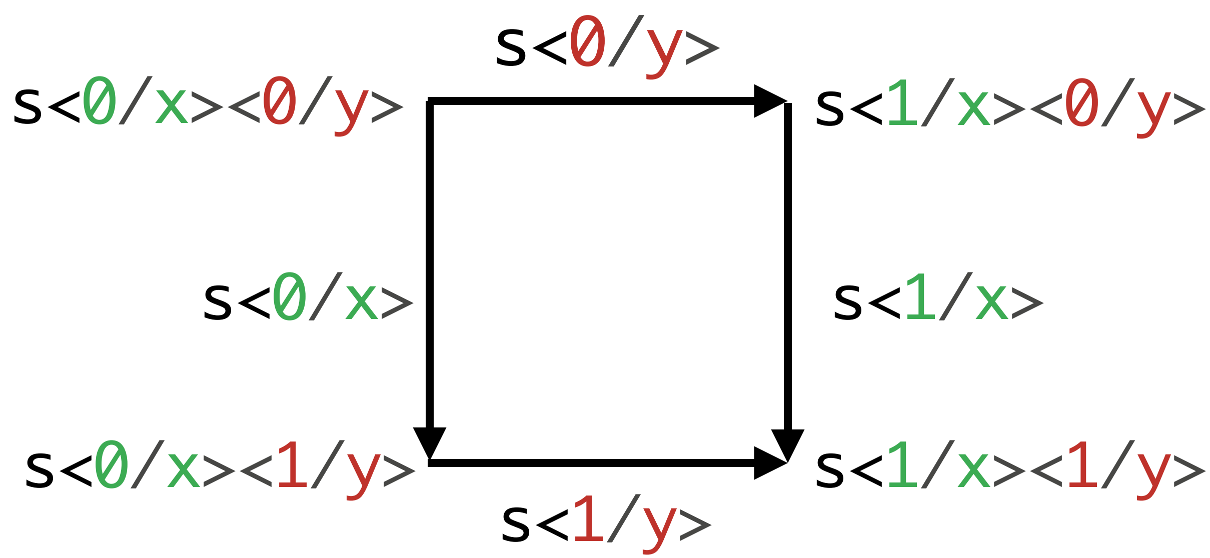
$$s\langle x \leftrightarrow y\rangle\langle 0/y\rangle = s\langle 0/x\rangle$$

