Categorical aspects of polar decomposition

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Abstract

Polar decomposition unquestionably provides a notion of factorization in the category of Hilbert spaces. But it does not fit existing categorical notions, mainly because its factors are not closed under composition. We observe that the factors are images of functors. This leads us to consider notions of factorization that emphasize reconstruction of the composite category from the factors. We find that the category of Hilbert spaces is a composition of a totally disconnected abelian groupoid and a finitely accessible category. This separates probabilistic and complementarity aspects in quantum mechanics.

1 Introduction

The rich theory of Hilbert spaces underpins much of modern functional analysis and therefore quantum physics [22], yet important parts of it have resisted categorical treatment. For example, the functional calculus of spectral theory clearly exhibits functorial aspects [9], which have nevertheless not been drawn out explicitly so far.

This article studies the notion of polar decomposition, which seems quite specific to morphisms between Hilbert spaces. Its purpose is twofold. First, it is an initial investigation into categorical behaviour of the spectral theory of Hilbert spaces. Polar decomposition is usually derived from the spectral theorem. Conversely, the former is also a crucial ingredient in order-theoretical proofs of the latter [18]. Secondly, polar decomposition provides a novel example of a factorization that is of interest in its own right categorically, since factorization has been a topic of quite intense study in category theory [11, 8, 13, 17, 23], not in the least because it is a fundamental part of abstract homotopy theory [21]. Although we are currently unaware of similar phenomena in other categories, we tentatively propose notions that simultaneously fit polar decomposition and established categorical notions of factorization system. The main novelty is that the two factors need no longer be subcategories, *i.e.* images of

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faithful bijective-on-objects functors, but can be images of general functors. In the case of polar decomposition, these are the L^2 and ℓ^2 functors. Incidentally, this separates continuous and discrete aspects of the category of Hilbert spaces, and separates probabilistic and complementarity aspects from a quantummechanical viewpoint. We call them *stratisfied factorization systems*, for lack of a better term (the more natural names *functorial factorization systems* or *image factorization systems* clashing with existing terminology).

We also consider the converse question: how to reconstruct a category from a notion of factorization that we will call a *refactorization system*. Every natural number n factors as $n = p_1 \cdots p_k$ into simpler (prime) numbers; similarly, with a (stratisfied) factorization system a category can be written as $\mathbf{C} = \mathcal{R} \circ \mathcal{L}$ for (images of) simpler categories $\mathcal{L}$ and $\mathcal{R}$. Conversely, given $p_1, \ldots, p_k$ we can reconstruct n, relying on the ambient semiring structure of $\mathbb{N}$. Similarly, we investigate what is needed to reconstruct a category $\mathbf{C}$ from factors $\mathcal{L}$ and $\mathcal{R}$, which turns out to be an analogue of commutativity of multiplication of natural numbers that we call a *refactoring law*. Orthogonality and even nonunique lifting properties of factorization systems seem not to play a role in this perspective. We will show that, in this sense, polar decomposition enables one to reconstruct the category of Hilbert spaces from two simpler categories: a totally disconnected abelian groupoid **Pos**, and a finitely accessible category **PInj**. (These are the categories on which the functor L^2 and ℓ^2 are defined, but the functors are not needed for the reconstruction.) This is somewhat surprising, since the reconstruction does not mention limits at all, and yet limit behaviour in the category of Hilbert spaces is much richer than in the two simpler categories. Also, in contrast to (weak) factorization systems, refactorization systems behave well under forming functor categories, which promises interesting consequences for functorial quantum field theories.

We proceed as follows. Section 2 recalls categorical notions of factorization system and polar decomposition, and shows that the latter is not an instance of the former because the factor classes of morphisms are not closed under composition. Sections 3 and 4 introduce the two functors L^2 and ℓ^2 into the category of Hilbert spaces, study the categories **Pos** and **PInj** on which they are defined, and observe that the two factor classes of polar decomposition are the images of these two functors. This leads us to consider stratisfied factorization systems in more generality in Section 5. In Section 6 we explore some choice issues, recall functorial notions of factorization system, and show that polar decomposition does not fit them. Section 7 takes the idea of a factorization as a reconstruction of the composite from the factors seriously, and studies the notion of refactorization system, naturally extending the stratisfied factorization systems of Section 5. Finally, Section 8 concludes and provides some context and outlook.

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2 Factorization systems

This section recalls factorization systems and polar decomposition, and discusses the extent to which latter is an instance of the former.

2.1 A *weak factorization system* on a category consists of a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms satisfying:

(i) every morphism f can be factored as $f = rl$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$;

(ii) every commutative square as below with $l \in \mathcal{L}$ and $r \in \mathcal{R}$ allows a *diagonal fill-in* d making both triangles commute;

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "l"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r] \arrow[ur, dashed, "d"] & \cdot \end{tikzcd}$$

(iii) every isomorphism belongs to both $\mathcal{L}$ and $\mathcal{R}$;

(iv) both $\mathcal{L}$ and $\mathcal{R}$ are closed under composition.

We speak of a *factorization system* when additionally

(ii') the diagonal fill-in in (ii) is unique; we denote this by $l \perp r$.

2.2 The following proposition establishes a one-to-one correspondence between (weak) factorization systems in equivalent categories. Therefore, as long as we are concerned with (weak) factorization systems, there is no loss of generality if we work with a skeleton rather than a given category itself.

2.3 Proposition If $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{C}'$ is an equivalence of categories, and $(\mathcal{L}, \mathcal{R})$ is a (weak) factorization system on $\mathbf{C}$, then there is a unique (weak) factorization system $(\mathcal{L}', \mathcal{R}')$ on $\mathbf{C}'$ satisfying $\mathcal{L} = F^{-1}(\mathcal{L}')$ and $\mathcal{R} = F^{-1}(\mathcal{R}')$.

Proof Define $\mathcal{L}' = G^{-1}(\mathcal{L})$ and $\mathcal{R}' = G^{-1}(\mathcal{R})$ and denote the unit and counit of the equivalence by $\eta_X : X \to GF(X)$ and $\varepsilon_X : FG(X) \to X$. Because the latter are natural isomorphisms, we have $GF(f) = \eta_Y f \eta_X^{-1}$ and $FG(f) = \varepsilon_Y^{-1} f \varepsilon_X$. Let $f : X \to Y$ be a morphism in $\mathbf{C}'$, and factorize $Gf = rl$ with $r \in \mathcal{R}$ and $l \in \mathcal{L}$. Then $f = r'l'$, where $r' = \varepsilon_Y^{-1}(Fr)$ and $l' = (Fl)\varepsilon_X$. Now $Gl' = (GFl)(G\varepsilon_X) = \eta_Y l \eta_X^{-1}(G\varepsilon_X)$, so that $l' \in \mathcal{L}'$ by 2.1(iii) and (iv) for $(\mathcal{L}, \mathcal{R})$. Similarly $r' \in \mathcal{R}'$. This establishes 2.1(i) for $(\mathcal{L}', \mathcal{R}')$.

As for property (ii), suppose that $gl' = fr'$ in $\mathbf{C}'$. Then there is a fill-in d with $d(Gl') = Gf$ and $(Gr')d = Gg$ in $\mathbf{C}$, leading to a fill-in $d' = \varepsilon_C(Fd)\varepsilon_B^{-1}$ in

$\mathbf{C}'$ as in the following diagram.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[ddd, "l'"'] & & C \arrow[r, equal] \arrow[d, "\varepsilon_C^{-1}"'] & C \arrow[ddd, "r'"] \\
& FG(A) \arrow[ul, "\varepsilon_A"] \arrow[r, "FGf"] \arrow[d, "FGl'"'] & FG(C) \arrow[ur, "\varepsilon_C"] \arrow[d, "FGr'"] & \\
& FG(B) \arrow[ur, "Fd"] \arrow[r, "FGg"'] \arrow[d, "\varepsilon_B"] & FG(D) & \\
B \arrow[r, equal] \arrow[ur, "\varepsilon_B^{-1}"] & B \arrow[rr, "g"'] & & D \arrow[ul, "\varepsilon_B^{-1}"']
\end{tikzcd}$$

If $(\mathcal{L}, \mathcal{R})$ has unique fill-ins, so does $(\mathcal{L}', \mathcal{R}')$: if $d''l' = f$ and $r'd'' = g$, then $Gd'' = d$, and therefore $d' = \varepsilon_C(Fd)\varepsilon_B^{-1} = \varepsilon_C(FGd'')\varepsilon_B^{-1} = d''$.

Properties (iii)–(iv) for $(\mathcal{L}', \mathcal{R}')$ follow directly from the corresponding properties for $(\mathcal{L}, \mathcal{R})$ by functoriality of G. One easily verifies the uniqueness claim.$\square$

The category of Hilbert spaces

2.4 Definition We are interested in the category **Hilb**, whose objects are complex Hilbert spaces, and whose morphisms are continuous linear functions. In fact, after this section we shall mostly be working with a skeleton $\mathbf{Hilb}_{\text{skel}}$ of **Hilb**.

2.5 The category **Hilb** has a *dagger*, *i.e.* a contravariant involutive functor $\dagger \colon \mathbf{Hilb}^{\text{op}} \to \mathbf{Hilb}$ that acts as the identity on objects. On a morphism $f \colon X \to Y$ it is given by the unique adjoint $f^\dagger \colon Y \to X$ satisfying $\langle f(x) \,|\, y\rangle = \langle x \,|\, f^\dagger(y)\rangle$. For example, an isomorphism u is unitary when $u^{-1} = u^\dagger$.

2.6 Furthermore, the usual tensor product of Hilbert spaces provides the category **Hilb** with symmetric monoidal structure. The monoidal unit is the 1-dimensional Hilbert space $\mathbb{C}$. In fact, **Hilb** has *dagger symmetric monoidal* structure, *i.e.* $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, and all coherence isomorphisms are unitaries.

2.7 Direct sums of Hilbert spaces provide the category **Hilb** with (finite) *dagger biproducts*. That is, $X \oplus Y$ is simultaneously a product and a coproduct, the projections are the daggers of the corresponding coprojections, and $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$.

2.8 Let us emphasize that we take continuous linear maps as morphisms between Hilbert spaces, rather than linear contractions. The category of Hilbert spaces with the latter morphisms is rather well-behaved, see *e.g.* [2]. However, it is the former choice of morphisms that is of interest in functional analysis and quantum physics. Unfortunately it also reduces the limit behaviour of the category **Hilb**, as the following lemma shows.

