Way below security

Dusko Pavlovic

Kestrel Institute and OUCL

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Outline

Outline

Security and domains?

Information systems, honesty, and guards

Domains for Bayesian inference and guessing

Summary

Security is complicated

Crypto system

Given the types

- $\mathcal{M}$ of *plaintexts*
- $\mathcal{C}$ of *cyphertexts*
- $\mathcal{K}$ of *keys*
- $\mathcal{R}$ of *random seeds*

Security is complicated

Crypto system

Given the types

- $\mathcal{M}$ of *plaintexts*
- $\mathcal{C}$ of *cyphertexts*
- $\mathcal{K}$ of *keys*
- $\mathcal{R}$ of *random seeds*

a *crypto-system* is a triple of algorithms:

- key generation $\langle k, \overline{k} \rangle : \mathcal{R} \longrightarrow \mathcal{K} \times \mathcal{K}$,
- encryption $\mathsf{E} : \mathcal{R} \times \mathcal{K} \times \mathcal{M} \longrightarrow \mathcal{C}$, and
- decryption $\mathsf{D} : \mathcal{K} \times \mathcal{C} \longrightarrow \mathcal{M}$,

Security is complicated

Crypto system

. . . that together provide

- unique decryption:

$$\mathsf{D}(\overline{k}, \mathsf{E}(k, m)) \quad = \quad m$$

Security is complicated

Crypto system

...that together provide

- unique decryption:

$$D(\overline{k}, E(k, m)) \quad = \quad m$$

- secrecy (IND-CCA):

$$\mathrm{Prob}\left(\begin{array}{l} c_0 \in \mathbb{A}_0,\ m = D(\overline{k}, c_0), \\ m_0, m_1 \in \mathbb{A}_1(c_0, m), c \in E(k, m_b) \\ c_1 \in \mathbb{A}_2(c_0, m, m_0, m_1, c^{\neq}), \widetilde{m} = D(\overline{k}, c_1) \end{array} \quad \vdash \quad b \in \mathbb{A}_3(c_0, m, m_0, m_1, c, c_1, \widetilde{m}) \right) \quad \leq \quad \frac{1}{2}$$

for any probabilistic algorithm $\mathbb{A} = \langle \mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3 \rangle$

Idea

Symbolically, the secret is guarded by a key

$$\forall \Theta. \quad \Theta,\ \mathsf{E}(k,m) \vdash m \quad \Longrightarrow \quad \Theta \vdash k$$

$$\Updownarrow$$

$$k \quad \text{guards} \quad m \quad \text{in} \quad \mathsf{E}(k,m)$$

Idea

Symbolically, the secret is guarded by a key

$$\forall \Theta. \quad \Theta,\ \mathsf{E}(k,m) \vdash m \quad \Longrightarrow \quad \Theta \vdash k$$

$$\Updownarrow$$

$$k \quad \text{guards} \quad m \quad \text{in} \quad \mathsf{E}(k,m)$$

Does that mean that the key is way-below the secret?

$$\forall J. \quad \bigsqcup J \sqsupseteq m \quad \Longrightarrow \quad J \sqsupseteq k$$

$$\Updownarrow$$

$$k \quad \ll \quad m$$

Outline

Information system of a protocol

Algebraic model

- algebra $\mathbb{T}$
  - equational presentation (Σ, E), generators V
- principals $\mathcal{W}$
  - $A \in \mathcal{W}$ owns $x \in \mathrm{V}_A$, a store, nonce or key
- fixed protocol run $\mathcal{Q}$
  - $A \in \mathcal{W}$ may send $t \in \mathbb{T}$, or receive into $x \in \mathrm{V}_A$

Information system of a protocol

Derivability

$$\Gamma \vdash \Theta \iff \forall t \in \Theta \;\; \exists \varphi \in \Sigma^* \;\; \exists \vec{g} \subseteq \Gamma. \;\; \varphi(\vec{g}) \stackrel{E}{=} t$$

