Simple
Monadic
Equational Reasoning

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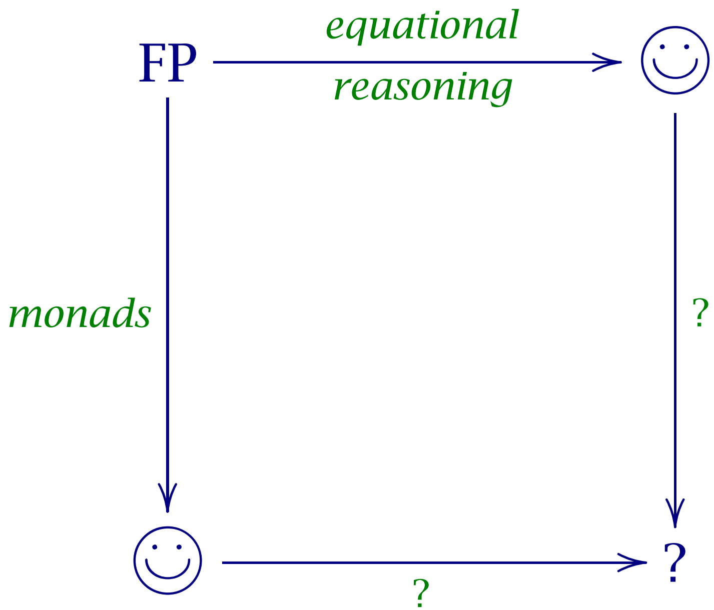
1. Reasoning with effects?



1.1. Seeing the wood through the trees

At TFP 2008, Hutton & Fulger discuss the 'correctness' of

$$relabel :: Tree\ a \rightarrow Tree\ Int$$

as an effectful (stateful) functional program.

I think they miss two opportunities for abstraction:

- from the specific *effects* (they expand the *State* monad to state-transforming functions), and
- from the *pattern of computation* (they use explicit induction on trees).

This is an attempt to address the first question.
(The second is a story for another time.)

2. Monads

'Ordinary' monads, with the usual laws:

$$\textbf{class}\ \mathit{Monad}\ m\ \textbf{where}$$
$$\quad \mathit{return} :: a \rightarrow m\ a$$
$$\quad (\gg\!=) \ :: m\ a \rightarrow (a \rightarrow m\ b) \rightarrow m\ b$$

Special cases:

$$\mathit{skip} :: \mathit{Monad}\ m \Rightarrow m\ ()$$
$$\mathit{skip} = \mathit{return}\ ()$$

$$(\gg) :: \mathit{Monad}\ m \Rightarrow m\ a \rightarrow m\ b \rightarrow m\ b$$
$$k \gg l = k \gg\!= \mathit{const}\ l$$

2.1. Fallibility

Computations may fail:

$$\textbf{class}\ \mathit{Monad}\ m \Rightarrow \mathit{MonadZero}\ m\ \textbf{where}$$
$$\quad \mathit{mzero} :: m\ a$$

such that

$$\mathit{mzero} \gg k = \mathit{mzero}$$

(I'm curious as to why it's not like this in Haskell 98. . .)

Often we just use

$$\mathit{mzero} = \bot$$

2.2. Guards

Define

$$guard :: MonadZero\ m \Rightarrow Bool \rightarrow m\ ()$$
$$guard\ b = \mathbf{if}\ b\ \mathbf{then}\ skip\ \mathbf{else}\ mzero$$

We'll write '$b!$' for '$guard\ b$'.

Familiar properties:

$$True! = skip$$
$$False! = mzero$$
$$(b_1 \wedge b_2)! = b_1! \gg b_2!$$
$$b_1! \gg k \gg b_2! = b_1! \gg k \quad \Longleftarrow \quad b_1 \Rightarrow b_2$$

2.3. Assertions

For $k :: MonadZero\ ()$, write '$k\ \{b\}$' for

$$\mathbf{do}\ \{\, k; b!\, \} = \mathbf{do}\ \{\, k\, \} \qquad (= k)$$

More generally, for $k :: MonadZero\ a$, define '$k\ \{b\}$' to be:

$$\mathbf{do}\ \{\, a \leftarrow k; b!; return\ a \} = \mathbf{do}\ \{\, a \leftarrow k; return\ a \}$$

By abuse of notation, extend to assertions about multiple statements: suppose statements $s_1; ...; s_n$ contain generators binding variables $v_1, ..., v_m$; write '$s_1; ...; s_n\ \{b\}$' for

$$\mathbf{do}\ \{\, s_1; ...; s_n; b!; return\ (v_1, ..., v_m)\, \} = \mathbf{do}\ \{\, s_1; ...; s_n; return\ (v_1, ..., v_m)\, \}$$

(A similar construction is used by Erkök and Launchbury (2000).)