2.9 Lemma The category **Hilb**:

(i) has (co)equalizers;

(ii) does not have infinite (co)products;

(iii) does not have directed (co)limits.

PROOF Part (i) holds because **Hilb** is enriched over abelian groups and has kernels [15]. For (ii), consider the following counterexample. Define an $\mathbb{N}$-indexed family $X_n = \mathbb{C}$ of objects of **Hilb**. Suppose the family (X_n) had a coproduct X with coprojections $\kappa_n \colon X_n \to X$. Define $f_n \colon X_n \to \mathbb{C}$ by $f_n(z) = n \cdot \|\kappa_n\| \cdot z$. These are bounded maps, since $\|f_n\| = n \cdot \|\kappa_n\|$. Then for all $n \in \mathbb{N}$ the norm of the cotuple $f \colon X \to \mathbb{C}$ of (f_n) must satisfy

$$n \cdot \|\kappa_n\| = \|f_n\| = \|f \circ \kappa_n\| \leq \|f\| \cdot \|\kappa_n\|,$$

so that $n \leq \|f\|$. This contradicts the boundedness and hence continuity of f. Finally, part (iii) follows from (ii) and [19, IX.1.1] $\square$

2.10 A standard example of a (weak) factorization system is given by taking $\mathcal{L}$ to be the surjections and $\mathcal{R}$ the injections in the category of sets and functions. More generally, taking $\mathcal{L}$ to be the epimorphisms and $\mathcal{R}$ to be the regular monomorphisms provides an example of a factorization system in any regular category. Here, we are mostly interested in the category **Hilb** of Hilbert spaces and continuous linear maps. Indeed, this category has a factorization system given by epimorphisms and kernels, which explicitly come down to linear maps with dense image and isometries [15].

Polar decomposition

2.11 There is another way than 2.10 to factorize morphisms in **Hilb**. To motivate it, first consider the complex field $\mathbb{C}$ as the morphisms of a category with one object, where composition is multiplication. Taking $\mathcal{L}$ to be the nonnegative real numbers, and $\mathcal{R}$ to be the unit circle, we get another example of a factorization system.

This idea of *polar* factorization extends to n by n complex matrices, and even to morphisms between Hilbert spaces of arbitrary dimension as in the following theorem. Recall that a morphism $p \colon X \to X$ in **Hilb** is *nonnegative* when $\langle px \,|\, x \rangle \geq 0$ for all $x \in X$, and *positive* when $\langle px \,|\, x \rangle > 0$, denoted by $p \geq 0$ and $p > 0$, respectively. A *partial isometry* is a map $i \colon X \to Y$ whose restriction to the orthogonal complement of its kernel is an isometry, or equivalently, $i = ii^\dagger i$.

2.12 Theorem [14, problem 134] For every morphism $f \colon X \to Y$ between Hilbert spaces, there exist a unique nonnegative map $p \colon X \to X$ and partial isometry $i \colon X \to Y$ satisfying $f = ip$ and $\ker(p) = \ker(i)$. $\square$

2.13 It will follow from 6.7 below that the factorization of 2.12 satisfies condition 2.1(ii) of a weak factorization system. However, the unicity condition $\ker(p) = \ker(i)$ in the previous theorem is something of a red herring. It should be understood as saying that both i and p are uniquely determined on the orthogonal complement of $\ker(f)$. On each point of $\ker(f)$, one of i and p must be zero, but the other's behaviour has no restrictions apart from being a partial isometry or positive map, respectively. If we alter i to be zero on $\ker(f)$, and p to be nonzero on $\ker(f)$, the factorization satisfies 2.1(ii'), as the following proposition shows. Therefore, we will refer to Theorem 2.12 as *weak polar decomposition*, and to the following proposition as *polar decomposition*. Incidentally, in the situation of 2.12, one easily derives that $\ker(f) = \ker(p) = \ker(i)$.

2.14 Proposition In the category **Hilb**, the classes $\mathcal{L}$ of positive maps and $\mathcal{R}$ of partial isometries satisfy (i) and (ii') of 2.1.

PROOF For 2.1(i), first apply 2.12 to get p and i. Since $\ker(f)$ is a closed subspace, functions defined on both $\ker(f)$ and $\ker(f)^\perp$ extend to a unique morphism on $\mathrm{dom}(f)$. Put $p' = p$ on $\ker(f)^\perp$ and $p' = \mathrm{id}$ on $\ker(f)$, and $i' = i$ on $\ker(f)^\perp$ and $i' = 0$ on $\ker(f)$. Then $p' > 0$ and $i' = i' i'^\dagger i'$, and $f = ip$.

As to 2.1(ii'), suppose that $bp = ia$ with $p > 0$ and i a partial isometry. Since p is positive, it is in particular invertible. Defining $d = p^{-1}a$ satisfies $a = pd$ and $id = b$, and furthermore is the unique such morphism. □

2.15 However, the classes $\mathcal{L}$ of positive maps and $\mathcal{R}$ of partial isometries do not satisfy the remaining conditions (iii) and (iv) of a factorization system. To see that the composition of partial isometries need not be a partial isometry, consider the partial isometries $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : \mathbb{C} \to \mathbb{C}^2$ and $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} : \mathbb{C}^2 \to \mathbb{C}$, whose composition is $(1/\sqrt{2}) \colon \mathbb{C} \to \mathbb{C}$. Also, it is not hard to see (using simultaneous spectral resolution [5]) that the composition of two positive maps is again positive if and only if they commute; for example, $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ are positive, but their composition $\begin{pmatrix} 3 & 1 \\ 2 & 6 \end{pmatrix}$ is not. For a counterexample to 2.1(iii), one only needs to consider the isomorphism $(-1) \colon \mathbb{C} \to \mathbb{C}$, which is neither positive nor a partial isometry.

2.16 Incidentally, systematic exploitation of the dagger in **Hilb** allows a 'right-handed' version of (weak) polar decomposition: every morphism of **Hilb** can be factorized as $f = pi$ for a positive (nonnegative) p and a partial isometry i.

3 The category of positive operators

This section observes that even though the class of positive operators in **Hilb** is not closed under composition, it is the image of the L^2 functor that we will define. The L^2 construction is quite fundamental and well-studied, but surprisingly enough functorial aspects seem not to have been considered before.

3.1 Even though the class of positive operators is not closed under the usual composition of continuous linear maps, polar decomposition provides a way to define another composition under which it is closed: the composition pq of positive operators p, q factorizes as a positive map $|pq|$ followed by a partial isometry. Unfortunately, that fails to be associative, $|p|qr|| \neq ||pq|r|$ for positive p, q, r, as can be seen by taking

$$p = q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad r = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

A more natural way out is to 'add more objects' for positive operators to act on, as in the following definition.

3.2 Definition The category **Pos** has as objects measure spaces, *i.e.* triples (S, Σ, μ) of a set S, a σ-algebra Σ on S, and a measure μ on (S, σ) [10]. An endomorphism on (S, Σ, μ) is an equivalence class of measurable functions $f\colon S \to \mathbb{R}^{>0}$ whose range is compact, where two such functions are identified when they differ only on a negligible set. Endomorphisms are composed by pointwise multiplication, and there are no nonendomorphisms. Identities are the constant functions 1.

3.3 By construction, the category **Pos** is totally disconnected, *i.e.* every morphism is an endomorphism. Moreover, every morphism f on (S, Σ, μ) is an isomorphism, with inverse determined by $f^{-1}(s) = f(s)^{-1}$, which is again a well-defined morphism. Hence **Pos** is a groupoid. Finally, since multiplication in $\mathbb{R}^{>0}$ is commutative, so is composition in **Pos**, making the latter a *totally disconnected abelian groupoid*.

3.4 The category **Pos** has more structure. For example, it has a *dagger*, simply by $f^\dagger = f$.

3.5 Furthermore, there are two dagger *symmetric monoidal* structures on **Pos**. First, we can act on objects by taking the product measure [10, Chapter 25]: the monoidal product of (S, Σ, μ) and (S', Σ', μ') has as carrier set the Cartesian product $S \times S'$ of sets, which extends to the σ-algebras and the measures. The monoidal unit $\mathbf{1}$ is the singleton set, with its unique σ-algebra and probability measure. On morphisms $f\colon S \to \mathbb{R}^{>0}$ and $f'\colon S' \to \mathbb{R}^{>0}$ one can choose to act either by product $(s, s') \mapsto f(s)f'(s')$ or by sum $(s, s') \mapsto f(s) + f'(s')$; both are again measurable with compact range. We denote the monoidal structure with the former choice on morphisms by $\otimes$. Notice that $\otimes$ is not a product in **Pos** and hence, by the dagger, not a coproduct either.

3.6 The second dagger symmetric monoidal structure on **Pos** is given by direct sum [10, 214K]. We denote it by $\oplus$. So that the carrier set of $(S, \Sigma, \mu) \oplus (S', \Sigma', \mu')$ is given by the disjoint union of S and S', which extends to the σ-algebras, the measures, and to morphisms. The monoidal unit is the empty set with its unique σ-algebra and measure. Notice that $\oplus$ is not a coproduct in **Pos** and hence not a product either.

The L^2 functor

3.7 Definition There is a functor $L^2 \colon \mathbf{Pos} \to \mathbf{Hilb}$, acting on objects as

$$L^2(S, \Sigma, \mu) = \{\varphi \colon S \to \mathbb{C} \mid \varphi \text{ measurable}, \int_S |\varphi|^2 \, \mathrm{d}\mu < \infty\},$$

subject to the identification of two such square-integrable functions when they differ only on a negligible set. This is a well-defined Hilbert space [10, 244N], with inner product $\langle \varphi \mid \psi \rangle = \int_S \overline{\varphi} \otimes \psi \, \mathrm{d}(\mu \otimes \mu)$. The action on morphisms takes $f \colon S \to \mathbb{R}^{>0}$ in **Pos** to the multiplication operator it induces:

$$(L^2 f)(\varphi)(s) = f(s)\varphi(s).$$

We also denote the induced functor $\mathbf{Pos} \to \mathbf{Hilb}_{\mathrm{skel}}$ by L^2.

