Causal Structure in Categorical Quantum Mechanics

Raymond Lal

Christ Church

University of Oxford

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Abstract

Categorical quantum mechanics is a way of formalising the structural features of quantum theory using category theory. It uses compound systems as the primitive notion, which is formalised by using symmetric monoidal categories. This leads to an elegant formalism for describing quantum protocols such as quantum teleportation. In particular, categorical quantum mechanics provides a graphical calculus that exposes the information flow of such protocols in an intuitive way. However, the graphical calculus also reveals surprising features of these protocols; for example, in the quantum teleportation protocol, information appears to flow 'backwards-in-time'. This leads to question of how causal structure can be described within categorical quantum mechanics, and how this might lead to insight regarding the structural compatibility between quantum theory and relativity.

This thesis is concerned with the project of formalising causal structure in categorical quantum mechanics. We begin by studying an abstract view of Bell-type experiments, as described by 'no-signalling boxes', and we show that under time-reversal no-signalling boxes generically become signalling. This conflicts with the underlying symmetry of relativistic causal structure. This leads us to consider the framework of categorical quantum mechanics from the perspective of relativistic causal structure. We derive the properties that a symmetric monoidal category must satisfy in order to describe systems in such a background causal structure. We use these properties to define a new type of category, and this provides a formal framework for describing protocols in spacetime. We explore this new structure, showing how it leads to an understanding of the counter-intuitive information flow of protocols in categorical quantum mechanics. We then find that the formal properties of our new structure are naturally related to axioms for reconstructing quantum theory, and we show how a reconstruction scheme based on purification can be formalised using the structures of categorical quantum mechanics. Finally, we discuss the philosophical aspects of using category theory to describe fundamental physics. We consider a recent argument that category-theoretic formulations of physics, such as categorical quantum mechanics, can be used to support a variant of structural realism. We argue against this claim. The work of this thesis suggests instead that the philosophy of categorical quantum mechanics is subtler than either operationalism or realism.

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Chapter 1

Introduction

This thesis is concerned with the foundations of quantum theory. The emergence of quantum information and computation has recently stimulated much work in quantum foundations. Moreover, there is a strand of current research in quantum information with a specifically *structural* focus, and this naturally leads to foundational questions. 'Structural' in this context means a focus on the logical relationships between physical properties of theories. A recent example of this is given by the work on understanding the device-independent resources required to perform certain cryptographic tasks [14, 49]. This structural turn has also led to a more abstract view of more traditional topics in quantum foundations such as quantum nonlocality. The development of 'foil theories' for quantum theory is an example of this, and one that is relevant to our concerns. Foil theories may exhibit nonlocality [13], and this provides a useful comparison for understanding the behaviour of nonlocality in quantum theory. In general, this requires developing a *formal framework* in which different physical theories can be formulated. Such a framework allows physical features such as the existence of a teleportation protocol to be identified in different theories. The logical relationship between these features can then be established within this framework: for example, whether the teleportation protocol and nonlocality are logically independent notions, which has been explored using the Spekkens toy theory as a foil for quantum theory [102].

This bring us to two themes of this thesis. Firstly, this broad 'information-theoretic' view is the setting for this thesis. However, our perspective will not be information-theoretic *per se*, but instead we shall consider notions of information—and, in particular, information flow—in a more abstract mathematical way than is usually considered. More specifically, our perspective will be one informed by theoretical computer science, in which we can view the standard quantum formalism (of density matrices and completely positive maps) as a kind of 'low-level' programming language. The development of simple quantum protocols such as quantum teleportation suggests that a *high-level language* can be extracted, which would allow protocols to be described in a simpler and more elegant way. Secondly, this thesis is largely concerned with building a framework; or rather, extending an existing framework. Our specific aim is to build a framework for studying causal structure in physical theories. This would allow causal structure, i.e. relativistic causal relations, to be defined and analysed in a wide variety of theories: quantum theory but also, for example, classical physics and foil theories.

Specific context. The work in this thesis largely uses the formalism of *categorical quantum mechanics (CQM)* [1]. This is an approach that uses category theory to study the abstract properties of quantum theory. The main idea of CQM is to discover the category-theoretic rules that correspond to quantum phenomena such as quantum teleportation. In broad terms, category theory describes the 'relational properties' of mathematical objects. A category contains mathematical objects of a certain kind, and morphisms between them, e.g. the category of vector spaces and linear maps. Category theory is the study of morphisms in a category, and how the algebra of these morphisms differs according to the type of objects in the category. In this way, category theory is an 'external' study of mathematical objects, via their morphisms to one another—as opposed to their internal structure. For example, as we shall see, the category of vector spaces and linear maps differs from the category of sets and functions in the way that bipartite processes are combined: sets combine with the cartesian product, but vector spaces combine with the tensor product. This corresponds to the difference between classical and quantum theory, since the tensor product leads to entanglement. Indeed, this relational aspect is at the *core* of CQM, since it is a formalism based on how physical systems and processes combine according to algebraic rules.

Thesis topic. The specific topic of this thesis is causal structure in CQM, and how this should be formalised. The motivation for the work in this thesis was originally to address the 'delicate balance' between special relativity and quantum theory.[1] For example, as many researchers have noted, the fact that quantum theory is nonlocal but does not violate the causal structure of relativity suggests a subtle relationship between the two theories. Our hope was that insight can be gained into this topic by studying it from a new perspective—in our case, from the perspective of CQM. However, the project evolved from this initial aim, for the simple reason that CQM required more development before notions such as causality could be formulated in it.

Thesis summary. The content of this thesis is therefore centred around abstract considerations of causal structure, in order to develop such a framework. We first study bipartite no-signalling devices, which have been widely studied as a means of gaining an abstract understanding of nonlocality [15]. We show that there is a certain type of time-asymmetry in their causality conditions (i.e. the no-signalling conditions). This leads us to consider causal structure in CQM from a similarly abstract perspective, and we show how physical considerations lead to certain categorical constraints when describing causal structure in CQM. For example, we derive terminality of the monoidal unit from the no-signalling assumption. We then formalise these constraints by defining the notion of a 'causal category', and we explore the structure of these categories. This leads us to consider the how causality is related to certain reconstructions of quantum theory. Finally we consider the philosophical implications of this, in the hope of understanding the philosophy of CQM.

We describe the content of this thesis in more detail as follows.

Thesis overview

This thesis is in three parts:

I. Background on CQM and causal structure.

II. Causal structure and probabilistic processes in CQM.

III. Reconstructing quantum theory from a categorical perspective.

Part I and Section 7.1 of Chapter 7 provide background material. The remainder is original research. Much of the original research is based on the author's work in the papers [26, 27, 28, 38, 39]; we give details of this below.

[1] The phrase 'delicate balance' and the motivation described here was first heard by the author in a lecture by Bob Coecke.

Part I

In Part I we describe the necessary background material for Parts II and III.

Chapter 2. We introduce the formalism of CQM. We start by introducing *symmetric monoidal categories*, which encode two ways that physical systems and processes can be combined. This corresponds to 'sequential' and 'parallel' composition: sequential composition is the ordinary categorical composition, and parallel composition is an additional type of composition defined for monoidal categories. Parallel composition corresponds to 'concurrent' or 'parallel' processes. We then introduce *dagger compact categories*. These are monoidal categories with an extra layer of structure: in a category that is interpreted physically, this extra structure corresponds to the existence of post-selected teleportation. We shall discuss how quantum theory provides an example of a dagger compact category with this interpretation. We then show how mixed states and operations can be formulated in CQM.

Chapter 3. We discuss two aspects of causal structure in foundational physics. Firstly, we discuss the theorems of Zeeman and Malament that allow a relativistic spacetime to be 'reconstructed' from its causal structure. These results underpin the foundational significance of the work in subsequent chapters, in which we shall describe a spacetime using discrete causal structure. Secondly, our focus is on understanding how causal structure is combined with what we could call 'process structure' in physics. That is, causal structure is often thought of as a network of causal relations between points in spacetime. However, there are also processes occurring in spacetime that are not explicitly modelled in a relativistic spacetime (or which are at least difficult to represent in the stress-energy tensor, for example), e.g. quantum unitary evolution. We discuss two approaches to combining quantum processes with spacetime, viz. algebraic quantum field theory and quantum causal histories. Both of these require using the standard quantum formalism, which is not easily combined with causal structure. However, CQM offers a potentially simplifying perspective, since it expresses information flow in an abstract way.

Part II

In Part II we consider causal structure from the perspective of CQM.

Chapter 4. We consider causal structure and time-symmetry. In particular, we shall consider *no-signalling devices* and the spacetime in which they reside. The causal structure of a relativistic spacetime exhibits a type of time-symmetry: reversing the time-orientation preserves spacelike separation of any two points. The main result of this Chapter is that bipartite no-signalling boxes do not share this time-symmetry. This presents a conflict between causality in probabilistic processes and causal structure in a relativistic spacetime. We also show how so-called 'possibilistic' reasoning can be used to study this phenomenon, which amounts to using the support of the probability distribution in question. We then discuss how time-reversed probability distributions that are signalling seem to violate λ-independence, at least in the classical case. This Chapter is based on the author's work in [38] and [28].

Chapter 5. In this Chapter we pose a problem for CQM: compact structure seems to violate causal structure when we assign light cones to the agents involved in teleportation. We use this to derive the properties of a category for which causal structure is consistently encoded. We base our notion of causal structure in a category on information flow, and we show that this corresponds to topological connectedness in the graphical language of

CQM. We also show that terminality of the monoidal unit is required for causal consistency, and that the monoidal product should be a partially-defined operation when considering the evolution of states in a protocol. The main results of this chapter are therefore this new understanding of how causal structure corresponds to a quite different type of category to the categories typically studied in CQM. This Chapter is based on the author's work in [26] and [39].

Chapter 6. Chapter 5 yields the structure required for CQM to encode causal structure. In this Chapter we formally define this structure, which we call a *causal category*. This is a new type of category, and we explore the formal properties of causal categories. For example, we show how these formal properties correspond to various physical properties, such as different types of causal structure, e.g. a Galilean spacetime. We then discuss methods of constructing causal categories. We provide two such methods, each of which is based on a different way of thinking about causal structure in a monoidal category. For example, one of these methods correspond to indexing the objects in a category with spacetime data (i.e. location). We give examples of the use of each of these methods. Both methods are quite intricate, and this serves to justify the definition of a causal category, since it concisely axiomatises these methods. This Chapter is based on the author's work in [39].

Part III

In the final Part we consider reconstructions of quantum theory from the perspective of CQM.

Chapter 7. We focus on recent work on reconstructing quantum theory based on using a purification postulate. This work uses graphical methods that are very similar to the graphical calculus of CQM. We explain how categorical definitions capture the graphical methods of this reconstruction. Moreover, we show a categorical scheme encodes some of the axioms, and in particular we show how the treatment of mixed states in CQM is closely related to the purification postulate. This yields insight into how CQM is related to reconstructions of quantum theory. This Chapter is based on the author's work in [27].

Chapter 8. Chapter 7 establishes a formal connection between CQM and operational approaches to reconstructing quantum theory. This suggests that it will be fruitful to develop a philosophy of CQM. To do so we consider an existing proposal. This concerns an apparent connection between category-theoretic formulations of physics and a type of realism that has been considered in the recent work on the philosophy of science. This type of realism is known as *structural realism*, and it appears to be related to the way that category theory is used in CQM. However, we argue that the arguments in favour of such a connection fail. We also argue that CQM is not strongly related to operationalism. CQM would therefore seem to embody a much subtler philosophy than either realism or operationalism.

Assumed knowledge. The intended audience of this thesis is a researcher in quantum foundations. Hence we shall assume knowledge of the standard quantum formalism and topics such as relativity and nonlocality. We shall introduce the required category theory. This is equivalent to Chapter 1 of [68], and amounts to the definitions of category, functor and natural transformation.

Chapter 2

Categorical quantum mechanics

Contents

Categorical quantum mechanics (CQM) was initiated by Abramsky and Coecke in [4]. The main body of the approach has been developed in several directions, often using tools from theoretical computer science, in particular the semantics of programming languages [97]. A sample of these further developments is Refs. [30, 34, 40], and an overview is given in [5]. The main idea of CQM is a conservative one, in the sense that its focus is on discarding the 'surplus structure' of Hilbert-space quantum theory. This can be summarised in the following steps:

1. Identify a particular physical phenomenon, such as a quantum information protocol;
2. Describe this phenomenon with conventional Hilbert-space quantum mechanics;
3. Find the 'compositional' rules of the Hilbert-space calculation when viewed in $\mathbf{fHilb}$, the category of finite-dimensional Hilbert spaces and linear maps.

The aim of this algorithm is to strip away 'unnecessary' structure when describing the original phenomenon, and allow it to be understood more intuitively, e.g. using the graphical calculus we describe below. This also leads to a more *general* formalism for studying the phenomena of quantum theory, in the sense that the rules describing such phenomena can be defined in categories other than $\mathbf{fHilb}$. These categories can also be considered to be *distinct* theories, or even 'foil' theories. By this approach, CQM has been applied in quantum foundations, such as the analysis of tripartite nonlocality in [36]. It has also been applied in quantum information, such as the classification of entangled states in [37].

2.1 Compact structure and teleportation

In this Section we shall introduce the formal structures for CQM that will be relevant in subsequent Chapters. Our aim is to define dagger compact categories and explain how they are used in the context of quantum theory. To reach this stage, we shall first introduce basic structures such as dagger symmetric monoidal categories, followed by a description of the graphical calculus associated with such categories.

2.1.1 Basic structures

To introduce the basic structures, we shall describe (i) monoidal categories; (ii) important examples; and (iii) dagger functors. We follow MacLane throughout [68].

To define the basic structures of CQM, we first recall some elements of category theory, in particular the definitions of category, functor and natural transformation. Informally, a category is collection of morphisms, i.e. maps, between mathematical objects such as vector spaces or just sets. Morphisms are denoted $f : A \to B$, where the objects A and B are the domain and codomain of the morphism f, denoted $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ respectively.

Definition 2.1. A *category* $\mathbf{C}$ consists of a class of objects, denoted $|\mathbf{C}|$, and a class of morphisms between each pair of objects A and B, denoted $\mathbf{C}(A, B)$ and called a *hom-set*, for which there exists a composition law:

$$\begin{aligned} - \circ - : \mathbf{C}(A,B) \times \mathbf{C}(B,C) &\longrightarrow \mathbf{C}(A,C) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

satisfying the following conditions:

- there exists an identity morphism 1_A for each object $A \in |\mathbf{C}|$, satisfying:

  $$1_A \circ f = f = f \circ 1_B$$

  for any morphism $f : A \to B$;
- composition is associative: for all objects A, B, C, D, and for all morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$ we have
  $$h \circ (g \circ f) = (h \circ g) \circ f.$$

Example 2.2. Consider a group $\mathcal{G}$. This is a category with one object, say A. Every element $g \in \mathcal{G}$ is a morphism $g : A \to A$ and the composition law $\circ$ of the category is given by the group multiplication. Every element $g_1 \in \mathcal{G}$ has an inverse, and so, viewing $\mathcal{G}$ as a category, every morphism $g_1 : A \to A$ is an *isomorphism*, meaning that g_1 has an inverse g_2 which satisfies $g_1 \circ g_2 = 1_A = g_2 \circ g_1$.

Example 2.3. We define the category $\mathbf{Set}$: objects $A \in |\mathbf{Set}|$ are sets, and a morphism $f : A \to B$ is a function with domain A and codomain B. The composition law $g \circ f$ for two functions f and g is defined by the usual function composition
$$(g \circ f)(x) := g(f(x)).$$

This composition is clearly associative, and there exists an identity function: hence $\mathbf{Set}$ is a category.

Definition 2.4. A *functor* $F : \mathbf{C} \longrightarrow \mathbf{D}$ between categories $\mathbf{C}$ and $\mathbf{D}$ is a function $A \mapsto F(A)$ from the objects $|\mathbf{C}|$ to the objects $|\mathbf{D}|$; and a function $f \mapsto F(f)$ from the morphisms of $\mathbf{C}$ to the morphisms of $\mathbf{D}$, satisfying, for all morphisms $f : A \to B$ and $g : B \to C$:

$$F(g \circ f) = F(g) \circ F(f)$$

and, for all objects A:

$$F(1_A) = 1_{F(A)}$$

Example 2.5. Consider two groups $\mathcal{G}$ and $\mathcal{H}$. Viewing each group as a category, a functor $F : \mathcal{G} \to \mathcal{H}$ is a group homomorphism, preserving the identity element and the group multiplication.

