Geometry of abstraction in quantum computation

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Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

Measurements

Future work

Outline

Introduction
- Quantum programming
- λ-abstraction

Graphical notation

Geometry of abstraction
- Abstraction with pictures
- Consequences

Geometry of $\ddagger$-abstraction
- $\ddagger$-monoidal categories
- Quantum objects
- Abstraction in $\ddagger$-monoidal categories
- Classical objects
- Base

Category of measurements

Future work

Outline

Introduction

- Quantum programming
- λ-abstraction

Graphical notation

Geometry of abstraction

- Abstraction with pictures
- Consequences

Geometry of ‡-abstraction

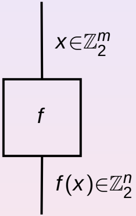
- ‡-monoidal categories
- Quantum objects
- Abstraction in ‡-monoidal categories
- Classical objects
- Base

Category of measurements

Future work

Introduction

What do quantum programmers do?



Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

Introduction

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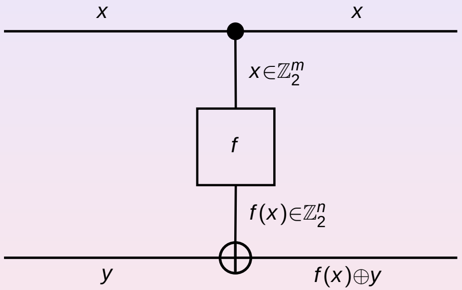
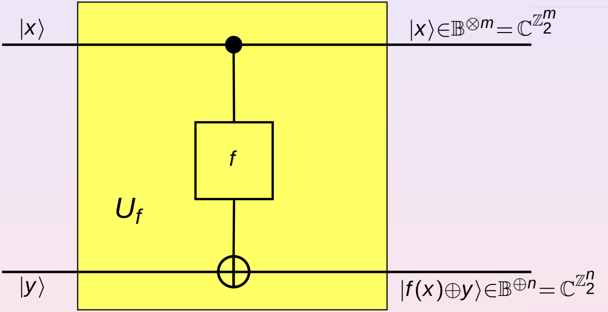


Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

Introduction

What do quantum programmers do?



Introduction
Quantum programming
λ-abstraction

Graphical notation

Geometry of abstraction

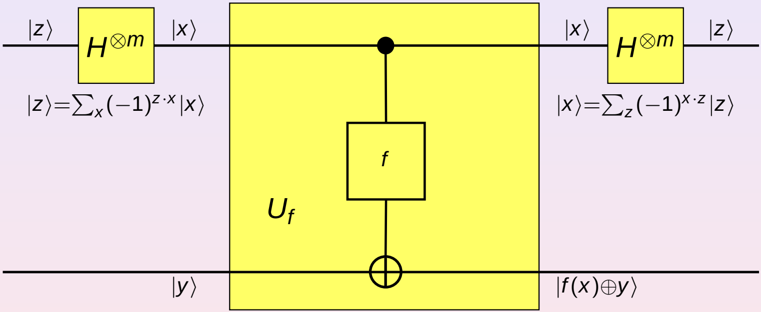
Geometry of ‡-abstraction

Measurements

Future work

Introduction

What do quantum programmers do?



Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

Introduction

What do quantum programmers do?

Simon's algorithm

$$\frac{\dfrac{f : \mathbb{Z}_2^m \longrightarrow \mathbb{Z}_2^n : x \mapsto f(x)}{f' : \mathbb{Z}_2^{m+n} \longrightarrow \mathbb{Z}_2^{m+n} : x, y \mapsto x, f(x) \oplus y}}{U_f : \mathbb{C}^{\mathbb{Z}_2^{m+n}} \longrightarrow \mathbb{C}^{\mathbb{Z}_2^{m+n}} : |x, y\rangle \mapsto |x, f(x) \oplus y\rangle}$$

$$\begin{aligned} Simon &= (H^{\otimes m} \otimes id) U_f (H^{\otimes m} \otimes id)\, |0, 0\rangle \\ &= \sum_{x,z \in \mathbb{Z}_2^m} (-1)^{x \cdot z} |z, f(x)\rangle \end{aligned}$$

Simon's algorithm

$$\frac{\dfrac{f : \mathbb{Z}_2^m \longrightarrow \mathbb{Z}_2^n : x \mapsto f(x)}{f' : \mathbb{Z}_2^{m+n} \longrightarrow \mathbb{Z}_2^{m+n} : x, y \mapsto x, f(x) \oplus y}}{U_f : \mathbb{C}^{\mathbb{Z}_2^{m+n}} \longrightarrow \mathbb{C}^{\mathbb{Z}_2^{m+n}} : |x, y\rangle \mapsto |x, f(x) \oplus y\rangle}$$

$$\begin{aligned} Simon &= (H^{\otimes m} \otimes id) U_f (H^{\otimes m} \otimes id) \, |0, 0\rangle \\ &= \sum_{x,z \in \mathbb{Z}_2^m} (-1)^{x \cdot z} |z, f(x)\rangle \end{aligned}$$

...to find a hidden subgroup

measurement $\rightsquigarrow$ find c such that $f(x + c) = f(x)$

Geometry of quantum abstraction
Dusko Pavlovic
Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

Introduction

What do quantum programmers do?

Shor's algorithm

$$\frac{\dfrac{f : \mathbb{Z}_q^m \longrightarrow \mathbb{Z}_q^n : x \mapsto a^x \mod q}{f' : \mathbb{Z}_q^{m+n} \longrightarrow \mathbb{Z}_q^{m+n} : x, y \mapsto x, a^x + y \mod q}}{U_f : \mathbb{C}^{\mathbb{Z}_q^{m+n}} \longrightarrow \mathbb{C}^{\mathbb{Z}_q^{m+n}} : |x, y\rangle \mapsto |x, a^x + y \mod q\rangle}$$

$$\begin{aligned} Shor &= (FT_m \otimes id) U_f (FT_m \otimes id)\, |0, 0\rangle \\ &= \sum_{x,z \in \mathbb{Z}_q^m} (-1)^{x \cdot z} |z, f(x)\rangle \end{aligned}$$

...to find a hidden subgroup

measurement $\rightsquigarrow$ find c such that $a^{x+c} = a^x \mod q$

Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

Introduction

What do quantum programmers do?

