Circuit timing analysis, linear maps, and semantic morphisms

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Tabula

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Outline

Timing analysis

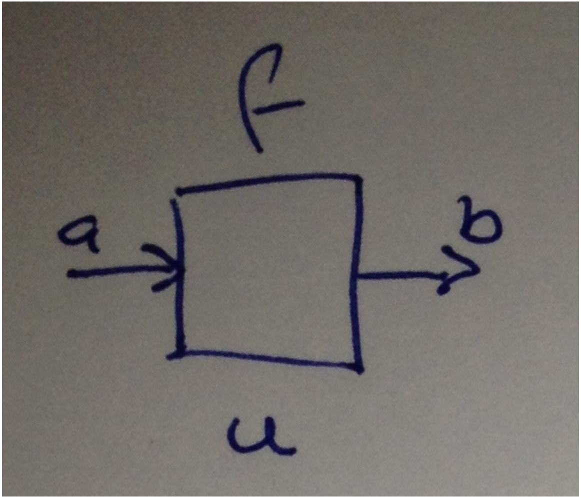
Linear transformations

Semantics and implementation

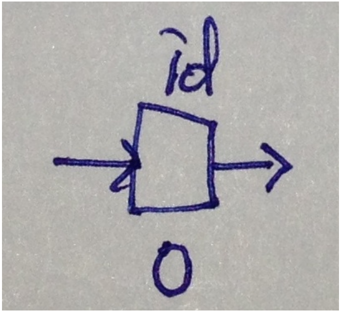
Timing analysis

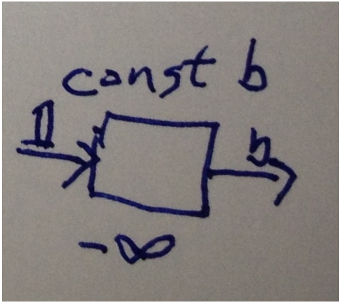
Simple timing analysis

Computation with a time delay:

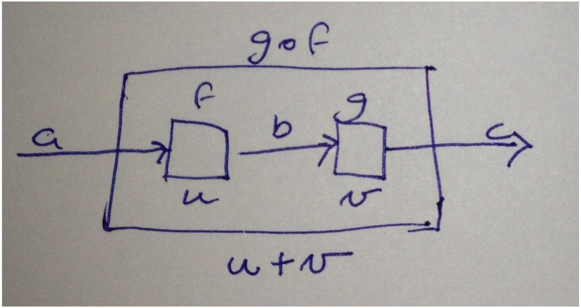


Trivial timings

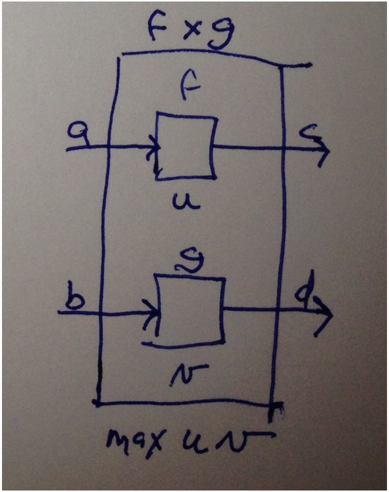




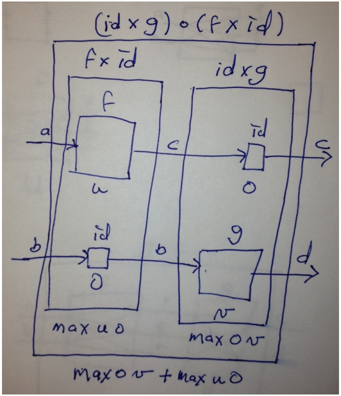
Sequential composition

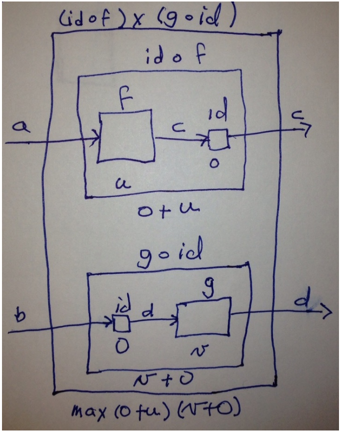


Parallel composition



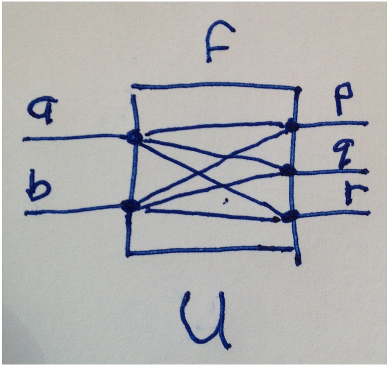
But ...





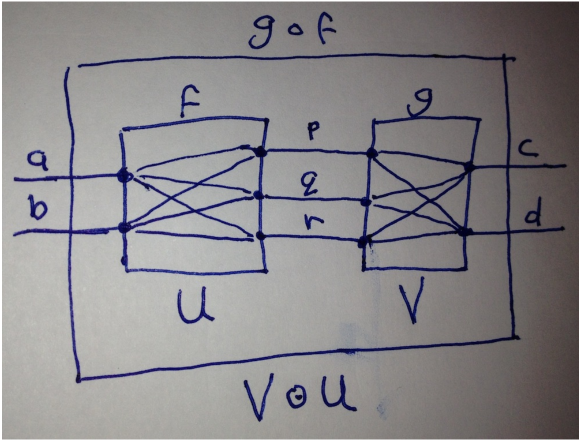
Oops: Same circuit $(f \times g)$, different timings.

Multi-path analysis

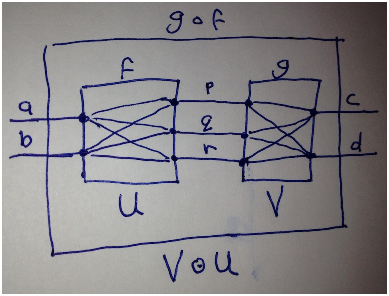


- Max delay for *each* input/output pair
- How do delays *compose*?

How do delays *compose*?



How do delays *compose*?



$V \odot U = W$, where

$$W_{i,k} = \operatorname*{Max}_j \left(U_{i,j} + V_{j,k}\right)$$

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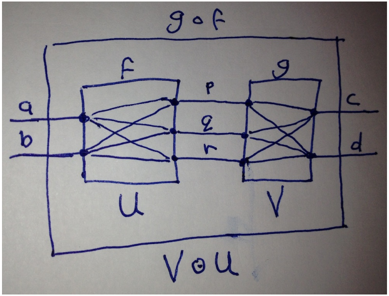
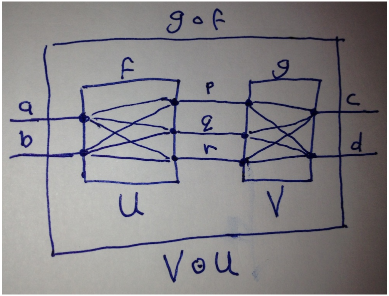


$V \odot U = W$, where

$$W_{i,k} = \operatorname*{Max}_j \left(U_{i,j} + V_{j,k}\right)$$

Look familiar?

How do delays *compose*?



$V \odot U = W$, where

$$W_{i,k} = \operatorname*{Max}_{j} \left(U_{i,j} + V_{j,k} \right)$$

Look familiar? Matrix multiplication?

MaxPlus algebra

type $Delay = MaxPlus\ Double$

data $MaxPlus\ a = MP\ a$

instance $Ord\ a \Rightarrow AdditiveGroup\ (MaxPlus\ a)$ **where**
  $MP\ a \mathbin{\hat{+}} MP\ b = MP\ (a\ \text{`}max\text{`}\ b)$

instance $(Ord\ a, Num\ a) \Rightarrow VectorSpace\ (MaxPlus\ a)$ **where**
  type $Scalar\ (MaxPlus\ a) = a$
  $a \cdot MP\ b = MP\ (a + b)$

Oops – We also need a zero.
VectorSpace is overkill. Module over a semi-ring suffices.