Given two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$, we define a *natural transformation* η as a family of morphisms $\eta_A : F(A) \to G(A)$ in $\mathbf{D}$ (but indexed by objects A in $\mathbf{C}$), such that for every morphism $f : A \to B$ in $\mathbf{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

If every morphism η_A is an isomorphism, then we call η a *natural isomorphism*.

The language of SMCs: initial definitions

The first layer of structure used in CQM is that of a *monoidal category*. A monoidal category is a category which has *two* types of composition: in addition to the usual composition in a category:

$$g \circ f : A \longrightarrow C$$

of morphisms $f : A \to B$ and $g : B \to C$, which is defined to exist only when 'types match' [i.e. when $\mathrm{cod}(f) = \mathrm{dom}(g)$], there is a *monoidal* composition

$$f \otimes g : A \otimes B \to B \otimes C.$$

As we shall show, the first kind of composition represents 'sequential composition', and the second kind represents 'parallel composition'.

Our formal exposition of these intuitive notions as follows.[1] Recall that a *product category* $\mathbf{C} \times \mathbf{D}$ is a category with objects given by pairs (A, B), where $A \in |\mathbf{C}|$ and $B \in |\mathbf{D}|$; and morphisms are given by pairs (f, g), where f is a morphism in $\mathbf{C}$ and g is a morphism in $\mathbf{D}$. The composition of morphisms (f_2, g_2) and (f_1, g_1) in $\mathbf{C} \times \mathbf{D}$ is given point-wise by

$$(f_2, g_2) \circ (f_1, g_1) := (f_2 \circ f_1, g_2 \circ g_1),$$

and the identity morphism is given by $(1_A, 1_B)$. We recall that a functor whose domain is a product category is called a *bifunctor.*

Definition 2.6. A *monoidal category* $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ is a category $\mathbf{C}$ with:

[1] When presented in applications to physics, monoidal categories are sometimes defined using the graphical calculus that we introduce later (e.g. as in [35]). However, for later chapters we shall need symbolic definitions. This will also clarify various technical details such as the formal status of the graphical calculus.

(i) a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$;

(ii) a *unit object* I; and

(iii) natural isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $\lambda_A : A \to I \otimes A$, $\rho_A : A \to A \otimes I$, all satisfying standard coherence conditions (e.g. see [68]), which we refer to as *structure morphisms*.

A *symmetric monoidal category* (SMC) is a monoidal category with a natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$, subject to further standard coherence conditions [68].

We shall elaborate on the suppressed coherence conditions satisfied by the structure morphisms shortly. First, note the following features of Definition 2.6:

- The way that the two types of composition interact is defined by specifying that $\otimes$ is a bifunctor, which means that the equations
$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$
and
$$1_A \otimes 1_B = 1_{A \otimes B},$$
are satisfied for all morphisms f, g, h, k and objects A, B. We refer to these equations as *bifunctoriality*.
- Monoidal composition is *unrestricted*, in the sense that the morphism $f \otimes g$ is always defined—unlike the composition $g \circ f$, which does not exist if $\mathrm{cod}(f) \neq \mathrm{dom}(g)$.

The latter point, together with the existence of a unit object, illustrates the use of the term 'monoidal', since it indicates that a monoidal category is a generalisation of a monoid. We might then expect that restricting $\otimes$ to the objects of a monoidal category $\mathbf{C}$ yields a monoid $(|\mathbf{C}|, \otimes, I)$. However, for most examples of a monoidal category this is not the case. As we shall see, if we were to require that a monoidal category restricts to a monoid on its objects, then many 'natural' mathematical examples would fail to form a monoidal category. Hence in order to provide enough generality, the structure morphisms α, λ, ρ in Definition 2.6 are only required to be isomorphisms rather than identities. The following example demonstrates this.

Example 2.7. Consider again the category $\mathbf{Set}$, whose objects and morphisms are sets and functions respectively, with composition given by function composition: $(g \circ f)(x) := g(f(x))$. The category $\mathbf{Set}$ is an SMC with $\otimes$ given by the cartesian product, and the unit object I given by the singleton set $\{*\}$ (or any other singleton—the unit object I is unique up to isomorphism).[2]. Hence for two functions $f : A \to C$ and $g : B \to D$, the monoidal product $f \otimes g$ is just the cartesian product of functions:

$$\begin{aligned} f \times g : A \times B &\longrightarrow C \times D \\ (a, b) &\longmapsto (f(a), g(b)) \end{aligned}$$

and the swap is the transposition function:

$$\begin{aligned} \sigma_{A,B} : A \times B &\longrightarrow B \times A \\ (a, b) &\longmapsto (b, a) \end{aligned}$$

But note that $(|\mathbf{Set}|, \times, I)$ is not a monoid: associativity fails because $(A \times B) \times C \neq A \times (B \times C)$, since

[2] As we shall see, many examples of a monoidal category can carry another monoidal product. The category $\mathbf{Set}$ is such as example, since it is also an SMC with $\otimes$ given by disjoint union, and the unit object given by the empty set $\emptyset$.

$((a, b), c) \neq (a, (b, c))$. Instead there is a 'rebracketing' isomorphism:

$$\begin{aligned}\alpha_{A,B,C} : (A \times B) \times C &\longrightarrow A \times (B \times C)\\ ((a, b), c) &\longmapsto (a, (b, c)).\end{aligned}$$

Similarly although $A \times \{*\} \neq A$, we have the isomorphism

$$\begin{aligned}\rho : A \times I &\longrightarrow A\\ (a, *) &\longmapsto a.\end{aligned}$$

Hence we see from Example 2.7 that the structure morphisms in condition (iii) of Definition 2.6 are weaker versions of the associativity and identity equations of a monoid, and are determined by the choice of $\otimes$. We shall now give an overview of two important theorems for monoidal categories relating to the structure morphisms: (i) the coherence theorem and (ii) the strictification theorem.

Now, let us focus on the role of the structure morphisms in more detail. A category usually contains a certain class of mathematical objects defined with respect to some underlying set-theoretic structure, e.g. the objects of the category of vector spaces $\mathbf{Vec}_{\mathbb{K}}$ over a field $\mathbb{K}$ are defined to have linear structure. An isomorphism in a category preserves these properties: in $\mathbf{Vec}_{\mathbb{K}}$, a categorical isomorphism is a $\mathbb{K}$-linear isomorphism. Hence, isomorphic objects 'behave the same' from the point of view of the underlying *set-theoretic* structure, is isomorphic vector spaces have the same vector-space properties, e.g. dimension. But isomorphic objects also have the same *categorical* properties. For example, categorical limits such as the categorical coproduct $A + B$ exist only up to isomorphism: an object C is isomorphic to $A + B$ if and only if C also a coproduct of A and B. The fact that categorical constructions are preserved under isomorphism is sometimes called the 'Principle of Isomorphism', a notion introduced by Makkai [74][3]. Hence, because the structure morphisms are isomorphisms, their use does not alter the results of any categorical computations we might do. For example, if we compute that $(A \otimes B) \otimes C$ satisfies a property such as being a limit of a certain type, then the *same is true* of $A \otimes (B \otimes C)$. So, in terms of categoric-theoretic computations, the parentheses in $(A \otimes B) \otimes C$ or $A \otimes (B \otimes C)$ can be ignored.

Moreover, since $\otimes$ is a functor, the question arises for morphisms as well, viz. whether the parentheses in $(f \otimes g) \otimes h$ and $f \otimes (g \otimes h)$ can also be ignored. Now, in standard category theory (as opposed to the theory of n-categories[4]) there is no notion of 'isomorphic morphisms'. Instead, we can consider the functors

$$(- \otimes -) \otimes - : (\mathbf{C} \times \mathbf{C}) \times \mathbf{C} \longrightarrow \mathbf{C}$$

and

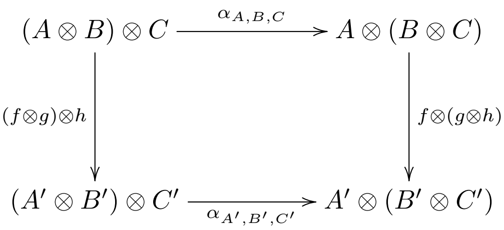
$$- \otimes (- \otimes -) : \mathbf{C} \times (\mathbf{C} \times \mathbf{C}) \longrightarrow \mathbf{C}$$

that are induced by the functor $\otimes$, and which yield the morphisms $(f \otimes g) \otimes h$ and $f \otimes (g \otimes h)$. The structure morphisms of a monoidal category $\mathbf{C}$ are *natural isomorphisms*, and in particular α is a natural isomorphism between the functors $(- \otimes -) \otimes -$ and $- \otimes (- \otimes -)$, which is the closest identification of a pair of functors that we can require if they are not identical. This means that for all objects A, B, C, A', B', C' and for all morphisms

[3] We have informally stated that this holds more generally, but a more formal statement can be made [74]. This proceeds by defining categorical constructions in an appropriate formal language, i.e. supplying axioms for a category and the construction at hand. Using this formal language, it then becomes a model-theoretic theorem that *any* categorical construction is invariant under isomorphism, see e.g. the invariance theorem of [74].

[4] For example, in a 2-category there are so-called *2-morphisms*, denoted $\alpha : f \Rightarrow g$, which are 'morphisms between morphisms'. So, in a 2-category, a pair of morphisms $f, g : A \to B$ are isomorphic if there exists a a 2-morphsm $\alpha : f \Rightarrow g$ which has a two-sided inverse.

f, g, h in $\mathbf{C}$, the following diagram commutes:



This implies that

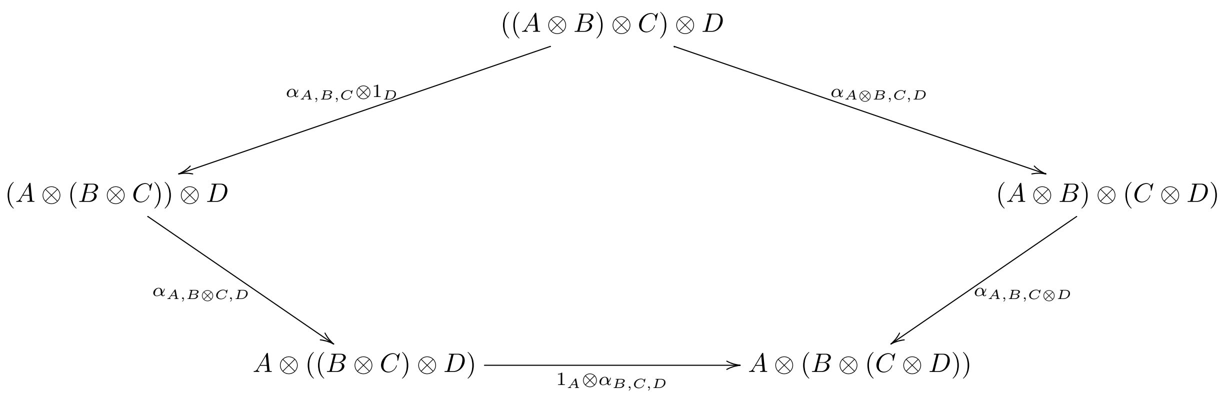
$$\alpha_{A',B',C'} \circ ((f \otimes g) \otimes h) \circ \alpha^{-1}_{A,B,C} = f \otimes (g \otimes h) \tag{2.1}$$

and so $(f \otimes g) \otimes h$ and $f \otimes (g \otimes h)$ are identical once the domain and codomain of $(f \otimes g) \otimes h$ have been rebracketed using α. As with any natural isomorphism, naturality of α ensures that Eq. 2.1 extends to all triples of morphisms in the category in a consistent way.

Hence it seems that we can elide parentheses when doing monoidal computations. However, so far we have only considered bracketing involving triples of objects or morphisms. Let us call a particular ‘parenthesisation’ of a monoidal product of objects a *bracketing*, so that $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are two different bracketings of the same triple of objects. Then, for triples there are only two bracketings, but for n-tuples of objects with $n > 3$, there are more than just two bracketings. For example, for $n = 4$ we have the bracketing $((A \otimes B) \otimes C) \otimes D$, but also four other bracketings. We can use the structure morphisms to build a morphism between any pair of these different bracketings, for example:

$$\alpha_{A,B,C} \otimes 1_D : ((A \otimes B) \otimes C) \otimes D \longrightarrow (A \otimes (B \otimes C)) \otimes D.$$

However, given one of the five different bracketings, there is not always a *unique* way of using the structure morphisms to rebracket it to another of the five different bracketings. For example, consider the *pentagon diagram*:



The pentagon diagram shows two routes from $((A \otimes B) \otimes C) \otimes D$ to $A \otimes (B \otimes (C \otimes D))$. In fact, one of the coherence conditions that we surpressed in Definition 2.6 is exactly that the condition that the pentagon diagram commutes, i.e. that the two routes are equal. Moreover, Mac Lane showed that the coherence conditions (which include another commutative diagram in addition to the pentagon diagram), ensure that this property extends to all n-tuples, i.e. not only $n = 3$ and $n = 4$, but any finite n.

Theorem 2.8. *Consider a monoidal category* $\mathbf{C}$ *and objects* $A_1, A_2, \ldots, A_n \in |\mathbf{C}|$. *Let* X *and* Y *be any two bracketings of* $A_1, A_2, \ldots, A_n$. *Also, let* $f, g : X \to Y$ *be any two isomorphisms obtained by composing the* *structure morphisms, and by a monoidal product with identity morphisms. Then $f = g$.*

Hence, not only are the different bracketings isomorphic, but the axioms of a monoidal category ensure that a stronger statement holds: when using the structure morphisms, there is always a *unique* isomorphism between any two bracketings.

Conceptually, both the preceding discussion and the coherence theorem show that the structure morphisms are essentially just *syntactical.* That is, the associator morphism α just implements a change of syntax, e.g. from $(A\otimes B)\otimes C$ to $A\otimes(B\otimes C)$, and so there is no *mathematically* significant difference between different bracketings.

This is given formal justification as follows. First by a *strict* monoidal category we mean a monoidal category in which the structure morphisms are identities, meaning that the natural isomorphisms α, λ and ρ are identity morphisms. A *monoidal equivalence* is a monoidal functor and an equivalence.[5] Then we have the following theorem [68].

Theorem 2.9 (Mac Lane)**.** *Every monoidal category $\mathbf{C}$ is monoidally equivalent to a strict monoidal category $\mathbf{D}$: there exist strong monoidal functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ providing an equivalence between $\mathbf{C}$ and $\mathbf{D}$.*

The significance of Theorem 2.9 is that it makes it easier to define a construction for an arbitrary monoidal category $\mathbf{C}$. We can define a construction for strict monoidal categories, and even if $\mathbf{C}$ is not strict, we know that the construction applies to the strict monoidal category that is monoidally equivalent to $\mathbf{C}$. We shall see an example of this in Chapter 6. Additionally, if we derive a property of strict monoidal categories, then it holds for all monoidal categories (since all categorical properties are preserved by equivalence). For example, consider the property that any strict monoidal category is equivalent to its graphical calculus: by Theorem 2.9, this also holds for non-strict monoidal categories.

Remark 2.10. As with the other structure morphisms, the symmetry morphisms in Example 2.7 cannot be identity morphisms, since $A \times B \neq B \times A$. Theorem 2.9 hence raises the question of whether we can also 'strictify' the symmetry morphism $\sigma_{A,B}$. By analogy with the other structure morphisms, this would correspond to providing an equivalence between any given symmetric monoidal category $\mathbf{C}$ and a *strictly-symmetric* monoidal category $\mathbf{D}$, for which

$$\sigma_{A,B} = 1_{A\otimes B} \tag{2.2}$$

for all pairs of objects (A, B). Moreover, the equivalence $F : \mathbf{C} \to \mathbf{D}$ that we are asking for would have two extra conditions to the equivalence of Theorem 2.9: the equivalence in firstly the strict symmetry of $\mathbf{D}$, and secondly that F is a *symmetric monoidal functor*. The latter condition is motivated by analogy with the notion of strictification in Theorem 2.9 which required that the equivalence preserves the structure morphisms α, ρ, λ that are to be strictified. Hence the equivalence that we are seeking should also preserve σ, since we this is the additional morphism that we are seeking to strictify. However, the strictification of σ is not possible: in Appendix A we show that such an equivalence does not exist, a question which has apparently not been addressed in the literature.