3.8 Because morphisms $f \colon S \to \mathbb{R}^{>0}$ in **Pos** are real-valued and hence satisfy $f(s) = \overline{f(s)}$, we have

$$\langle (L^2 f)(\varphi) \mid \psi \rangle = \int_S \overline{\varphi(s)} f(s) \psi(s) \, \mathrm{d}\mu(s) = \langle \varphi \mid (L^2(f^\dagger))(\psi) \rangle,$$

and therefore the functor L^2 preserves daggers.

3.9 Furthermore, the functor L^2 is symmetric (strong) monoidal with respect to $\otimes$ on **Pos** and $\otimes$ on **Hilb**. The required natural isomorphism $\mathbb{C} \to L^2(\mathbf{1}) = \{\varphi \colon 1 \to \mathbb{C}\}$ is obvious, and the required natural isomorphism $L^2(S, \Sigma, \mu) \otimes L^2(S', \Sigma', \mu') \to L^2((S, \Sigma, \mu) \otimes (S', \Sigma', \mu'))$ is determined by mapping (φ, ψ) to the function $(s, s') \mapsto \varphi(s)\psi(s)$. The required coherence diagrams are readily verified.

3.10 Finally, the functor L^2 is also symmetric (strong) monoidal with respect to $\oplus$ on **Pos** and the direct sum $\oplus$ on **Hilb**. There is a canonical natural isomorphism between the 0-dimensional Hilbert space, which contains only the zero vector, and $L^2(\emptyset)$, which consists only of the empty function. The required natural isomorphism $L^2(S, \Sigma, \mu) \oplus L^2(S', \Sigma', \mu') \to L^2((S, \Sigma, \mu) \oplus (S', \Sigma', \mu'))$ is also clear once one realizes that a square integrable function on $S + S'$ simply comes down to a pair of a square integrable function on S and one on S'.

3.11 Proposition A morphism $p \colon X \to X$ in $\mathbf{Hilb}_{\mathrm{skel}}$ is positive if and only if $p = L^2 f$ for a morphism f in **Pos**.

Proof Clearly, for a morphism $f \colon S \to \mathbb{R}^{>0}$ in **Pos**, we have $\langle (L^2 f)(\varphi) \mid \varphi \rangle = \int_S \overline{f(s)} |\varphi(s)|^2 \, \mathrm{d}\mu(s) > 0$ so that $L^2 f$ is positive.

Conversely, let $p \colon X \to X$ be a positive operator in **Hilb**. Since p is self-adjoint, the spectral theorem [22] guarantees the existence of a measure space (S, Σ, μ), a unitary isomorphism $u \colon X \xrightarrow{\cong} L^2(S, \Sigma, \mu)$, and a measurable function $f \colon S \to \mathbb{C}$ whose range is the spectrum $\sigma(p)$ of p, such that $p = u^{-1} \circ m \circ u$, where m is the multiplication operator induced by f. Because p is positive, f must take values in $\mathbb{R}^{>0}$. And because the spectrum of any bounded operator is compact, so is the range of f. Hence $m = L^2 f$. Finally, being in a skeletal category, we are neglecting isomorphisms, and therefore $p = L^2 f$. $\square$

4 The category of partial isometries

This section observes that even though the class of partial isometries in **Hilb** is not closed under composition, it is the image of a functor.

4.1 As in 3.1, one could try to use polar decomposition to get a different composition of partial isometries. But again, associativity fails: for a counterexample, take the partial isometries given by

$$i = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1 & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and take k the partial isometry got by factorizing $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

It should be noted that other compositions of partial isometries, that do make them into a category, are known [16]. However, we follow a similar strategy as in Section 3, and 'split out objects' as follows.

4.2 Definition A *partial injection* is a partial function that is injective, wherever it is defined. More precisely, it(s graph) is a relation $R \subseteq X \times Y$ such that for every x there is precisely on y with $(x, y) \in R$, and for every y there is precisely on x with $(y, x) \in R$. Sets and partial injections form a category that we denote by **PInj**.

4.3 Notationally, a partial injection $f\colon X \to Y$ can be conveniently represented as a span $(\, X \twoheadleftarrow f_1 \prec F \succ f_2 \twoheadrightarrow Y \,)$ of monics in **Set**. Here, f_1 is (the inclusion of) the domain of definition of f, and f_2 is its (injective) action on that domain. Composition in this representation is by pullback. We will also write $\mathrm{Dom}(f) = f_1(F)$ for the domain of definition, and $\mathrm{Im}(f) = f_2(F)$ for the range of f.

If it wasn't already, the span notation immediately makes it clear that **PInj** is a *dagger* category: $(\, X \twoheadleftarrow f_1 \prec F \succ f_2 \twoheadrightarrow Y \,)^\dagger = (\, Y \twoheadleftarrow f_2 \prec F \succ f_1 \twoheadrightarrow X \,)$.

4.4 The category **PInj** has two dagger *symmetric monoidal* structures. The first one, that we denote by $\otimes$, acts as the Cartesian product on objects. Because the Cartesian product of injections is again injective, $\otimes$ is well-defined on morphisms of **PInj** as well. The monoidal unit is a singleton set **1**. Notice that $\otimes$ is not a product, and hence not a coproduct either.

The second dagger symmetric monoidal structure on **PInj**, denoted by $\oplus$, is given by disjoint union on objects. It is easy to see that a disjoint union of injections is again injective, making $\oplus$ well-defined on morphisms of **PInj**. The monoidal unit is the empty set. Notice that $\oplus$ is not a coproduct, and hence not a product either.

4.5 Lemma The category **PInj**:

(i) has (co)equalizers;

(ii) has a zero object;

(iii) does not have finite (co)products;

Proof The equalizer of $f, g\colon X \to Y$ is the inclusion of

$$\big\{x \in X \mid x \notin (\mathrm{Dom}(f) \cup \mathrm{Dom}(g)) \vee \big(x \in (\mathrm{Dom}(f) \cap \mathrm{Dom}(g)) \wedge f(x) = g(x)\big)\big\}$$

into X. The empty set is a zero object in **PInj**.

Towards (iii), notice that if $(X \xrightarrow{\kappa_X} X + Y \xleftarrow{\kappa_Y} Y)$ were a coproduct in **PInj**, then one must have $\mathrm{Dom}(\kappa_X) = X$, $\mathrm{Dom}(\kappa_Y) = Y$ and $\mathrm{Im}(\kappa_X) \cap \mathrm{Im}(\kappa_Y) = \emptyset$, because otherwise unique existence of mediating morphisms is violated. Hence any coproduct must contain the disjoint union of X and Y. Let $f\colon X \to Z$ and $g\colon Y \to Z$ be any morphisms. Then a mediating morphism $m\colon X + Y \to Z$ has to satisfy $m(x) = f(x)$ for $x \in \mathrm{Dom}(f)$ and $m(y) = g(y)$ for $y \in \mathrm{Dom}(g)$. But such an m is not unique, unless $\mathrm{Dom}(f) = X$ and $\mathrm{Dom}(g) = Y$. In fact, it is not even a partial injection unless $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \emptyset$. We conclude that **PInj** does not have binary (co)products. □

4.6 In fact, part (ii) of Lemma 4.5 follows from the existence of directed colimits, which we now work towards. Recall that a category has directed colimits if and only if it has colimits of chains, *i.e.* colimits of well-ordered diagrams [2, Corollary 1.7]. Observe that for a chain $D\colon I \to \mathbf{PInj}$, if $c_i\colon D(i) \to X$ is a cocone on D, then $\mathrm{Dom}(c_i) \subseteq \mathrm{Dom}(D(i \le j))$ for all $j \ge i$. To see this, notice that $c_i = c_j \circ D(i \le j)$ since c_i is a cocone, and therefore

$$\mathrm{Dom}(c_i) = \mathrm{Dom}(c_j \circ D(i \le j)) \subseteq \mathrm{Dom}(D(i \le j)).$$

This observation suggests that the colimit of a well-ordered diagram in **PInj** should consist of all 'infinite paths'. The following proposition shows that this is indeed a colimit.

4.7 Proposition The category **PInj** has directed colimits.

Proof Let $D\colon I \to \mathbf{PInj}$ be a chain. Define

$$X = \{x \in \coprod_i D(i) \mid \forall_{j \ge i}[x \in \mathrm{Dom}(D(i \le j))]\}/\sim,$$

where the coproduct is taken in **Set**, and the equivalence relation $\sim$ is generated by $x \sim D(i \le j)(x)$ for all $i \le j$ in I and $x \in \mathrm{Dom}(D(i \le j))$. For $i \in I$, define $c_i\colon D(i) \to X$ by

$$\mathrm{Dom}(c_i) = \{x \in D(i) \mid \forall_{j \ge i}[x \in \mathrm{Dom}(D(i \le j))]\},$$

and $c_i(x) = [x]$.

First of all, let us show that the c_i form a cocone. One has:

$$\begin{aligned}
&\mathrm{Dom}(c_j \circ D(i \le j)) \\
&= \{x \in D(i) \mid x \in \mathrm{Dom}(D(i \le j)) \wedge D(i \le j)(x) \in \mathrm{Dom}(c_j)\} \\
&= \{x \in D(i) \mid x \in \mathrm{Dom}(D(i \le j)) \wedge \forall_{k \ge j}[D(i \le j)(x) \in \mathrm{Dom}(D(j \le k))]\}.
\end{aligned}$$

The well-orderedness of I implies that

$$\forall_{k \geq i}[P(k)] \Leftrightarrow \forall_{k \geq j}[P(k)] \wedge P(j)$$

for any property P on the objects of I, whence

$$\mathrm{Dom}(c_j \circ D(i \leq j)) = \{x \in D(i) \mid \forall_{k \geq i}[x \in \mathrm{Dom}(D(i \leq k))]\} = \mathrm{Dom}(c_i).$$

Moreover $c_j \circ D(i \leq j)(x) = [D(i \leq j)(x)] = [x] = c_i(x)$ for $x \in \mathrm{Dom}(c_i)$, by definition of the equivalence relation.