Consistent sets

$$\Gamma_A^{\mathcal{Q}} = A\text{'s environment in } \mathcal{Q}$$

$$\mathsf{Con}_A^{\mathcal{Q}} = \{\Theta \in \wp_{fin}\mathbb{T} \mid \Gamma_A^{\mathcal{Q}} \vdash \Theta\}$$

$$\mathsf{Con}^{\mathcal{Q}} = \bigcup_{A \in \mathcal{W}} \mathsf{Con}_A^{\mathcal{Q}}$$

Example: Encryption protocol

- $\Sigma = \{\mathsf{E}, \mathsf{D} : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T}\}$
- $E = \{\mathsf{D}(x, \mathsf{E}(x, y)) = y\}$
- $\mathcal{Q} = \{A \xrightarrow{\mathsf{E}(k,m)} B\}$
  - $k \in \Gamma_X \iff X \in \{A, B\}$
  - $m \in \Gamma_X \iff X = A$

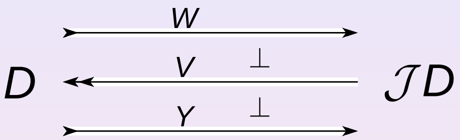
Domain of a protocol

$$D^{\mathcal{Q}} = \{a \in \wp\mathbb{T} \mid \forall \Theta \underset{fin}{\subseteq} a.\ \Theta \in \mathsf{Con}^{\mathcal{Q}} \wedge \Theta \vdash \Gamma \Rightarrow \Gamma \subseteq a\}$$

Order ideals

$$\mathcal{J}D \;=\; \Big\{ J \in D \;\;\big|\;\; \forall a \subseteq b \in J \Rightarrow a \in J \\ \wedge\;\; \forall ab \in J\ \exists c \in J.\ a, b \subseteq c \Big\}$$

Continuity = left adjoint to $\bigsqcup$



$$
\begin{aligned}
Y(a) &= \{x \sqsubseteq a\} \\
V(J) &= \bigsqcup J \\
W(a) &= \{x \ll a\}
\end{aligned}
$$

Continuity = left adjoint to $\bigsqcup$

Intuition

- $W(a) = \{x \ll a\}$ are the *key elements* of a
  - if $\bigsqcup J \sqsupseteq a$ is a "computation" of a
  - then $k \ll a$ means $k \in J$ for every such computation.
- $VW(a) = a$ means that a can be computed from its key elements.

Example: Encryption protocol

- $W(m) = \{m\}$

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- $\mathrm{Con}^{\mathcal{Q}}$ contains some sets that never occur
  - they cover the honesty assumptions

Example: Encryption protocol

- $W(m) = \{m\}$:(
  - although m is never sent in the clear
  - and no principal knows it without k
- $\mathrm{Con}^{\mathcal{Q}}$ contains some sets that never occur
  - they cover the honesty assumptions
- culprit: $\forall a \subseteq b \in J \Rightarrow a \in J$
  - every derivable term is derivable on its own

Honesty system of a protocol

Derivability

$$\Gamma \vdash \Theta \iff \forall t \in \Theta \;\; \exists \varphi \in \Sigma^* \;\; \exists \vec{g} \subseteq \Gamma. \;\; \varphi(g) \stackrel{E}{=} t$$

Honest sets

$$\begin{aligned}
\Gamma_A^{\mathcal{Q}} &= A\text{'s environment in } \mathcal{Q} \\
\mathrm{Hon}_A^{\mathcal{Q}} &= \{\Theta \in \wp_{fin}\mathbb{T} \mid \Gamma_A^{\mathcal{Q}} \subseteq \Theta \wedge \Gamma_A^{\mathcal{Q}} \vdash \Theta\} \\
\mathrm{Hon}^{\mathcal{Q}} &= \bigcup_{A \in \mathcal{W}} \mathrm{Hon}_A^{\mathcal{Q}}
\end{aligned}$$