As pointed out in [29], a more intuitive way to understand the problem with strict symmetry is by considering an informal way that this would affect the physical interpretation of SMCs. Consider morphisms $f : A \to C$ and $g : B \to D$ in a strict-symmetric monoidal category $\mathbf{C}$. Then because $\mathbf{C}$ is strict-symmetric, Eq. 2.2 is satisfied

[5] Recall that categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent* if there exists functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\epsilon : 1_{\mathbf{C}} \to G \circ F$ and $\eta : F \circ G \to 1_{\mathbf{D}}$. For monoidal categories $(\mathbf{C}, \otimes, 1)$ and $(\mathbf{D}, \bullet, 1)$, a *strong monoidal functor* is a functor $F : \mathbf{C} \to \mathbf{D}$, such that there is a natural isomorphism $\phi : F(A \otimes B) \to FA \bullet FB$, in addition to certain conditions for the preservation of structure morphisms, which can be found in [68].

for all $A, B \in |\mathbf{C}|$. Hence we have $A \otimes B = B \otimes A$ and $C \otimes D = D \otimes C$, and so:

$$f \otimes g = g \otimes f : A \otimes B \to C \otimes D.$$

Now, later we will want to interpret SMCs spatiotemporally. Hence let us consider $A \otimes B$ and $C \otimes D$ to be 'spacelike hypersurfaces' at times t_1 and t_2 respectively, we can think of A and C as being at position x_1 and C and D as being at position x_2. Then f and g should be interpreted as *evolution* between the slices. But now we see that, with this spatiotemporal interpretation, letting $f \otimes g = g \otimes f$ identifies f and g at either of the positions x_1 or x_2, as depicted in Figure 2.1. Hence each point x could only have one type of evolution, which is not useful for formulating a physical theory. In fact, despite this intuitive appeal, in Appendix A we show that this informal argument *does not* hold formally in a strict-symmetric SMC. However, this intuitive argument *can* be formalised in the structure we develop in Chapters 5 and 6, the aim of which is to use SMCs in a precise spatiotemporal way.

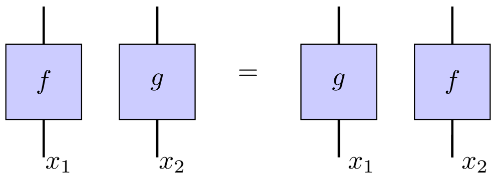


Figure 2.1: Spatiotemporal interpretation of strict symmetry isormorphism.

Example 2.11. Note that, despite the lack of strictification described in Remark 2.10, there exist categories for which the symmetry isomorphism is the identity. Consider the category $\mathbf{Nat}$: this has as objects the natural numbers $n \in \mathbb{N}$, and hom-sets are defined using the order $\leq$ on $\mathbb{N}$, i.e. a morphism $f : n_1 \to n_2$ exists when $n_1 \leq n_2$. The monoidal product is given by multiplication and so strictness of σ corresponds to commutativity, i.e. $n_1 \otimes n_2 = n_1.n_2 = n_2.n_1 = n_2 \otimes n_1$.

Important examples

What we have described so far defines the language and some technical aspects of SMCs. We shall now begin to expose the physical reasoning that they can be used for. We first need to introduce the motivating example for CQM.

Example 2.12. The category $\mathbf{fHilb}$ is defined to have complex finite-dimensional Hilbert spaces as objects and linear maps as morphisms, and composition is given by functional composition. This is an SMC $(\mathbf{fHilb}, \otimes, \mathbb{C})$ with the tensor product $\otimes$ of Hilbert spaces as the monoidal product, and $\mathbb{C}$ as the monoidal unit. However, $\mathbf{fHilb}$ is also a monoidal category with a *different* choice of monoidal product: if we choose the direct sum $\oplus$ then we obtain a monoidal category $(\mathbf{fHilb}, \oplus, 0)$ using the trivial Hilbert space 0 as the monoidal unit. We shall comment on the difference soon, although we emphasise that, unless otherwise indicated, $\mathbf{fHilb}$ refers to the monoidal category $(\mathbf{fHilb}, \otimes, \mathbb{C})$.

Even before introducing further structure, the language of SMCs is useful in CQM for generating examples of mathematical structures which are formally similar to $\mathbf{fHilb}$. An initial example of this is as follows. Consider again the category $\mathbf{Set}$. As a category, it would seem that the elements $a \in A$ for any object A are not represented categorically in $\mathbf{Set}$, since a category only defines the 'external' structure of objects A, i.e. the morphisms to other objects. However, elements $a \in A$ are in bijective correspondence with morphisms

$$\begin{aligned} \psi_a : \{*\} &\longrightarrow A \\ * &\longmapsto a. \end{aligned}$$

The morphisms ψ_a fully represent the elements of A, since

$$b = f(a) \iff \psi_b = f \circ \psi_a$$

where, for some function $f : A \to B$, we write the set-theoretic calculation on the left of the biconditional, and the category-theoretic one on the right[6]. Hence we can 'externalise' the internal structure of the objects in $\mathbf{Set}$ in this way. The pattern of this example is a crucial method for building the formalism of CQM, since it translates a set-theoretic notion into a category-theoretic one. Indeed, we can repeat this for **fHilb**: the morphisms $\mathbb{C} \to \mathcal{H}$ are in bijective correspondence with elements $|\psi\rangle \in \mathcal{H}$, since the morphisms in **fHilb** are linear, and by linearity ψ is determined by the value $\psi(1)$:

$$\begin{aligned} \psi : \mathbb{C} &\longrightarrow \mathbb{C} \\ 1 &\longmapsto |\psi\rangle. \end{aligned}$$

Hence a notion of state can be defined in exactly the same way for both $\mathbf{Set}$ and **fHilb**, and also to further examples, such as the following.

Example 2.13. The category $\mathbf{Rel}$ has sets as objects, relations as morphisms, and composition of morphisms $R : A \to B$ and $S : B \to C$ given by

$$S \circ R := \{(a, c) \,|\, a \in A, c \in C, \exists b \in B : (a, b) \in R \wedge (b, c) \in S\}.$$

That is, composition in $\mathbf{Rel}$ is given by pairs of elements (a, c) for which there is an 'intermediate' element b in the set B. The category $\mathbf{Rel}$ is an SMC $(\mathbf{Rel}, \times, \{*\})$ with $\otimes$ given by the cartesian product, and as for $\mathbf{Set}$ it also has the singleton as the unit object. The states in $\mathbf{Rel}$ are morphisms $p : \{*\} \to A$, which are just subsets of A. However, just as for Example 2.12, the category $\mathbf{Rel}$ carries another monoidal product: using the disjoint union $+$ and the empty set $\emptyset$ we obtain the monoidal category $(\mathbf{Rel}, +, \emptyset)$.

We can now also identify some physical consequences when considering these categories as 'toy models'.

Remark 2.14. Example 2.7 can actually be thought of as a 'classical' (as opposed to a quantum) monoidal category, in the sense that the monoidal product $\otimes$ for $\mathbf{Set}$ is given by the categorical product $\times$: when this occurs the category is called *cartesian*. Since the categorical product in $\mathbf{Set}$ is the usual cartesian product, we can think of cartesian categories as being $\mathbf{Set}$-like. Recall that the categorical product of A and B is defined as the triple

[6] We describe the concrete computation using elements of sets as 'set-theoretic' but this is imprecise, since category theory in its usual formalisation is also ultimately set-theoretic. Hence 'set-theoretic' in this context should be read as 'set-based'.

$(A \times B, \pi_1 : A \times B \to A, \pi_2 : A \times B \to B)$, satisfying the universal property:

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{f_1} & \Big\downarrow{\scriptstyle \langle f_1, f_2 \rangle} & \searrow^{f_2} & \\ A & \xleftarrow[\pi_1]{} & A \times B & \xrightarrow[\pi_2]{} & B \end{array} \tag{2.3}$$

The classicality of a cartesian category can be seen in various ways. One way is that cartesian categories contain morphisms which provide uniform cloning operation. This arises from the fact that the morphism $\langle 1_A, 1_A \rangle$ (defined using Eq. 2.3) induces a natural transformation, which we call a *natural diagonal*:

$$\Delta : A \longrightarrow A \otimes A$$

Naturality of Δ means that:

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & A \otimes A \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f \otimes f} \\ B & \xrightarrow[\Delta_B]{} & B \otimes B \end{array} \tag{2.4}$$

But this equation corresponds exactly to the existence of a uniform cloning operation. For example, consider the morphism Δ in $\mathbf{Set}$. Let $A = \{*\}$ and $f = \psi$ be an arbitrary state of B. Then Eq. 2.4 becomes:

$$\begin{array}{ccc} I & \xrightarrow{\Delta_I} & I \otimes I \\ {\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle p \otimes p} \\ B & \xrightarrow[\Delta_B]{} & B \otimes B \end{array}$$

which implies that there is an operation on the 'state space' B such that, for all states $p : I \to B$ we have (up to isomorphism):

$$p \otimes p = \Delta_B \circ p. \tag{2.5}$$

Hence if cartesian categories are interpreted as categories of pure states, then they must be classical in the sense that they allow uniform cloning.

The following example is not classical in this sense.

Example 2.15. Suppose the natural diagonal existed for the category **fHilb**: setting $A = \mathcal{H}$, and with a chosen basis $\{|i\rangle\}_i$, we would expect Δ to be

$$\begin{aligned} \Delta_{\mathcal{H}} : \mathcal{H} &\longrightarrow \mathcal{H} \otimes \mathcal{H} \\ |i\rangle &\longmapsto |i\rangle|i\rangle \end{aligned}$$

Then, for a quantum state $|\psi\rangle = |0\rangle + |1\rangle$, the naturality square Eq. 2.4 reproduces the original Wootters-Zurek no-cloning proof [108], since we obtain the analogous equation to Eq. 2.5:

$$|\psi\rangle \otimes |\psi\rangle = \Delta_{\mathcal{H}} \circ |\psi\rangle. \tag{2.6}$$

But the right-hand side is the Bell state $|00\rangle + |11\rangle$, and so Eq. 2.6 is not satisifed for all $|\psi\rangle$. Hence Eq. 2.4 fails for $\mathbf{fHilb}$. As shown by Abramsky in [2], similar problems occur for other choices of Δ, and so this category *has no natural diagonal*, which corresponds to the *no-cloning theorem.* Now, consider the category $\mathbf{Vec}_{\mathbb{K}}$, whose objects are vector spaces over a field $\mathbb{K}$ and whose morphisms are linear maps. This is an SMC, with the monoidal product is given by the usual tensor product, and the unit object is given by the one-dimensional vector space $\mathbb{K}$. The fact that the usual no-cloning theorem is just a consequence of linearity is represented here by the fact that $\mathbf{Vec}_{\mathbb{K}}$ also has no natural diagonal. In fact, the generality of using SMCs allows us to translate the same result to further mathematical models, in which no-cloning holds even though the models do not use vector spaces.

Example 2.16. Exactly the same reasoning as above shows that no natural diagonal exists for $(\mathbf{Rel}, \times, \{*\})$, where $\times$ is the cartesian product, and hence this gives some indication of its quantum-like structure. Indeed, we can expose two interesting features of SMCs at this point. Firstly, we mentioned in Example 2.13 that $\mathbf{Rel}$ carries two monoidal structures: $(\mathbf{Rel}, \times, \{*\})$ and $(\mathbf{Rel}, +, 0)$, where $+$ is the disjoint union. Now, the lack of diagonal for $(\mathbf{Rel}, \times, \{*\})$ does *not* carry over for $(\mathbf{Rel}, +, 0)$: the latter is a cartesian category. Hence this illustrates the importance of the *choice* of monoidal product for a given category, since although the objects and morphisms in each category are the same, $(\mathbf{Rel}, \oplus, 0)$ is classical in a way that $(\mathbf{Rel}, \otimes, \{*\})$ is not, i.e. the former allows uniform cloning. Secondly, we can now also see how CQM is useful for generating new models in which to understand physical phenomena, and moreover models which are quantum-like. The reason for this is that, as we mentioned in Example 2.12, the category $\mathbf{fHilb}$ also carries two possible monoidal products, in addition to the usual tensor product which gives $(\mathbf{fHilb}, \otimes, \mathbb{C})$, the direct sum yields a monoidal category $(\mathbf{fHilb}, \oplus, 0)$. Just as with $(\mathbf{Rel}, +, 0)$, $(\mathbf{fHilb}, \oplus, 0)$ carries a natural diagonal, i.e. it allows uniform cloning. Hence we see a strong formal analogy between $\mathbf{fHilb}$ and $\mathbf{Rel}$: the direct sum and the disjoint union make each category respectively 'classical', and the tensor product and the cartesian product makes each category respectively 'quantum'.

Let us now consider the notion of classicality of a cartesian category in more detail. In fact, cartesian categories seem to have *two* notions of classicality: the first is the property of uniform cloning as just described. (We note that, in addition to cloning given by the natural diagonal Δ, the projections π_i in Eq. 2.3 can be seen as uniform deleting operations [8].) The second notion of classicality is as follows. Eq. 2.3 says, loosely speaking, that given morphisms f and g, there is a unique composite morphism $\langle f, g\rangle$ with 'marginals' f and g. Indeed, this can be stated more precisely, since if we write $f = \langle f_1, f_2\rangle$, then it is straightforward to show that

$$f = \langle \pi_1 \circ f, \pi_2 \circ f\rangle \tag{2.7}$$

This encodes the idea that a product morphism f is *determined by its parts*, i.e. the projections $\pi_1 \circ f$ and $\pi_2 \circ f$ (in fact the categorical product can also be defined using this property, e.g. see [7]). So a cartesian category clearly captures only *product states*, and for a theory to express quantum features such as entanglement we require a different notion, e.g. the tensor product of Hilbert spaces.

Now, Remark 2.14 can be generalised to show that all cartesian categories allow uniform cloning (as in Eq. 2.7). On the other hand, the preceding discussion shows the existence of projections in cartesian categories leads to a classical-like product (as in Eq. 2.7). The question then arises: do either or both of these notions suffice to characterise cartesian categories? This is answered by the following result.

Proposition 2.17. *Let* $\mathbf{C}$ *be a monoidal category. The monoidal product* $\otimes$ *of* $\mathbf{C}$ *is the categorical product if and only if* $\mathbf{C}$ *carries a natural diagonal and natural projections.*

The significance of this is that the physical properties (cloning and projections) 'track' the categorical structure (the categorical product).

Remark 2.18. Our discussion of no-cloning illustrates some interesting features of CQM. Firstly, it shows that—in the categorical setting—classicality is a *special case* of monoidal structure, since not all monoidal categories are cartesian. That is, the monoidal product generalises the categorical product, since categorical products can be thought of as monoidal products with extra structure, viz. they have diagonals and projections. This extra structures ensures classicality through Proposition 2.17 above. Hence if we were to approach building a formalism for quantum theory (or any physical theory) starting from category theory instead of the existing formalism of classical physics, we might find it natural that systems are not combined using the cartesian product. Secondly, the categorical framework allows us to consider the two aspects of classicality discussed above, viz. cartesian product and cloning of pure states, as logically independent notions. That is, *a priori* they do not coincide, and the formalism of CQM allows for this possibility. For example, there exist categories in which diagonals exist but not projections, and so a category may have uniform cloning but still not be cartesian. This is useful because it clarifies the logical relationships between fundamental notions. These abstract relationships can now be applied to toy models: e.g. toy models in categories that have both cloning but also entangled states.

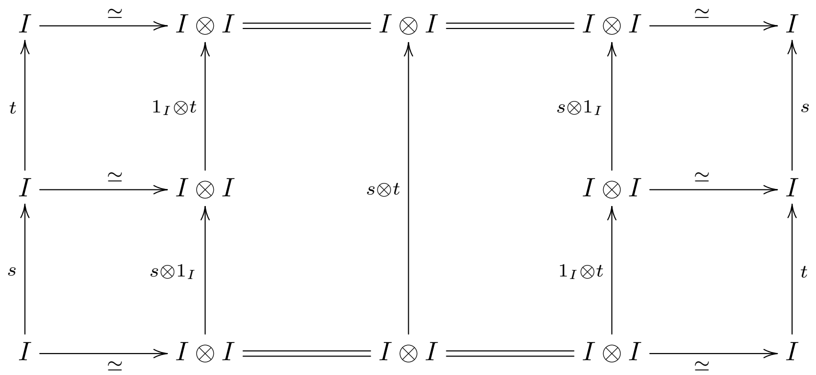
With some examples now at hand, we can now exhibit the role of the monoidal unit in SMCs more fully. Consider again Example 2.15, and the endomorphisms $s : \mathbb{K} \to \mathbb{K}$. As with the case previously discussed of states in $\mathbf{fHilb}$, since all morphisms in $\mathbf{Vec}_{\mathbb{K}}$ are linear by definition, the endomorphisms on $\mathbb{K}$ are in bijective correspondence with elements $\sigma \in \mathbb{K}$, since they are determined by $s(1)$:

$$\begin{aligned} s : \mathbb{K} &\longrightarrow \mathbb{K} \\ 1 &\longmapsto \sigma \end{aligned}$$

Moreover, the endomorphisms are in a multiplicative isomorphism with $\mathbb{K}$, since $(s \circ t)(1) = s(1) \times t(1)$, where '$\times$' denotes the multiplication in $\mathbb{K}$. Therefore the endomorphisms of the monoidal unit capture much of the structure of elements of the unit object, and accordingly we call them *scalars*. This definition of scalars reflects a general pattern in category theory of 'externalising' properties of mathematical objects that are usually defined and calculated using the internal set-theoretic structure. The scalars in a category are subject to the following result by Kelly and Laplaza [64].