Next, we show that c_i is universal. Let $d_i \colon D(i) \to Y$ be any cocone, and define $m \colon X \to Y$ by

$$\mathrm{Dom}(m) = \{[x] \mid x \in \mathrm{Dom}(d_i)\}$$

and $m([x]) = d_i(x)$ for $x \in D(i)$; this is well-defined since d_i is a cocone. Then

$$\begin{aligned} \mathrm{dom}(m \circ c_i) &= \{x \in D(i) \mid x \in \mathrm{Dom}(c_i) \wedge m_i(x) \in \mathrm{Dom}(m)\} \\ &= \{x \in D(i) \mid \forall_{j \geq i}[x \in \mathrm{Dom}(D(i \leq j))] \wedge x \in \mathrm{Dom}(d_i)\} \\ &= \mathrm{Dom}(d_i), && \text{(by 4.6)} \end{aligned}$$

and $m \circ c_i(x) = m([x]) = d_i(x)$ for $x \in D(i)$. Thus $m \circ c_i = d_i$, so m is indeed a mediating morphism.

Finally, if $m' \colon X \to Y$ satisfies $m \circ c_i = d_i$, then it follows from the above considerations that $\mathrm{Dom}(m') = \mathrm{Dom}(m)$ and $m'(x) = m(x)$ for $x \in \mathrm{Dom}(m)$. Hence m is the unique mediating morphism. $\square$

4.8 Recall that an object X in a category $\mathbf{C}$ is called *finitely presentable* when the hom-functor $\mathbf{C}(X, -) \colon \mathbf{C} \to \mathbf{Set}$ preserves directed colimits. Explicitly, this means that for any directed poset $D \colon I \to \mathbf{C}$, any colimit cocone $d_i \colon D(i) \to Y$ and any morphism $f \colon X \to Y$, there are $j \in I$ and a morphism $g \colon X \to D(j)$ such that $f = d_j \circ g$. Moreover, this morphism g is essentially unique, in the sense that if $f = d_i \circ g = d_i \circ g'$, then $D(i \to i') \circ g = D(i \to i') \circ g'$ for some $i' \in I$.

$$\begin{tikzcd}
D(i) \arrow[r] \arrow[dr, "d_i"'] & D(i') \arrow[r] \arrow[d, "d_{i'}"'] & D(i'') \arrow[r] \arrow[dl, "d_{i''}"'] & \cdots \arrow[r] & D(j) \arrow[r] \arrow[dlll, "d_j"'] & \cdots \\
& Y & & & X \arrow[lll, "f"] \arrow[u, dashed, "g"'] &
\end{tikzcd}$$

A category is called *finitely accessible* [2] when it has directed colimits and every object is a directed colimit of finitely presentable objects.

4.9 Lemma A set is finitely presentable in $\mathbf{PInj}$ if and only if it is finite.

Proof The only thing, in the situation of 4.8 with X finite, is to notice that if a partial injection g is to exist, we must have $\mathrm{Dom}(g) = \mathrm{Dom}(f)$. The rest follows from [2, 1.2.1]. $\square$

4.10 Theorem The category **PInj** is finitely accessible.

Proof It suffices to prove that every set in **PInj** is a directed colimit of finite ones. But that is easy: X is the colimit of the directed diagram consisting of its finite subsets. $\square$

The ℓ^2 functor

4.11 Definition There is a functor $\ell^2 \colon \mathbf{PInj} \to \mathbf{Hilb}$, acting on a set X as

$$\ell^2(X) = \{\varphi\colon X \to \mathbb{C} \mid \sum_{x\in X} |\varphi(x)|^2 < \infty\},$$

which is a well-defined Hilbert under the inner product $\langle \varphi \mid \psi \rangle = \sum_{x\in X} \overline{\varphi(x)}\psi(x)$. The action on morphisms sends a partial injection $(\, X \twoheadleftarrow f_1 \rightarrowtail F \rightarrowtail f_2 \twoheadrightarrow Y \,)$ to the linear function $\ell^2 f \colon \ell^2(X) \to \ell^2(Y)$ determined informally by $\ell^2 f = (\_) \circ f^\dagger$. Explicitly,

$$(\ell^2 f)(\varphi)(y) = \sum_{x \in f_2^{-1}(y)} \varphi(f_1(x)).$$

We also denote the induced functor $\mathbf{PInj} \to \mathbf{Hilb}_{\mathrm{skel}}$ by ℓ^2.

4.12 In verifying that $\ell^2 f$ is indeed a well-defined morphism of **Hilb**, it is essential that f is a (partial) injection.

$$\begin{aligned}\sum_{y\in Y} \left|(\ell^2 f)(\varphi)(y)\right|^2 &= \sum_{y\in Y} \Big| \sum_{x\in f_2^{-1}(y)} \varphi(f_1(x))\Big|^2 \leq \sum_{y\in Y} \sum_{x\in f_2^{-1}(y)} |\varphi(f_1(x))|^2 \\ &= \sum_{x\in F} |\varphi(f_1(x))|^2 \leq \sum_{x\in X} |\varphi(x)|^2 < \infty.\end{aligned}$$

That this breaks down for functions f in general, instead of (partial) injections, was first noticed in [4]. Functoriality of ℓ^2 is easy to verify.

4.13 The following calculation shows that the ℓ^2 functor preserves daggers. For a partial injection $(\, X \twoheadleftarrow f_1 \rightarrowtail F \rightarrowtail f_2 \twoheadrightarrow Y \,)$, $\varphi \in \ell^2(X)$ and $\psi \in \ell^2(Y)$:

$$\begin{aligned}\langle (\ell^2 f)(\varphi) \mid \psi \rangle_{\ell^2(Y)} &= \sum_{y\in Y} \overline{(\ell^2 f)(\varphi)(y)} \cdot \psi(y) = \sum_{y\in Y} \sum_{x\in f_2^{-1}(y)} \overline{\varphi(f_1(x))} \cdot \psi(y) \\ &= \sum_{x\in F} \overline{\varphi(f_1(x))} \cdot \psi(f_2(x)) = \sum_{x\in X} \sum_{x'\in f_1^{-1}(x)} \overline{\varphi(x)} \cdot \psi(f_2(x')) \\ &= \sum_{x\in X} \overline{\varphi(x)} \cdot \Big(\sum_{x'\in f_1^{-1}(x)} \psi(f_2(x'))\Big) = \langle \varphi \mid \ell^2(f^\dagger)(\psi) \rangle_{\ell^2(X)}.\end{aligned}$$

4.14 The functor ℓ^2 preserves the tensor product $\otimes$, *i.e.* it is symmetric (strong) monoidal. There is a canonical isomorphism $\mathbb{C} \cong \ell^2(\mathbf{1})$. The required natural morphisms $\ell^2(X) \otimes \ell^2(Y) \to \ell^2(X \otimes Y)$ are given by mapping (φ, ψ) to the function $(x, y) \mapsto \varphi(x)\psi(y)$. That there are inverses is seen when one realizes that $\ell^2(X \otimes Y)$ is the Cauchy-completion of the set of functions $X \times Y \to \mathbb{C}$ with finite support. The required coherence diagrams follow easily.

4.15 Also, the ℓ^2 functor is symmetric (strong) monoidal with respect to $\oplus$. There is a canonical isomorphism between the 0-dimensional Hilbert space and the set $\ell^2(\emptyset)$ consisting only of the empty function. The natural morphisms $\ell^2(X) \oplus \ell^2(Y) \to \ell^2(X \oplus Y)$ map (φ, ψ) to the cotuple $[\varphi, \psi]\colon X \oplus Y \to \mathbb{C}$. One sees that these are isomorphisms by recalling that $\ell^2(X \oplus Y)$ is the closure of the span of the Kronecker functions δ_x and δ_y for $x \in X$ and $y \in Y$, on which the inverse acts as the appropriate coprojection. Coherence properties readily follow.

4.16 By now, the reader may be wondering why we are talking about **PInj** as a 'category of partial isometries'. The following proposition makes precise that the partial isometries indeed form precisely the image of the ℓ^2 functor. This comes down to the fact that the functor ℓ^2 is part of an equivalence between **PInj** and a category of Hilbert spaces with a chosen orthonormal basis whose morphisms are partial injections of those chosen bases [1].

4.17 Proposition A morphism $i\colon X \to Y$ in $\mathbf{Hilb}_{\mathrm{skel}}$ is a partial isometry if and only if $i = \ell^2 f$ for a morphism f in **PInj**.

Proof Clearly a map of the form $\ell^2 f$ is a partial isometry. Conversely, suppose that $i\colon X \to Y$ is a partial isometry. Choose an orthonormal basis $A \subseteq X$ for its initial space $\ker(i)^\perp$, and choose an orthonormal basis $A' \subseteq X$ for $\ker(i)$, so that $X \cong \ell^2(A \oplus A')$. Let $B = i(A) \subseteq Y$. Then B will be an orthonormal basis for the final space $\ker(i^\dagger)^\perp$ because i acts isometrically on A. Choose an orthonormal basis $B' \subseteq Y$ for $\ker(i^\dagger)$, so that $Y \cong \ell^2(B \oplus B')$. Now, if we define $f = (\,A \oplus A' \leftarrowtail A \overset{i}{\rightarrowtail} B \oplus B'\,)$, then $i = \ell^2 f$. $\square$

4.18 It is noteworthy that the category mentioned in 4.16, denoted by **HStar**, can be defined algebraically. Its objects are morphisms $\delta\colon X \to X \otimes X$ of **Hilb** satisfying the so-called H*-axioms [1]. Such objects correspond precisely to a chosen orthonormal basis on the Hilbert space X, given by $\{x \in X \mid \delta(x) = x \otimes x\} \backslash \{0\}$. Morphisms in **HStar** are morphisms f in **Hilb** satisfying both $(f \otimes f)\delta = \delta f$ and $(f^\dagger \otimes f^\dagger)\delta = \delta f^\dagger$.

To be precise we should talk about **HStar**[**Hilb**], and mention that this construction is a functorial. In fact, **HStar**$[-]$ is an idempotent comonad on the category of dagger symmetric monoidal categories, whose counit is the obvious forgetful functor **HStar**[**C**] $\to$ **C**.

5 Stratified factorization systems

Combining the observations that positive operators are precisely the image of the L^2 functor (see 3.11), and that partial isometries are precisely the image of the ℓ^2 functor (see 4.17), with polar decomposition (as in 2.14), we end up in the situation tentatively axiomatized as follows. This section considers some direct consequences of the following definition. Many are straight adaptations of standard proofs (see *e.g.* [7, Section 5.5]), but there are some deviations.