MaxPlus algebra

$$\textbf{type } Delay = MaxPlus\ Double$$

$$\textbf{data } MaxPlus\ a = -\infty \mid Fi\ a$$

$$\textbf{instance } Ord\ a \Rightarrow AdditiveGroup\ (MaxPlus\ a)\ \textbf{where}$$
$$0 = -\infty$$
$$MP\ a \mathbin{\hat{+}} MP\ b = MP\ (a \text{ `}max\text{` } b)$$
$$-\infty \mathbin{\hat{+}} \_ = -\infty$$
$$\_ \mathbin{\hat{+}} -\infty = -\infty$$

$$\textbf{instance } (Ord\ a, Num\ a) \Rightarrow VectorSpace\ (MaxPlus\ a)\ \textbf{where}$$
$$\textbf{type } Scalar\ (MaxPlus\ a) = a$$
$$a \cdot MP\ b = MP\ (a + b)$$
$$\_ \cdot -\infty = -\infty$$

Linear transformations

Representation?

How might we represent linear maps/transformations $a \multimap b$?

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How might we represent linear maps/transformations $a \multimap b$?

- Matrices
- Functions
- What else?

Matrices

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

Static typing?

Statically sized matrices

$$\textbf{type}\ Mat\ m\ n\ a = Vec\ m\ (Vec\ n\ a)$$

$$(\circ) :: (IsNat\ m, IsNat\ o) \Rightarrow$$
$$\quad Mat\ n\ o\ D \rightarrow Mat\ m\ n\ D \rightarrow Mat\ o\ m\ D$$
$$no \circ mn = crossF\ dot\ (transpose\ no)\ mn$$

$$crossF :: (IsNat\ m, IsNat\ o) \Rightarrow$$
$$\quad (a \rightarrow b \rightarrow c) \rightarrow Vec\ o\ a \rightarrow Vec\ m\ b \rightarrow Mat\ o\ m\ c$$
$$crossF\ f\ as\ bs = (\lambda a \rightarrow f\ a \mathbin{<\$>} bs) \mathbin{<\$>} as$$

$$dot :: (Ord\ a, Num\ a) \Rightarrow$$
$$\quad Vec\ n\ a \rightarrow Vec\ n\ a \rightarrow a$$
$$u\ \text{`}dot\text{`}\ v = sum\ (zipWithV\ (*)\ u\ v)$$

Generalizing

$$\textbf{type}\ Mat\ m\ n\ a = m\ (n\ a)$$

$$\begin{aligned}&(\circ) :: (Functor\ m, Applicative\ n, Traversable\ n, Applicative\ o) \Rightarrow\\&\qquad Mat\ n\ o\ D \to Mat\ m\ n\ D \to Mat\ o\ m\ D\\&no \circ mn = crossF\ dot\ (sequenceA\ no)\ mn\end{aligned}$$

$$\begin{aligned}&crossF :: (Functor\ m, Functor\ o) \Rightarrow\\&\qquad\qquad (a \to b \to c) \to o\ a \to m\ b \to Mat\ o\ m\ c\\&crossF\ f\ as\ bs = (\lambda a \to f\ a \mathbin{<\$>} bs) \mathbin{<\$>} as\end{aligned}$$

$$\begin{aligned}&dot :: (Foldable\ n, Applicative\ n, Ord\ a, Num\ a) \Rightarrow\\&\qquad n\ a \to n\ a \to a\\&u\ `dot`\ v = sum\ (liftA2\ (*)\ u\ v)\end{aligned}$$

Represent via type family (old)

$$\textbf{class}\ \mathit{VectorSpace}\ v \Rightarrow \mathit{HasBasis}\ v\ \textbf{where}$$
$$\quad \textbf{type}\ \mathit{Basis}\ v :: *$$
$$\quad \mathit{coord} :: v \rightarrow (\mathit{Basis}\ v \rightarrow \mathit{Scalar}\ v)$$

Linear map as memoized function from basis:

$$\textbf{newtype}\ a \multimap b = L\ (\mathit{Basis}\ a \rightarrow^{M} b)$$

See *Beautiful differentiation* (ICFP 2009).