Proposition 2.19. *For any monoidal category* $\mathbf{C}$*, the scalar monoid* $(\mathbf{C}(I, I), \circ, 1_I)$ *is commutative.*

Proof. Consider arbitrary $s, t \in \mathbf{C}(I, I)$. Then commutativity of $\mathbf{C}(I, I)$ holds if the following diagram commutes:



The inner two rectangles commute via bifunctoriality, so that for the right inner rectangle we have

$$s \otimes t = (s \circ 1_I) \otimes (1_I \circ t) = (s \otimes 1_I) \circ (1_I \otimes t)$$

and similarly for the left inner rectangle. The four outer squares commute by naturality of the unit isomorphisms $\lambda_I : I \to I \otimes I$. Therefore the the entire diagram commutes. □

Example 2.20. As an example of a scalar monoid consider the category $\mathbf{Set}$. The unit object is the singleton set $\{*\}$, hence the scalar monoid is trivial since there is only the identity function $1_{\{*\}}$ on $\{*\}$. However, consider the category $\mathbf{Rel}$. The unit object for $\mathbf{Rel}$ is also $\{*\}$ but there are *two* relations on $\{*\}$: the identity and empty relations, and hence there are two scalars in $\mathbf{Rel}(I, I)$. The set of scalars $\mathbf{Rel}(I, I)$ is actually the Boolean semiring $\mathbb{B} = (\wedge, \vee, 0, 1)$. For example, let s_0 and s_1 be the scalars corresponding to the empty relation and the identity relation respectively. Then using the definition of composition $\mathbf{Rel}$, we have $s_1 \circ s_0 = s_0 \circ s_1 = s_0 \circ s_0 = s_0$, and also $s_1 \circ s_1 = s_1$. This proves the multiplicative isomorphism $\phi : \mathbf{Rel}(I, I) \to \mathbb{K}$ that satisfies $\phi(s_0) = 0$ and $\phi(s_1) = 1$, and the remaining semiring properties of $\mathbf{Rel}(I, I)$ are shown similarly.

Remark 2.21. It is important to note that the scalars in a monoidal category provide an action on the *whole category*, since it provides scalar multiplication on any morphism $f : A \to B$. This is defined using using the monoidal product $\otimes$ and the structure morphisms:

$$s \bullet f := \; A \xrightarrow{\lambda_A} I \otimes A \xrightarrow{s \otimes f} I \otimes B \xrightarrow{\lambda_B^{-1}} B$$

This means in particular that hom-sets are *enriched* with the monoid multiplication provided by the scalars $s : I \to I$, meaning that the hom-sets carry extra structure. For example, the hom-sets in **fHilb** are closed under multiplication by elements of $\mathbb{C}$, i.e. for two Hilbert spaces $\mathcal{H}, \mathcal{K}$ and a linear maps $L : \mathcal{H} \to K$, we can define multiplication by a complex scalar s that yields another linear map $s \bullet L : \mathcal{H} \to \mathcal{K}$.

Dagger structure

So far we have only explored SMCs without any further conditions. We shall now introduce a crucial piece of extra structure.

Definition 2.22 (†-SMC)**.** A *dagger category* is a category $\mathbf{C}$ with an involutive, identity-on-objects, contravariant endofunctor $\dagger : \mathbf{C} \to \mathbf{C}$ (the *dagger functor*). A *dagger symmetric monoidal category* (†-SMC) is an SMC which is also a dagger category, and for which the † functor satisfies the following conditions:

(i) for all $f, g \in \mathrm{Arr}(\mathbf{C})$, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$; and

(ii) the natural isomorphisms of the symmetric monoidal structure are unitary: for all $A, B, C \in |\mathbf{C}|$, $\alpha^\dagger_{A,B,C} = \alpha^{-1}_{A,B,C}$, $\lambda^\dagger_A = \lambda^{-1}_A$ and $\sigma^\dagger_{A,B} = \sigma^{-1}_{A,B}$.

Condition (i) in Definition 2.22 means that the dagger preserves the monoidal structure 'on-the-nose' (i.e. not just up to isomorphism). Condition (ii) means that the † provides the inverses for the structure morphisms.

Example 2.23. The category $\mathbf{fHilb}$ is a dagger category, with the dagger functor given by the linear-algebraic adjoint:

$$\begin{aligned} \dagger : \mathbf{fHilb}^{\mathrm{op}} &\longrightarrow \mathbf{fHilb} \\ \mathcal{H} &\longmapsto \mathcal{H} \\ (f : \mathcal{H} \to \mathcal{K}) &\longmapsto (f^H : \mathcal{K} \to \mathcal{H}) \end{aligned}$$

where f^H is the adjoint operator of f. From now on we denote this as $f^\dagger$; it will be clear from the context whether $f^\dagger$ refers to an abstract dagger of f, or its linear-algebraic adjoint.

Example 2.24. The category $\mathbf{Set}$ is not a †-SMC. Intuitively, we might not expect it to be, since if a function $f : A \to B$ is not injective, then by analogy with Example 2.23, we would assume $f^\dagger$ to be given by the 'multi-valued mapping' $f^\dagger : B \to A$. But this could only be a relation, and not a function; hence it is not in $\mathbf{Set}$. We can state this more precisely as follows. For any objects A, B in a dagger category $\mathbf{C}$, the † functor establishes a bijection between the hom-sets $\mathbf{C}(A, B)$ and $\mathbf{C}(B, A)$. But no such bijection exists in $\mathbf{Set}$: e.g. for any set A such that $A \neq I$ and $I \neq A$ we have

$$|\mathbf{Set}(A, I)| \neq |\mathbf{Set}(I, A)|,$$

because $|\mathbf{Set}(A, I)| = 1$ but $|\mathbf{Set}(I, A)| = |A|$.

Example 2.25. The category $\mathbf{Rel}$ is a †-SMC, with the † functor given by the relational converse: for a relation $R : A \to B$, the dagger $R^\dagger : B \to A$ is the relation

$$R^\dagger := \{(b, a) \mid (a, b) \in R\}$$

Another important example of a †-SMC is $\mathbf{fRel}$, the category of *finite* sets and relations, for which the monoidal product is again the cartesian product of sets, and the † is given by relational converse.

We have already defined states in an SMC: a *state* ψ of a system A as an arrow $\psi : I \to A$. We define effects similarly. An *effect* ϕ in an SMC is a morphism $\phi : A \to I$. Now, in a general †-SMC, the † functor establishes a bijection between states and effects. Then given two states $\psi, \phi : I \to A$ we can define their *inner product* as the composition $\phi^\dagger \circ \psi : I \to I$. Hence we obtain the inner product structure of the usual formalism for quantum theory through the dagger functor, and in $\mathbf{FdHilb}$ this recovers the usual seqsuilinear inner product:

$$\phi^\dagger \circ \psi = \langle \phi | \psi \rangle$$

Note also that a †-category has an involution on the scalars, given by $s \mapsto s^\dagger$. In $\mathbf{fHilb}$ this is complex conjugation. In this way, adding more structure to an SMC adds more structure to the scalars. For example, SMCs which are enriched over commutative monoids are often studied, where *enrichment* here means that each hom-set carries the structure of a commutative monoid. In this case, the scalars form a semi-ring: the multiplication and addition arising from composition and monoid enrichment respectively.

Remark 2.26. Much recent work in the foundations of quantum theory has focussed on generalised probabilistic theories [13]. These are theories which may be nonlocal but which have a different level of nonlocality from quantum theory, as measured by violation of a Bell-type inequality. For example, the maximal violation of the CHSH inequality, produced by the Popescu-Rohrlich box [89], can be formalised as a square state in space in $\mathbb{R}^2$. This is known as *boxworld*, and was introduced in [13]. This is to be contrasted with the 3-dimensional Bloch sphere of normalised states in $\mathbb{R}^3$. As shown in [98], using the maximal tensor product, boxworld does not have a bijection between states and effects for compound systems, and indeed has no entangled effects. Hence, although the maximal tensor product will yield an SMC for boxworld, it will *not* be a †-SMC since a dagger functor cannot be defined for it. This illustrates the way that CQM can classify theories using categorical structure.

To summarise briefly our exposition so far: we have shown that diagonals and projections correspond to copying and deleting maps, as in the examples of in $(\mathbf{Set}, \times, \{*\})$ and $(\mathbf{Rel}, +, \emptyset)$. Hence the categorical product will not provide a useful structure for quantum theory, given the no-cloning and no-deleting theorems. This dovetails with the fact that monoidal structure allows us to express non-classical correlations between systems, whereas the categorical product—which is an abstraction of the cartesian product used to describe compound *classical* systems—cannot capture this 'holism', since the state of a compound system is always a product of the states of each subsystem.

The examples we have discussed so far are:

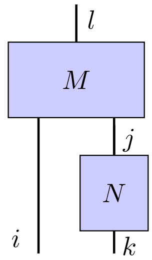
Category	Monoidal product	Unit	Cartesian?
$\mathbf{Set}$	$\times$	$\{*\}$	✓
$\mathbf{Rel}$	$\times$	$\{*\}$	
$\mathbf{Rel}$	$+$	$\{*\}$	✓
$\mathbf{fHilb}$	$\otimes$	$\mathbb{C}$	
$\mathbf{fHilb}$	$\oplus$	0	✓

2.1.2 Graphical calculus

Much of the power of CQM arises from the graphical calculus that exists for SMCs (or for weaker structures such as braided monoidal categories; a comprehensive survey of which can be found in [95]). The use of a purely graphical language to describe quantum systems is perhaps a deep feature of CQM, as evidenced by Theorem 2.29 below. But note that an identical graphical notation can also be inferred from a tensor calculus without any reference to category theory. This kind of reasoning originated with Penrose [86]. Interestingly, this led to the development of spin networks, a formalism which depicts the representations of a gauge group in particle physics using a graph. But initially spin networks had a spatiotemporal interpretation, a similar aim to ours in Chapter 6. For example, consider multi-linear maps $G : A \otimes B \to C$, $M : A \otimes B \to C$ and $N : B \to B$. Then consider the equation $G = M \circ (1_A \otimes N)$ in component form. With a basis chosen for each vector space, the components satisfy

$$G^l_{ik} = M^l_{ij} N^j_k,$$

where we use the Einstein summation convention. As Penrose noted, in more complicated equations the contraction becomes difficult to determine visually. Hence we can use a graphical notation:



There are two further steps that bring us essentially to the graphical calculus of SMCs. Firstly, it is desirable to reflect the basis-independence of this calculation notationally, and so instead of the indices i, j, k we can display the spaces A, B, C. Secondly, as Penrose observed, this graphical notation can also be used if the terms A, B, C *do not represent vector spaces*, but some other type of mathematical object [86]. The development of this idea awaited work in category-theory [62, 64], which we now describe.

To demonstrate the importance of the graphical language, it will be useful to formally distinguish between graphical and symbolic representations. Hence we shall first define more precisely the symbolic language.[7]

Definition 2.27 (Symbolic language)**.** By an *object formula* in the symbolic language of an SMC we mean any expression involving objects and the monoidal product of objects. By a *(well-formed) morphism formula* in the symbolic language of an SMC we mean any expression involving morphisms, sequential composition, and parallel composition thereof, for which sequential composition only occurs for morphisms with matching types.

Consider an object formula $A_1 \otimes A_2$ in a category **C**, with $A = A_1 \otimes A_2$. We shall be careful to distinguish between A and $A_1 \otimes A_2$, since there may exist objects B_1 and B_2 such that $A = B_1 \otimes B_2$. Hence, the object formula $A_1 \otimes A_2$ contains more information than its corresponding object in the category A does: namely, it shows how it was formed. The same applies to morphism formulae, which also contain more information than the corresponding morphism in the category. For example, $(h \otimes k) \circ (f \otimes g)$ and $(h \circ f) \otimes (k \circ g)$ are distinct morphism formulae, but are the same morphism in a monoidal category (by bifunctoriality). We shall notationally distinguish the object language and objects as follows:

- Object formulae will be denoted by calligraphic capital letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$
- Objects will be denoted by Roman-font capital letters $A, B, C, \ldots$

Similarly we shall distinguish between the morphism language and morphisms as follows:

- Morphism formulae will be denoted by calligraphic capital letters $\mathcal{F}, \mathcal{G}, \mathcal{H}, ...$
- Morphisms will be denoted by Roman font $f, g, h, ...$

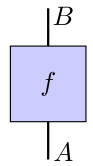
Each morphism formula $\mathcal{F}$ has an object formula as its input and output, which we specify by writing $\mathcal{F} : \mathcal{A} \to \mathcal{B}$. Now, we define the *corresponding object* A for an object formulae $\mathcal{A}$ to be the object in the category denoted by the formula $\mathcal{A}$. For example, when both $A = A_1 \otimes A_2$ and $A = B_1 \otimes B_2$ are true, then A is the corresponding object for both of the formulae $A_1 \otimes A_2$ and $B_1 \otimes B_2$. Similarly, we can associate to $\mathcal{F}$ a *corresponding morphism* $f : A \to B$, simply by evaluating the compositions expressed within the object formulae and morphism formula. We will sometimes use the notation $\mathcal{F} : A \to B$ to mean that in $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ we have the object A corresponding

[7] The symbolic language we define here is known as the *internal language* of a category. This involves using *type theory* to define the allowed syntactical constructs. In order to minimise the technical exposition, we have omitted the type-theoretic description.

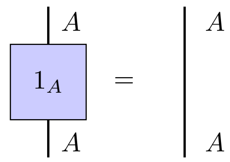
to the object formula $\mathcal{A}$, and similarly B corresponding to $\mathcal{B}$. An equation $\mathcal{F} = \mathcal{G}$ means that the corresponding morphisms are equal, i.e. $f = g$. The physical intuition behind this is that several 'physical scenarios' or 'experimental protocols', while distinct in their implementation details, may have exactly the same overall effect. This intuition motivates the following definition, which will be important for Chapter 6.

Definition 2.28 (Operational terminology)**.** A *protocol* is a morphism formula in the symbolic language of SMCs.

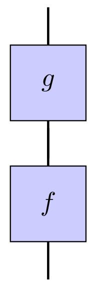
We shall now describe the graphical language: after doing so we will see that a protocol can be equivalently defined as a diagram in the graphical language. For each morphism formula $\mathcal{F}$ there is a corresponding *diagram* in the graphical language. We depict a morphism $f : A \to B$ by using a line to represent the objects and a box to represent the morphism (our convention is to draw diagrams directed from bottom to top):



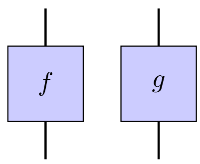
For an identity morphism $1_A : A \to A$, we shall omit both the box and the morphism label:



The usefulness of the graphical calculus will be in describing which boxes (i.e. morphisms) are connected to each other, and so object labels are unnecessary. Hence from now on we shall not display the names of objects in a diagram. To depict the composition of morphisms $f : A \to B$ and $g : B \to C$, we denote $g \circ f : A \to C$ by joining the output of the box representing f to the input of the box for g:



To depict the monoidal product of morphisms $f : A \to B$ and $g : C \to D$, we denote the morphism formula $f \otimes g : A \otimes C \to B \otimes D$ by placing the boxes representing f and g side-by-side:

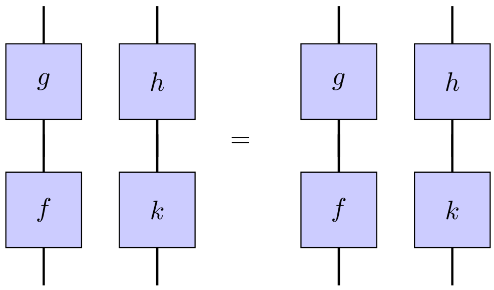


If we think of the morphisms of an SMC as representing physical processes, then it is natural to consider the vertical direction to be informally regarded as the time dimension.