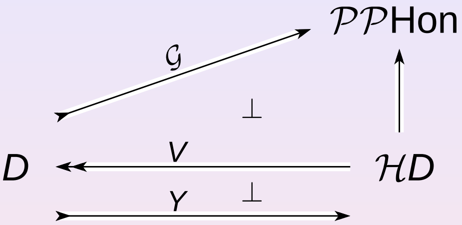
Domain of a protocol

$$D^{\mathcal{Q}} = \{a \in \wp\mathbb{T} \mid \forall \Xi \underset{fin}{\subseteq} a\ \exists \Theta \in \mathsf{Hon}^{\mathcal{Q}}.\ \Xi \subseteq \Theta \wedge \Xi \vdash \Gamma \Rightarrow \Gamma \subseteq a\}$$

Honest ideals

$$\mathcal{H}D = \Big\{ H \in D \;\Big|\; \forall a \subseteq b \in H$$
$$(\exists \Theta \in \mathrm{Hon}.\ \Theta \subseteq a) \Rightarrow a \in H$$
$$\wedge\ \forall ab \in H\ \exists c \in H.\ a, b \subseteq c \Big\}$$

Guards = left multi-adjoint to $\bigsqcup$

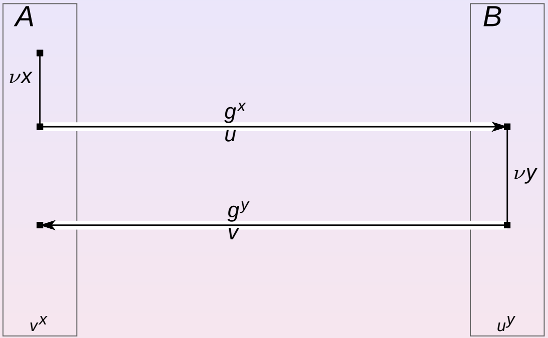


$$a \sqsubseteq V(H) \iff \exists G \in \mathcal{G}(a).\ G \sqsubseteq H$$

Example: Encryption protocol

- $\mathcal{G}(m) = \{\{k\}\}$:)

Example: Diffie-Hellman Key Agreement



Example: Diffie-Hellman Key Agreement

- $W(g^{xy}) = \{g^{xy}\}$:(
- $\mathcal{G}(g^{xy}) = \{\{g^x, y\}, \{g^y, x\}\}$:)

Algebraic theory of secrecy

(Meadows & DP)

$$\mathcal{G} \ \text{guards}_{\Upsilon}\Theta \quad = \quad \forall t \in \Theta\ \forall \Xi \subseteq \Upsilon\ \exists \Gamma \in \mathcal{G}.\ \Xi \vdash t \Rightarrow \Xi \vdash \Gamma$$

Algebraic theory of secrecy

(Meadows & DP)

$$\begin{aligned}
\mathsf{Have}(\Theta; G) &= \forall X \in G.\ \Gamma_X \vdash \Theta \\
\mathsf{Only}(\Theta; G) &= \forall X \in \mathcal{W}\ \forall t \in \Theta.\ \Gamma_X \vdash t \Rightarrow X \in G \\
\mathsf{Secr}(\Theta; G) &= \mathsf{Have}(\Theta; G) \wedge \mathsf{Only}(\Theta; G) \\
&= \forall X \in \mathcal{W}\ \forall t \in \Theta.\ \Gamma_X \vdash t \Leftrightarrow X \in G
\end{aligned}$$