Hallgren's algorithm

$$\frac{\dfrac{h : \mathbb{Z}^m \longrightarrow \mathbb{Z}^n : x \mapsto I_x \text{ (fraction ideal)}}{h' : \mathbb{Z}^{m+n} \longrightarrow \mathbb{Z}^{m+n} : x, y \mapsto x, y - h(x)}}{U_h : \mathbb{C}^{\mathbb{Z}^{m+n}} \longrightarrow \mathbb{C}^{\mathbb{Z}^{m+n}} : |x, y\rangle \mapsto |x, y - h(x)\rangle}$$

$$\begin{aligned} Hallgren &= (FT_m \otimes id) U_h (FT_m \otimes id)\, |d, \widetilde{d}\rangle \\ &= \sum_{x,z \in \mathbb{Z}^m} (-1)^{x \cdot z} |z, h(x)\rangle \end{aligned}$$

. . . to find a hidden subgroup

measurement $\rightsquigarrow$ find R such that $h(x + R) = h(x)$

Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

Introduction

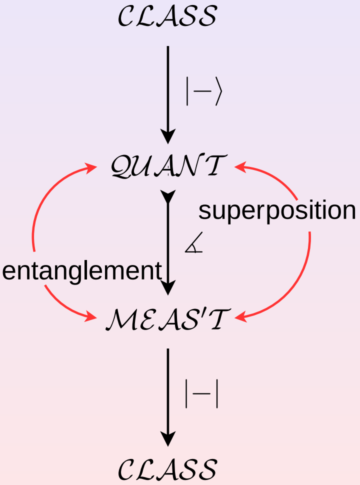
General design pattern of quantum software engineering ;)

$$
\begin{array}{c}
\mathcal{CLASS} \\
\Big\downarrow \; |-\rangle \\
\mathcal{QUANT} \\
\Big\downarrow \; \measuredangle \\
\mathcal{MEAS'T} \\
\Big\downarrow \; |-| \\
\mathcal{CLASS}
\end{array}
$$

Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

Introduction

Quantum prog. = functional prog. + superposition + entanglement



Introduction

Standard type universes where quantum programmers work

$$\mathcal{CLASS} \rightsquigarrow \mathsf{FSet},\ \mathsf{FSet}^{op}_{\wp\wp},\ \mathsf{FFMod}_R \ldots$$

$$\big\downarrow\ |-\rangle$$

$$\mathcal{QUANT} \rightsquigarrow \mathsf{FHilb},\ \mathsf{CPM}(\mathsf{FHilb}) \ldots$$

$$\big\downarrow\ \measuredangle$$

$$\mathcal{MEAS'T} \rightsquigarrow \mathsf{FHilb},\ \mathsf{CPM}(\mathsf{FHilb}) \ldots$$

$$\big\downarrow\ |-|$$

$$\mathcal{CLASS}$$

Introduction

Function abstraction in quantum programming

$$\mathcal{CLASS}[x] \rightsquigarrow \mathrm{FSet}[x], \mathrm{FSet}^{op}_{\wp\wp}[x], \ \mathrm{FFMod}_R[x]\ldots$$

$$\downarrow |-\rangle$$

$$\mathcal{QUANT}[x] \rightsquigarrow \mathrm{FHilb}[x], \ \mathrm{CPM(FHilb)}[x]\ldots$$

$$\downarrow \measuredangle$$

$$\mathcal{MEAS'T}\{x\} \rightsquigarrow \mathrm{FHilb}\{x\}, \ \mathrm{CPM(FHilb)}\{x\}\ldots$$

$$\downarrow |-|$$

$$\mathcal{CLASS}$$

Introduction

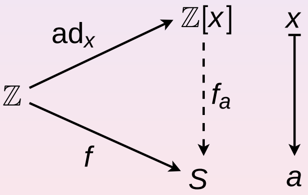
Function abstraction in quantum programming

$$
\begin{array}{ccc}
\mathcal{CLASS}[x] & \cong & \mathcal{KL}(X\times) \\
\Big\downarrow |-\rangle & & \\
\mathcal{QUANT}[x] & \cong & \mathcal{KL}(X\otimes) \\
\Big\downarrow \measuredangle & & \\
\mathcal{MEAS'T}\{x\} & \simeq & \mathcal{EM}(X\otimes) \\
\Big\downarrow |-| & & \\
\mathcal{CLASS} & &
\end{array}
$$

Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

λ-abstraction

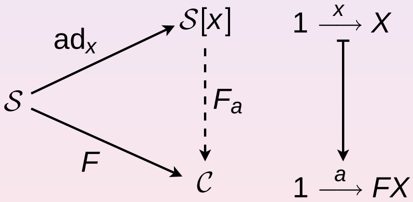
$$\frac{2x^3 + 3x + 1 \text{ in } \mathbb{Z}[x]}{\lambda x.\, 2x^3 + 3x + 1 \text{ in } \mathbb{Z} \longrightarrow \mathbb{Z}}$$



Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

λ-abstraction in cartesian closed categories

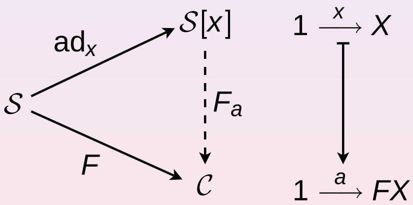
$$\frac{p(x) : B \quad \text{in } \mathcal{S}[x : X]}{\lambda x.\, p(x) : B^X \ \text{in } \mathcal{S}}$$



Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

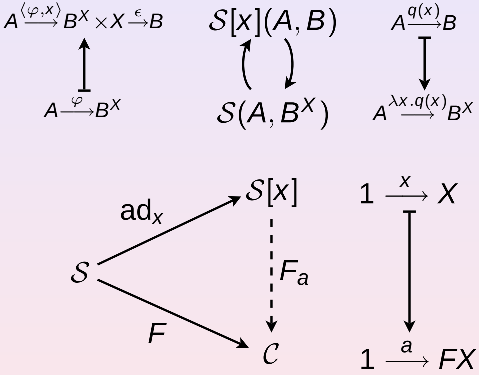
λ-abstraction in cartesian closed categories

$$\frac{A \xrightarrow{q(x)} B \quad \text{in } \mathcal{S}[x : X]}{A \xrightarrow{\lambda x.q(x)} B^X \text{ in } \mathcal{S}}$$



Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

λ-abstraction in cartesian closed categories



Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

λ-abstraction in cartesian closed categories

Theorem (Lambek, Adv. in Math. 79)

Let $\mathcal{S}$ be a cartesian closed category, and $\mathcal{S}[x]$ the free cartesian closed category generated by $\mathcal{S}$ and $x : 1 \longrightarrow X$.

λ-abstraction in cartesian closed categories

Theorem (Lambek, Adv. in Math. 79)

Let $\mathcal{S}$ be a cartesian closed category, and $\mathcal{S}[x]$ the free cartesian closed category generated by $\mathcal{S}$ and $x : 1 \to X$.

Then the inclusion $\mathrm{ad}_x : \mathcal{S} \longrightarrow \mathcal{S}[x]$ has a right adjoint $\mathrm{ab}_x : \mathcal{S}[x] \longrightarrow \mathcal{S} \; : \; A \mapsto A^X$ and the transpositions

$$\begin{array}{ccc} A \xrightarrow{\langle \varphi, x\rangle} B^X \times X \xrightarrow{\epsilon} B & \mathcal{S}[x](\mathrm{ad}_x A, B) & A \xrightarrow{q(x)} B \\ \uparrow & \uparrow\downarrow & \downarrow \\ A \xrightarrow{\varphi} B^X & \mathcal{S}(A, \mathrm{ab}_x B) & A \xrightarrow{\lambda x. q(x)} B^X \end{array}$$

model λ-abstraction and application.

Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

λ-abstraction in cartesian closed categories

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$$\begin{array}{ccc} A \xrightarrow{\langle \varphi, x\rangle} B^X \times X \xrightarrow{\epsilon} B & \mathcal{S}[x](\mathrm{ad}_x A, B) & A \xrightarrow{q(x)} B \\ \big\uparrow & \big\uparrow \; \big\downarrow & \big\downarrow \\ A \xrightarrow{\varphi} B^X & \mathcal{S}(A, \mathrm{ab}_x B) & A \xrightarrow{\lambda x. q(x)} B^X \end{array}$$

model λ-abstraction and application.

$\mathcal{S}[x]$ is isomorphic with the Kleisli category for the power monad $(-)^X$.

Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

κ-abstraction in cartesian categories

Theorem (Lambek, Adv. in Math. 79)

Let $\mathcal{S}$ be a cartesian category, and $\mathcal{S}[x]$ the free cartesian category generated by $\mathcal{S}$ and $x : 1 \to X$.

Then the inclusion $\mathrm{ad}_x : \mathcal{S} \longrightarrow \mathcal{S}[x]$ *has a* left *adjoint* $\mathrm{ab}_x : \mathcal{S}[x] \longrightarrow \mathcal{S} \; : \; A \mapsto X \times A$ *and the transpositions*

$$\begin{array}{ccc} A \xrightarrow{\langle x, id\rangle} X\times A \xrightarrow{\varphi} B & \mathcal{S}[x](A, \mathrm{ad}_x B) & A \xrightarrow{f_x} B \\ \Big\uparrow & \big\uparrow \;\; \big\downarrow & \Big\downarrow \\ X\times A \xrightarrow{\varphi} B & \mathcal{S}(\mathrm{ab}_x A, B) & X \times A \xrightarrow{\kappa x.\, f_x} B \end{array}$$

model first order abstraction and application.

$\mathcal{S}[x]$ is isomorphic with the Kleisli category for the product comonad $X \times (-)$.

κ-abstraction in *monoidal* categories

Theorem (DP, MSCS 95)

Let $\mathcal{C}$ be a monoidal *category, and $\mathcal{C}[x]$ the free* monoidal *category generated by $\mathcal{C}$ and $x : 1 \to X$.*

Then the strong adjunctions $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ are in one-to-one correspondence with the internal comonoid structures on X. The transpositions

$$\begin{array}{ccc} A \xrightarrow{x \otimes A} X \otimes A \xrightarrow{\varphi} B & \mathcal{C}[x](A, \mathrm{ad}_x B) & A \xrightarrow{f_x} B \\ \uparrow & \uparrow \downarrow & \downarrow \\ X \otimes A \xrightarrow{\varphi} B & \mathcal{C}(\mathrm{ab}_x A, B) & X \otimes A \xrightarrow{\kappa x. f_x} B \end{array}$$

model action abstraction and application.

$\mathcal{C}[x]$ is isomorphic with the Kleisli category for the comonad $X \otimes (-)$, induced by any of the comonoid structures.

κ-abstraction in monoidal categories

Task

Extend this to $\ddagger$-monoidal categories.

κ-abstraction in monoidal categories

Task

Extend this to $\ddagger$-monoidal categories.

Problem

Lots of complicated diagram chasing.

κ-abstraction in monoidal categories

Task

Extend this to $\ddagger$-monoidal categories.

Problem

Lots of complicated diagram chasing.

Solution?

What does abstraction mean graphically?

Outline

Introduction
Quantum programming
λ-abstraction

Graphical notation

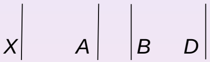
Geometry of abstraction
Abstraction with pictures
Consequences

Geometry of ‡-abstraction
‡-monoidal categories
Quantum objects
Abstraction in ‡-monoidal categories
Classical objects
Base

Category of measurements

Future work

Objects



$$X \otimes A \otimes B \otimes D$$

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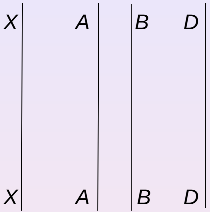
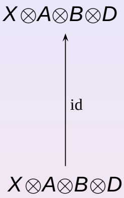
Introduction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

Identities





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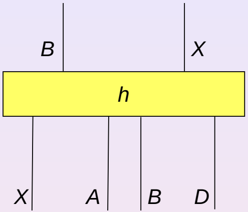
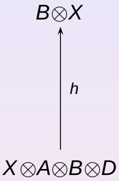
Morphisms





Geometry of quantum abstraction

Dusko Pavlovic

Introduction

Graphical notation

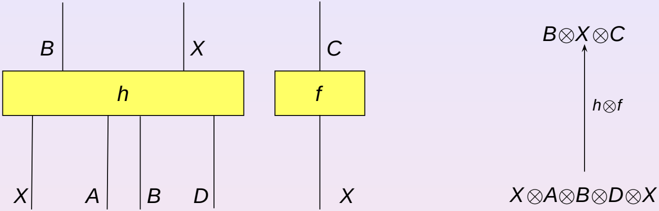
Geometry of abstraction

Geometry of ‡-abstraction

Measurements

Future work

Tensor (parallel composition)



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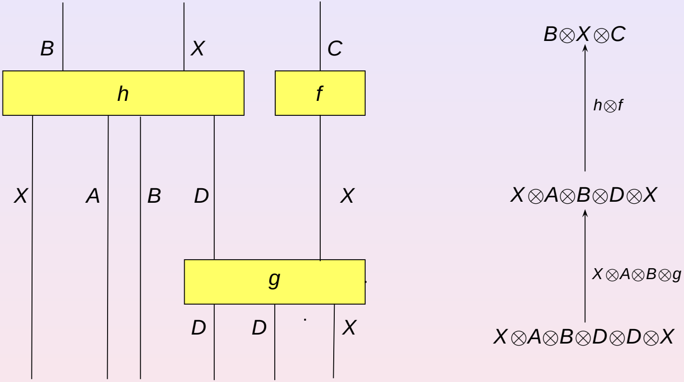
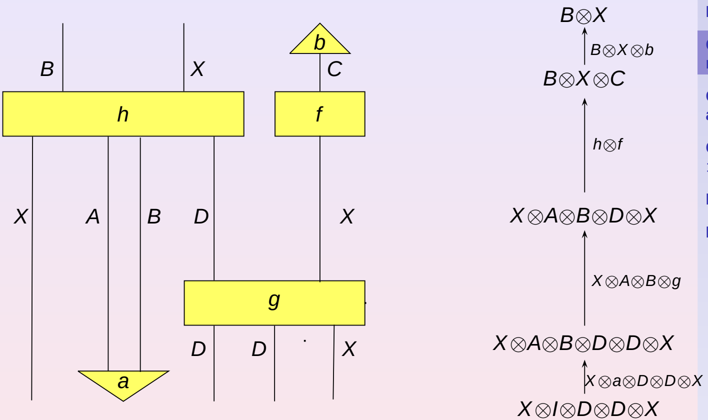
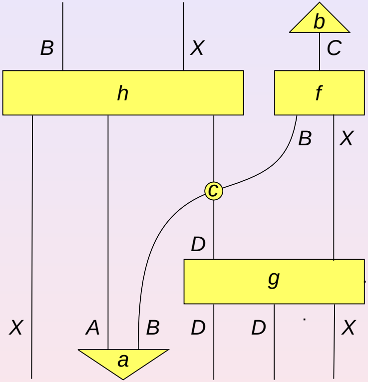
Sequential composition