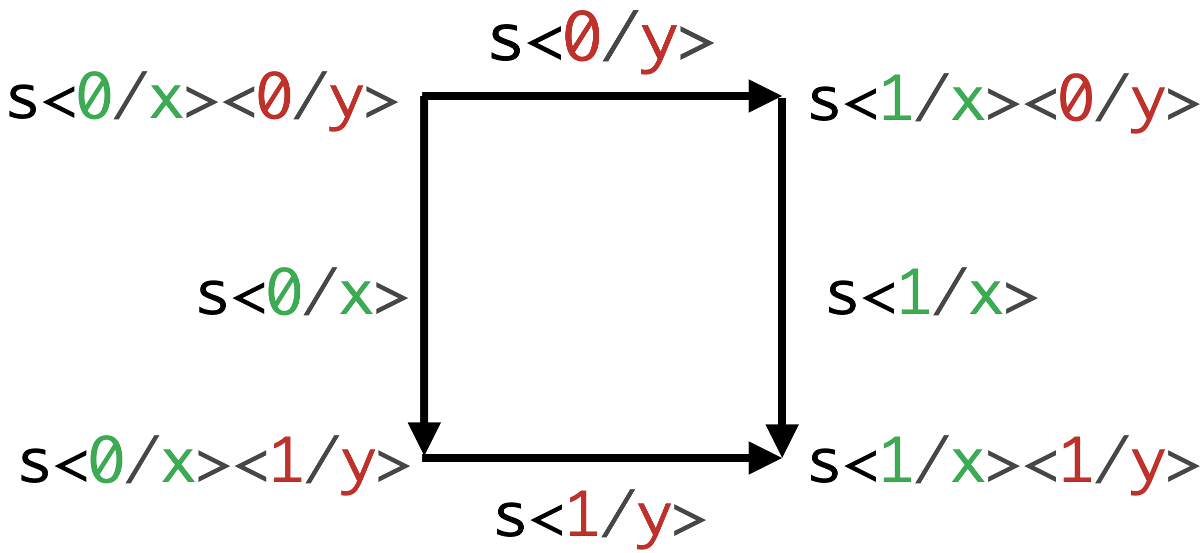
Symmetries



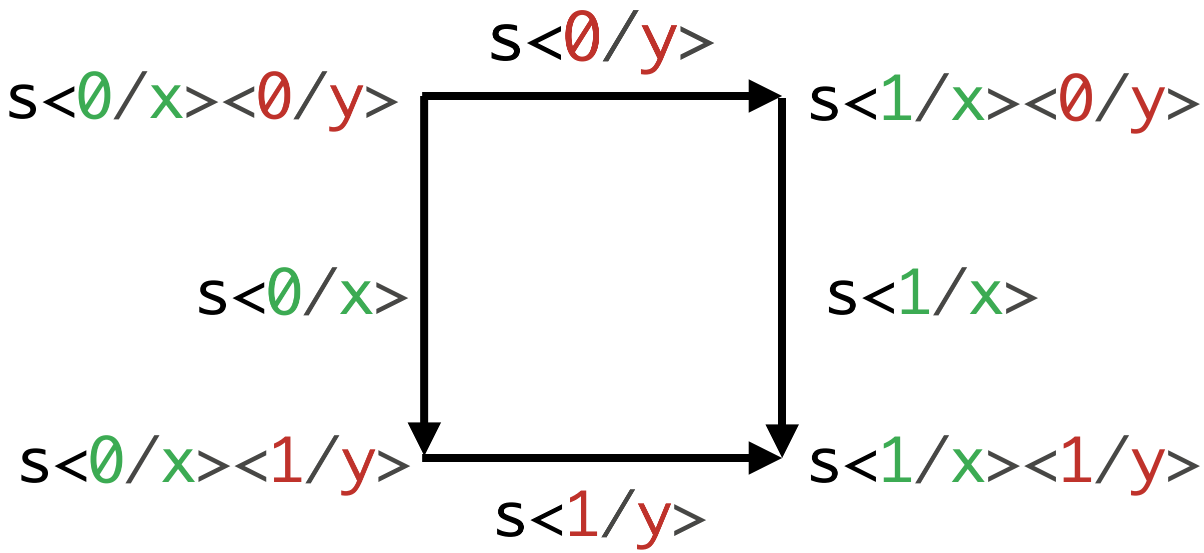
s<x↔y><0/x> = s<0/y>

s<x↔y><0/y> = s<0/x>


s<0/y>
s<0/x><0/y>
s<1/x><0/y>
s<0/x>
s<1/x>
s<0/x><1/y>
s<1/x><1/y>