Algebraic theory of secrecy

(Meadows & DP)

$$\frac{\mathrm{Have}(\Xi; G) \qquad \Xi \vdash_G \Theta}{\mathrm{Have}(\Theta; G)}$$

$$\frac{\mathrm{Only}(\Xi_i; G_i)\ \Big|_{i=1}^{n} \qquad \{\Xi_i\}_{i=1}^{n} \text{ guards } \Theta}{\mathrm{Only}(\Theta; \bigcup_{i=1}^{n} G_i)}$$

$$\frac{\mathrm{Secr}(\Xi_i; G_i)\ \Big|_{i=1}^{n} \quad \Xi_i \vdash_{G_i} \Theta\ \Big|_{i=1}^{n} \quad \{\Xi_i\}_{i=1}^{n} \text{ guards } \Theta}{\mathrm{Secr}(\Theta; \bigcup_{i=1}^{n} G_i)}$$

Security is complicated

Crypto system

- unique decryption:

$$\mathsf{D}(\overline{k}, \mathsf{E}(k, m)) \quad = \quad m$$

- secrecy (IND-CCA):

$$\mathrm{Prob}\left(\begin{array}{l} c_0 \in \mathbb{A}_0,\ m = \mathsf{D}(\overline{k}, c_0), \\ m_0, m_1 \in \mathbb{A}_1(c_0, m), c \in \mathsf{E}(k, m_b) \\ c_1 \in \mathbb{A}_2(c_0, m, m_0, m_1, c^{\neq}), \widetilde{m} = \mathsf{D}(\overline{k}, c_1) \end{array} \quad \vdash \quad b \in \mathbb{A}_3(c_0, m, m_0, m_1, c, c_1, \widetilde{m}) \right) \quad \leq \quad \frac{1}{2}$$

for any probabilistic algorithm $\mathbb{A} = \langle \mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3 \rangle$

Outline

Idea

- assume that the algebra $\mathbb{T}$ is given with
  - an implementation

$$\begin{aligned} [\![-]\!] &: \mathcal{T} \to \mathcal{L} \\ \surd &: \mathcal{L} \rightharpoonup \mathcal{T} \end{aligned}$$

    such that $\surd [\![t]\!] = t$

  - monoid of feasible maps $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{L}}$
  - frequency distribution

$$\mathsf{Prob} \ : \ \mathbb{T} \longrightarrow [0,1]$$

Idea

- generalize
  - from algebraic derivability

$$\begin{array}{rccc} - \vdash - \ : & D \times D & \longrightarrow & \{0,1\} \\ & \Gamma, \Theta & \longmapsto & \Gamma \vdash \Theta \end{array}$$

  - to Bayesian inference

$$\begin{array}{rccc} (- \vdash -) \ : & D \times D & \longrightarrow & [0,1] \\ & \Gamma, \Theta & \longmapsto & \mathrm{Prob}\,(\Gamma \vdash \Theta) \end{array}$$

  - and guessing (cryptanalysis)

$$\begin{array}{rccc} [- \vdash -] \ : & D \times D & \longrightarrow & [0,1] \\ & \Gamma, \Theta & \longmapsto & \bigvee_{\mathbb{A} \in \mathcal{F}} \mathrm{Prob}\,(\Gamma \vdash \Theta \in \mathbb{A}(\Gamma)) \end{array}$$

Hope

Develop

- a manageable formalization of guessing
- using information systems
- *enriched* over the ordered monoid $\left([0,1],\ \cdot,\ 1,\ \leq\ \right)$
- treat $\left(\Gamma \vdash \Theta\right)$ and $\left[\Gamma \vdash \Theta\right]$ as hom-objects in $[0,1]$
  - the states $\Gamma, \Theta \ldots$ can now be viewed as sets of equations *and inequalities*

Problem

The Bayesian inference and guessing are **not** transitive:

$$\left(\Xi \vdash \Gamma\right)\cdot\left(\Gamma \vdash \Theta\right) \quad \not\leq \quad \left(\Xi \vdash \Theta\right)$$
$$\left[\Xi \vdash \Gamma\right]\cdot\left[\Gamma \vdash \Theta\right] \quad \not\leq \quad \left[\Xi \vdash \Theta\right]$$