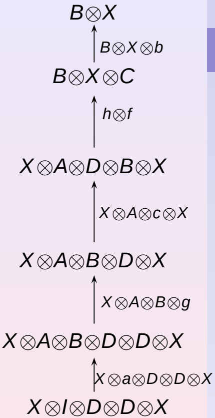


Elements (vectors) and coelements (functionals)

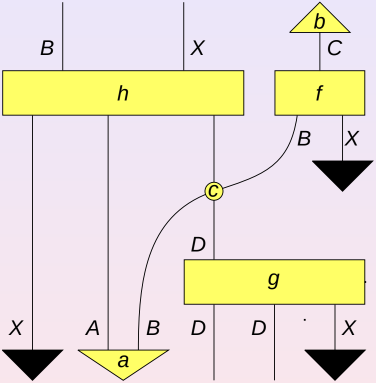


Symmetry


B
X
b
C
h
f
B
X
c
D
g
X
A
B
D
D
X
a


$B \otimes X$
$B \otimes X \otimes b$
$B \otimes X \otimes C$
$h \otimes f$
$X \otimes A \otimes D \otimes B \otimes X$
$X \otimes A \otimes c \otimes X$
$X \otimes A \otimes B \otimes D \otimes X$
$X \otimes A \otimes B \otimes g$
$X \otimes A \otimes B \otimes D \otimes D \otimes X$
$X \otimes a \otimes D \otimes D \otimes X$
$X \otimes I \otimes D \otimes D \otimes X$

Polynomials



$$
\begin{array}{l}
B\otimes X\\
\uparrow\ B\otimes X\otimes b\\
B\otimes X\otimes C\\
\uparrow\ h\otimes f\\
X\otimes A\otimes D\otimes B\otimes X\\
\uparrow\ \mathrm{id}\otimes x\\
X\otimes A\otimes D\otimes B\otimes I\\
\uparrow\ X\otimes A\otimes c\otimes r\\
X\otimes A\otimes B\otimes D\\
\uparrow\ X\otimes A\otimes B\otimes g\\
X\otimes A\otimes B\otimes D\otimes D\otimes X\\
\uparrow\ x\otimes a\otimes D\otimes D\otimes x\\
I\otimes I\otimes D\otimes D\otimes I
\end{array}
$$

Introduction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
Measurements
Future work

Outline

Introduction

- Quantum programming
- λ-abstraction

Graphical notation

Geometry of abstraction

- Abstraction with pictures
- Consequences

Geometry of ‡-abstraction

- ‡-monoidal categories
- Quantum objects
- Abstraction in ‡-monoidal categories
- Classical objects
- Base

Category of measurements

Future work

Abstraction with pictures

Theorem (again)

Let $\mathcal{C}$ be a symmetric monoidal category, and $\mathcal{C}[x]$ the free symmetric monoidal category generated by $\mathcal{C}$ and $x : 1 \to X$.

Then there is a one-to-one correspondence between

- *adjunctions* $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *satisfying*
  1. $\mathrm{ab}_x(A \otimes B) = \mathrm{ab}_x(A) \otimes B$
  2. $\eta(A \otimes B) = \eta(A) \otimes B$
  3. $\eta_I = x$

  and
- *commutative comonoids on* X.

Introduction

Graphical notation

Geometry of abstraction

Abstraction with pictures

Consequences

Geometry of ‡-abstraction

Measurements

Future work

Abstraction with pictures

Theorem (again)

Let $\mathcal{C}$ be a symmetric monoidal category, and $\mathcal{C}[x]$ the free symmetric monoidal category generated by $\mathcal{C}$ and $x : 1 \to X$.

Then there is a one-to-one correspondence between

- *adjunctions* $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *satisfying*
  1. $\mathrm{ab}_x(A \otimes B) = \mathrm{ab}_x(A) \otimes B$
  2. $\eta(A \otimes B) = \eta(A) \otimes B$
  3. $\eta_I = x$

  and
- *commutative comonoids on X.*

$\mathcal{C}[x]$ is isomorphic with the Kleisli category for the commutative comonad $X \otimes (-)$, induced by any of the comonoid structures.

Introduction

Graphical notation

Geometry of abstraction

Abstraction with pictures

Consequences

Geometry of ‡-abstraction

Measurements

Future work

Proof (↓)

Given $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$, conditions 1.-3. imply

- $\mathrm{ab}_x(A) = X \otimes A$
- $\eta(A) = x \otimes A$

Introduction

Graphical notation

Geometry of abstraction

Abstraction with pictures

Consequences

Geometry of ‡-abstraction

Measurements

Future work

Proof (↓)

Therefore the correspondence

$$\mathcal{C}\left(\mathrm{ab}_x(A), B\right) \rightleftarrows \mathcal{C}[x]\left(A, \mathrm{ad}_x(B)\right)$$

Introduction

Graphical notation

Geometry of abstraction

Abstraction with pictures

Consequences

Geometry of ‡-abstraction

Measurements

Future work

Proof (↓)

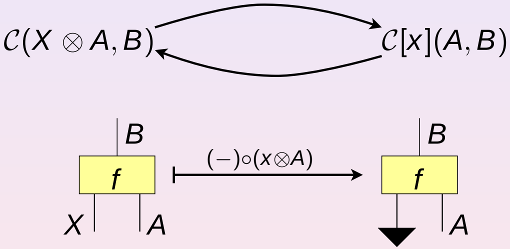
…is actually

$$\mathcal{C}(X \otimes A, B) \rightleftarrows \mathcal{C}[x](A, B)$$

Introduction
Graphical notation
Geometry of abstraction
Abstraction with pictures
Consequences
Geometry of ‡-abstraction
Measurements
Future work