s<1/y>

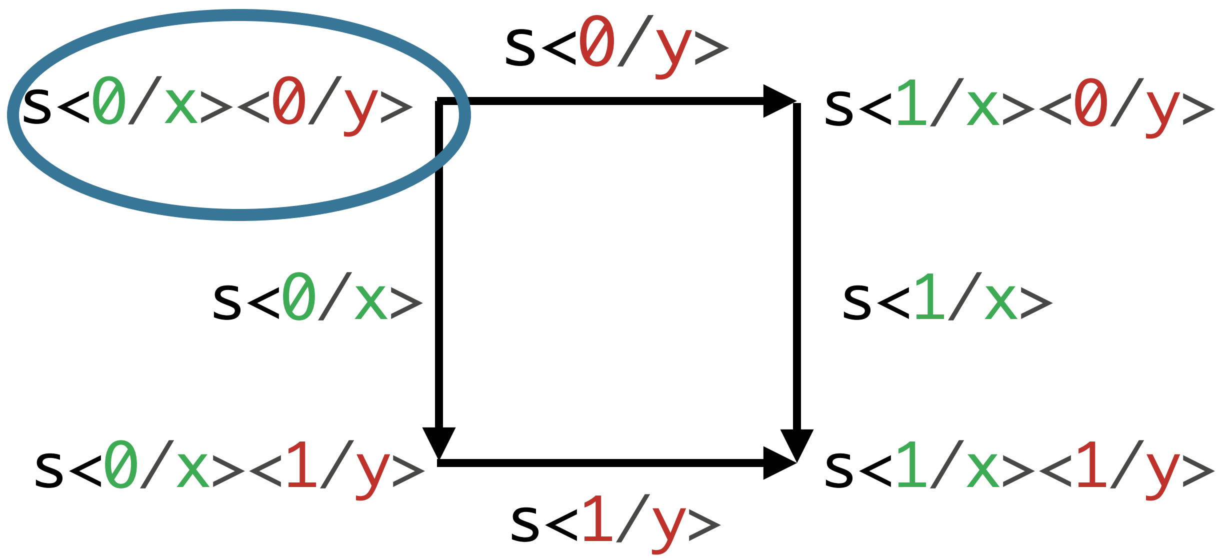


s<x/y> is a line ({x}-cube)



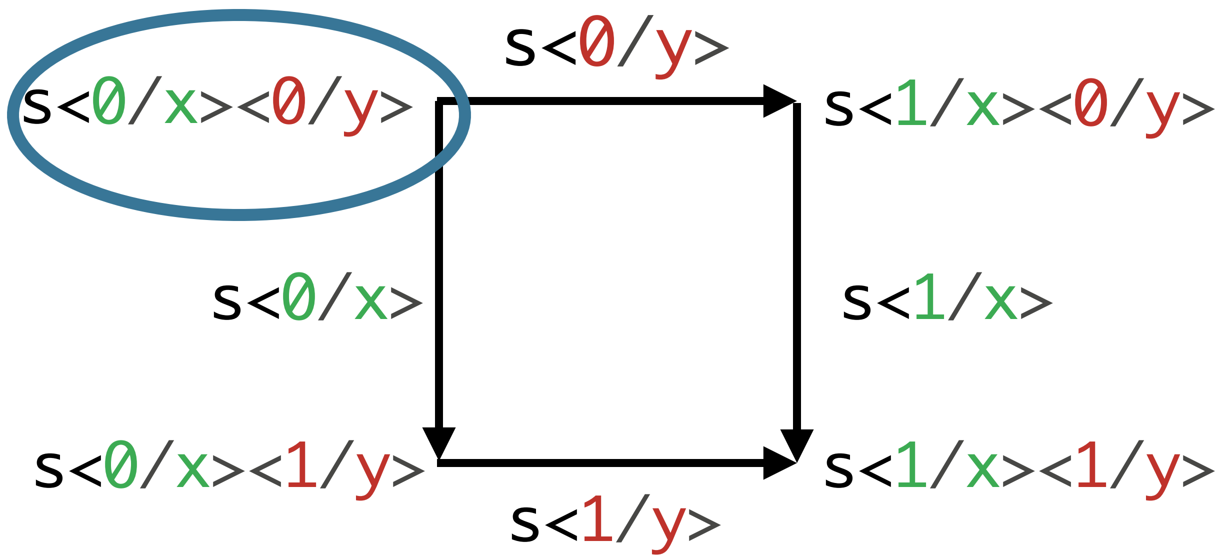
s<x/y> is a line ({x}-cube)

s<x/y><0/x> = s<0/x><0/y>



s<x/y> is a line ({x}-cube)