5.1 Definition A *weak stratisfied factorization system* on a category $\mathbf{C}$ consists of functors $L\colon \mathbf{L} \to \mathbf{C}$ and $R\colon \mathbf{R} \to \mathbf{C}$ such that:

(i) every morphism f in $\mathbf{C}$ can be factored as $f = (Rr)(Ll)$ with l in $\mathbf{L}$ and r in $\mathbf{R}$;

(ii) every commutative square as below with l in $\mathbf{L}$ and r in $\mathbf{R}$ allows a diagonal fill-in d making both triangles commute;

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "Ll"'] & \cdot \arrow[d, "Rr"] \\ \cdot \arrow[r] \arrow[ur, dashed, "d"] & \cdot \end{tikzcd}$$

(iii) both L and R reflect isomorphisms.

If the diagonal fill-in is unique, (L, R) is dubbed a *stratisfied factorization system.*

5.2 Every (weak) factorization system $(\mathcal{L}, \mathcal{R})$ is an example of a (weak) stratisfied factorization system. After all, by virtue of 2.1(iv), both $\mathcal{L}$ and $\mathcal{R}$ are subcategories, and hence we can define L and R to be the inclusion functors.

Polar decomposition, too, provides an example of a stratisfied factorization system on $\mathbf{Hilb}_{\text{skel}}$, when we define L and R as the L^2 and ℓ^2 functors.

5.3 The examples $(\mathcal{L}, \mathcal{R})$ and (L^2, ℓ^2) of 5.2 enjoy additional properties over those axiomatized in 5.1. For example, in both cases the functors L and R are faithful, amnestic, and reflect identities. Moreover, as we have seen, the functors L^2 and ℓ^2 preserve daggers, and are symmetric (strong) monoidal with respect to both $\oplus$ and $\otimes$. We refrain from incorporating such properties into the definition, cautioned by the dearth of examples.

5.4 The functors L and R of a stratisfied factorization system need not determine each other, as do the classes $\mathcal{L}$ and $\mathcal{R}$ in a factorization system. In the latter case, one has $f \in \mathcal{L}$ if and only if $f \perp r$ for all $r \in \mathcal{R}$. But in the former case, it is not true that $f = Ll$ for some l in $\mathbf{L}$ if and only if $f \perp Rr$ for all r in $\mathbf{R}$. A counterexample is given by -1 in polar decomposition: $-1 \perp \ell^2 f$ for all partial injections f, but -1 is not positive.

5.5 Many factorization systems $(\mathcal{L}, \mathcal{R})$ correspond to reflective subcategories; these are characterized by the property that $g \in \mathcal{L}$ whenever $fg \in \mathcal{L}$ and $f \in \mathcal{L}$. A stratisfied version of this property does not hold for right-handed polar decomposition (see 2.16): if $fg = Ll$ and $g = Ll'$ for some l, l' in $\mathbf{PInj}$, then a standard proof shows that $f \perp \mathcal{R}$. But this does not imply that $f = Ll''$ for some l'' in **PInj**, basically because of the obstruction of 5.4. For a precise counterexample, consider $f = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; then $g = fg$ is a partial isometry, but f is not.[1] Nevertheless, the category $\mathbf{L}$ is of course always closed under composition.

5.6 Lemma In a stratified factorization system, factorization is unique up to an isomorphism of $\mathbf{C}$.

Proof Suppose that $(Rr)(Ll) = f = (Rr')(Ll')$. The diagonal fill-in of 5.1(ii) then gives morphisms u, v as in the diamond below.

$$\begin{tikzcd}
 & LB = RC \arrow[dr, "Rr"] \arrow[dd, dashed, shift right, "u"'] & \\
LA = LA' \arrow[ur, "Ll"] \arrow[dr, "Ll'"'] & & RD = RD' \\
 & LB' = RC' \arrow[ur, "Rr'"'] \arrow[uu, dashed, shift right, "v"'] &
\end{tikzcd}$$

Hence the following two squares commute.

$$\begin{tikzcd}
LA \arrow[r, "Ll"] \arrow[d, "Ll"'] & LB = RC \arrow[d, "Rr"] \arrow[dl, shift right, "vu"'] \arrow[dl, shift left, "\mathrm{id}"] \\
LB = RC \arrow[r, "Rr"'] & RD
\end{tikzcd}
\qquad
\begin{tikzcd}
LA' \arrow[r, "Ll'"] \arrow[d, "Ll'"'] & LB' = RC' \arrow[d, "Rr'"] \arrow[dl, shift right, "uv"'] \arrow[dl, shift left, "\mathrm{id}"] \\
LB' = RC' \arrow[r, "Rr'"'] & RD'
\end{tikzcd}$$

Therefore, by unicity of diagonal fill-ins, $uv = \mathrm{id}$ and $vu = \mathrm{id}$. $\square$

5.7 For functors $L\colon \mathbf{L} \to \mathbf{C}$ and $R\colon \mathbf{R} \to \mathbf{C}$ in general, let us recall their pullback in the category **Cat** of (small) categories. Its objects are pairs (A, B) of objects A of $\mathbf{L}$ and B of $\mathbf{R}$, such that $LA = RB$. A morphism $(A, B) \to (A', B')$ is a pair (l, r) of morphisms $l\colon A \to A'$ in $\mathbf{L}$ and $r\colon B \to B'$ in $\mathbf{R}$ such that $Ll = Rr$.

$$\begin{tikzcd}
 & \mathbf{L} \times_{\mathbf{C}} \mathbf{R} \arrow[dl] \arrow[dr] & \\
\mathbf{L} \arrow[dr, "L"'] & & \mathbf{R} \arrow[dl, "R"] \\
 & \mathbf{C} &
\end{tikzcd}$$

For a (weak) factorization system $(\mathcal{L}, \mathcal{R})$, the category $\mathbf{L} \times_{\mathbf{C}} \mathbf{R}$ consists of the same objects as $\mathbf{C}$, and morphisms that are in both subcategories $\mathbf{L}$ and $\mathbf{R}$. These are precisely the isomorphisms of $\mathbf{C}$.

In the case of polar decomposition, the category $\mathbf{G} = \mathbf{Pos} \times_{\mathbf{Hilb}_{\mathrm{skel}}} \mathbf{PInj}$ has as objects L^2-spaces equipped with a chosen orthonormal basis, and as

[1]However, at least in finite dimensions, $g > 0$ and $fg > 0$ do imply $f > 0$. To see this, use the fact that $A > 0$ iff $A_{ii} > 0$, $\det(A) > 0$ and $A^\dagger = A$ in any basis. The claim then follows by working in a basis of eigenvectors for f and using multiplicativity of the determinant.

morphisms positive partial isometries of the right form. Unwinding definitions, we see that $\mathbf{G}$ is in fact a *discrete* category, *i.e.* its only morphisms are identities! These morphisms are mapped precisely onto the unitaries in $\mathbf{Hilb}_{\text{skel}}$.[2]

The following lemma points out that this is indeed a general phenomenon for stratisfied factorization systems.

5.8 Incidentally, notice that if we work with $\mathbf{Hilb}$ instead of $\mathbf{Hilb}_{\text{skel}}$, the objects of $\mathbf{Pos} \times_{\mathbf{Hilb}} \mathbf{PInj}$ would be Hilbert spaces that are simultaneously of the form $L^2(S, \Sigma, \mu)$ and $\ell^2(X)$, *i.e.* not just isomorphic, but literally identical sets. This can only happen when $S = X$, Σ is the powerset of X, and μ is the counting measure. In this case, too, the pullback is discrete.

5.9 Lemma If (L, R) is a stratisfied factorization system, $\mathbf{L} \times_{\mathbf{C}} \mathbf{R}$ is a groupoid.

Proof Consider a morphism m in $\mathbf{L} \times_{\mathbf{C}} \mathbf{R}$; by definition m is of the form (l, r) for some morphisms l in $\mathbf{L}$ and r in $\mathbf{R}$ such that $Ll = Rr$. In other words, the following diagram commutes.

$$\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "Ll"'] & \cdot \arrow[d, "Rr"] \\
\cdot \arrow[r, equal] \arrow[ur, dashed, "d"] & \cdot
\end{tikzcd}$$

Hence there is a diagonal fill-in d making both triangles commute. In other words, $f = Ll = Rr$ is an isomorphism. Since L reflects isomorphisms, l is an isomorphism, too. Finally, m is an isomorphism as the functor $\mathbf{L} \times_{\mathbf{C}} \mathbf{R} \to \mathbf{L}$ also reflects isomorphisms. $\square$

5.10 The classes $\mathcal{L}$ and $\mathcal{R}$ of a factorization system enjoy closure under several operations apart from composition. For example, $\mathcal{L}$ is closed under colimits, and $\mathcal{R}$ under limits [7, 5.5]. This need not be the case for stratisfied factorization systems: 4.5 shows that $\mathbf{PInj}$ has no nontrivial products, whereas $\mathbf{Hilb}$ does have finite products by 2.9. Hence R cannot create or reflect limits. However, it turns out R does preserve certain limits.

5.11 Lemma If (L, R) is a stratisfied factorization system, then L preserves coequalizers and R preserves equalizers.

Proof Suppose that e is an equalizer of morphisms $r_1, r_2 \colon X \to Y$ in $\mathbf{R}$. Then $(Rr_1)(Re) = (Rr_2)(Re)$. Suppose that also $(Rr_1)f = (Rr_2)f$.

$$\begin{tikzcd}
RE \arrow[rr, "Re"] & & RX \arrow[r, shift left, "Rr_1"] \arrow[r, shift right, "Rr_2"'] & RY \\
& LB = RC \arrow[ul, dashed, "Rm"'] \arrow[ur, "Rr"] & & \\
LA \arrow[ur, "Ll"] \arrow[uurr, bend right, "f"'] & & &
\end{tikzcd}$$

[2]Similarly, positive partial isometries are precisely orthogonal projections, *i.e.* morphisms p with $p^\dagger = p = pp$.

Factorize $f = (Rr)(Ll)$. Since e is an equalizer, there is a unique morphism m in $\mathbf{R}$ with $r = em$. But then $(Rm)(Ll)$ is the unique morphism in $\mathbf{C}$ with $f = (Re)(Rm)(Ll)$ by virtue of Lemma 5.6. Hence Re is an equalizer of Rr_1 and Rr_2 in $\mathbf{C}$. A symmetric argument shows that L preserves coequalizers. $\square$

6 Choice issues

This section explores issues surrounding the apparent choice of basis that is made in (weak) polar decomposition. In doing so, we compare stratisfied factorization systems with functorial notions of factorization [13, 17].