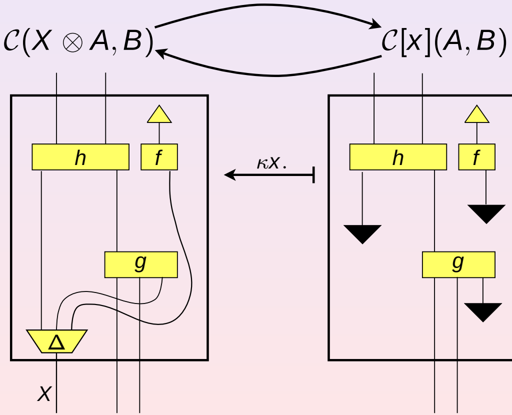
Proof (↓)

…with



Proof (↓)

. . . and



Proof (↓)

The bijection corresponds to the conversion:

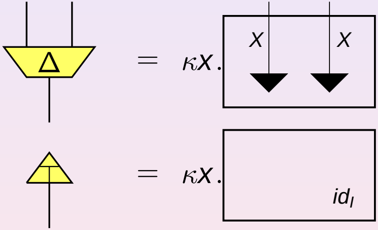
$$\mathcal{C}(X \otimes A, B) \underset{\kappa x.}{\overset{(-)\circ(x\otimes A)}{\rightleftarrows}} \mathcal{C}[x](A, B) \qquad \cong$$

$$\big(\kappa x.\ \varphi(x)\big) \circ \big(x \otimes A\big) = \varphi(x) \qquad (\beta\text{-rule}$$

$$\kappa x.\ \big(f \circ (x \otimes A)\big) = f \qquad (\eta\text{-rule}$$

Proof (↓)

The comonoid structure $(X, \Delta, \top)$ is



Introduction

Graphical notation

Geometry of abstraction

Abstraction with pictures

Consequences

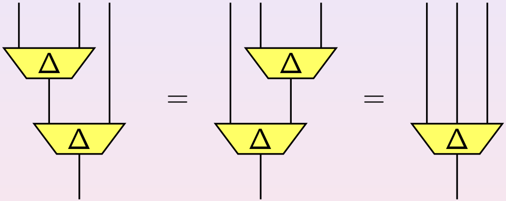
Geometry of ‡-abstraction

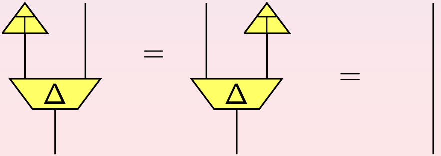
Measurements

Future work

Proof (↓)

The conversion rules imply the comonoid laws

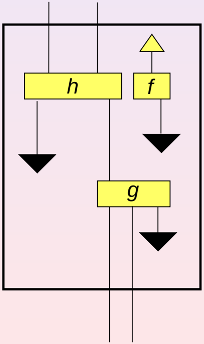


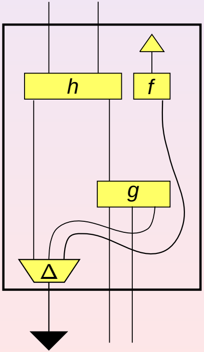


Proof (↑)

Given $(X, \Delta, \top)$, use its copying and deleting power, and the symmetries, to normalize every $\mathcal{C}[x]$-arrow:

$$\varphi(x) = \overline{\varphi} \circ (x \otimes A)$$





Introduction

Graphical notation

Geometry of abstraction

Abstraction with pictures

Consequences

Geometry of ‡-abstraction

Measurements

Future work

Proof (↑)

Then set $\kappa x.\, \varphi(x) = \overline{\varphi}$ to get

$$\mathcal{C}(X \otimes A, B) \underset{\kappa x.}{\overset{(-)\circ(x\otimes A)}{\rightleftarrows}} \mathcal{C}[x](A, B) \qquad (\cong)$$

$$\big(\kappa x.\, \varphi(x)\big) \circ \big(x \otimes A\big) = \varphi(x) \qquad (\beta\text{-rule}$$

$$\kappa x.\, \big(f \circ (x \otimes A)\big) = f \qquad (\eta\text{-rule}$$

Remark

- $\mathcal{C}[x] \cong \mathcal{C}_{X\otimes}$ and $\mathcal{C}[x, y] \cong \mathcal{C}_{X\otimes Y\otimes} = \mathcal{KL}(X \otimes Y\otimes)$, reduce the finite polynomials to the Kleisli morphisms.

Remark

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- But the extensions $\mathcal{C}[\mathcal{X}]$, where $\mathcal{X}$ is large are also of interest.

Remark

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- But the extensions $\mathcal{C}[\mathcal{X}]$, where $\mathcal{X}$ is large are also of interest.
  - Cf. $\mathbb{N}[\mathbb{N}]$, Set[Set], and $CPM(\mathcal{C})$.

Interpretation

Introduction

Graphical notation

Geometry of abstraction

Abstraction with pictures

Consequences

Geometry of ‡-abstraction

Measurements

Future work

Interpretation

Upshot

In symmetric monoidal categories,
abstraction applies just to copiable and deletable data.

Interpretation

Upshot

In symmetric monoidal categories,
abstraction applies just to copiable and deletable data.

Definition

A vector $\varphi \in \mathcal{C}(I, X)$ is a *base vector* (or a *set-like element*) with respect to the abstraction operation κx if it can be copied and deleted in $\mathcal{C}[x]$

$$(\kappa x.x \otimes x) \circ \varphi \quad = \quad \varphi \otimes \varphi$$
$$(\kappa x.\mathrm{id}_I) \circ \varphi \quad = \quad \mathrm{id}_I$$

Interpretation

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In symmetric monoidal categories,
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$$(\kappa x. \mathrm{id}_I) \circ \varphi \quad = \quad \mathrm{id}_I$$

Proposition

$\varphi \in \mathcal{C}(I, X)$ is a *base vector* with respect to κx if and only if it is a homomorphism for the comonoid structure $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ corresponding to κx.

Introduction

Graphical notation

Geometry of abstraction

Abstraction with pictures

Consequences

Geometry of ‡-abstraction

Measurements

Future work

Introduction

Graphical notation

Geometry of abstraction

Abstraction with pictures

Consequences

Geometry of ‡-abstraction

Measurements

Future work

Interpretation

Upshot

In other words,
only the base vectors can be substituted for variables.

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Substitution is a structure preserving ioof $\mathcal{C}[x] \longrightarrow \mathcal{C}$.

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Upshot

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Definition

Substitution is a structure preserving ioof $\mathcal{C}[x] \longrightarrow \mathcal{C}$.

Corollary

The substitution functors $\mathcal{C}[x] \longrightarrow \mathcal{C}$ are in one-to-one correspondence with the base vectors of type X.

Outline

Introduction
Quantum programming
λ-abstraction

Graphical notation

Geometry of abstraction
Abstraction with pictures
Consequences

Geometry of ‡-abstraction
‡-monoidal categories
Quantum objects
Abstraction in ‡-monoidal categories
Classical objects
Base

Category of measurements

Future work

‡-monoidal categories

Definitions

A ‡-*category* $\mathcal{C}$ is given with an involutive ioof $\ddagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

Definitions

A *‡-category* $\mathcal{C}$ is given with an involutive ioof $‡ : \mathcal{C}^{op} \longrightarrow \mathcal{C}$.