Represent as GADT

$$
\begin{aligned}
&\textbf{data}\ a \multimap b\ \textbf{where} \\
&\quad \mathit{Dot} :: \mathit{InnerSpace}\ b \Rightarrow \\
&\qquad\qquad b \to (b \multimap \mathit{Scalar}\ b) \\
&\quad (:\triangle) :: \mathit{VS}_3\ a\ c\ d \Rightarrow \qquad \text{-- vector spaces with same scalar field} \\
&\qquad\qquad (a \multimap c) \to (a \multimap d) \to (a \multimap c \times d)
\end{aligned}
$$

Semantics and implementation

Semantics

$$
\begin{aligned}
[\![\cdot]\!] &:: (a \multimap b) \to (a \to b) \\
[\![Dot\ b]\!] &= dot\ b \\
[\![f :\!\vartriangle g]\!] &= [\![f]\!] \vartriangle [\![g]\!]
\end{aligned}
$$

where, on functions,

$$(f \vartriangle g)\ a = (f\ a, g\ a)$$

Recall:

$$
\begin{aligned}
&\textbf{data}\ a \multimap b\ \textbf{where} \\
&\quad Dot :: InnerSpace\ b \Rightarrow b \to (b \multimap Scalar\ b) \\
&\quad (:\!\vartriangle) :: VS_3\ a\ c\ d \quad \Rightarrow (a \multimap c) \to (a \multimap d) \to (a \multimap c \times d)
\end{aligned}
$$

Semantic type class morphisms

Category instance specification:	*Arrow* instance specification:
$\llbracket id \rrbracket \equiv id$ $\llbracket g \circ f \rrbracket \equiv \llbracket g \rrbracket \circ \llbracket f \rrbracket$	$\llbracket f \vartriangle g \rrbracket \equiv \llbracket f \rrbracket \vartriangle \llbracket g \rrbracket$ $\llbracket f \times g \rrbracket \equiv \llbracket f \rrbracket \times \llbracket g \rrbracket$

where

$$
\begin{aligned}
(\vartriangle) &:: Arrow\ (\leadsto) \Rightarrow (a \leadsto c) \to (a \leadsto d) \to (a \leadsto c \times d) \\
(\times) &:: Arrow\ (\leadsto) \Rightarrow (a \leadsto c) \to (b \leadsto d) \to (a \times b \leadsto c \times d)
\end{aligned}
$$

The *Category* and *Arrow* laws then follow.

Deriving a *Category* instance

One case:

$$
\begin{aligned}
& [\![(f :\!\vartriangle g) \circ h]\!] \\
\equiv\ & ([\![f]\!] \vartriangle [\![g]\!]) \circ [\![h]\!] \\
\equiv\ & [\![f]\!] \circ [\![h]\!] \vartriangle [\![g]\!] \circ [\![h]\!] \\
\equiv\ & [\![f \circ h \vartriangle g \circ h]\!]
\end{aligned}
$$

(where $f \circ h \vartriangle g \circ h \equiv (f \circ h) \vartriangle (g \circ h)$). Uses:

$$(f \vartriangle g) \circ h \equiv f \circ h \vartriangle g \circ h$$

Implementation:

$$(f :\!\vartriangle g) \circ h = f \circ h :\!\vartriangle g \circ h$$

Deriving a *Category* instance

$$
\begin{aligned}
&[\![Dot\ s \circ Dot\ b]\!] \\
\equiv\ & dot\ s \circ dot\ b \\
\equiv\ & dot\ (s \cdot b) \\
\equiv\ & [\![Dot\ (s \cdot b)]\!]
\end{aligned}
$$

$$
\begin{aligned}
&[\![Dot\ (a, b) \circ (f :\!\vartriangle g)]\!] \\
\equiv\ & dot\ (a, b) \circ ([\![f]\!] \vartriangle [\![g]\!]) \\
\equiv\ & add \circ (dot\ a \circ [\![f]\!] \vartriangle dot\ b \circ [\![g]\!]) \\
\equiv\ & dot\ a \circ [\![f]\!] \mathbin{\hat{+}} dot\ b \circ [\![g]\!] \\
\equiv\ & [\![Dot\ a \circ f \mathbin{\hat{+}} Dot\ b \circ g]\!]
\end{aligned}
$$