2.4. Queries

A special class of monadic operations, particularly amenable to manipulation.

A *query* q has no side-effects:

$$\mathbf{do}\ \{a \leftarrow q; k\} = \mathbf{do}\ \{k\} \quad \text{-- when } k \text{ doesn't depend on } a$$

and is consistent:

$$\mathbf{do}\ \{a_1 \leftarrow q; a_2 \leftarrow q; k\ a_1\ a_2\} = \mathbf{do}\ \{a \leftarrow q; k\ a\ a\}$$

(They're not just the pure operations, ie those of the form $return\ a$. Consider $get :: State\ s\ s$ of the state monad.)

3. A counter example

A counting monad:

$$\textbf{class}\ \mathit{Monad}\ m \Rightarrow \mathit{MonadCount}\ m\ \textbf{where}$$
$$\quad \mathit{tick} \ \ :: m\ ()$$
$$\quad \mathit{total} :: m\ \mathit{Int}$$

where *total* is a query, and

$$n \leftarrow \mathit{total}; \mathit{tick}; n' \leftarrow \mathit{total}\ \{n' = n + 1\}$$

(exploiting our abuse of notation).

3.1. Towers of Hanoi—specification

Given this program:

$$
\begin{aligned}
&\mathit{hanoi} :: \mathit{MonadCount}\ m \Rightarrow \mathit{Int} \rightarrow m\ ()\\
&\mathit{hanoi}\ 0 \qquad\quad\ = \mathit{skip}\\
&\mathit{hanoi}\ (n+1) = \mathbf{do}\ \{\mathit{hanoi}\ n; \mathit{tick}; \mathit{hanoi}\ n\}
\end{aligned}
$$

we claim:

$$t \leftarrow \mathit{total}; \mathit{hanoi}\ n; u \leftarrow \mathit{total}\ \{2^n - 1 = u - t\}$$

Proof by induction on n. The base case is immediate. Inductive step. . .

3.2. Reasoning

$$
\begin{array}{ll}
& \mathbf{do}\ \{t \leftarrow \mathit{total}; \mathit{hanoi}\ (n+1); u \leftarrow \mathit{total}; (2^{n+1} - 1 = u - t)!\} \\
= & [\![\ \text{definition of } \mathit{hanoi}\ ]\!] \\
& \mathbf{do}\ \{t \leftarrow \mathit{total}; \mathit{hanoi}\ n; \mathit{tick}; \mathit{hanoi}\ n; u \leftarrow \mathit{total}; (2^{n+1} - 1 = u - t)!\} \\
= & [\![\ \text{inserting some queries}\ ]\!] \\
& \mathbf{do}\ \{t \leftarrow \mathit{total}; \mathit{hanoi}\ n; u' \leftarrow \mathit{total}; \mathit{tick}; t' \leftarrow \mathit{total}; \\
& \quad \mathit{hanoi}\ n; u \leftarrow \mathit{total}; (2^{n+1} - 1 = u - t)!\} \\
= & [\![\ \text{inductive hypothesis; } \mathit{tick}\ ]\!] \\
& \mathbf{do}\ \{t \leftarrow \mathit{total}; \mathit{hanoi}\ n; u' \leftarrow \mathit{total}; (2^{n} - 1 = u' - t)!; \mathit{tick}; t' \leftarrow \mathit{total}; \\
& \quad (t' = u' + 1)!; \mathit{hanoi}\ n; u \leftarrow \mathit{total}; (2^{n} - 1 = u - t')!; (2^{n+1} - 1 = u - t)!\} \\
= & [\![\ \text{arithmetic: } 2^{n+1} - 1 = u - t \text{ follows from other guards}\ ]\!] \\
& \mathbf{do}\ \{t \leftarrow \mathit{total}; \mathit{hanoi}\ n; u' \leftarrow \mathit{total}; (2^{n} - 1 = u' - t)!; \mathit{tick}; t' \leftarrow \mathit{total}; \\
& \quad (t' = u' + 1)!; \mathit{hanoi}\ n; u \leftarrow \mathit{total}; (2^{n} - 1 = u - t')!\} \\
= & [\![\ \text{redundant guards, definition of } \mathit{hanoi}\ ]\!] \\
& \mathbf{do}\ \{t \leftarrow \mathit{total}; \mathit{hanoi}\ (n+1); u \leftarrow \mathit{total}\}
\end{array}
$$