A morphism f in a ‡-category $\mathcal{C}$ is called *unitary* if $f^‡ = f^{-1}$.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

Definitions

A *‡-category* $\mathcal{C}$ is given with an involutive ioof $\ddagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$.

A morphism f in a ‡-category $\mathcal{C}$ is called *unitary* if $f^{\ddagger} = f^{-1}$.

A (symmetric) monoidal category $\mathcal{C}$ is *‡-monoidal* if its monoidal isomorphisms are unitary.

‡-monoidal categories

Using the monoidal notations for:

- vectors: $\mathcal{C}(A) = \mathcal{C}(I, A)$
- scalars: $\mathbb{I} = \mathcal{C}(I, I)$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

Using the monoidal notations for:

- vectors: $\mathcal{C}(A) = \mathcal{C}(I, A)$
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in every ‡-monoidal category we can define

- *abstract inner product*

$$\begin{array}{rcl} \langle - | - \rangle_A \ : \ \mathcal{C}(A) \times \mathcal{C}(A) & \longrightarrow & \mathbb{I} \\ (\varphi, \psi : I \longrightarrow A) & \longmapsto & \left(I \xrightarrow{\varphi} A \xrightarrow{\psi^\ddagger} I \right) \end{array}$$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

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- *partial inner product*

$$\begin{array}{rcl} \langle -|-\rangle_{AB} \ : \ \mathcal{C}(A \otimes B) \times \mathcal{C}(A) & \longrightarrow & \mathcal{C}(B) \\ (\varphi : I \to A \otimes B, \psi : I \to A) & \longmapsto & \left(I \xrightarrow{\varphi} A \otimes B \xrightarrow{\psi^\ddagger \otimes B} B \right) \end{array}$$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

Using the monoidal notations for:

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- *entangled vectors* $\eta \in \mathcal{C}(A \otimes A)$, such that $\forall \varphi \in \mathcal{C}(A)$

$$\langle \eta | \varphi \rangle_{AA} \quad = \quad \varphi$$

Introduction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
‡-monoidal categories
Quantum objects
Abstraction in ‡-monoidal categories
Classical objects
Base
Measurements
Future work

‡-monoidal categories

Using

- entangled vectors $\eta A : I \longrightarrow A \otimes A$ and, $\eta B : I \longrightarrow B \otimes B$ and
- their adjoints $\varepsilon A = \eta^{\ddagger} A : A \otimes A \longrightarrow I$ and $\varepsilon B = \eta^{\ddagger} B : B \otimes B \longrightarrow I$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

Using

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- their adjoints $\varepsilon A = \eta^{\ddagger} A : A \otimes A \longrightarrow I$ and $\varepsilon B = \eta^{\ddagger} B : B \otimes B \longrightarrow I$

we can define for every $f : A \longrightarrow B$

- the *dual* $f^* : B \longrightarrow A$

$$f^* \quad = \quad B \xrightarrow{B\eta} BAA \xrightarrow{BfA} BBA \xrightarrow{\varepsilon A} A$$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

Using

- entangled vectors $\eta A : I \longrightarrow A \otimes A$ and, $\eta B : I \longrightarrow B \otimes B$and
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$$f^* \quad = \quad B \xrightarrow{B\eta} BAA \xrightarrow{BfA} BBA \xrightarrow{\varepsilon A} A$$

- the *conjugate* $f_* : A \longrightarrow B$

$$f_* \quad = \quad f^{*\ddagger} \; = \; f^{\ddagger *}$$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

Proposition

For every object A in a ‡-monoidal category $\mathcal{C}$ holds

(a) $\iff$ (b) $\Longleftarrow$ (c),

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

Proposition

For every object A in a ‡-monoidal category $\mathcal{C}$ holds (a) $\iff$ (b) $\Longleftarrow$ (c), where

(a) $\eta \in \mathcal{C}(A \otimes A)$ is entangled

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

Proposition

For every object A in a ‡-monoidal category $\mathcal{C}$ holds (a) $\iff$ (b) $\Longleftarrow$ (c), where

(a) $\eta \in \mathcal{C}(A \otimes A)$ is entangled

(b) $\varepsilon = \eta^\ddagger \in \mathcal{C}(A \otimes A, I)$ internalizes the inner product

$$\varepsilon \circ (\psi_* \otimes \varphi) \quad = \quad \langle \varphi | \psi \rangle$$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

‡-monoidal categories

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(b) $\varepsilon = \eta^{\ddagger} \in \mathcal{C}(A \otimes A, I)$ internalizes the inner product

$$\varepsilon \circ (\psi_* \otimes \varphi) \quad = \quad \langle \varphi | \psi \rangle$$

(c) (η, ε) realize the self-adjunction $A \dashv A$, in the sense

$$A \xrightarrow{\eta \otimes A} A \otimes A \otimes A \xrightarrow{A \otimes \varepsilon} A \quad = \quad \mathrm{id}_A$$

$$A \xrightarrow{A \otimes \eta} A \otimes A \otimes A \xrightarrow{\varepsilon \otimes A} A \quad = \quad \mathrm{id}_A$$

The three conditions are equivalent if I generates $\mathcal{C}$.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

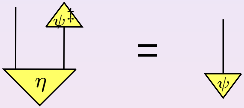
Future work

‡-monoidal categories

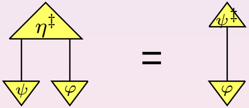
Proposition in pictures

For every object A in a ‡-monoidal category $\mathcal{C}$ holds (a) $\iff$ (b) $\Longleftarrow$ (c), where

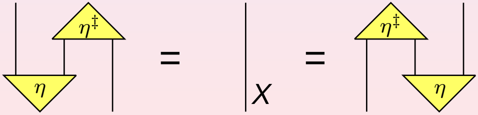
(a)



(b)



(c)



Introduction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
‡-monoidal categories
Quantum objects
Abstraction in ‡-monoidal categories
Classical objects
Base
Measurements
Future work

Quantum objects

Definition

A *quantum object* in a ‡-monoidal category is an object equipped with the structure from the preceding proposition.

Quantum objects

Definition

A *quantum object* in a ‡-monoidal category is an object equipped with the structure from the preceding proposition.