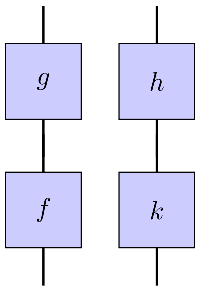
The graphical language has many useful features. Recall that bifunctoriality of the monoidal product yields the equation:

$$(g \otimes h) \circ (f \otimes k) = (g \circ f) \otimes (h \circ k).$$

When depicted graphically, this equation becomes a tautology:

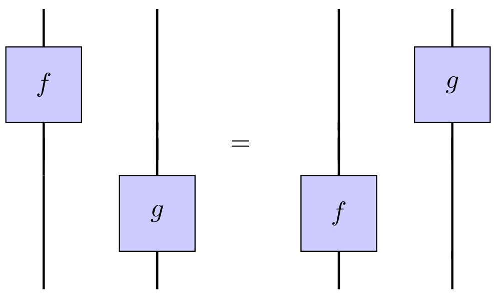


Why does this 'tautology' occur? Part of the reason is that the graphical language is a two-dimensional notation. The x-axis represents the monoidal product $\otimes$, and the y-axis represents composition $\circ$. This means that graphical expressions conveys more information than a symbolic expression that contains no parentheses. For example, the symbolic expression $f \otimes g \circ h \otimes k$ is ambiguous, but an analogous graphical expression, such as

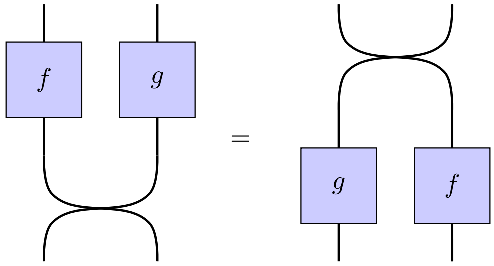


is not. The symbolic expression could correspond to either $(f \otimes g) \circ (h \otimes k)$ or $f \otimes (g \circ h) \otimes k$. The graphical expression could correspond only to $(g \otimes f) \circ (h \otimes k)$ [or to $(g \circ f) \otimes (h \circ k)$, but by bifunctoriality both morphism formulae denote the same morphism]. So the graphical language excludes some possible ambiguities: it encodes information about the morphisms involved, in particular it encodes the sequential composition of morphisms using $\circ$ using *topological connectedness*. In contrast, a morphism formula has to include more information than its corresponding diagram in the graphical language, viz. the bracketing.

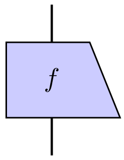
These sort of features make the graphical calculus an efficient notation for SMCs. For example, in Proposition 2.19, to show the commutativity of the scalars $\mathbf{C}(I, I)$, we used the fact that bifunctoriality implies $f \otimes g = (f \otimes 1_D) \circ (1_A \otimes g) = (1_B \otimes g) \circ (f \otimes 1_C)$. This is subsumed graphically, since it corresponds to the ability to 'slide' boxes along lines:



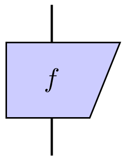
This shows how, from a physical perspective, the extra information in a morphism formula is in fact redundant. This example exhibits the general rule that guides the use of the graphical language: diagrams are defined up to graph isomorphism, and the graph-theoretic properties of diagrams preserve the algebraic structure of the corresponding SMC. This is also demonstrated by the diagram corresponding to the condition in Definition 2.6 that the symmetry morphism σ is an isomorphism:



For a †-SMC we must also reflect the dagger structure in the graphical notation. Since the dagger functor is the identity function on objects, to reflect its properties we need only account for its swapping of the domain and codomain of a morphism. Hence we use asymmetric boxes to represent the morphism $f : A \to B$:

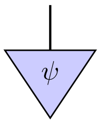


so that $f^\dagger : B \to A$ is depicted as

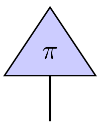


Note that the label for the box is not $f^\dagger$ but f, since the former label would make the asymmetry of the box redundant.

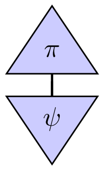
The graphical representation of a SMC has an intriguing interpretation as a kind of generalised Dirac notation. Instead of $|\psi\rangle$ we denote a state $\psi : I \to A$ as



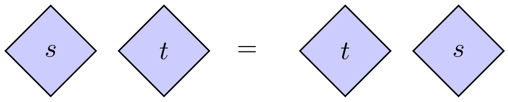
and instead of $\langle\pi|$ we denote an effect $\pi : A \to I$ as



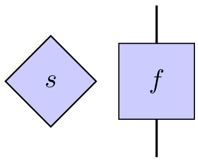
We denote the composition $\pi \circ \psi$ (corresponding to $\langle \pi \mid \psi \rangle$) as



and hence scalars are represented by graphical elements without inputs or outputs. This elegantly encapsulates the commutativity of scalars given by Proposition 2.19, since these elements can be moved freely in the plane:



Note also that the scalar action on arbitrary morphisms is also subsumed by the graphical calculus. This means that placing a closed diagram s next to any other diagram f will not change the input and output types of the latter diagram:



which represents $s \bullet f$ (as defined in Remark 2.21). Hence the graphical language is a 'rotated' Dirac notation, which, in order to account for the monoidal structure, is *two-dimensional*.

Now, we have seen from the description above that the graphical calculus is expressive enough to describe features of SMCs, and that it also subsumes symbolic equations such as bifunctoriality.

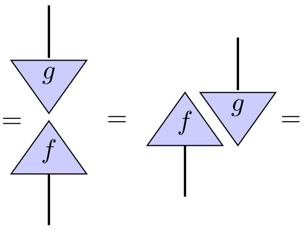
The power of the graphical calculus, as opposed to the symbolic language, is made clear by the following theorem due to Joyal and Street [62], which implicitly defines what we actually mean by 'graphical calculus'.[8]

Theorem 2.29. *A well-typed equation between morphisms in the language of symmetric monoidal †-categories is provable from the axioms of †-SMCs if and only if it holds, up to graph isomorphism, in the graphical language.*

Hence Theorem 2.29 justifies using the graphical language in lieu of symbolic calculations. Therefore we have two languages at our disposal: the symbolic language and the graphical language. From now on we shall make use of the efficiency of the graphical language by using it to make certain *symbolic* equations superfluous: we will treat morphism formulae up to equivalence in the diagrammatic representation. For example, consider the following ambiguity in our use of parallel vs. sequential composition as defined *symbolically*. While parallel composition *always* leads to topological disconnectedness in the graphical language, sequential composition may lead to either a connected or a disconnected diagram. In particular, when we compose over the monoidal unit I, the two modes

[8] In addition to Joyal and Street's original paper, the survey by Selinger [95] provides a clear exposition of the Joyal-Street theorem.

of composition coincide:

$g \circ f =$  $= \quad f \otimes g$

Our use of diagrammatic equivalence classes resolves this ambiguity, since we can always represent a composition over the monoidal unit, i.e. a disconnected diagram, by a symbolic formula that uses '$\otimes$' instead of '$\circ$'.

We shall also assume that all our morphism formulae contain only 'atomic' expressions—those which do not contain, in the corresponding graphical representation, topologically disconnnected components. To define this symbolically, we first define a *generalised symmetry morphism* to be a morphism that is either the identity morphism, or is the vertical or parallel composition of symmetry morphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$ or identity morphisms. Generalised symmetry morphisms σ therefore have the form:

$$\sigma = (\sigma_1 \otimes \sigma_2 \otimes \dots \sigma_n) \circ (\sigma'_1 \otimes \sigma'_2 \otimes \dots \sigma'_n) \circ \dots \circ (\sigma''_1 \otimes \sigma''_2 \otimes \dots \sigma''_n) \tag{2.8}$$

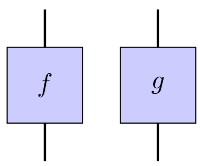
for $\sigma_i, \sigma'_i, \sigma''_i \in \{\sigma_{A,B}, 1_{A,B} \,|\, \forall A, B \in \mathbf{C}\}$. Note that there is no ambiguity in our use of 'σ', since the generalised symmetry morphism σ is not annotated with objects (which are instead inferred from the composition in which they appear), unlike the usual symmetry isomorphism $\sigma_{A,B}$.

Definition 2.30 (Non-trivial parallel composition; atomic morphism)**.** The *non-trivial parallel composition* of morphisms $g_1 : A_1 \to B_1$ and $g_2 : A_2 \to B_2$ is a morphism $f = g_1 \otimes g_2$, where neither g_1 nor g_2 is a scalar (i.e. of type $I \to I$). A morphism $f : A \to B$ is *atomic* if, for all post or pre–compositions of generalised symmetry morphisms, i.e. for all g such that

$$g = \sigma \circ f \circ \sigma$$

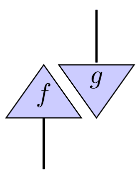
g cannot be written as a non-trivial parallel composition of other morphisms.

Examples of non-atomic morphisms are:



$: A \otimes B \to C \otimes D$

and



$: A \otimes I \to I \otimes B$

So non-atomic morphisms in the diagrammatic language always consist of two non-trivially typed (i.e. non-scalar) subcomponents. Definition 2.30 will be important in Chapter 5, but elsewhere we will not need to be so careful about the graphical and symbolic languages.

2.1.3 Compact structure

In order to abstractly express quantum or quantum-like features such as entanglement-based protocols, we now define a crucial further structure which a †-SMC can carry. We introduce this by the following physical argument.

Consider the quantum teleportation protocol as described by the usual formalism, i.e. a calculation in $\mathbf{fHilb}$. The protocol occurs in three stages. First, we have agents Alice and Bob, whose Hilbert spaces are $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. They share the Bell state $|\Psi_1\rangle = |00\rangle + |11\rangle$, where $|\Psi_1\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ of entangled qubits (we ignore normalisation factors throughout). Alice also has a qubit $|\psi\rangle \in \mathcal{H}_A$ which she will teleport to Bob, which we write as $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$. Hence the state of the joint system of Alice and Bob before teleportation is $|\psi\rangle \otimes |\Psi_1\rangle$.

Consider the Bell basis $\{|\Psi_i\rangle\}_i$, where

$$\begin{array}{rcl} |\Psi_1\rangle & = & |00\rangle + |11\rangle \\ |\Psi_2\rangle & = & |00\rangle - |11\rangle \\ |\Psi_3\rangle & = & |01\rangle + |10\rangle \\ |\Psi_4\rangle & = & |01\rangle - |10\rangle. \end{array}$$

Each of the states in the Bell basis can be obtained from the state $|\Psi_1\rangle$ by the Pauli operations. We let $U_1 = 1$, $U_2 = \sigma_Z$, $U_3 = \sigma_X$ and $U_4 = \sigma_Z\sigma_X$. Then we have $|\Psi_i\rangle = 1 \otimes U_i|\Psi_1\rangle$ for each i.

The second stage of the protocol is that Alice measures her two qubits in the Bell basis. Consider the measurement branch corresponding to $|\Psi_1\rangle$. Straightforward calculation gives that the joint state after Alice's measurement is then

$$(|\Psi_1\rangle\langle\Psi_1| \otimes 1)(|\psi\rangle \otimes |\Psi_1\rangle) = |\Psi_1\rangle \otimes |\psi\rangle.$$

The state $|\psi\rangle$ has been transferred to Bob. However for the other three possible outcomes ($i = 2, 3, 4$) the projection operator $|\Psi_i\rangle\langle\Psi_i|$ will be related by a unitary operation to $|\Psi_1\rangle\langle\Psi_1|$, and Bob will accordingly have to apply a unitary correction to obtain the state $|\psi\rangle$, which is the third stage of the protocol. (Hence Alice must send two bits worth of classical information to Bob, informing him which the four outcomes of her measurement has occurred). For example, for the measurement outcome $i = 4$, the resulting state of the joint system of Alice and Bob is

$$(|\Psi_4\rangle\langle\Psi_4| \otimes 1)(|\psi\rangle \otimes |\Psi_1\rangle) = |\Psi_4\rangle \otimes (c_0|1\rangle - c_1|0\rangle),$$

so that Bob must apply $1 \otimes U_4$ to his state $c_0|1\rangle - c_1|0\rangle$ to give him $|\psi\rangle$.

Consider this sequence of operations expressed in the graphical calculus, as shown in Figure 2.2. Let us graphically distinguish the state $|\Psi_1\rangle$ from other states by depicting it with a cup or cap figure inside the usual triangle (and the unitary correction is represented by the trapezoid).

Now, since the above description occurred in $\mathbf{fHilb}$, we can rewrite the computation using the graphical calculus. As we discussed in our overview of the graphical calculus, this amounts to rotating the bras and ket of the computation. We observe that the quantum teleportation protocol, when depicted using the graphical calculus (which itself corresponds to a SMC), exhibits a literal connection between Alice and Bob. The protocol is defined by the wire depicted in the first expression on the left. When this wire is 'yanked' straight, we obtain the result of the protocol, as depicted in the in last diagram: an identity wire from Alice to Bob, which represents teleportation of the qubit. So, if the 'cup' and 'cap' wires represent entangled states, then we see that it is the topological properties of the graphical calculus which are important for calculating the result of a quantum protocol. We can now define

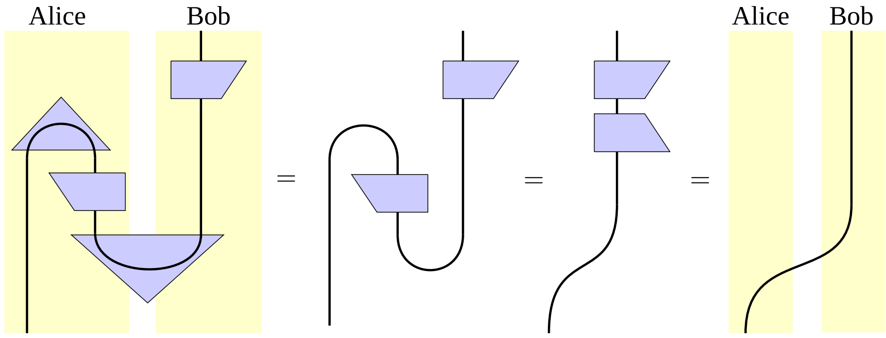
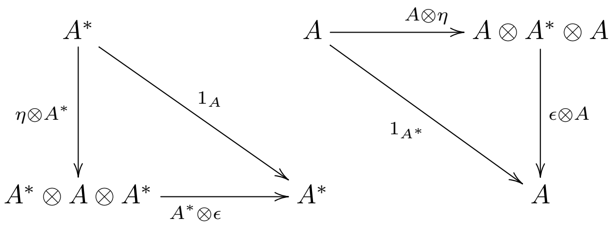


Figure 2.2: The quantum teleportation protocol as described by CQM.

the algebraic structure that corresponds to this topological property of the graphical language. We obtained it by studying the graphical scheme that corresponds to the usual calculations in $\mathbf{FdHilb}$, but we now assume that any category carrying it can abstractly model quantum teleportation.

Definition 2.31. A *compact structure* on an object A of a SMC $\mathbf{C}$ is a quadruple $(A, A^*, \epsilon : A \otimes A^* \to I, \eta : I \to A^* \otimes A)$ consisting of A, its *dual object* A^*, the *unit* η_A and the *counit* ϵ_A, such that the following diagrams commute:

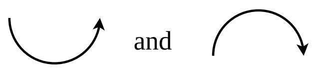


Remark 2.32. Compact closed categories were first studied systematically in representation theory [45, 82]. This also explains the use of the term 'compact', since the category of representations $\mathrm{Rep}(\mathcal{G})$ of a compact group $\mathcal{G}$ forms (what is now called) a compact closed category. Several interesting results concerning this type of representation category have been obtained, in particular the Doplicher-Roberts theorem [46, 47], which shows that $\mathrm{Rep}(\mathcal{G})$ and $\mathcal{G}$ are categorically equivalent. This theorem is described comprehensively in [52], especially its significance from the point of view of algebraic quantum field theory.