4. Tree relabelling

A monad for generating fresh symbols:

type *Symbol* = ...
instance *Eq Symbol* **where** ...

class *Monad m* ⇒ *MonadGensym m* **where**
  fresh :: *m Symbol*
  used :: *m* (*Set Symbol*)

such that *used* (only used in reasoning) is a query, and

$$x \leftarrow used; n \leftarrow fresh; y \leftarrow used\ \{x \subseteq y \land n \in y - x\}$$

4.1. Specification

Tree relabelling:

$$\mathbf{data}\ \mathit{Tree}\ a = \mathit{Tip}\ a \mid \mathit{Bin}\ (\mathit{Tree}\ a)\ (\mathit{Tree}\ a)$$

$$\begin{aligned}
&\mathit{relabel} :: \mathit{MonadGensym}\ m \Rightarrow \mathit{Tree}\ a \to m\ (\mathit{Tree}\ \mathit{Symbol})\\
&\mathit{relabel}\ (\mathit{Leaf}\ a) = \mathbf{do}\ \{ n \leftarrow \mathit{fresh}; \mathit{return}\ (\mathit{Leaf}\ n) \}\\
&\mathit{relabel}\ (\mathit{Bin}\ t\ u) = \mathbf{do}\ \{ t' \leftarrow \mathit{relabel}\ t; u' \leftarrow \mathit{relabel}\ u; \mathit{return}\ (\mathit{Bin}\ t'\ u') \}
\end{aligned}$$

(in fact, an idiomatic *traverse*), satisfies

$$x \leftarrow \mathit{used}; t' \leftarrow \mathit{relabel}\ t; y \leftarrow \mathit{used}\ \{\mathit{distinct}\ t' \wedge \mathit{labels}\ t' \subseteq y - x\}$$

where

$$\begin{aligned}
&\mathit{distinct} :: \mathit{Tree}\ \mathit{Symbol} \to \mathit{Bool}\\
&\mathit{labels}\ \ :: \mathit{Tree}\ \mathit{Symbol} \to \mathit{Set}\ \mathit{Symbol}
\end{aligned}$$

(written d and l below, for short).

4.2. Reasoning: base case

$$
\begin{array}{ll}
 & \textbf{do}\ \{x \leftarrow \mathit{used}; v \leftarrow \mathit{relabel}\ (\mathit{Leaf}\ a); y \leftarrow \mathit{used}; (d\ v \wedge l\ v \subseteq y - x)!\} \\
= & [\![\ \text{definition of } \mathit{relabel}\ ]\!] \\
 & \textbf{do}\ \{x \leftarrow \mathit{used}; n \leftarrow \mathit{fresh}; \textbf{let}\ v = \mathit{Leaf}\ n; y \leftarrow \mathit{used}; (d\ v \wedge l\ v \subseteq y - x)!\} \\
= & [\![\ \text{definition of } d, l\ ]\!] \\
 & \textbf{do}\ \{x \leftarrow \mathit{used}; n \leftarrow \mathit{fresh}; \textbf{let}\ v = \mathit{Leaf}\ n; y \leftarrow \mathit{used}; (\mathit{True} \wedge \{n\} \subseteq y - x)!\} \\
= & [\![\ \text{axiom for } \mathit{fresh}\ ]\!] \\
 & \textbf{do}\ \{x \leftarrow \mathit{used}; n \leftarrow \mathit{fresh}; \textbf{let}\ u = \mathit{Leaf}\ n; y \leftarrow \mathit{used}\} \\
= & [\![\ \text{folding definitions}\ ]\!] \\
 & \textbf{do}\ \{x \leftarrow \mathit{used}; v \leftarrow \mathit{relabel}\ (\mathit{Leaf}\ a); y \leftarrow \mathit{used}\}
\end{array}
$$