Remark

The subcategory of quantum objects in any ‡-monoidal category is ‡-compact (strongly compact) — with all objects self-adjoint.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Abstraction in ‡-monoidal categories

Theorem

Let $\mathcal{C}$ be a ‡-monoidal category, and $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ a comonoid that induces $\mathrm{ab}_X \dashv \mathrm{ad}_X : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Abstraction in ‡-monoidal categories

Theorem

Let $\mathcal{C}$ *be a ‡-monoidal category,*
and $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ *a comonoid that induces*
$\mathrm{ab}_X \dashv \mathrm{ad}_X : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

Then the following conditions are equivalent:

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Abstraction in ‡-monoidal categories

Theorem

Let $\mathcal{C}$ be a ‡-monoidal category, and $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ a comonoid that induces $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

Then the following conditions are equivalent:

(a) $\mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *creates* $\ddagger : \mathcal{C}[x]^{op} \longrightarrow \mathcal{C}[x]$ *such that* $\langle x|x\rangle = x^{\ddagger} \circ x = \mathrm{id}_I$.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Abstraction in ‡-monoidal categories

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(b) $\eta = \Delta \circ \bot$ *and* $\varepsilon = \eta^{\ddagger} = \nabla \circ \top$ *realize* $X \dashv X$.

Abstraction in ‡-monoidal categories

Theorem

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such that $\langle x|x\rangle = x^{\ddagger} \circ x = \mathrm{id}_I$.

(b) $\eta = \Delta \circ \bot$ *and* $\varepsilon = \eta^{\ddagger} = \nabla \circ \top$ *realize* $X \dashv X$.

(c) $(X \otimes \nabla) \circ (\Delta \otimes X) = \Delta \circ \nabla = (\nabla \otimes X) \circ (X \otimes \Delta)$

Abstraction in ‡-monoidal categories

Theorem

Let $\mathcal{C}$ be a ‡-monoidal category, and $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ a comonoid that induces $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

Then the following conditions are equivalent:

(a) $\mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *creates* $\ddagger : \mathcal{C}[x]^{op} \longrightarrow \mathcal{C}[x]$ *such that* $\langle x|x\rangle = x^\ddagger \circ x = \mathrm{id}_I$.

(b) $\eta = \Delta \circ \bot$ *and* $\varepsilon = \eta^\ddagger = \nabla \circ \top$ *realize* $X \dashv X$.

(c) $(X \otimes \nabla) \circ (\Delta \otimes X) = \Delta \circ \nabla = (\nabla \otimes X) \circ (X \otimes \Delta)$

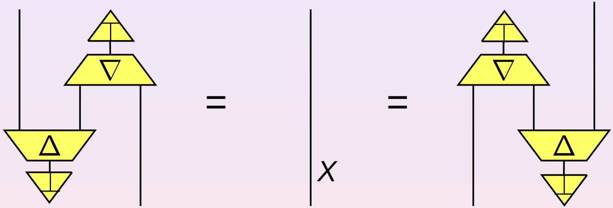
where $X \otimes X \xrightarrow{\nabla} X \xleftarrow{\bot} I$ *is the induced monoid*

$$\begin{aligned} \nabla &= \Delta^\ddagger \\ \bot &= \top^\ddagger \end{aligned}$$

Introduction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
‡-monoidal categories
Quantum objects
Abstraction in ‡-monoidal categories
Classical objects
Base
Measurements
Future work

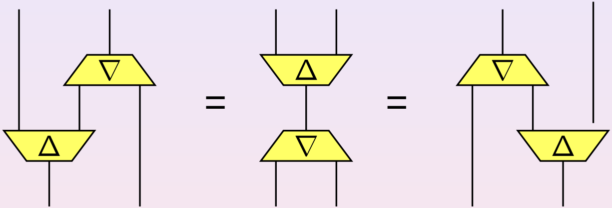
Abstraction in ‡-monoidal categories

Theorem in pictures

(b) 

Abstraction in ‡-monoidal categories

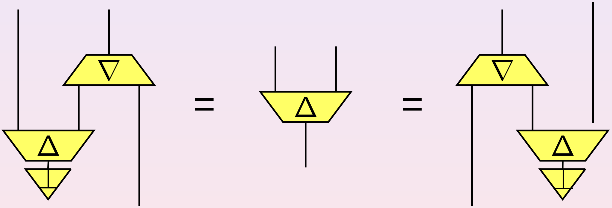
Theorem in pictures

(c) 

Proof of (b)⟹(c)

Lemma 1

If (b) holds then



Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

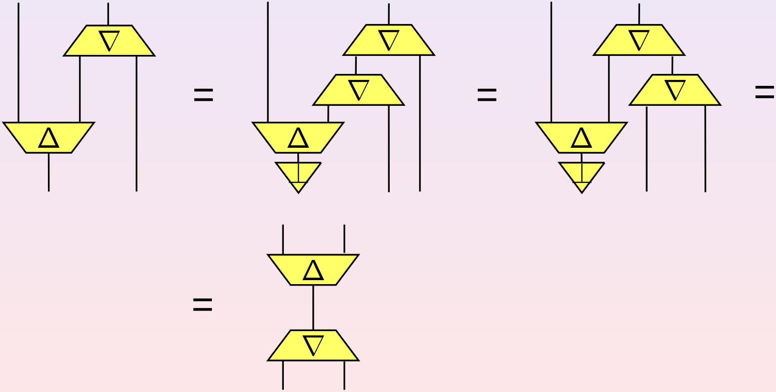
Base

Measurements

Future work

Proof of (b)⟹(c)

Then (c) also holds because



Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

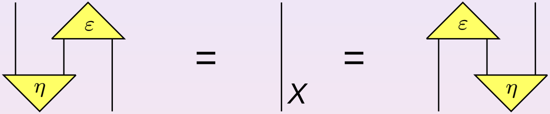
Measurements

Future work

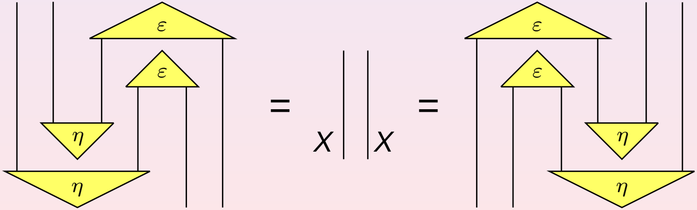
Proof of Lemma 1

Lemma 2

If



then



Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

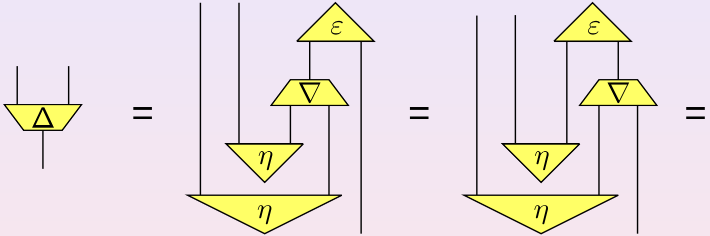
Base

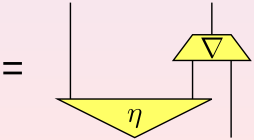
Measurements

Future work

Proof of Lemma 1

Using Lemma 2, and the fact that (b) implies $\nabla = \Delta^{\ddagger} = \Delta^{*}$, we get





The message of the proof

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

The message of the proof

There is more to categories than just diagram chasing.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

The message of the proof

There is more to categories than just diagram chasing.