6.1 Both functors L^2 and ℓ^2 of the stratisfied factorization system given by polar decomposition are forgetful in the sense that they forget the chosen basis on objects. Let us now make precise the intuition that 'choosing bases is unnatural', by remarking that the 'forgetful' functor $\ell^2 \colon \mathbf{PInj} \to \mathbf{Hilb}_{\mathrm{skel}}$ cannot have a (functorial) converse, even though one can choose an orthonormal basis for every Hilbert space. It is perhaps also worth mentioning that ℓ^2 is not a fibration in the technical sense of the word, not even a nonsplit or noncloven one, as the reader might perhaps think; Cartesian liftings in general do not exist because 'choosing bases is unnatural'.

6.2 Lemma The functor $\ell^2 \colon \mathbf{PInj} \to \mathbf{Hilb}$ has no adjoints.

Proof By 4.5, the only objects in $\mathbf{PInj}$ that have a (co)product are zero objects. Suppose ℓ^2 had a left adjoint F. Then it would preserve coproducts. Since $\mathbf{Hilb}$ has finite coproducts, the image of F would have coproducts, and therefore would contain only zero objects. But then $\{0\} = \mathbf{PInj}(0, S) \cong \mathbf{PInj}(FX, S) \cong \mathbf{Hilb}(X, \ell^2 S)$ for any Hilbert space X and set S, which is a contradiction. Because of the presence of the dagger, a right adjoint cannot exist either. $\square$

6.3 In the situation $(\mathbf{L} \xrightarrow{L} \mathbf{C} \xleftarrow{R} \mathbf{R})$ of a stratisfied factorization system, one might think of functors $(\mathbf{L} \xleftarrow{A} \mathbf{C} \xrightarrow{B} \mathbf{R})$ in the opposite direction making the choice of factorization functorial, in the sense that $f = (RBf)(LAf)$ for every morphism f in $\mathbf{C}$. But requiring so forces all morphisms of $\mathbf{C}$ to be endomorphisms. Indeed, since $\mathrm{id}_X = (RB\mathrm{id}_X)(LA\mathrm{id}_X)$, we have $X = \mathrm{dom}(\mathrm{id}_X) = \mathrm{dom}(LA\mathrm{id}_X)$ and $X = \mathrm{cod}(\mathrm{id}_X) = \mathrm{cod}(RB\mathrm{id}_X)$, whence $X = LAX = RBX$ for all objects X. But for a morphism $f \colon X \to Y$ then $X = RBX = \mathrm{dom}(RBX) = \mathrm{cod}(LAY) = LAY = Y$.

6.4 Write $\mathbf{2}$ for the ordinal $2 = (0 \leq 1)$, regarded as a category, and define the category $\mathbf{3}$ similarly. Then the category $\mathbf{C}^{\mathbf{2}}$ is the *arrow category* of $\mathbf{C}$: its objects are morphisms of $\mathbf{C}$, and its morphisms are pairs of morphisms of $\mathbf{C}$ making the square commute. Similarly, objects of $\mathbf{C}^{\mathbf{3}}$ are composable pairs of morphisms. Recall that a *functorial factorization* is a functor $F \colon \mathbf{C}^{\mathbf{2}} \to \mathbf{C}^{\mathbf{3}}$ that splits the composition functor.

Polar decomposition, as in Proposition 2.14, provides a functorial factorization system, simply because positive maps are isomorphisms. We now show that weak polar decomposition, as in Theorem 2.12, also provides a functorial factorization system.

6.5 Lemma Suppose that the outer rectangle of the diagram in **Hilb** below commutes, where p, p' are nonnegative maps, u, u' are partial isometries, and a, b are arbitrary morphisms, $\ker(p) = \ker(u)$ and $\ker(p') = \ker(u')$.

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "a"'] & X \arrow[r, "u"] \arrow[d, dashed, "m"] & Y \arrow[d, "b"] \\ X' \arrow[r, "p'"'] & X' \arrow[r, "u'"'] & Y' \end{tikzcd}$$

Then one can find a morphism m making both inner squares commute.

PROOF It suffices to define m on $\ker(p)^\perp = \overline{\mathrm{Im}(p)}$ and $\ker(p)$. For $x \in \ker(p)$, put $m(x) = 0$. Let $x \in \overline{\mathrm{Im}(p)}$, say $x = \lim_n x_n$ with $x_n \in \mathrm{Im}(p)$. Since nonnegative maps are positive and hence bijective when restricted to the orthogonal complement of their kernel, there are unique $w_n \in \mathrm{Im}(p) \subseteq X$ with $p(x_n) = w_n$. Defining $m(x) = \lim_n p'a(w_n)$ therefore gives a well-defined linear operation.

We verify that this m makes the right square commute. For $x \in \ker(p) = \ker(u)$, obviously $bu(x) = 0 = u'm(x)$. And for $x \in \overline{\mathrm{Im}(p)}$, say $x = \lim_n p(w_n)$, we have $bu(x) = \lim_n bup(w_n) = \lim_n u'p'a(w_n) = u'm(x)$.

As to the left square, if $w \in \overline{\mathrm{Im}(p)} \subseteq X$, then $mp(w) = p'a(w)$ by construction of m. Finally, if $x \in \ker(p)$, then clearly $mp(w) = 0$. Also $u'p'a(w) = bup(w) = 0$. Since $\ker(up) = \ker(p)$, this implies that $p'a(w) = 0$. □

6.6 The mediating morphism in the previous proposition is not unique. For a counterexample, take $X = Y = X' = Y' = \mathbb{C}$, $a = b = \mathrm{id}$, and $p = u = p' = u' = 0$. Then any m makes both squares commute; this is the essence of the red herring of 2.13. Nevertheless, the assignment of the previous proposition makes weak polar decomposition a functorial factorization system, as we now show.

6.7 Proposition Weak polar decomposition forms a functorial factorization system $F \colon \mathbf{Hilb^2} \to \mathbf{Hilb^3}$, whose action on objects is given by Theorem 2.12, and whose action on morphisms is given by Lemma 6.5.

PROOF For well-definedness, we are to show that the action on morphisms preserves identities and composition, which is not directly obvious because of 6.6. However, notice that (the proof of) Lemma 6.5 does give a constructive prescription for m (in terms of uniquely defined elements w). That identities are preserved is trivial: if $a = b = p = u = p' = u' = \mathrm{id}$, then Lemma 6.5 prescribes that $m(x) = x$. Now suppose that $(a, b) \colon up \to u'p'$ gets mapped to k and $(c, d) \colon u'p' \to u''p''$ to m. Then mk is, by prescription, the following map: $mk(x) = 0$ if $x \in \ker(p)$, and otherwise

$$mk(x) = m(\lim_n p'a(w_n)) = \lim_n mp'a(w_n) = \lim_n p''ca(w_n),$$

where $x = \lim_n p(w_n)$. But this is precisely the mediating morphism that (ca, db) gets mapped to, so the action on morphisms preserves composition. □

6.8 Let us briefly recall the notion of a *natural weak factorization system*; for details we refer to [13]. It consists of a functorial factorization $F\colon \mathbf{C^2} \to \mathbf{C^3}$ with extra structure. By postcomposing F with the functor $\mathbf{C^3} \to \mathbf{C^2}$ induced by the functor $\mathbf{2} \to \mathbf{3}$ given by $n \mapsto n+1$, we get a functor $R\colon \mathbf{C^2} \to \mathbf{C^2}$ that takes the right morphism of the factorization F. Similarly we get a functor $L\colon \mathbf{C^2} \to \mathbf{C^2}$ that takes the left morphism of the factorization F. There is also a canonical natural transformation $\Lambda\colon \mathrm{Id}_{\mathbf{C^2}} \Rightarrow R$ whose component at f is

$$\begin{tikzcd} X \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[dr, phantom, "\Lambda_f"] & \cdot \arrow[d, "Rf"] \\ Y \arrow[r, equal] & Y. \end{tikzcd}$$

The functor L is similarly copointed. The extra structure needed to make F a natural weak factorization system is an extension of the pointed endofunctor (R, Λ) to a monad on $\mathbf{C^2}$, and an extension of the copointed endofunctor L to a comonad.

In the case of polar decomposition, notice that $R \circ R = R$. To see that both functors act the same on objects, first notice that $Rf = (Rf)\mathrm{id}$ where id is a positive map; by uniqueness, it follows that $RRf = Rf$. On morphisms (a, b), the functor R gives the right square in the diagram left below, and RR gives the right square in the diagram on the right.

$$\begin{tikzcd} X \arrow[r, "Lf"] \arrow[d, "a"'] & X \arrow[r, "Rf"] \arrow[d, dotted, "m=(Lg)a(Lf)^{-1}"'] & Y \arrow[d, "b"] \\ X' \arrow[r, "Lg"'] & Y' \arrow[r, "Rg"'] & Z'. \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "m"'] & X \arrow[r, "Rf"] \arrow[d, dotted, "n"] & Y \arrow[d, "b"] \\ X' \arrow[r, "\mathrm{id}"'] & X' \arrow[r, "Rg"'] & Y' \end{tikzcd}$$

Indeed $n = m$, and so $R \circ R = R$. This argument can be adapted to show that $RR = R$ for the weak polar decomposition of Proposition 6.7, too, by distinguishing $X = \ker(Lf) \oplus \ker(Lf)^\perp$.

Weak polar decomposition does not provide a natural weak factorization system. It already fails to satisfy the required monad unit law $\Pi \circ R\Lambda = \Pi \circ \Lambda_R = \mathrm{id}\colon R \Rightarrow R$, irrespective of any monad multiplication Π one could endow R with. The first equation does hold: writing Kf for the 'image' object of f under F, one finds that Λ_R and $R\Lambda\colon R \Rightarrow R \circ R = R$ have components

$$\begin{tikzcd} Kf \arrow[r, "L(Rf)"] \arrow[d, "Rf"'] \arrow[dr, phantom, "\Lambda_{Rf}"] & K^2 f = Kf \arrow[d, "Rf"] \\ Y \arrow[r, equal] & Y, \end{tikzcd} \qquad \begin{tikzcd} Kf \arrow[r, "R(Lf)"] \arrow[d, "Rf"'] \arrow[dr, phantom, "R(\Lambda_f)"] & K^2 f = Kf \arrow[d, "Rf"] \\ Y \arrow[r, equal] & Y, \end{tikzcd}$$

and equal each other since $\ker(Lf) = \ker(Rf)$ and hence $L(Rf) = 0 \oplus \mathrm{id} = R(Lf)\colon \ker(Lf)^\perp \oplus \ker(Lf) \to Kf$. However, there is no morphism g in **Hilb** with $g(RLf) = \mathrm{id}_{Kf}$, since the former in general has rank strictly lower than the latter. Hence there no Π such that $\Pi \circ (Rf) = \mathrm{id} \colon R \Rightarrow R$, so R cannot extend to a monad, and weak polar decomposition does not provide a natural weak factorization system.