4.3. Reasoning: inductive step

$$
\begin{array}{ll}
& \mathbf{do}\ \{x \leftarrow used; v \leftarrow relabel\ (Bin\ t\ u); z \leftarrow used; (d\ v \wedge l\ v \subseteq z - x)!\} \\
= & [\![\ \text{definition of } relabel\ ]\!] \\
& \mathbf{do}\ \{x \leftarrow used; t' \leftarrow relabel\ t; u' \leftarrow relabel\ u; \mathbf{let}\ v = Bin\ t'\ u'; z \leftarrow used; \\
& \qquad (d\ v \wedge l\ v \subseteq z - x)!\} \\
= & [\![\ \text{definition of } d, l\ ]\!] \\
& \mathbf{do}\ \{x \leftarrow used; t' \leftarrow relabel\ t; u' \leftarrow relabel\ u; \mathbf{let}\ v = Bin\ t'\ u'; z \leftarrow used; \\
& \qquad (d\ t' \wedge d\ u' \wedge l\ t' \cap l\ u' = \emptyset \wedge l\ t' \cup l\ u' \subseteq z - x)!\} \\
= & [\![\ \text{induction}\ ]\!] \\
& \mathbf{do}\ \{x \leftarrow used; t' \leftarrow relabel\ t; y \leftarrow used; (d\ t' \wedge l\ t' \subseteq y - x)!; \\
& \qquad u' \leftarrow relabel\ u; z \leftarrow used; (d\ u' \wedge l\ u' \subseteq z - y)!; \mathbf{let}\ v = Bin\ t'\ u'; \\
& \qquad (d\ t' \wedge d\ u' \wedge l\ t' \cap l\ u' = \emptyset \wedge l\ t' \cup l\ u' \subseteq z - x)!\} \\
= & [\![\ \text{queries, redundant guards, folding definitions}\ ]\!] \\
& \mathbf{do}\ \{x \leftarrow used; v \leftarrow relabel\ (Bin\ t\ u); z \leftarrow used\}
\end{array}
$$

5. Towers of Hanoi, more directly

Hoare-style reasoning is a bit painfully long-winded:
repeat the program on every line, gradually discharging guards.

Sometimes a more direct approach works. In fact,

$$hanoi\ n = rep\ (2^n - 1)\ tick$$

where

$$\begin{aligned} & rep :: Monad\ m \Rightarrow Int \to m\ () \to m\ () \\ & rep\ 0\ \quad\quad\ ma = skip \\ & rep\ (n+1)\ ma = ma \gg rep\ n\ ma \end{aligned}$$

In particular, note that

$$rep\ (m + n)\ ma = rep\ m\ ma \gg rep\ n\ ma$$

5.1. More direct proof

...by induction on n. Base case is trivial. For inductive step,

$$
\begin{aligned}
& \mathit{hanoi}\ (n+1) \\
= \quad & [\![\ \text{definition of } \mathit{hanoi}\ ]\!] \\
& \mathit{hanoi}\ n \gg \mathit{tick} \gg \mathit{hanoi}\ n \\
= \quad & [\![\ \text{inductive hypothesis}\ ]\!] \\
& \mathit{rep}\ (2^n - 1)\ \mathit{tick} \gg \mathit{tick} \gg \mathit{rep}\ (2^n - 1)\ \mathit{tick} \\
= \quad & [\![\ \text{composition}\ ]\!] \\
& \mathit{rep}\ ((2^n - 1) + 1 + (2^n - 1))\ \mathit{tick} \\
= \quad & [\![\ \text{arithmetic}\ ]\!] \\
& \mathit{rep}\ (2^{n+1} - 1)\ \mathit{tick}
\end{aligned}
$$

But I don’t see how to do tree relabelling in this more direct style...

6. Probabilistic computations

Probability distributions form a monad (Giry, Jones, Ramsey, Erwig. . .).

For simplicity, only finitely-supported distributions here:

$$\mathbf{class}\ \mathit{Monad}\ m \Rightarrow \mathit{MonadProb}\ m\ \mathbf{where}$$
$$\quad \mathit{choice} :: \mathit{Rational} \to m\ a \to m\ a \to m\ a$$

where the rationals are constrained to the unit interval.

Following Hoare, let's write '$mx \triangleleft p \triangleright my$' for '$\mathit{choice}\ p\ mx\ my$'.