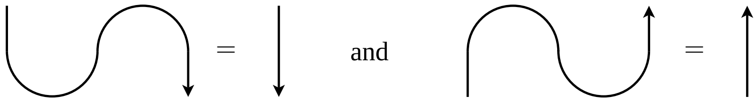
In a †-SMC we define a *Bell state* (A, A^*, η) to be a compact structure $(A, A^*, \eta^\dagger \circ \sigma_{A,A^*}, \eta)$. In $\mathbf{fHilb}$ a Bell state corresponds to the state $|\,\Psi_1\rangle$, since we can define the unit as

$$\begin{array}{rccc} \eta_A : & \mathbb{C} & \longrightarrow & A^* \otimes A \\ & 1 & \longmapsto & \sum\limits_{i \in I, j \in J} |\, a_i a_j \rangle \end{array}$$

We can see how compact structure encapsulates the teleportation protocol by considering the extension of the graphical calculus (defined so far only for †-SMCs) to include it. We represent dual objects A^* by lines directed from top to bottom, and the η and ϵ morphisms by a cup and cap respectively:



Then the commutative diagrams of Definition 2.31 become



Hence compact structure formalises the 'yanking' aspect in Figure 2.2. Given the intuition that the vertical direction of the graphical language represents time, the information flow in compact structure seems to run forwards and backwards in time (since dual objects run backwards). We shall return to this problem in Chapter 5. However, we shall mostly omit showing the directions on the lines.

Definition 2.33. A *dagger compact category* is a †-SMC for which each object has a chosen Bell state.

Kelly and Laplaza [64] proved the coherence theorem for compact closed categories (i.e. the analogous theorem to to Theorem 2.29). As well as **fHilb**, the category **fRel** is dagger compact. However, the category **Set** is not. This is because for any two sets A, B, there is a unique function $e : A \otimes B \to I$. In particular e is a constant function

$$e : A \times B \longrightarrow \{*\}$$
$$(a, b) \longmapsto *$$

Hence, let us take $s : I \to B \otimes A$ to be an arbitrary state, and potentially a Bell state. Then we define a function

$$t := (e \otimes 1_A) \circ (1_A \otimes s) : A \longrightarrow A$$

If compact structure exists, then there must exist functions e and s such that $t = 1_A$. But since e is a constant function, t is a constant function. Hence compact structure does not exist in **Set**, or in **fSet**, the category of finite sets and functions.

Remark 2.34. The categories $\mathbf{Vec}_{\mathbb{K}}$ and **Hilb** are not compact. For any infinite dimensional vector space V, the composition $\epsilon_V \circ \eta_V$ is not defined, because this is the trace of the identity morphism. For example, if V is in $\mathbf{Vec}_{\mathbb{K}}$, then $\epsilon_V \circ \eta_V$ is not a scalar $s \in \mathbb{K}$.

Remark 2.35 (Methodology)**.** As we briefly described at the beginning of this Chapter, the reasoning we used to extract compact structure from the teleportation protocol exhibits a general algorithm for finding abstract counterparts to notions in the standard quantum formalism. The algorithm is:

1. Translate the quantum protocol from the conventional formalism to the categorical one (in the case above: teleportation using η etc.);
2. Write this graphically if possible (yanking);
3. Find an algebraic relationship that captures the set of graphical rules (compact structure).

There is a strong mathematical relationship between †-compact categories and **fHilb**. Consider a formula $\mathcal{T}$ in the symbolic language of dagger compact categories. An *interpretation* of $\mathcal{T}$, denoted $[\![\mathcal{T}]\!]$, consists of an assignment of objects and morphisms in a given category to the object and morphism formulae respectively in $\mathcal{T}$. For example, if $\mathcal{T} = f \otimes g$, with types $f : A \to B$ and $g : C \to D$, then one interpretation in **fHilb** is $[\![\mathcal{T}]\!] = 1_{\mathbb{C}^2} \otimes U$, where $[\![A]\!] = [\![B]\!] := \mathbb{C}^2$, $[\![C]\!] = [\![D]\!] := \mathbb{C}^4$, $[\![f]\!] := 1_{\mathbb{C}^2}$ and $[\![g]\!] := U$ for some unitary on $\mathbb{C}^4$. Selinger showed the following [97].

Theorem 2.36 (Selinger)**.** *Let $\mathcal{T}_1, \mathcal{T}_2$ be two terms in the language of †-compact categories . If $[\![\mathcal{T}_1]\!] = [\![\mathcal{T}_2]\!]$ for all interpretations in* **fHilb***, then $\mathcal{T}_1 = \mathcal{T}_2$.*

Theorem 2.36 says that **fHilb** is the most expressive model for dagger compact categories. To elaborate: the surprising part of the theorem is that if an equation (in the language of dagger compact categories) holds for all interpretations in **fHilb**, then it holds in all dagger compact categories. The conceptual significance of Theorem 2.36 for quantum foundations has not been explored. However, *prima facie* it bears an intriguing resemblance to the recent reconstructions of quantum theory [23, 43, 56]. For example, Theorem 2.36 could be interpreted as saying that complex Hilbert spaces are the optimal way of representing the operational notions encoded in the graphical calculus. We shall leave such speculation for future work.

2.2 CPM construction

The formalism of CQM that has been described so far only applies to pure states. For example, when using the †-compact category **fHilb**, a point $\psi : I \to \mathcal{H}$ is the state $|\psi\rangle \in \mathcal{H}$. To capture quantum theory more fully we need to also describe mixed states and operations.

As a starting point, we can define a category **Mix** whose objects are finite-dimensional Hilbert spaces, and whose morphisms are completely positive maps for the appropriate domain and codomain. More precisely, denoting the set of linear operators on $\mathcal{H}$ as $L(\mathcal{H})$ we define the hom-sets as:

$$\mathbf{Mix}(\mathcal{H}_1, \mathcal{H}_2) := \{f : L(\mathcal{H}_1) \to L(\mathcal{H}_2) \mid f \text{ is completely positive}\}.$$

Note that the morphisms defined with reference to $L(\mathcal{H})$, even though the domain and codomain are Hilbert spaces $\mathcal{H}$ (which is consistent with our definition of the objects in the category as Hilbert spaces $\mathcal{H}$, as opposed to operator spaces $L(\mathcal{H})$. Since completely positive maps are closed under composition this is a category, and it is a monoidal category, with the monoidal product again given by the tensor product of Hilbert spaces. In fact **Mix** is also a compact category: compact structure is just given by the Choi-Jamiołkowski (CJ) isomorphism as follows [24, 60]. We define $|\mathcal{B}\rangle := \Sigma_i |i\rangle \otimes |i\rangle$ for a fixed orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$ (i.e. the maximally entangled state for $\mathcal{H}_1 \otimes \mathcal{H}_2$). Then the morphism η is given by the operator $|\mathcal{B}\rangle\langle\mathcal{B}|$, and this provides a map ϕ from completely positive maps $f : L(\mathcal{H}_1) \to L(\mathcal{H}_2)$ to positive operators M on $L(\mathcal{H}_1 \otimes \mathcal{H}_2)$, given by

$$\phi :: f \longmapsto (f \otimes 1_{L(\mathcal{H}_2)}) \circ |\mathcal{B}\rangle\langle\mathcal{B}| \tag{2.9}$$

and whose inverse is

$$\phi^{-1} :: M \longmapsto \mathrm{Tr}_{\mathcal{H}_1}[(1_{\mathcal{H}_2} \otimes (-)^T)M].$$

The category **Mix** contains the usual formalism of quantum information. However, on its own it is of limited use for CQM; it is defined as a concrete example, but we would hope to identify an abstract structure of categories of mixed states (in the same way that **fHilb** is an example of compact category). Hence it is desirable to have a construction which provides the following features:

- An abstraction of **Mix**, which allow us to construct categories of mixed states that are analogous to **Mix**, yielding other toy models. This could apply, for example, to **Rel** (so that the construction may require compact structure, but not necessarily a field of scalars).
- A construction that distinguishes the pure from the mixed operations, e.g. by producing the mixed category

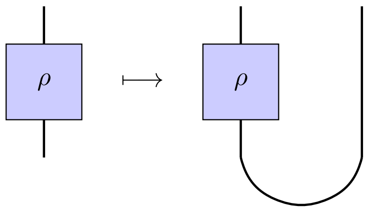
from a pure category. This is useful for understanding whether e.g. the compact structure of the mixed category arises from the pure states (as is the case in $\mathbf{Mix}$).

In the spirit of Remark 2.35, we shall now provide such a construction by constructing $\mathbf{Mix}$ from $\mathbf{fHilb}$. By doing so with graphical reasoning we will obtain a construction that applies to any †-compact category [96].

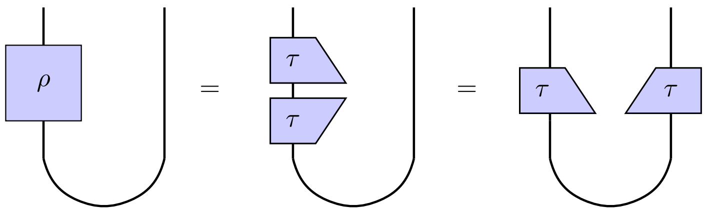
We first note that we can define positivity in any †-SMC.

Definition 2.37. In a †-SMC a *positive morphism* is a morphism $f : A \to A$ for which there exists an object B and a morphism $g : A \to B$ such that $f = g^\dagger \circ g$.

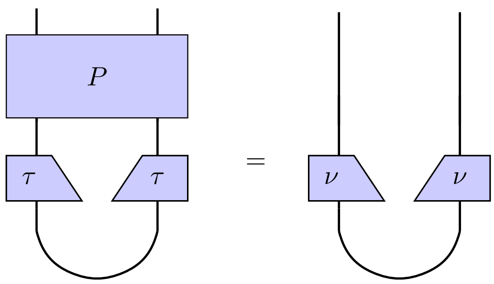
In $\mathbf{fHilb}$ positive morphisms are positive operators. Now, for the category we are constructing, $\mathbf{Mix}$, the morphisms are superoperators, and these act on the states $\rho \in L(\mathcal{H})$. However, the states ρ correspond to operators $\rho : \mathcal{H} \to \mathcal{H}$ in $\mathbf{fHilb}$. But since $\mathbf{fHilb}$ is a compact category, we can use the compact structure to 'unfold' operators to states. For example, using Eq. 2.9 an operator $\rho : \mathcal{H} \to \mathcal{H}$ is mapped to a state $\tilde{\rho} = (\rho \otimes 1_{\mathcal{H}}) \circ \eta_{\mathcal{H}}$:



If ρ is a positive operator , then for some morphism τ this becomes a morphism $\tilde{\rho}$, which we call a *positive matrix*:



Moreover, we can now define *positive maps* as those morphisms P which send positive matrices to positive matrices. Graphically, these are morphisms P such that, for all τ, there exists a morphism ν satisfying:

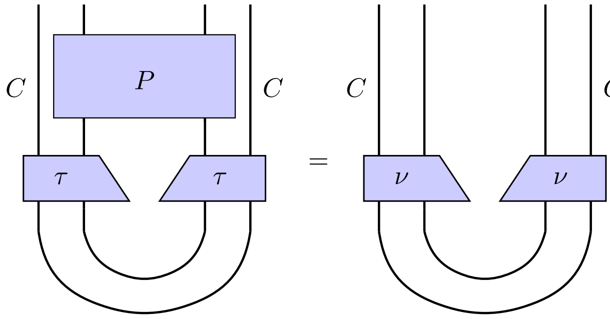


(2.10)

In order to express the requirement of *complete* positivity, we simply expand the domain and codomain of the positive state $\tilde{\rho}$ in Eq. 2.10 to have an ancillary space $\mathcal{K}$, so that it now has types $\rho : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$. Before stating this definition, we note that to provide this form of complete positivity, we have only used the fact that $\mathbf{fHilb}$ has a dagger functor (to express positivity) and that it is compact (to map operators to states). Hence we can do this for any †-compact category as follows.

Definition 2.38. In a †-compact category a morphism P is *completely positive* if, for all positive matrices $\rho =$

$\tau \circ \tau^\dagger$ and all objects C, there exists a morphism ν satisfying:

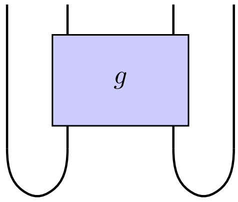


(2.11)

Hence the requirement that a map CP is *completely positive* is expressed by the quantification over all objects C, which in the case of **fHilb** we denoted above as $\mathcal{K}$.

However, so far we do not have a way of constructing the completely positive morphisms for a †-compact category—we just know that completely positive morphisms P must satisfy Eq. 2.11. To construct the completely positive morphisms for a †-compact category, we need to know what form they take. The following chain of reasoning will provide this.

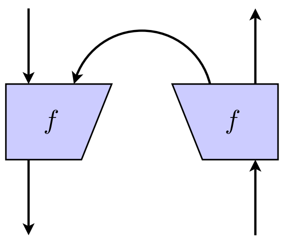
Lemma 2.39 (Selinger, [96])**.** *A morphism g in a †-compact category is completely positive if and only if the morphism ξ_g:*

$\xi_g :=$ 

is a positive matrix.

As pointed out in [96], Lemma 2.39 is a categorical version of Choi's theorem [24]. It will allow us to prove the following.

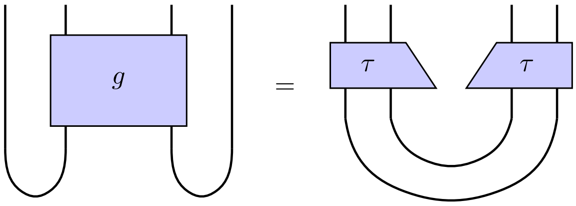
Proposition 2.40. *A morphism g in a †-compact category is completely positive if and only if it has the form*



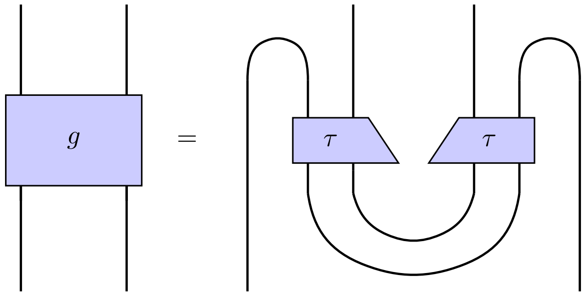
for some f.

Proof. (⇒) Let g be a completely positive morphism, and ξ_g be the corresponding morphism as defined in the statement of Lemma 2.39. Now, using Lemma 2.39 we have that ξ_g is a positive matrix, and hence there exists τ

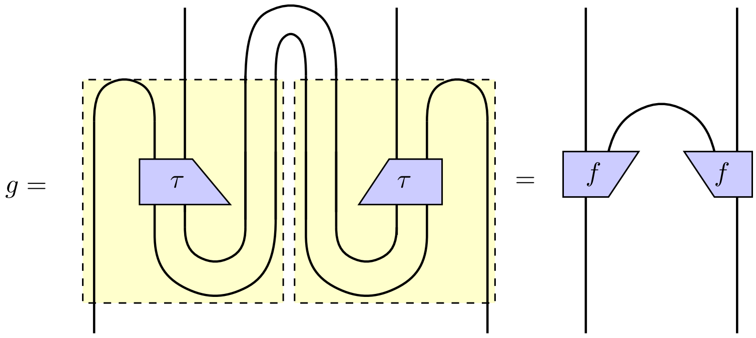
such that:



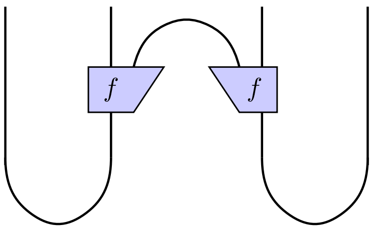
Since ξ_g is a morphism in a †-compact category then, by compactness, ξ_g exists if and only if g has the form



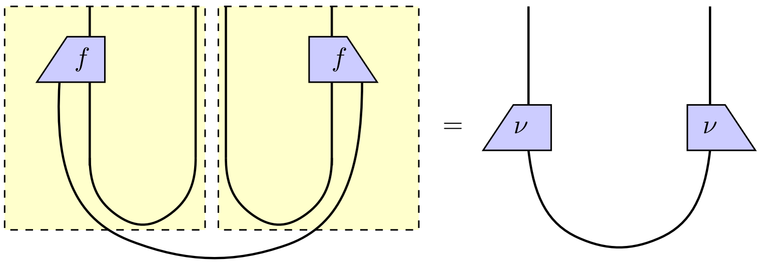
Using compactness again we have



where the morphism f is defined as the morphism denoted by the dotted box, and hence g has the required form.
($\Leftarrow$) Conversely, if g has the stated form then the morphism ξ_g is:

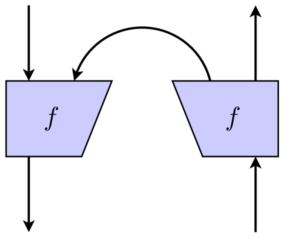


Using compactness this becomes



where ν is defined by the dotted box. Hence ξ_g is positive and using Lemma 2.39 this implies that g is completely positive. □

Proposition 2.40 identifies the form of completely positive morphisms in a †-compact category as



(2.12)

Hence we can now define a construction for mixed states using any † compact category $\mathbf{C}$, by restricting to these morphism f in $\mathbf{C}$.