For polar decomposition the monad unit laws are satisfied, since for any morphism f we have $RLf = \mathrm{id} = LRf$. Hence $\Lambda R = R\Lambda$, so that if (R, Λ) would extend to a monad, it would be idempotent. But that would imply that F comes from a weak factorization system, which we know not to be the case from 2.15. We conclude that polar decomposition, weak or not, is not a natural weak factorization system.

6.9 For a stratisfied factorization system (L, R), consider the functors $\mathrm{cod} \circ L^{\mathbf{2}} \colon \mathbf{L}^{\mathbf{2}} \to \mathbf{C}$ and $\mathrm{dom} \circ R^{\mathbf{2}} \colon \mathbf{R}^{\mathbf{2}} \to \mathbf{C}$. Their pullback $\mathbf{L}^{\mathbf{2}} \times_{\mathbf{C}} \mathbf{R}^{\mathbf{2}}$ has as objects pairs (l, r) of morphisms in $\mathbf{L}$ and $\mathbf{R}$ for which $\mathrm{cod}(Ll) = \mathrm{dom}(Rr)$. Morphisms consist of triples (a, b, c) of morphisms in $\mathbf{L}$, $\mathbf{L} \times_{\mathbf{C}} \mathbf{R}$ and $\mathbf{R}$, respectively, making the following diagram commute.

$$\begin{tikzcd}
\cdot \arrow[r, "Ll"] \arrow[d, "La"'] & \cdot \arrow[r, "Rr"] \arrow[d, "Lb=Rb"] & \cdot \arrow[d, "Rc"] \\
\cdot \arrow[r, "Ll'"'] & \cdot \arrow[r, "Rr'"'] & \cdot
\end{tikzcd}$$

There is an induced functor $\mathbf{L}^{\mathbf{2}} \times_{\mathbf{C}} \mathbf{R}^{\mathbf{2}} \to \mathbf{C}^{\mathbf{3}}$, since the latter category is the pullback of $\mathrm{dom}, \mathrm{cod} \colon \mathbf{C}^{\mathbf{2}} \to \mathbf{C}$. Hence we can speak of composition as a functor $\mathbf{L}^{\mathbf{2}} \times_{\mathbf{C}} \mathbf{R}^{\mathbf{2}} \to \mathbf{C}^{\mathbf{2}}$. This prompts the following definition.

6.10 Definition A stratisfied factorization system (L, R) is *natural* when there exists a functor $F \colon \mathbf{C}^{\mathbf{2}} \to \mathbf{L}^{\mathbf{2}} \times_{\mathbf{C}} \mathbf{R}^{\mathbf{2}}$ that splits the composition functor $\mathbf{L}^{\mathbf{2}} \times_{\mathbf{C}} \mathbf{R}^{\mathbf{2}} \to \mathbf{C}^{\mathbf{2}}$.

6.11 The morphisms of the pullback in the previous definition impose the following condition: all morphisms f in $\mathbf{C}$ can be written as $f = (Rr)(Ll)$ for morphisms l and r of $\mathbf{L}$ and $\mathbf{R}$, respectively, and if $f = (Rr')(Ll')$, then there is a morphism m in $\mathbf{L} \times_{\mathbf{C}} \mathbf{R}$ making the following diagram commute.

$$\begin{tikzcd}
& \cdot \arrow[dd, dashed, "m"] \arrow[dr, "Rr"] & \\
\cdot \arrow[ur, "Ll"] \arrow[dr, "Ll'"'] & & \cdot \\
& \cdot \arrow[ur, "Rr'"'] &
\end{tikzcd}$$

In other words, factorization is unique up to morphisms in $\mathbf{L} \times_{\mathbf{C}} \mathbf{R}$.

For factorization systems, this means that factorizations are unique up to isomorphism, which is true. Polar decomposition is an example, too, with factorization unique up to unitary. (See also Lemma 5.9). However, for weak polar decomposition the previous definition would mean that factorization is unique up to projection, which does not hold. For a counterexample, consider $f = \pi_2 \colon \mathbb{C}^2 \to \mathbb{C}^2$, $l = \pi_2$, $r = \pi_2$, $l' = \mathrm{id}$ and $r' = \pi_2$; then $f = rl = r'l'$, but there is no projection m such that $l' = ml$.

7 Refactorization systems

A weak factorization system could be seen as a decomposition $\mathbf{C} = \mathcal{R} \circ \mathcal{L}$ into subcategories. Conversely, the current section takes this analogy seriously, and studies under which circumstances $\mathbf{C}$ can be recovered as a 'composition' of $\mathcal{L}$ and $\mathcal{R}$. We will more generally phrase this in terms of stratisfied factorization systems, instead of weak factorization systems.

7.1 Recall that the category **Cat** of categories is cocomplete. Colimits in it can be computed as follows [12]:

1. calculate the colimit of underlying graphs;
2. take the free category on the resulting graph;
3. quotient out the smallest congruence making the graph homomorphisms of the colimit cone into functors.

For the diagram $\mathbf{A} \xleftarrow{F} \mathbf{G} \xrightarrow{H} \mathbf{B}$ in **Cat** with **G** discrete, say with underlying reflexive graphs $A \xleftarrow{f} G \xrightarrow{h} B$, step 1 results in the graph with vertices $(V_A + V_B)/\sim$, where $a \sim b$ iff $a = f(g)$ and $b = h(g)$ for some vertex $g \in V_G$ of G, and edges $(E_A + E_B)/\sim$, where $e \sim e'$ iff $e = f(e'')$ and $e' = h(e'')$ for some edge $e'' \in E_G$ of G. Step 2 is then performed by taking all paths in the graph as morphisms, after which step 3 identifies the images of $f(e'')$ and $h(e'')$ for all $e'' \in E_G$.

In particular, applying this recipe to the situation

$$\begin{tikzcd} & \mathbf{G} \arrow[dl, "P"'] \arrow[dr, "I"] & \\ \mathbf{Pos} & & \mathbf{PInj}, \end{tikzcd}$$

we find that the pushout has objects

$$(\{(S, \Sigma, \mu) \text{ in } \mathbf{Pos}\} + \{E \textbf{ in Set}\})/\sim$$

where $\sim$ is generated by $(S, \Sigma, \mu) \sim E$ iff $L^2(S, \Sigma, \mu) = \ell^2 E$. Morphisms $[E]_\sim \to [E']_\sim$ are composable sequences of alternating positive maps and partial isometries between $\ell^2 E$ and $\ell^2 E'$, which we denote with parentheses:

$$(i_1, p_1, \ldots, i_n, p_n),$$

where i_k is a partial isometry that is not strictly positive and p_k is a strictly positive map that is not a partial isometry. We emphasize that such a sequence need not begin with partial isometry, and need not end with a positive map.

7.2 Since **Pos**, **PInj** and $\mathbf{Hilb}_{\text{skel}}$ are dagger categories, and the functors L^2 and ℓ^2 preserve daggers, one might think of taking the pushout in the category **DagCat** of dagger categories instead of in **Cat**. The recipe of 7.1 would then be extended by taking the free dagger category on the resulting category, and quotienting out the smallest congruence making the colimit cone functors preserve daggers. Since the pushout in **Cat** is already a dagger category by

$$(i_1, p_1, \ldots, i_n, p_n)^\dagger = (p_n^\dagger, i_n^\dagger, \ldots, p_1^\dagger, i_1^\dagger),$$

the colimit cone functors already preserves daggers, and nothing is added: the pushouts in **Cat** and **DagCat** are the same.

7.3 There is a functor F from the pushout of **Pos** and **PInj** to $\mathbf{Hilb}_{\text{skel}}$ given by $F[E] = \ell^2 E$ and $F(p_1, i_1, \ldots, p_n, i_n) = i_n \circ p_n \circ \cdots \circ i_1 \circ p_1$. One would like to use polar decomposition to define a functor G in the other direction by $G\ell^2 E = E$ and $Gf = (i, p)$, but unfortunately that is not functorial. What is needed is a way to change a factorization $f = ip$ into one of the form $f = p'i'$, prompting the following definition.

But first, let us recall the pullback category $\mathbf{L^2} \times_{\mathbf{P}} \mathbf{R^2}$, where $\mathbf{P}$ is the pushout of two given functors ($\mathbf{L} \xleftarrow{A} \mathbf{G} \xrightarrow{B} \mathbf{R}$). Its objects are pairs (l, r) of morphisms $l \in \mathbf{L^2}$ and $r \in \mathbf{R^2}$ that are composable in $\mathbf{P}$. Morphisms $(l, r) \to (l', r')$ consist of three morphisms of $\mathbf{P}$ coming from $a \in \mathbf{L^2}$, $c \in \mathbf{R^2}$, and b from both $\mathbf{L^2}$ and $\mathbf{R^2}$, making the following diagram commute.

$$\begin{tikzcd}
\cdot \arrow[r, "l"] \arrow[d, "a"'] & \cdot \arrow[r, "r"] \arrow[d, "b"] & \cdot \arrow[d, "c"] \\
\cdot \arrow[r, "l'"'] & \cdot \arrow[r, "r'"'] & \cdot
\end{tikzcd}$$

Notice that there *is* a canonical functor $\mathbf{L^2} \times_{\mathbf{C}} \mathbf{R^2} \to \mathbf{L^2} \times_{\mathbf{P}} \mathbf{R^2}$ in the situation where $\mathbf{G}$ arises as a pullback of $L \colon \mathbf{L} \to \mathbf{C}$ and $R \colon \mathbf{R} \to \mathbf{C}$.