Uses:

$$
\begin{aligned}
dot\ (a, b) &\equiv add \circ (dot\ a \times dot\ b) \\
(k \times h) \circ (f \vartriangle g) &\equiv k \circ f \vartriangle h \circ g \\
[\![f \mathbin{\hat{+}} g]\!] &\equiv [\![f]\!] \mathbin{\hat{+}} [\![g]\!]
\end{aligned}
$$

Deriving an *Arrow* instance

$$
\begin{aligned}
& [\![f \vartriangle g]\!] \\
\equiv\ & [\![f]\!] \vartriangle [\![g]\!] \\
\equiv\ & [\![f :\!\vartriangle g]\!]
\end{aligned}
$$

$$
\begin{aligned}
& [\![f \times g]\!] \\
\equiv\ & [\![f]\!] \times [\![g]\!] \\
\equiv\ & [\![f]\!] \circ \mathit{fst} \vartriangle [\![g]\!] \circ \mathit{snd} \\
\equiv\ & [\![\mathit{compFst}\ f]\!] \vartriangle [\![\mathit{compSnd}\ g]\!] \\
\equiv\ & [\![\mathit{compFst}\ f :\!\vartriangle \mathit{compSnd}\ g]\!]
\end{aligned}
$$

assuming

$$
\begin{aligned}
[\![\mathit{compFst}\ \ f]\!] &\equiv [\![f]\!] \circ \mathit{fst} \\
[\![\mathit{compSnd}\ g]\!] &\equiv [\![g]\!] \circ \mathit{snd}
\end{aligned}
$$

Composing with *fst* and *snd*

$$compFst :: VS_3\ a\ b\ c \Rightarrow a \multimap c \rightarrow a \times b \multimap c$$
$$compSnd :: VS_3\ a\ b\ c \Rightarrow b \multimap c \rightarrow a \times b \multimap c$$

Derivation:

$$dot\ a \circ fst \equiv dot\ (a, 0)$$
$$(f \vartriangle g) \circ fst \equiv f \circ fst \vartriangle g \circ fst$$

Implementation:

$$compFst\ (Dot\ a) = Dot\ (a, 0)$$
$$compFst\ (f :\vartriangle g) = compFst\ f \vartriangle compFst\ g$$

$$compSnd\ (Dot\ b) = Dot\ (0, b)$$
$$compSnd\ (f :\vartriangle g) = compSnd\ f \vartriangle compSnd\ g$$

Adding linear maps

$$
\begin{aligned}
& [\![Dot\ b \mathbin{\hat{+}} Dot\ c]\!] \\
\equiv\ & dot\ b \mathbin{\hat{+}} dot\ c \\
\equiv\ & dot\ (b \mathbin{\hat{+}} c) \\
\equiv\ & [\![Dot\ (b \mathbin{\hat{+}} c)]\!]
\end{aligned}
$$

$$
\begin{aligned}
& [\![(f :\!\vartriangle g) \mathbin{\hat{+}} (h :\!\vartriangle k)]\!] \\
\equiv\ & ([\![f]\!] \vartriangle [\![g]\!]) \mathbin{\hat{+}} ([\![h]\!] \vartriangle [\![k]\!]) \\
\equiv\ & ([\![f]\!] \mathbin{\hat{+}} [\![h]\!]) \vartriangle ([\![g]\!] \mathbin{\hat{+}} [\![k]\!]) \\
\equiv\ & [\![(f \mathbin{\hat{+}} h) \vartriangle (g \mathbin{\hat{+}} k)]\!]
\end{aligned}
$$

Other cases don't type-check.
Uses (on functions):

$$(f \vartriangle g) \mathbin{\hat{+}} (h \vartriangle k) \equiv (f \mathbin{\hat{+}} h) \vartriangle (g \mathbin{\hat{+}} k)$$

What next?

- Fancier timing analysis
- What else is linear?
- More examples of semantic type class morphisms