Definition 2.41. Let $\mathbf{C}$ be a †-compact category . We define the category $CPM(\mathbf{C})$ to be the subcategory of $\mathbf{C}$ with

- the same objects as $\mathbf{C}$, i.e. $|\mathbf{C}| = |CPM(\mathbf{C})|$;
- the morphisms $g : A \to B$ are the completely positive morphisms of $\mathbf{C}$, i.e. of the form Eq. 2.12, hence

$$g = (1_B \otimes \epsilon_C \otimes 1_{B^*}) \circ (f \otimes f_*) \circ (1_A \otimes 1_{A^*}) \tag{2.13}$$

  in $\mathbf{C}$.

It can be easily checked that Definition 2.41 is consistent, i.e. that it is a category.

Remark 2.42. Note that there exists an embedding functor:

$$\mathcal{I} : \mathbf{C} \longrightarrow CPM(\mathbf{C})$$
$$f \longmapsto f \otimes f_*$$

Indeed, the morphisms $f \otimes f_*$ are included in the CPM construction: the symbolic form of completely positive morphisms is given by Eq. 2.13, and we obtain $f \otimes f_*$ by taking $C := I$. In particular, we note that the image of $\mathcal{I}$ is a category, denoted $\mathcal{I}[\mathbf{C}]$. The category $\mathcal{I}[\mathbf{C}]$ is the category of pure states $\mathbf{C}$, but written in 'density matrix form'. This is because $\mathcal{I}[\mathbf{C}]$ consists only of morphisms in $\mathbf{C}$ which have been 'doubled', from f to $f \otimes f_*$. In particular, when $f = \psi$, this is the mapping (after using compactness):

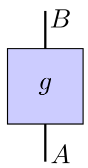
$$|\psi\rangle \longmapsto |\psi\rangle\langle\psi|$$

Hence $\mathcal{I}[\mathbf{C}] \cong \mathbf{C}$, and this shows what we mean by describing $\mathcal{I}[\mathbf{C}]$ as the 'density matrix' version of $\mathbf{C}$.

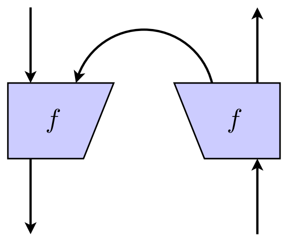
Theorem 2.43 (Selinger)**.** *The category $CPM(\mathbf{C})$ is dagger compact.*

Theorem 2.43 is proven by showing that the embedding $\mathcal{I}$ provides both a dagger and compact structure.

Remark 2.44. It is important to note that, since $CPM(\mathbf{C})$ is a dagger compact category (and therefore an SMC), it has its own graphical calculus. That is to say, we have so far been describing the morphisms in $CPM(\mathbf{C})$ using the graphical calculus of $\mathbf{C}$, but if we do not need to expose the underlying morphisms, then we can simply use the graphical calculus of $CPM(\mathbf{C})$. For example, a morphism in $g : A \to B$ in $CPM(\mathbf{C})$ is displayed as



and, as we have discussed, the morphism that this corresponds to in $\mathbf{C}$ is displayed as

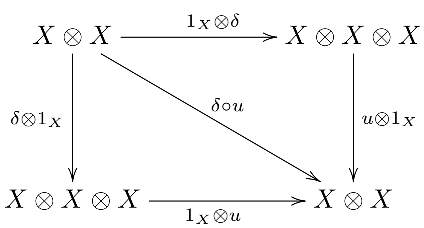


Example 2.45. As desired, the category $CPM(\mathbf{fHilb})$ has finite-dimensional complex Hilbert spaces as objects and completely positive linear maps as morphisms. States $\rho : I \to \mathcal{H}$ are density operators.

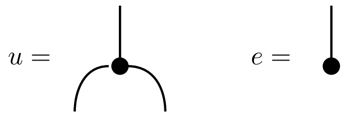
2.3 Classical data

In order to give an account of classical data we shall introduce a particular type of internal algebraic structure for an object X in a †-SMC.

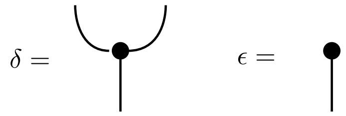
Definition 2.46. A *Frobenius algebra* in an SMC is an internal monoid $(X, e : I \to X, u : X \otimes X \to X)$ and an internal comonoid $(X, \epsilon : X \to I, \delta : X \to X \otimes X)$ which together satisfy the *Frobenius law*:



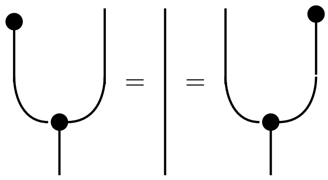
A Frobenius algebra is called *special* if $u \circ \delta = \mathrm{id}_X$ and *commutative* if $\sigma \circ \delta = \delta$. This will allow us to introduce 'classical interfaces' with the quantum processes modelled in a †-SMC. The conceptual reason for why Frobenius algebras model classical data is that Definition 2.46 axiomatises the *copying* of data. As we discussed above, in many †-SMCs there is not a natural diagonal $\Delta : X \to X \otimes X$. This would provide a uniform cloning operation. However, Frobenius algebras will provide a *non-uniform* cloning operation. We can see this by using the graphical expressions corresponding to the Frobenius conditions. The composition u and unit e of the monoid are given by:



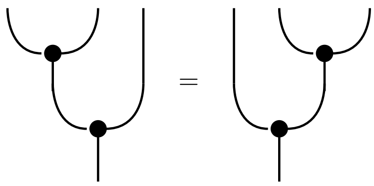
and cocomposition and counit of the comonoid is



Now, the coassociative law for the comonoid is

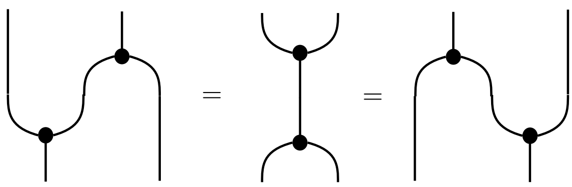


Note that the coassociative law expresses the condition that applying δ, and then applying it a second time to output 1 gives the same result as applying it a second time to output 2. This is a property that we would expect of a copying operation, since in this case output 1 and output 2 are each equal to the input. The counit law for the comonoid is:



and, interpreting ϵ as a deleting operation, this expresses the notion that copying the output and then deleting either output 1 or output 2 just leaves the original input.

We note that the graphical form of the associative and unit laws for the monoid are given by the dagger of the diagrams for the coassociative and the counit law, i.e. they are flipped upside-down. We also note for Chapter 6 that the commutative diagram in Definition 2.46 has a diagrammatic representation as:



The connection between the usual formulation of observables, at least at the level of orthonormal bases, and CQM is made as follows.

Definition 2.47. A *classical structure* (X, δ, ϵ) in a †-SMC is a commutative special Frobenius algebra for which $\epsilon = e^\dagger$ and $\delta = u^\dagger$.

In [41] it was observed that an orthonormal basis $\{| \psi_i\rangle\}_i$ for a Hilbert space $\mathcal{H}$ induces a classical structure through maps which copy and delete the basis vectors:

$$\delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$$
$$|\psi_i\rangle \longmapsto |\psi_i \psi_i\rangle$$

and

$$\epsilon : \mathcal{H} \longrightarrow \mathbb{C}$$
$$|\psi_i\rangle \longmapsto 1.$$

For example, we can write δ as

$$\delta = |\psi_i \psi_i\rangle\langle\psi_I| \tag{2.14}$$

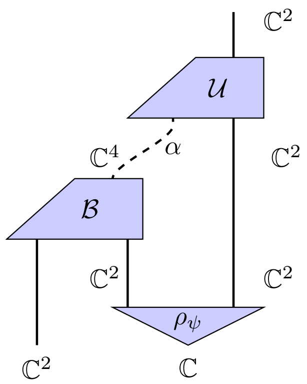
In fact, all classical structures arise in this way: Definition 2.47 is justified by the following result due to Coecke, Pavlovic and Vicary [42].

Theorem 2.48. *In* **fHilb** *classical structures are in bijective correspondence with orthonormal bases.*

Hence classical information, i.e. (finite-dimensional) observables, can be captured abstractly in CQM.

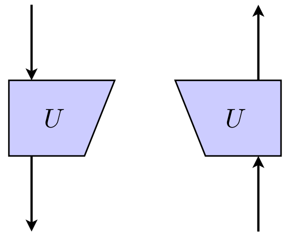
Remark 2.49. This can be seen as an example of the operational character of CQM. The no-cloning theorem in quantum mechanics states that the *uniform* copying of states is forbidden. On the other hand, the comonoid in a classical structure distinguishes classical from quantum information by expressing the fact that the states representing the former can be copied. So CQM distinguishes classical from quantum data by information-theoretic behaviour. We shall discuss the conceptual significance of this in Chapter 8.

With classical structures defined, it is possible to define teleportation with classical communication in an arbitrary †-SMC. We describe this schematically as follows. We first note that, the category $CPM(\mathbf{fHilb})$ has the following morphism

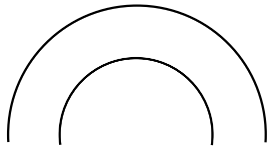


(2.15)

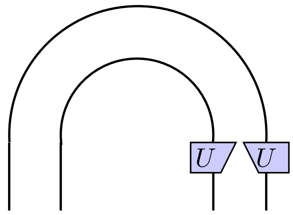
where $\rho_\psi = \mathcal{I}(\psi)$ is the embedding of the Bell state ψ (using the embedding defined in Remark 2.42). Now, $\mathcal{B}$ and $\mathcal{U}$ represent the Bell measurement and unitary correction repsectively, with α the classical communication from Alice to Bob (denoted with a dotted line). These are all completely positive maps, and so will exist in $CPM(\mathbf{fHilb})$. For example, a single unitary will be depicted as



A single Bell effect will be depicted as



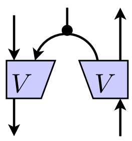
Then each of the four Bell effects will be obtained by composing with a unitary, which is depicted as follows:



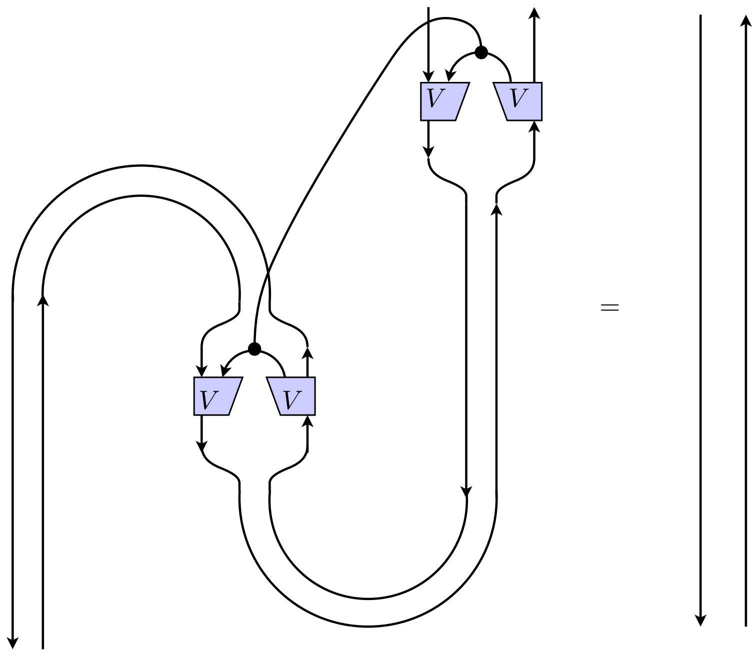
It remains to describe the classical indexing: i.e. sum over Bell effects, and the outcome which control the unitary to be applied by Bob. This is classical data, and to use classical structures to graphically show how unitaries are controlled by the measurement outcome, we simply need expand the Bell effects and unitaries to include an ancillary space which provides an index. From Eq. 2.14 we see that the morphism δ in $\mathbf{fHilb}$ is a sum over basis elements. Then consider a morphism $V : \mathcal{H} \to \mathcal{H} \otimes X$ in $\mathbf{fHilb}$ for which X, such that

$$(1_{\mathcal{H}} \otimes \langle i|)) \circ V = U_i$$

where $\{|i\rangle\}_i$ is a basis for X, and $\{U_i\}_i$ are Pauli matrices. The dimension of the basis for X is the number of classical outcomes, so that for our teleportation example $X = \mathbb{C}^4$ (as also shown in Eq. 2.15). Then we can use the classical structure $\delta : X \to X \otimes X$ in $\mathbf{fHilb}$ to represent this indexing. Graphically this appears as:



Putting these various graphical components together, we can depict the teleportation scheme of Eq. 2.15 as follows:



Note that this is a diagram of a morphism in $CPM(\mathbf{fHilb})$, but depicted using the graphical calculus of $\mathbf{fHilb}$ (cf. the distinction discussed in Remark 2.44). More details concerning this description of teleportation with classical communication are given in [35].

Chapter summary. We have introduced the formalism of CQM. We described monoidal categories, and the graphical calculus. We then described dagger compact categories, and motivated their structure by considering post-selected teleportation in $\mathbf{fHilb}$. We then introduced mixed states and classical data.

Chapter 3

Aspects of causal structure

Contents

In this Chapter we shall give an overview of two aspects of causal structure. The first notion we shall discuss is the *global* causal structure that arises in relativity. The second notion is that of describing quantum systems combined with this background relativistic causal structure. The aim of this Chapter is to provide the context for subsequent Chapters, in which we shall consider the compatibility of these two notions, primarily from the point of view of CQM. This Chapter will also illustrate the foundational significance of the work in later chapters.

3.1 Relativistic causal structure

The causal structure of Minkowski spacetime $\mathbb{M}$ is an intuitively clear notion: the light cones that exist at each point $p \in \mathbb{M}$. However, it will be useful to discuss relativistic causal structure more formally. This is for two reasons. Firstly, our later constructions will not be restricted to Minkowski spacetime. Secondly, a formal perspective will help to reveal the importance of causal structure from the point of view of theory-building, which will be demonstrated primarily by Corollary 3.4 below. This will show that, under certain conditions, the causal structure of a spacetime determines 'most' of the metric structure. We shall refer to such results as 'reconstruction results' (indeed, these are analogous to the reconstructions of quantum theory). Hence for these reasons we shall consider the causal structure of a relativistic spacetime in more precise and general terms than the informal notion of a network of light cones. To do so, we follow Nakahara [83] for differential geometry and notation; and Wald [106] for general relativity.

A relativistic spacetime $(\mathcal{M}, g)$ is a differentiable manifold $\mathcal{M}$ equipped with a Lorentzian metric tensor g. This means that at each point $p \in \mathcal{M}$ there is a bilinear, symmetric and nondegenerate function

$$g_p : T_p\mathcal{M} \times T_p\mathcal{M} \longrightarrow \mathbb{R}$$

where $T_p\mathcal{M}$ is the tangent space at p. The metric tensor g is the smooth assignment $g :: p \mapsto g_p$ for each point $p \in \mathcal{M}$, and in our setting g is Lorentzian, meaning that it has signature $(- + \cdots +)$ (using Wald's convention

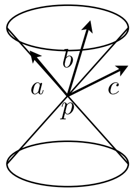


Figure 3.1: Vectors a, b and c are null, time-like and space-like respectively, in the tangent space $T_p\mathcal{M}$.

[106]). We shall assume that $\mathcal{M}$ has dimension greater than two (although this is only necessary for Hawking's theorem below). Now, a relativistic spacetime contains several layers of structure, which are defined in sequence as

$$\text{topological} \rightsquigarrow \text{differentiable} \rightsquigarrow \text{metric}.$$

We shall use the metric to define causal structure. Causal structure is arguably a simpler notion than metric structure, since the latter requires topology and differentiable structure to be defined; in addition to the *existence* of such structure, various *constructions* are needed to define the metric, e.g. the tangent spaces at each point of the spacetime. In short, the metric needs a 'tower of constructions' to support it, to use the phrase of [81]. On the other hand, as we shall see, causal structure can exist without these or other prior definitions, since it can be defined more abstractly. Another advantage of focussing on causal structure is a conceptual reason: causality is arguably an operational notion in physics, since whether two parts of a spacetime are causally connected is observable in a more direct way than the metric (see [75] for more discussion on this topic). This lends conceptual significance to the reconstruction theorems below. Accordingly, the idea of these theorems below is to *reverse* the usual direction of dependence, the usual direction being from metric to causal structure. These results will allow a spacetime $\mathcal{M}$ to be *identified* (up to a conformal factor) using an abstract causal structure, i.e. a set of binary relations that are defined *without* reference to the metric.