Problem

The Bayesian inference and guessing are **not** transitive:

$$(\Xi \vdash \Gamma)\cdot(\Gamma \vdash \Theta) \quad \not\leq \quad (\Xi \vdash \Theta)$$
$$[\Xi \vdash \Gamma]\cdot[\Gamma \vdash \Theta] \quad \not\leq \quad [\Xi \vdash \Theta]$$

e.g., for

- $\Gamma = \emptyset$, thus $(\Xi \wedge \Gamma) = (\Xi)$
  - $(\Xi \vdash \Gamma) = \frac{(\Xi \wedge \Gamma)}{(\Xi)} = 1$
- $\Theta = \neg\Xi$
  - $(\Gamma \vdash \Theta) = (\emptyset \vdash \neg\Xi) = 1 - (\Xi)$
  - $(\Xi \vdash \Theta) = \frac{(\Xi \wedge \neg\Xi)}{(\Xi)} = 0$

Problem

…but they do satisfy

$$(\Xi \vdash \Gamma)\cdot(\Xi,\Gamma \vdash \Theta) \;=\; (\Xi \vdash \Gamma,\Theta)$$
$$[\Xi \vdash \Gamma]\cdot[\Xi,\Gamma \vdash \Theta] \;=\; [\Xi \vdash \Gamma,\Theta]$$

because

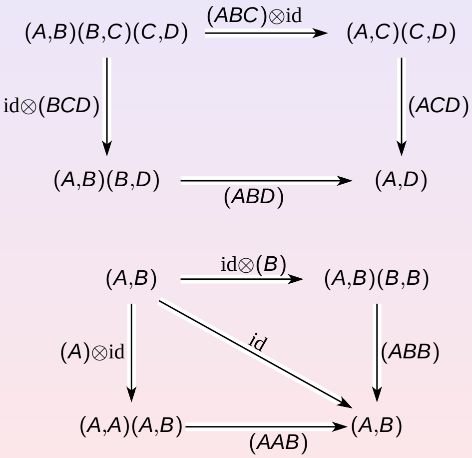
$$\frac{(\Xi\Gamma)}{(\Xi)}\cdot\frac{(\Xi\Gamma\Theta)}{(\Xi\Gamma)} \;=\; \frac{(\Xi\Gamma\Theta)}{(\Xi)}$$

$\mathbb{V}$-categories

- $(\mathbb{V}, \otimes, I)$
  - monoidal category, abbreviate $k \otimes \ell$ to $k\ell$
- $\mathbb{C} = \{A, B, \ldots\}$
  - class of objects
- $(A, B) \in \mathbb{V}$
  - hom-objects, for every $A, B \in \mathbb{C}$
- $(ABC) : (A, B) \otimes (B, C) \longrightarrow (A, C)$
  - composition, for every $A, B, C \in \mathbb{C}$
- $(A) : I \longrightarrow (A, A)$
  - identities, for every $A \in \mathbb{C}$

$\mathbb{V}$-categories

... satisfying



$\mathbb{V}$-categories

Examples

- $\mathbb{V} = \left(\{0,1\}, \wedge, 1\right)$ — preorders
- $\mathbb{V} = \left(\mathsf{Set}, \times, 1\right)$ — categories
- $\mathbb{V} = \left([0,\infty], +, 0\right)$ — metric spaces

$\mathbb{V}$-dependencies

- $(\mathbb{V}, \otimes, I)$
  - monoidal category, abbreviate $k \otimes \ell$ to $k\ell$
- $(\mathbb{C}, \cdot, \top)$
  - abelian monoid of objects, abbreviate $A \cdot B$ to AB
- $(A, B) \in \mathbb{V}$
  - hom-objects, for every $A, B \in \mathbb{C}$
- $(ABC) : (A, B) \otimes (AB, C) \longrightarrow (A, BC)$
  - composition, for every $A, B, C \in \mathbb{C}$
- $\pi(AB) : I \longrightarrow (AB, B)$ and
  $\delta(AB) : (A, B) \longrightarrow (A, BB)$
  - projections and diagonals, for every $A, B \in \mathbb{C}$