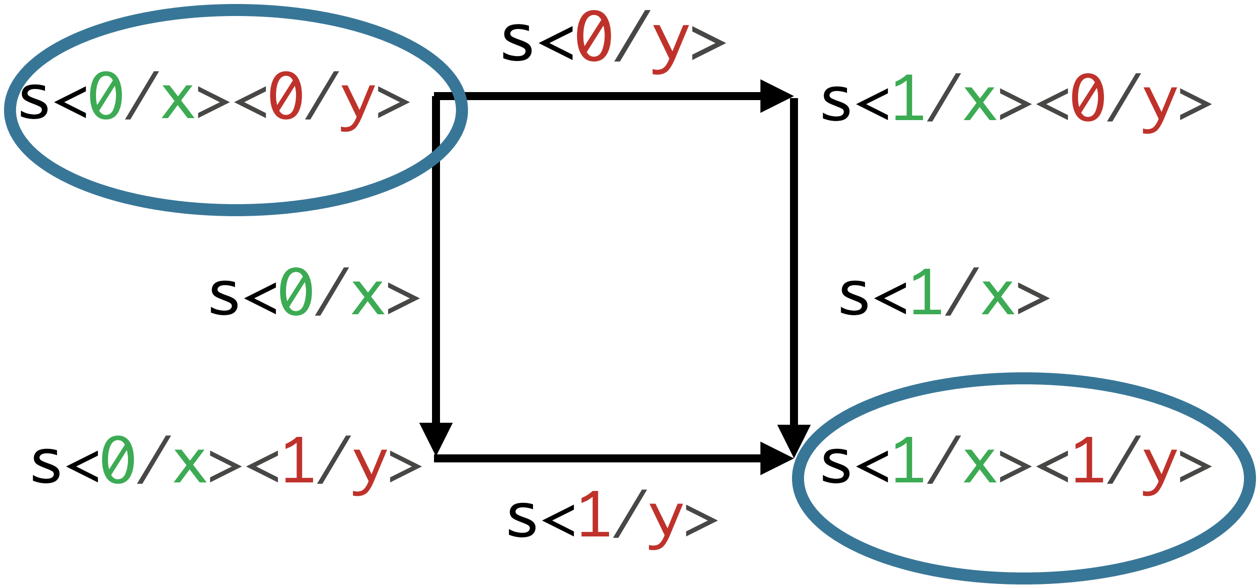
s<x/y><0/x> = s<0/x><0/y>



s<x/y> is a line ({x}-cube)