6.1. Laws of choice

Unit, idempotence, commutativity:

$$
\begin{aligned}
mx \lhd 0 \rhd my &= my \\
mx \lhd 1 \rhd my &= mx \\
mx \lhd p \rhd mx &= mx \\
mx \lhd p \rhd my &= my \lhd 1 - p \rhd mx
\end{aligned}
$$

A kind of associativity:

$$
\begin{aligned}
& mx \lhd p \rhd (my \lhd q \rhd mz) = (mx \lhd r \rhd my) \lhd s \rhd mz \\
& \quad \Longleftarrow p = r\, s \wedge (1 - s) = (1 - p)\,(1 - q)
\end{aligned}
$$

Bind distributes over choice, in both directions:

$$
\begin{aligned}
mx \gg\!= \lambda a \to (k_1\ a) \lhd p \rhd (k_2\ a) &= (mx \gg\!= k_1) \lhd p \rhd (mx \gg\!= k_2) \\
mx \lhd p \rhd my \gg\!= k &= (mx \gg\!= k) \lhd p \rhd (my \gg\!= k)
\end{aligned}
$$

6.2. Normal form

Finite mappings from outcomes to probabilities (ignore order, disregard weightless entries, weights sum to one, amalgamate duplicates):

$$\mathbf{newtype}\ \mathit{Distribution}\ a = D\{\ \mathit{unD} :: [\,(a, \mathit{Rational})\,]\,\}$$

All you need to interpret a distribution is *choice*:

$$\begin{aligned}&\mathit{fromDist} :: \mathit{MonadProb}\ m \Rightarrow \mathit{Distribution}\ a \to m\ a\\&\mathit{fromDist}\ d = \mathit{fst}\ (\mathit{foldr}1\ \mathit{combine}\ [\,(\mathit{return}\ a, p) \mid (a, p) \leftarrow \mathit{unD}\ d, p > 0\,])\\&\quad \mathbf{where}\ \mathit{combine}\ (\mathit{mx}, p)\ (\mathit{my}, q) = (\mathit{mx} \lhd {}^{p}/_{p+q} \rhd \mathit{my}, p + q)\end{aligned}$$

For example,

$$\begin{aligned}&\mathit{uniform} :: \mathit{MonadProb}\ m \Rightarrow [\,a\,] \to m\ a\\&\mathit{uniform}\ x = \mathit{fromDist}\ (D\,[\,(a, p) \mid a \leftarrow x\,])\ \mathbf{where}\ p = 1\ /\ \mathit{length}\ x\end{aligned}$$

6.3. Implementation

Moreover, *Distribution* itself is a fine instance of *MonadProb*:

$$
\begin{aligned}
&\textbf{instance}\ \mathit{Monad}\ \mathit{Distribution}\ \textbf{where}\\
&\quad \mathit{return}\ a = D\,[\,(a,1)\,]\\
&\quad \mathit{px} \gg\!= f\ \ = D\,[\,(b, p \times q) \mid (a,p) \leftarrow \mathit{unD}\ \mathit{px}, (b,q) \leftarrow \mathit{unD}\ (f\ a)\,]\\
&\textbf{instance}\ \mathit{MonadProb}\ \mathit{Distribution}\ \textbf{where}\\
&\quad \mathit{ma} \lhd p \rhd \mathit{mb} = D\ (\mathit{scale}\ p\ (\mathit{unD}\ \mathit{ma}) +\!\!+ \mathit{scale}\ (1-p)\ (\mathit{unD}\ \mathit{mb}))\\
&\qquad \textbf{where}\ \mathit{scale}\ r\ \mathit{pas} = [\,(a, r \times p) \mid (a,p) \leftarrow \mathit{pas}\,]
\end{aligned}
$$

(Kidd points out that *Distribution* = *WriterT Rational* (*ListT Identity*), using the writer monad from the monoid of rationals with multiplication.)

6.4. Monty Hall

$$\mathbf{data}\ \mathit{Door} = A \mid B \mid C\ \mathbf{deriving}\ (\mathit{Eq}, \mathit{Show})$$
$$\mathit{doors} = [A, B, C]$$

$$\mathit{hide} :: \mathit{MonadProb}\ m \Rightarrow m\ \mathit{Door}$$
$$\mathit{hide} = \mathit{uniform}\ \mathit{doors}$$

$$\mathit{pick} :: \mathit{MonadProb}\ m \Rightarrow m\ \mathit{Door}$$
$$\mathit{pick} = \mathit{uniform}\ \mathit{doors}$$

$$\mathit{tease} :: \mathit{MonadProb}\ m \Rightarrow \mathit{Door} \to \mathit{Door} \to m\ \mathit{Door}$$
$$\mathit{tease}\ h\ p = \mathit{uniform}\ (\mathit{doors} \setminus\!\setminus [h, p])$$