7.4 Definition A *refactoring law* between ($\mathbf{L} \xleftarrow{A} \mathbf{G} \xrightarrow{B} \mathbf{R}$) is an assignment Γ of an object in $\mathbf{L^2} \times_{\mathbf{P}} \mathbf{R^2}$ to each object of $\mathbf{R^2} \times_{\mathbf{P}} \mathbf{L^2}$, where $\mathbf{P}$ is the pushout of A and B in **Cat**, such that $rl = l'r'$ when $\Gamma(r, l) = (l', r')$. When the assignment extends to a functor $\Gamma \colon \mathbf{R^2} \times_{\mathbf{P}} \mathbf{L^2} \to \mathbf{L^2} \times_{\mathbf{P}} \mathbf{R^2}$ under which the composition functor is invariant, we call the refactoring law *natural.*

7.5 Definition A *(natural) refactorization system* consists of two categories, $\mathbf{L}$ and $\mathbf{R}$, equipped with the following constraint data:

(on objects) two functors, $A \colon \mathbf{G} \to \mathbf{L}$ and $B \colon \mathbf{G} \to \mathbf{R}$, from a groupoid $\mathbf{G}$;

(on morphisms) a (natural) refactoring law Γ between A and B.

7.6 Proposition Every (natural) weak stratified factorization system gives rise to a (natural) refactorization system.

Proof Given a stratified factorization system (L, R), take $\mathbf{G}$ to be the pullback $\mathbf{L} \times_{\mathbf{C}} \mathbf{R}$, with $A = L^{-1}(R)$ and $B = R^{-1}(L)$ the induced functors. As to the refactoring law, define $\Gamma(r, l) = (l', r')$, where $lr = r'l'$ is given by 5.1(i). If the stratified factorization system was natural, there is a functor $F\colon \mathbf{C^2} \to \mathbf{L^2} \times_{\mathbf{C}} \mathbf{R^2}$ that we can use to define Γ functorially as the following composite, where all nonlabeled functors arise canonically from universality of pushouts or pullbacks.

$$\mathbf{R^2} \times_{\mathbf{P}} \mathbf{L^2} \longrightarrow \mathbf{P^3} \longrightarrow \mathbf{C^3} \xrightarrow{\circ} \mathbf{C^2} \xrightarrow{F} \mathbf{L^2} \times_{\mathbf{C}} \mathbf{R^2} \longrightarrow \mathbf{L^2} \times_{\mathbf{P}} \mathbf{R^2}$$

Hence in that case the refactorization system is also natural. □

7.7 Definition A span $(\mathbf{L} \xrightarrow{L} \mathbf{C} \xleftarrow{R} \mathbf{R})$ *satisfies the (natural) refactoring law* Γ between A and B when $LA = RB$ and $r'l' = lr$ for $\Gamma(r,l) = (l', r')$.

We also say that $\mathbf{C}$ is a *composition* of $\mathbf{L}$ and $\mathbf{R}$, and write $\mathbf{C} = \mathbf{R} \circ \mathbf{L}$. The following theorem shows that, given a refactorization system, its composition exists. It is the universal (and hence unique up to isomorphism) category in which every morphism f can be written as $f = (Rr)(Ll)$.

7.8 Theorem Compositions of refactorization systems exist universally. More precisely: given a refactorization system Γ there exists a category $\mathbf{C}$ that is a composition of $\mathbf{L}$ and $\mathbf{R}$, and for any other composition $\mathbf{C}'$ there is a unique functor $\mathbf{C} \to \mathbf{C}'$ making the following diagram commute.

$$\begin{tikzcd}
 & \mathbf{G} \arrow[dl, "A"'] \arrow[dr, "B"] & \\
\mathbf{L} \arrow[r, "L"] \arrow[dr, "L'"'] & \mathbf{C} \arrow[d, dashed] & \mathbf{R} \arrow[l, "R"'] \arrow[dl, "R'"] \\
 & \mathbf{C}' &
\end{tikzcd}$$

If the refactorization system arose from a stratified factorization system on $\mathbf{C}$, then its composition is (isomorphic to) $\mathbf{C}$.

Proof The category $\mathbf{C}$ is constructed as follows. Its objects are those of the pushout $\mathbf{P}$ of A and B. Morphisms are those morphisms of $\mathbf{P}$ of the form (l, r) with l from $\mathbf{L}$ and r from $\mathbf{R}$. The composition of (l_1, r_1) and (l_2, r_2) is $(l'l_1, r_2r')$, where $\Gamma(r_2, l_2) = (l', r')$. Because composition (in $\mathbf{P}$) is invariant under Γ, composition in $\mathbf{C}$ is indeed associative. The identity is $(\mathrm{id}, \mathrm{id})$.

Next, the functor L acts on objects as the functor $\mathbf{L} \to \mathbf{P}$ induced by $\mathbf{P}$ being a pushout. On morphisms, $L(l) = (l, \mathrm{id})$. The functor $R\colon \mathbf{R} \to \mathbf{C}$ is defined similarly by $R(r) = (\mathrm{id}, r)$.

By construction, (L, R) satisfies the refactoring law Γ, so that $\mathbf{C}$ is a composition of $\mathbf{L}$ and $\mathbf{R}$.

Finally, if $\mathbf{C}'$ is a composition of $\mathbf{L}$ and $\mathbf{R}$, then there is a unique functor $\mathbf{P} \to \mathbf{C}'$ from the pushout. Because $\mathbf{C}'$ satisfies the refactoring law, this functor factors through $\mathbf{C}$. □

7.9 From the previous theorem, a converse to Proposition 7.6 almost follows. Given a (natural) refactorization system, the previous theorem constructs the required data ($\mathbf{L} \xrightarrow{L} \mathbf{C} \xleftarrow{R} \mathbf{R}$) for a (natural) weak stratisfied factorization system. Moreover, that data satisfies 5.1(i) by construction, and is also easily seen to satisfy 5.1(iii). However, the existence of diagonal fill-ins as in 5.1(ii) seems to be lost. One could investigate what extra condition on a refactorization system accounts for (unique) diagonal fill-ins. We refrain from doing so here, because that is not so interesting from the viewpoint of reconstructing a category from factors.

7.10 The situation is not quite that of an amalgamation of categories [20], and the functors L^2 and ℓ^2 are not embeddings. Nevertheless, it would be interesting to see for what refactorization systems the category $\mathbf{C}$ coincides with the pushout category $\mathbf{P}$, *i.e.* when the square

$$\begin{tikzcd} & \mathbf{G} \arrow[dl] \arrow[dr] & \\ \mathbf{L} \arrow[dr] & & \mathbf{R} \arrow[dl] \\ & \mathbf{C} & \end{tikzcd}$$

is both a pullback and a pushout. In a sense, this would mean that there is only one way to factorize a morphism of $\mathbf{C}$, and factors cannot be factorized further in a nontrivial way.

7.11 Since (co)limits in functor categories are computed pointwise, a refactorization system for $\mathbf{C}$ induces one for any category of functors with values in $\mathbf{C}$. In this sense, refactorization systems are more useful than factorization systems, which are not stable under taking functor categories.

In the case of polar decomposition, we automatically get a refactorization system for categories of the form $\mathbf{Hilb}_{\mathrm{skel}}^{\mathbf{C}}$. This can have interesting consequences for (topological or unitary conformal) quantum field theories, which can be formulated as functors into $\mathbf{Hilb}_{\mathrm{skel}}$ (see [3] and [24], respectively): apparently, such quantum field theories can be reconstructed from a functor into **Pos** and a functor into **PInj**.

7.12 Connecting to the analogy with prime factorization of natural numbers in the introduction, notice that in principle composition of categories as in Definition 7.7 need not be restricted to two categories. One could consider an arbitrary number of factors. In fact, one could consider colimits of other shapes than (wide) pushouts in **Cat**. The price to pay is increased complexity of the refactoring law.

8 Concluding remarks

Roughly, the current article can be summed up in a single equation:

$$\mathbf{Hilb}_{\text{skel}} = \mathbf{PInj} \circ \mathbf{Pos}. \tag{1}$$

We have mostly studied the situation abstractly; more examples of stratisfied factorization systems and refactorization systems seem needed. Nevertheless there are several concrete issues that deserve further attention.

- The category **Pos** is totally disconnected and hence a disjoint union of monoids. By Theorem 4.10, the category **PInj** is finitely accessible, and can be seen as several copies of a directed complete partial order. Hence (1) is interesting from the viewpoint of the intuition that categories are common generalization of monoids and posets. One could optimistically imagine a structure theorem exhibiting every category as equivalent to a composition of a monoid and preorder, as given by a refactorization system.

- Equation (1) could be seen as making first categorical inroads into spectral theory. The fact that **Pos** is a totally disconnected groupoid could be taken as the basis for a definition of 'spectral categories' as those that are a composition of a groupoid and another category. The structure of such categories certainly deserves to be studied further. As a first step, one might look for a construction of the category **Pos** from the dagger monoidal category **Hilb** similar to 4.18, in order to use polar decomposition as a 'spectral assumption' on arbitrary dagger monoidal categories. This seems to suggest that the category **Rel** of sets and relations does not have a satisfactory spectral calculus, even though one can see a relation as a matrix of zeroes and ones; after all, factoring a relation as a bijection followed by another relation does not reduce the latter class of relations.

- From the viewpoint of quantum mechanics, too, (1) is quite interesting. It separates the category of state spaces cleanly into a 'quantitative' and a 'qualitative' part. Remarkably, these parts also cleanly separate 'continuous' aspects from 'discrete' ones. The former part is concerned just with probabilistic weights, whereas the latter part captures complementarity issues. Section 4 provides an initial investigation of the latter category, but the complementarity aspects can be explicated much further: what structure does the totality of changes of bases form? This is especially exciting in connection to quantum field theories (see 7.11), whose quantitative parts routinely need 'renormalizing' to have any rigorous meaning.

- Some aspects of refactorization systems as we have studied them are not quite satisfactory. For example, we had to work with a skeleton $\mathbf{Hilb}_{\text{skel}}$ rather than **Hilb** itself to make polar decomposition work at all. Hopefully, considering bilimits in the 2-category **Cat** instead of limits will shed more light on the goings-on. Similarly, connections to [6] could be investigated. Finally, the role of the dagger could be cleared up by considering bilimits in the 2-category **DagCat** of dagger categories.

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