$\mathbb{V}$-dependencies

...satisfying

$$
\begin{array}{ccc}
(A,B)(AB,C)(ABC,D) & \xrightarrow{(ABC)\otimes \mathrm{id}} & (A,BC)(ABC,D) \\
{\scriptstyle \mathrm{id}\otimes((AB)CD)}\big\downarrow & & \big\downarrow{\scriptstyle (A(BC)D)} \\
(A,B)(AB,CD) & \xrightarrow[(AB(CD))]{} & (A,BCD)
\end{array}
$$

$$
\begin{array}{ccc}
(A,B) & & \\
{\scriptstyle \pi(A\top)\otimes \mathrm{id}}\big\downarrow & \searrow^{\mathrm{id}} & \\
(A,\top)(A,B) & \xrightarrow[(A\top B)]{} & (A,B)
\end{array}
\qquad
\begin{array}{ccc}
(A,B) & \xrightarrow{\mathrm{id}\otimes\pi(AB)} & (A,B)(AB,B) \\
 & \searrow^{\delta(AB)} & \big\downarrow{\scriptstyle (ABB)} \\
 & & (A,BB)
\end{array}
$$

$\mathbb{V}$-dependencies

Examples

- $\mathbb{V} = (\{0,1\}, \wedge, 1)$ — semilattices

$$a \leq b \wedge ab \leq c \iff a \leq bc$$

- $\mathbb{V} = (\mathrm{Set}, \times, 1)$ — dependent types (RCCCs)

$$\begin{array}{rrcl} & (x:A & \rhd & f(x) : B(x)) \\ \times & (x:A, y:B(x) & \rhd & g(x,y) : C(x,y)) \\ \longrightarrow & (x:A & \rhd & \langle f(x), g(x,f(x))\rangle : B(x) \times C(x,f(x))) \end{array}$$

- $\mathbb{V} = ([0,1], \cdot, 1)$ — Bayesian nets

$$(A \vdash B) \cdot (AB \vdash C) = (A \vdash BC)$$

Domain theory of Bayesian inference

Definition

Let $\mathbb{D}$ be a $[0,1]$-dependency. A *Bayesian ideal* is a map $\varphi : \mathbb{D} \longrightarrow [0,1]$ such that

$$(\Xi \vdash \Theta) \cdot \varphi(\Xi\Theta) \quad \leq \quad \varphi(\Xi)$$

We denote by $\mathcal{J}\mathbb{D}$ the dependency of Bayesian ideals, with the monoid and hom-object structure

$$\begin{aligned} \varphi \cdot \psi(\Theta) &= \varphi(\Theta) \cdot \psi(\Theta) \\ (\varphi \vdash \psi) &= \bigwedge_{\Theta \in \mathbb{D}} \left(\frac{\varphi(\Theta)}{\psi(\Theta)} \wedge \frac{\psi(\Theta)}{\varphi(\Theta)} \right) \end{aligned}$$

Domain theory of guessing

Definition

Let $\mathbb{D}$ be a $[0,1]$-dependency. A *guessing ideal* is an algorithm $\Phi : \mathbb{D} \longrightarrow [0,1]$ such that

$$\left[\Xi \vdash \Theta\right]\cdot\Phi(\Xi\Theta) \quad \leq \quad \Phi(\Xi)$$

We denote by $\mathcal{H}\mathbb{D}$ the dependency of guessing ideals, with the monoid and hom-object structure

$$\begin{aligned}\Phi\cdot\Psi(\Theta) \quad &= \quad \Phi(\Theta)\cdot\Psi(\Theta)\\ \left[\Phi \vdash \Psi\right] \quad &= \quad \bigwedge_{\Theta\in\mathbb{D}}\left(\frac{\Phi(\Theta)}{\Psi(\Theta)} \wedge \frac{\Psi(\Theta)}{\Phi(\Theta)}\right)\end{aligned}$$