$$\mathit{switch} :: \mathit{MonadProb}\ m \Rightarrow \mathit{Door} \to \mathit{Door} \to m\ \mathit{Door}$$
$$\mathit{switch}\ p\ t = \mathit{return}\ (\mathit{head}\ (\mathit{doors} \setminus\!\setminus [p, t]))$$

$$\mathit{stick} :: \mathit{MonadProb}\ m \Rightarrow \mathit{Door} \to \mathit{Door} \to m\ \mathit{Door}$$
$$\mathit{stick}\ p\ t = \mathit{return}\ p$$

6.5. The whole story

Monty's script:

$$
\begin{array}{l}
play :: MonadProb\ m \Rightarrow (Door \to Door \to m\ Door) \to m\ Bool \\
play\ strategy = \\
\quad \mathbf{do} \\
\quad\quad h \leftarrow hide \qquad\qquad \text{-- host hides the car behind door } h \\
\quad\quad p \leftarrow pick \qquad\qquad \text{-- you pick door } p \\
\quad\quad t \leftarrow tease\ h\ p \qquad \text{-- host teases you with door } t\ (\not\equiv h, p) \\
\quad\quad s \leftarrow strategy\ p\ t \quad \text{-- you choose, based on } p \text{ and } t \text{ (but not } h\text{!)} \\
\quad\quad return\ (s \equiv h) \qquad \text{-- you win iff your choice } s \text{ equals } h
\end{array}
$$

6.6. In support of Marilyn Vos Savant

It's a straightforward proof by equational reasoning that

$$\begin{aligned} \mathit{play\ switch} &= \mathit{uniform}\ [\mathit{True}, \mathit{True}, \mathit{False}] \\ \mathit{play\ stick} &= \mathit{uniform}\ [\mathit{False}, \mathit{False}, \mathit{True}] \end{aligned}$$

The key is that separate uniform distributions are independent:

$$\mathbf{do}\ \{a \leftarrow \mathit{uniform}\ x; b \leftarrow \mathit{uniform}\ y; \mathit{return}\ (a, b)\} = \mathit{uniform}\ (\mathit{cp}\ x\ y)$$

where

$$\begin{aligned} &\mathit{cp} :: [a] \rightarrow [b] \rightarrow [(a, b)] \\ &\mathit{cp}\ x\ y = [(a, b) \mid a \leftarrow x, b \leftarrow y] \end{aligned}$$

(Ask me over a beer. . .)

7. Combining probability and nondeterminism

Nobody said that Monty has to play fair.
He has a free choice in hiding the car, and in teasing you.

To model this, we need to combine probabilism with nondeterminism:

$$\textbf{class}\ \mathit{MonadZero}\ m \Rightarrow \mathit{MonadPlus}\ m\ \textbf{where}$$
$$\quad \mathit{mplus} :: m\ a \rightarrow m\ a \rightarrow m\ a$$

such that *mzero* and *mplus* form a monoid, and

$$(m\ \text{`}\mathit{mplus}\text{`}\ n) \gg\!= k = (m \gg\!= k)\ \text{`}\mathit{mplus}\text{`}\ (n \gg\!= k)$$

Happily, although monads do not compose in general, $[\,\mathit{Distribution}\ a\,]$ is a monad. Moreover, it is a *MonadProb* and a *MonadPlus* too.

(So is $\mathit{Distribution}\ [\,a\,]$, but I think that doesn't help.)

(There's a nice tale in terms of monad transformers.)

7.1. A simple example: mixing choices

A fair coin:

$$\mathit{coin} :: \mathit{MonadProb}\ m \Rightarrow m\ \mathit{Bool}$$
$$\mathit{coin} = (\mathit{return}\ \mathit{True}) \lhd {}^{1}/_{2} \rhd (\mathit{return}\ \mathit{False})$$

An arbitrary choice:

$$\mathit{arb} :: \mathit{MonadPlus}\ m \Rightarrow m\ \mathit{Bool}$$
$$\mathit{arb} = \mathit{return}\ \mathit{True}\ \text{`}\mathit{mplus}\text{`}\ \mathit{return}\ \mathit{False}$$

Two combinations:

$$\mathit{arbcoin}, \mathit{coinarb} :: (\mathit{MonadPlus}\ m, \mathit{MonadProb}\ m) \Rightarrow m\ \mathit{Bool}$$
$$\mathit{arbcoin} = \mathbf{do}\ \{ a \leftarrow \mathit{arb}; c \leftarrow \mathit{coin}; \mathit{return}\ (a == c) \}$$
$$\mathit{coinarb} = \mathbf{do}\ \{ c \leftarrow \mathit{coin}; a \leftarrow \mathit{arb}; \mathit{return}\ (a == c) \}$$

What do you think they do?