There is also **picture chasing.**

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Classical objects

Definition

A *classical object* in a ‡-monoidal category $\mathcal{C}$ is a comonoid $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ satisfying the equivalent conditions from the preceding theorem.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Classical objects

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Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Classical objects

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Answer: classical elements = base vectors

- $\dashrightarrow$ is neither injective or surjective

Geometry of quantum abstraction

Dusko Pavlovic

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Consequences

Upshot

The Frobenius condition (c) assures the preservation of the abstraction operation under ‡.

Consequences

Upshot

The Frobenius condition (c) assures the preservation of the abstraction operation under $\ddagger$.

This leads to entanglement.

Consequences

Proposition

The vectors $\mathcal{C}(X)$ of any classical object X form a $\star$-algebra.

Consequences

Proposition

The vectors $\mathcal{C}(X)$ of any classical object X form a $\star$-algebra.

$$\begin{aligned} \varphi \cdot \psi &= \nabla \circ (\varphi \otimes \psi) \\ \epsilon &= \bot \\ \varphi^{\star} &= \varphi^{\ddagger *} = \varphi^{* \ddagger} \end{aligned}$$

Base

Definition

Two vectors $\varphi, \psi \in \mathcal{C}(A)$ in a ‡-monoidal category are *orthonormal* if their inner product is idempotent:

$$\langle \varphi \mid \psi \rangle \quad = \quad \langle \varphi \mid \psi \rangle^2$$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Base

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Two vectors $\varphi, \psi \in \mathcal{C}(A)$ in a $\ddagger$-monoidal category are *orthonormal* if their inner product is idempotent:

$$\langle \varphi \mid \psi \rangle \quad = \quad \langle \varphi \mid \psi \rangle^2$$

Proposition

Any two base vectors are orthonormal.
In particular, any two variables in a polynomial category are orthonormal.

Base

Definition

A classical object X is *standard* if it is (regularly) generated by its base vectors

$$\mathcal{B}(X) \quad = \quad \{\varphi \in \mathcal{C}(X) | \; (\kappa x.\, x \otimes x)\varphi = \varphi \otimes \varphi \\ \wedge \; (\kappa x.\, \mathrm{id}_I)\varphi = \mathrm{id}_I\}$$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

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in the sense

$$\forall f, g \in \mathcal{C}(X, Y).\ (\forall \varphi \in \mathcal{B}(X).\, f\varphi = g\varphi) \Longrightarrow f = g$$

A base is *regular* if $\mathcal{C}(X, Y) \rightarrowtail \mathcal{C}(Y)^{\mathcal{B}(X)}$ splits.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Base

Proposition 1.

All standard classical structures, that an object $X \in \mathcal{C}$ may carry, induce the bases with the same number of elements.

Introduction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
‡-monoidal categories
Quantum objects
Abstraction in ‡-monoidal categories
Classical objects
Base
Measurements
Future work

Base

Proposition 1.

All standard classical structures, that an object $X \in \mathcal{C}$ may carry, induce the bases with the same number of elements.

Proposition 2.

Let $X \in \mathcal{C}$ be a classical object with a regular base. Then the equipotent regular bases on any $Y \in \mathcal{C}$ are in one-to-one correspondence with the unitaries $X \longrightarrow Y$.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Base

Definition

A *qubit type* in an arbitrary ‡-monoidal category $\mathcal{C}$ is a classical object $\mathbb{B}$ with a unitary H of order 2. The induced bases are usually denoted by $|0\rangle, |1\rangle$, and $|+\rangle, |-\rangle$.

Base

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Computing with qubits

A ‡-monoidal category with $\mathbb{B}$ suffices for the basic quantum algorithms.

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Base

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Computing with qubits

A $\ddagger$-monoidal category with $\mathbb{B}$ suffices for the basic quantum algorithms.
A Klein group of unitaries on $\mathbb{B}$ suffices for all teleportation and dense coding schemes.

Base

Proposition (Coecke, Vicary, P)

Every classical object X in **FHilb** is regular, and $X \cong \mathbb{C}^n$. The classical structure is induced by a base $\mathcal{B}(X) = \{|i\rangle \mid i \leq n\}$, with

$$\begin{aligned} \Delta|i\rangle &= |ii\rangle \\ \top &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle \end{aligned}$$

Introduction
Graphical notation
Geometry of abstraction
Geometry of ‡-abstraction
‡-monoidal categories
Quantum objects
Abstraction in ‡-monoidal categories
Classical objects
Base
Measurements
Future work

Base

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$$\Delta|i\rangle = |ii\rangle$$

$$\top = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle$$

Moreover,

$$\mathbf{FHilb}_\Delta \simeq \mathbf{FSet}$$

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

‡-monoidal categories

Quantum objects

Abstraction in ‡-monoidal categories

Classical objects

Base

Measurements

Future work

Proof

A $\star$-algebra in **FHilb** is a $C^\star$-algebra.

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A $\star$-algebra in **FHilb** is a $C^\star$-algebra.

Thus for a classical $X \in$ **FHilb**,

$$\begin{array}{rcccl} \nabla & : & \mathbf{FHilb}(X) & \longrightarrow & \mathbf{FHilb}(X,X) \\ & & \left(I \xrightarrow{\varphi} X\right) & \longmapsto & \left(X \xrightarrow{\varphi \otimes X} X \otimes X \xrightarrow{\nabla} X\right) \end{array}$$

is a representation of a commutative $C^\star$-algebra.

Proof

Working through the Gelfand-Naimark duality, we get

$$X \quad \cong \quad \mathbb{C}^n$$

Proof

Working through the Gelfand-Naimark duality, we get

$$X \cong \mathbb{C}^n$$

— because the spectrum of a commutative finitely dimensional C^*-algebra is a discrete set n of minimal central projections, while the representing spaces are the full matrix algebras $\mathbb{C}(1)$

Outline

Introduction
Quantum programming
λ-abstraction

Graphical notation

Geometry of abstraction
Abstraction with pictures
Consequences

Geometry of ‡-abstraction
‡-monoidal categories
Quantum objects
Abstraction in ‡-monoidal categories
Classical objects
Base

Category of measurements

Future work

Category of measurements

((this was not presented))

Introduction

Graphical notation

Geometry of abstraction

Geometry of ‡-abstraction

Measurements

Future work

Outline

Introduction
Quantum programming
λ-abstraction
Graphical notation
Geometry of abstraction
Abstraction with pictures
Consequences
Geometry of ‡-abstraction
‡-monoidal categories
Quantum objects
Abstraction in ‡-monoidal categories
Classical objects
Base
Category of measurements
Future work

Future work

Claim: Simple quantum algorithms have simple categorical semantics.

Task: Implement and analyze quantum algorithms in nonstandard models: network computation, data mining.