Ideals are Cauchy sequences

Theorem

Bayesian (resp. guessing) ideals over a $[0,1]$*-dependency* $\mathbb{D}$ *correspond to the sequences of events (resp. guesses)* $\langle \Theta_i \rangle_{i=1}^{\infty}$ *such that*

$$\forall k \in \mathbb{N}\ \exists N \in \mathbb{N}\ \forall n > N\ \forall m \in \mathbb{N}.$$
$$\left(\Theta_1, \Theta_2, \ldots, \Theta_n \vdash \Theta_{n+m}\right) \quad \geq \quad e^{-\frac{1}{k}}$$

$$\exists N \in \mathcal{F}\ \forall k \in \mathbb{N}\ \forall n > N(k)\ \forall m \in \mathbb{N}.$$
$$\left[\Theta_1, \Theta_2, \ldots, \Theta_n \vdash \Theta_{n+m}\right] \quad \geq \quad e^{-\frac{1}{k}}$$

Guessing by adjoints

Fact.

$$\mathbb{D} \xrightarrow{Y} \mathcal{H}\mathbb{D}$$

$$Y(\Theta) = [- \vdash \Theta]$$

Guessing by adjoints

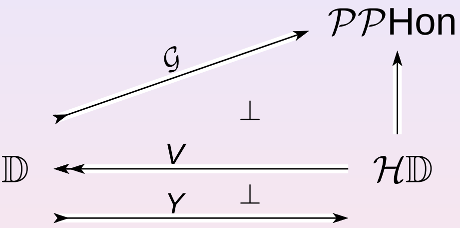
Proposition

$$\mathbb{D} \underset{Y}{\overset{V}{\twoheadleftarrow\!\!\!\rightarrowtail}} \mathcal{H}\mathbb{D} \qquad (V \dashv Y:\ V : \mathcal{H}\mathbb{D} \twoheadrightarrow \mathbb{D},\ Y : \mathbb{D} \rightarrowtail \mathcal{H}\mathbb{D},\ \bot)$$

$$\begin{aligned} Y(\Theta) &= [- \vdash \Theta] \\ V(\Theta_i) &= \lim_{i \to \infty} \Theta_i \end{aligned}$$

Guessing by adjoints

Proposition



$$
\begin{aligned}
Y(\Theta) \quad &= \quad \left[- \vdash \Theta\right] \\
V(\Theta_i) \quad &= \quad \lim_{i \to \infty} \Theta_i \\
\mathcal{G} \text{ guards } \Theta \quad &\iff \quad \forall \Xi.\ \left[\Xi \vdash \Theta\right] = \sum_{\Gamma \in \mathcal{G}} \left[\Xi \vdash \Gamma\right]\left[\Xi\Gamma \vdash \Theta\right]
\end{aligned}
$$

Way below IND-CCA

$$
\begin{array}{rcl}
 & & [c_0] \\
\cdot[c_0, m = \mathsf{D}(\overline{k}, c_0) & \vdash & m_0, m_1] \\
\cdot[c_0, m = \mathsf{D}(\overline{k}, c_0), m_0, m_1, c \in \mathsf{E}(k, m_b) & \vdash & c_1 \neq c] \\
\cdot[c_0, m = \mathsf{D}(\overline{k}, c_0), m_0, m_1, c \in \mathsf{E}(k, m_b), & & \\
c_1 \neq c, \widetilde{m} = \mathsf{D}(\overline{k}, c_1) & \vdash & b] \quad \leq \quad [b]
\end{array}
$$

Outline

Summary

- an algebraic theory of guessing can be presented as an algebraic theory of approximation
- a probabilistic theory of guessing can be presented by extending $\{0, 1\}$-domains to $[0, 1]$-domains