7.2. ...as sets of distributions

Define

$$\textbf{type}\ \mathit{NondetProb}\ a = [\,\mathit{Distribution}\ a\,]$$

Then (with suitable *shows*):

$$*\mathit{Main}\rangle\ \mathit{arbcoin} :: \mathit{NondetProb}\ \mathit{Bool}$$
$$[\,[\,(\mathit{True}, {}^{1}/_{2}), (\mathit{False}, {}^{1}/_{2})\,],$$
$$[\,(\mathit{False}, {}^{1}/_{2}), (\mathit{True}, {}^{1}/_{2})\,]\,]$$

$$*\mathit{Main}\rangle\ \mathit{coinarb} :: \mathit{NondetProb}\ \mathit{Bool}$$
$$[\,[\,(\mathit{True}, {}^{1}/_{2}), (\mathit{False}, {}^{1}/_{2})\,],$$
$$[\,(\mathit{True}, {}^{1}/_{2}), (\mathit{True}, {}^{1}/_{2})\,],$$
$$[\,(\mathit{False}, {}^{1}/_{2}), (\mathit{False}, {}^{1}/_{2})\,],$$
$$[\,(\mathit{False}, {}^{1}/_{2}), (\mathit{True}, {}^{1}/_{2})\,]\,]$$

7.3. . . . as expectations

$$\textbf{class}\ \mathit{MonadProb}\ m \Rightarrow \mathit{MonadExpect}\ m\ \textbf{where}$$
$$\quad \mathit{expect} :: (\mathit{Ord}\ n, \mathit{Fractional}\ n) \Rightarrow m\ a \to (a \to n) \to n$$

$$\textbf{instance}\ \mathit{MonadExpect}\ \mathit{NondetProb}\ \textbf{where} \qquad \text{-- morally}$$
$$\quad \mathit{expect}\ \mathit{px}\ h = \mathit{minimum}\ (\mathit{map}\ (\mathit{mean}\ h \circ \mathit{unD})\ \mathit{px})\ \textbf{where}$$
$$\qquad \mathit{mean}\ h\ \mathit{aps} = \mathit{sum}\,[\,p \times f\ a \mid (a, p) \leftarrow \mathit{aps}\,]\ /\ \mathit{sum}\ (\mathit{map}\ \mathit{snd}\ \mathit{aps})$$

Your reward is 1 if the booleans agree, and 0 otherwise:

$$\mathit{reward}\ b = \textbf{if}\ b\ \textbf{then}\ 1\ \textbf{else}\ 0$$

Then:

$$*\mathit{Main}\rangle\ \mathit{expect}\ (\mathit{arbcoin} :: \mathit{NondetProb}\ \mathit{Bool})\ \mathit{reward}$$
$$^{1}/_{2}$$
$$*\mathit{Main}\rangle\ \mathit{expect}\ (\mathit{coinarb} :: \mathit{NondetProb}\ \mathit{Bool})\ \mathit{reward}$$
$$0$$

7.4. Back to nondeterministic Monty...

We could define instead:

$$hide :: MonadPlus\ m \Rightarrow m\ Door$$
$$hide = arbitrary\ doors$$

$$tease :: MonadPlus\ m \Rightarrow Door \rightarrow Door \rightarrow m\ Door$$
$$tease\ h\ p = arbitrary\ (doors \setminus\setminus [h, p])$$

where

$$arbitrary :: MonadPlus\ m \Rightarrow [a] \rightarrow m\ a$$
$$arbitrary = foldr\ mplus\ mzero \circ map\ return$$

I believe that the calculation carries through just as before: still

$$play\ switch = uniform\ [True, True, False]$$
$$play\ stick = uniform\ [False, False, True]$$

8. Summary

- axiomatic approach to reasoning with effects
- simple and generic
- smacks of ‘algebraic theories of effects’ (Plotkin & Power, Lawvere) (in particular, partiality and continuations do not arise from algebraic theories)
- IO is uninteresting?
- more examples wanted!