s<x/y><0/x> = s<0/x><0/y>

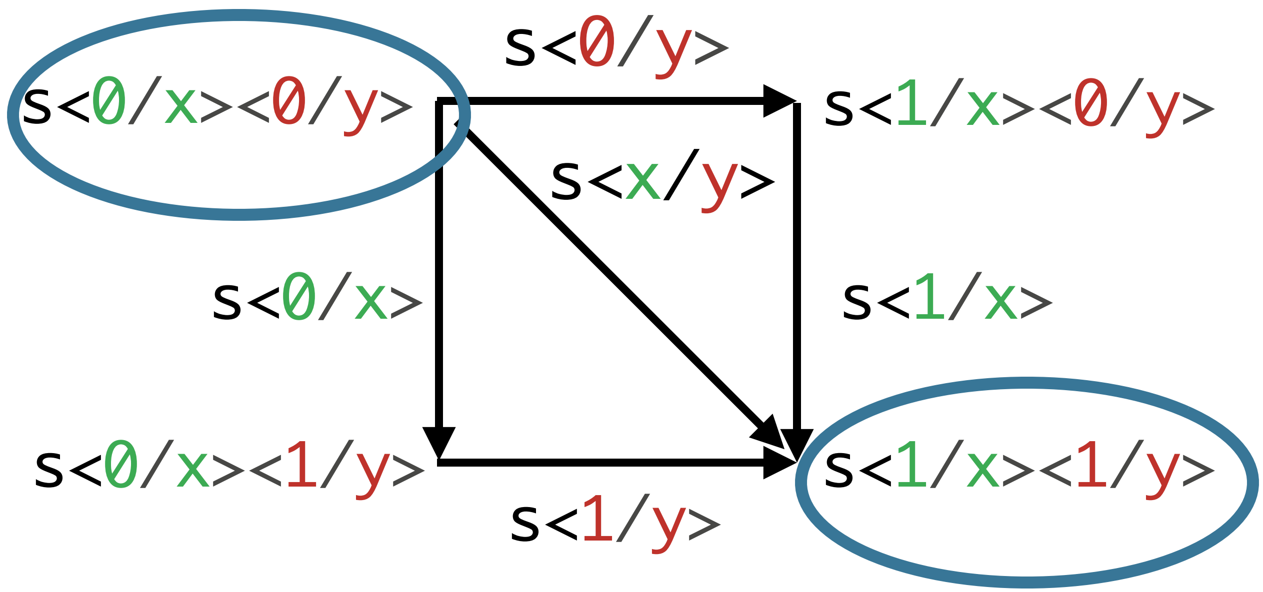
s<x/y><1/x> = s<1/x><1/y>



s<x/y> is a line ({x}-cube)

s<x/y><0/x> = s<0/x><0/y>

s<x/y><1/x> = s<1/x><1/y>

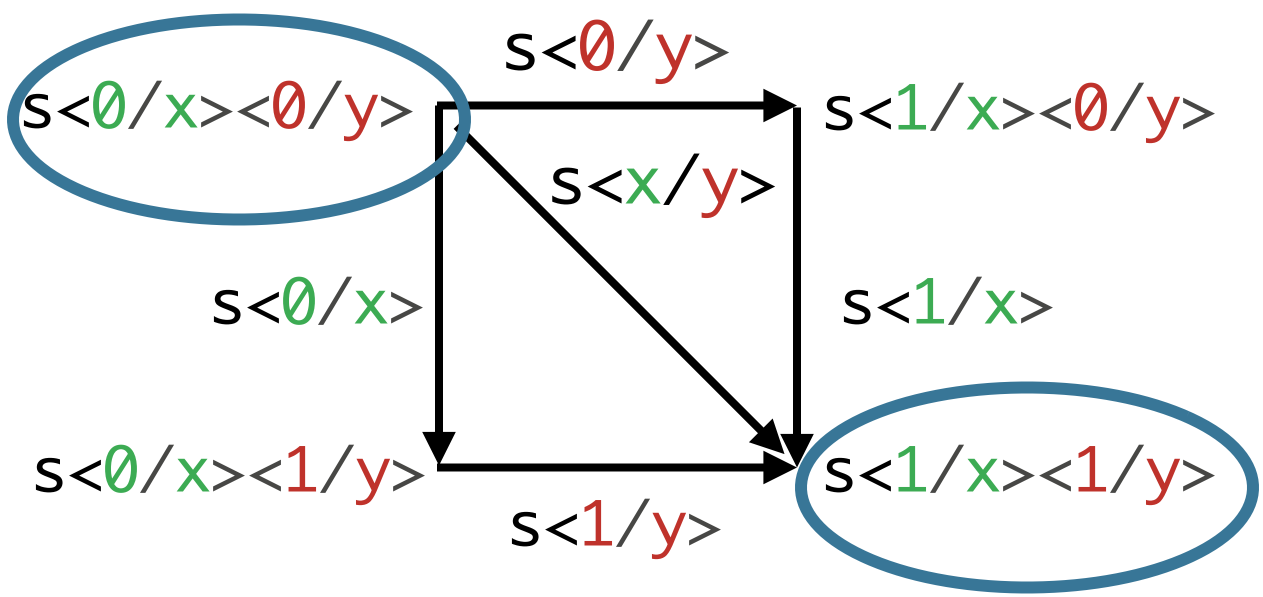


s<x/y> is a line ({x}-cube)

s<x/y><0/x> = s<0/x><0/y>

s<x/y><1/x> = s<1/x><1/y>

Diagonals



s<x/y> is a line ({x}-cube)

s<x/y><0/x> = s<0/x><0/y>

s<x/y><1/x> = s<1/x><1/y>

Dimensions [Coquand,Pitts]

- n-dimensional cube has n dimension names free
- α-equivalence: make {x,...}-cube into {x’,...}-cube
- Substitution of 0 or 1: faces
- Weakening: degeneracy/reflexivity
- Exchange: symmetry
- Contraction: diagonal

Properties are *cubical identities*