In a general curved spacetime, we begin with the tangent space $T_p\mathcal{M}$ at a point $p \in \mathcal{M}$, which is the set of tangent vectors that are tangent to curves through p. Since we are considering Lorentzian spacetimes we have $T_p\mathcal{M} \cong \mathbb{M}$, i.e. $\mathcal{M}$ is *locally* isomorphic to Minkowski spacetime. With the signature defined above, the non-zero tangent vectors $v \in T_p\mathcal{M}$ are called:

$$\begin{array}{rcl} \textit{timelike} & \text{if} & g(v,v) < 0 \\ \textit{null} & \text{if} & g(v,v) = 0 \\ \textit{spacelike} & \text{if} & g(v,v) > 0. \end{array} \tag{3.1}$$

This classification is depicted in Figure 3.1. If a tangent vector is timelike or null then we say that it is *causal*, and by using the term 'light cone' at p, we mean the collection of all causal tangent vectors in $T_p\mathcal{M}$. Now, since $T_p\mathcal{M}$ is a real vector space, for every causal tangent vector v there is a causal tangent vector $-v$, and hence the light cone is split into two halves, as depicted in Figure 3.1. Then we can assign the labels 'future' and 'past' to each half of the light cone at each point $p \in \mathcal{M}$. We say that a spacetime is *time-orientable* if a continuous such choice can be made over all of $\mathcal{M}$. In what follows we shall only consider time-orientable spacetimes.

In time-orientable spacetimes we can use the classification of tangent vectors (which are directional derivatives) in Eq. 3.1 to identify those curves which are 'physically realisable', meaning that such curves can be interpreted as the paths of particles. Recall that a curve is a smooth map:

$$\gamma : \iota \longrightarrow \mathcal{M}$$

where ι is a real interval $[a, b] \subseteq \mathbb{R}$[1] Note that there can be *closed* curves $\gamma : \iota \to \mathcal{M}$, for which there exist $p, p' \in \iota$ such that $\gamma(p) = \gamma(p')$. Now, if a curve γ has a tangent vector which is timelike, null or spacelike at all points in the image $\gamma[\iota]$ of the curve, then the curve is *timelike*, *null* or *spacelike* respectively. A curve is *causal* if it is timelike or null. A causal curve γ is *future-directed* if its tangent vector is always in the future light cone at each point $p \in \gamma[\iota]$; similarly it is *past-directed* if its tangent vector is always in the past light cone at each point $p \in \gamma[\iota]$. The physical meaning of causal curves is, as usual, that timelike curves are the paths of massive particles, and null curves are the paths of massless particles.

Using these classes of curves in $\mathcal{M}$, we obtain several types of binary relation on the points $p \in \mathcal{M}$; these are defined using the type of curve that can connect a pair of points. In particular, we define two binary relations as follows. We write $p < q$ if there exists a future-directed causal curve from p to q, and we write $p \ll q$ if there exists a future-directed timelike curve from p to q. By the 'causal structure' of a spacetime we mean this pair of binary relations; we shall focus in particular on the relation $\ll$. Note that time-orientability does not prohibit closed timelike curves, and so these relations can be symmetric, e.g. $p \ll q$ and $q \ll p$ can both be satisfied (for example, the Gödel spacetime is time-orientable [50], despite having closed timelike curves through every point). Hence without further conditions the causal structure of a spacetime is not in general a partial order. It is useful to define the *timelike future* of a point p as the set $\mathcal{I}^+(p) := \{q \,|\, p \ll q\}$, and *timelike past* of p as the set $\mathcal{I}^-(p) := \{q \,|\, q \ll p\}$.

Consider two spacetimes $(\mathcal{M}, g)$ and $(\mathcal{M}', g)$. Let us consider two different types of structure-preserving maps $\phi : \mathcal{M} \to \mathcal{M}'$. Firstly, a *causal isomorphism* is a bijection $\phi : \mathcal{M} \to \mathcal{M}'$ such that

$$p \ll q \iff \phi(p) \ll \phi(q).$$

The second type of structure-preservation concerns preservation of the metric. Now, if two manifolds $\mathcal{M}$ and $\mathcal{M}'$ are diffeomorphic then they isomorphic as differentiable manifolds, i.e. they can be smoothly deformed into one another. But a diffeomorphism need not preserve the metric. A *smooth isometry* $\phi : \mathcal{M} \to \mathcal{M}'$ is a diffeomorphism that does preserve the metric, i.e.

$$\phi_*(g) = g',$$

where ϕ_* is the pushforward, which maps the metric g (acting on pairs of tangent vectors in $T_p\mathcal{M}$) to the metric $\phi_*(g)$ (acting on pairs of tangent vectors in $T_{\phi(p)}\mathcal{M}'$).[2] A weaker notion is a *smooth conformal isometry* $\phi : \mathcal{M} \to \mathcal{M}'$, which is a diffeomorphism for which there exists a smooth map $\tilde{\Omega} : \mathcal{M}' \to \mathbb{R}$, such that

$$\phi_*(g) = \Omega^2 g',$$

where Ω is defined as the smooth mapping

$$\Omega :: p \longmapsto \Omega_p(v, w) := \tilde{\Omega}(p),$$

so that the domain of $\tilde{\Omega}$ is lifted from $\mathcal{M}'$ to $T_p\mathcal{M}' \times T_p\mathcal{M}'$. In other words, a smooth conformal isometry is a

[1] Note that we denote an arbitrary real interval with 'ι', and not 'I' as is usually used, since the latter already denotes the unit object in a monoidal category.

[2] We have used the pushforward ϕ_* on the metric g, but the pushforward is usually defined on vectors, specifically between tangent spaces $\phi_* : T_p\mathcal{M} \to T_{\phi(p)}\mathcal{M}'$. However, since ϕ is a diffeomorphism, we can use the pushforward of the inverse ϕ^{-1}, i.e. the map $[\phi^{-1}]_* : T_{\phi(p)}\mathcal{M}' \to T_p\mathcal{M}$, to define ϕ_* on the metric, even though this is not a vector in $T_p\mathcal{M}$. Specifically, we define the action of $\phi_*(g)$ on tangent vectors $v, w \in T_{\phi(p)}\mathcal{M}'$ as follows. We write $\overline{g} := \phi_*(g)$, and then define:

$$\overline{g}_p(v, w) := g_p([\phi^{-1}]_*(v), [\phi^{-1}]_*(w)).$$

See Appendix C of [106] for further details.

diffeomorphism that preserves the metric up to a conformal factor, i.e. a position-dependent positive scalar factor. (The equivalence class of such *conformally equivalent* metrics is sometimes referred to as the 'conformal metric', i.e. the equivalence class $[g] = \{\tilde{g} \,|\, \tilde{g} = \Omega^2 g\}$.) This means that a smooth conformal isometry preserves the angles between vectors in the tangent space, but not the length of vectors. Moreover, because a conformal isometry ϕ provides a *positive* scaling of the metric, ϕ also preserves causal structure. This is because the classification in Eq. 3.1 is just whether $g(v, v)$ is negative, zero, or positive in $\mathbb{R}$. Hence this classification is invariant under multiplication by any $s \in \mathbb{R}^+$. The aim of the results below is to establish the extent to which the converse is true:

If ϕ is a causal isomorphism, is ϕ a conformal isometry?

Now, for the specific case of causal *automorphisms* on Minkowski spacetime, we can answer this question without any further conditions. To do so, we define two symmetry groups of $\mathbb{M}$. Firstly, let $\mathcal{C}$ be the group of causal automorphisms $\phi : \mathbb{M} \to \mathbb{M}$ of Minkowski spacetime. To define the second symmetry group, we first say that a mapping of spacetimes is *orthochronous* if it preserves time-orientation. Then consider the group $\mathcal{G}$ which is defined as the group generated by the following orthochronous linear transformations on $\mathbb{M}$:

- boosts, rotations and spatial inversion [generating the orthochronous Lorentz group $O^+(3, 1)$], as well as translations (generating the orthochronous Poincaré group);
- dilatations, i.e. conformal transformations for which $\tilde{\Omega}$ is a constant function.

Now, all these transformations are conformal isometries, and as we discussed above, such transformations must preserve causal structure, and so smooth isometries are causal automorphisms. Indeed, to state this more conceptually (and as discussed by Malament [75] and Winnie [107]), we could say that the reason why different but conformally equivalent metrics correspond to the same causal structure, is that the latter cannot fix the scale of spacetime geometry. (i.e. the relation $p \ll q$ contains no numerical information). Hence we would expect the group of all causal automorphisms to contain all of $\mathcal{G}$, and not only, for example, the orthochronous Poincaré group without dilatations. Therefore $\mathcal{G} \subseteq \mathcal{C}$. The converse, i.e. whether $\mathcal{C} \subseteq \mathcal{G}$ is true, is not obvious because, for example, causal automorphisms need not even be linear. Or, causal isomorphisms might include position-dependent conformal transformations. However, the following result precludes these possibilities.

Theorem 3.1 (Alexandrov [10]; Zeeman [110])**.** $\mathcal{C} = \mathcal{G}$.

In other words, the orthochronous Poincaré group augmented by dilatations can be defined as the group of causal automorphisms of Minkowski spacetime. This therefore answers the question raised above: causal automorphisms are indeed smooth conformal isometries, which are, moreover, constant.

To extend Theorem 3.1 to more general spacetimes, the focus of subsequent work has been on characterising causal isomorphisms between general spaces $\mathcal{M}$ and $\mathcal{M}'$, rather than automorphisms of a given spacetime $\mathcal{M}$. This generalisation is with the proviso that it requires supplying certain 'chronology' conditions that, to varying degress, prohibit closed time-like curves (CTCs). The weakest condition that has been considered in this context is a spacetime which is *chronological*, meaning that it contains no CTCs. Note that, in this case, $\ll$ is a partial order: timelike curves are transitive in a general time-orientable spacetime, but for a chronological spacetime, $\ll$ is also anti-symmetric. Similarly $<$ is a partial order. However, the chronological condition is part of a hierarchy of causality conditions that describe, to varying extents, the lack of CTCs or approximations thereof. Hence we can consider stronger conditions than the chronological condition as follows. We say that a space-time $(\mathcal{M}, g)$ is *future-distinguishing* if, for all $p, q \in \mathcal{M}$:

$$\mathcal{I}^+(p) = \mathcal{I}^+(q) \Longrightarrow p = q.$$

Similarly we say that a space-time $(\mathcal{M}, g)$ is *past-distinguishing* if for all $p, q \in \mathcal{M}$:

$$\mathcal{I}^-(p) = \mathcal{I}^-(q) \Longrightarrow p = q.$$

The significance of future- and past-distinguishability lies in an equivalent condition relating to CTCs. A space-time $(\mathcal{M}, g)$ is future- (respectively, past-) distinguishing if and only if for all $p \in \mathcal{M}$ and all open sets O for which $p \in O$, there exists an open set O' satisfying $p \in O' \subseteq O$, such that no future- (respectively, past-) directed time-like curve through p that leaves O' returns to O'. This means that, at any point $p \in \mathcal{M}$ and open set O, we can always find a small enough open set O' contained by O for which there are no timelike curves starting at p that are 'approximately' CTCs, in the sense of returning to O' and 'almost' intersecting p. We depict this schematically in Figure 3.2. Let us call a spacetime that is both future- and past-distinguishing a *distinguishing* spacetime. Distinguishing spacetimes are at the stronger end of the hierarchy of causality conditions. A stronger condition, that of a 'strongly causal' spacetime, satisfies a similar property to distinguishing spacetimes. This condition is that, given $p \in \mathcal{M}$ and an open set O, we can always find a small enough open set contained by O for which there are no timelike curves that are 'approximately' CTCs, *even* for curves not passing through p (which distinguishability is not required to satisfy). However, the theorems below only use the causality conditions of distinguishability (see [75] for a concise survey of the other causality conditions).

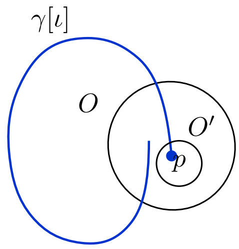


Figure 3.2: Schematic diagram of the distinguishability condition.

Theorem 3.2 (Hawking [57])**.** *Let $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ be distinguishing spacetimes. Let $\phi : \mathcal{M} \to \mathcal{M}'$ be a homeomorphism such that ϕ and ϕ^{-1} preserve future-directed continuous timelike curves. Then ϕ is a smooth conformal isometry.*

Theorem 3.2 means that homeomorphisms (satisfying the stated condition) are conformal isometries. In other words, the differential and conformal structure follow from the topological structure.

Theorem 3.3 (Malament [75])**.** *Let (M, g) and (M', g') be distinguishing spacetimes. Let $\phi : \mathcal{M} \to \mathcal{M}'$ be a causal isomorphism. Then ϕ is a homeomorphism such that ϕ and ϕ^{-1} preserve future-directed continuous timelike curves.*

Stated informally, Theorem 3.3 shows that the causal structure of a spacetime $\mathcal{M}$, defined as the relation $\ll$ of smooth timelike curves between points, determines the topology of $\mathcal{M}$. Theorems 3.2 and 3.3 yield the following.

Corollary 3.4. *Let $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ be distinguishing spacetimes. If a function $\phi : \mathcal{M} \to \mathcal{M}'$ is a causal isomorphism, then ϕ is a smooth conformal isometry.*

We note parenthetically that if we let $\mathcal{M} = \mathcal{M}' = \mathbb{M}$, then we recover a weaker version of the Alexandrov-Zeeman theorem for Minkowski spacetime, since Theorem 3.1 yields the stronger result that causal automorphisms are *constant* conformal isometries (whereas Corollary 3.4 only yields arbitrary smooth conformal isometries). Now, we can view Corollary 3.4 as a 'reconstruction' of a relativistic spacetime; in short, it says that from the causal structure we obtain the conformal metric. This interpretation of Corollary 3.4 is the motivation for the causal set programme [101], which aims to obtain a theory of quantum gravity by using a discrete analogue of a relativistic spacetime. We shall also regard Corollary 3.4 as underpinning the significance of causal structure for relativity. This justifies interpreting our work in subsequent chapters as capturing some part of relativity, since we shall define causal structure for CQM.

However, consider the following concern with this supposed significance of Malament's theorem. The notion of causal structure used in Malament's theorem is the order $\ll$. But the relation $p \ll q$ is defined to hold when there exists a *continuous* timelike curve from p and q. That is, one needs the class of continuous timelike curves to be *given data*. This requires knowing which curves are continuous. However, this seems to place Theorem 3.3 on the precipice of circularity. For, the significance of Theorem 3.3 is that the topology of $\mathcal{M}$ is derived from the the relation $\ll$, but $\ll$ is defined using the notion of continuity, i.e. the topology of $\mathcal{M}$. Let us consider Sklar's response to this apparent circularity [99]. Malament's result is best seen as showing that the 'full' topology of the spacetime $\mathcal{M}$ can be derived from the 'partial' topology of continuous timelike curves γ in $\mathcal{M}$. If the latter can be specified without specifying the former, then Theorem 3.3 would avoid circularity. It would only appear to be because of the way continuous timelike curves are defined mathematically, viz. we define the topology of a spacetime, and then the notions of curves and continuous curves.

But how might the 'partial topology' of continuous timelike curves be specified, in practice, without giving the entire topology of $\mathcal{M}$? Sklar argues that this can be understood in in an 'operational' way. For example, we can imagine that an experimenter can determine the continuous timelike curves in a spacetime (or a large enough portion of it) by *observing* the paths of massive particles. Then Malament's result is that from this data, the experimenter can determine the topology of the entire spacetime. This provides a useful interpretation of Malament's theorem in our context. It crucially uses an operational notion of the continuity of a curve, which can be known without the data of the full topology of the spacetime.

There are other results that provide the reconstruction of a spacetime from either causal structure or a related notion. To provide context for Malament's theorem, it is useful to briefly discuss the other main result. Malament's theorem improved upon a theorem by Hawking, King and McCarthy (HKM) [57]. This is based on a non-standard topology on a spacetime $\mathcal{M}$, called the *path topology*, which is defined using timelike curves in $\mathcal{M}$. Analogously to Corollary 3.4, HKM show that the homeomorphisms of the path topology for a spacetime are the same as the smooth conformal isometries. Now, as with Sklar's argument, HKM argue that the path topology is physically significant because it is based on the paths of physical particles, and in particular the paths that observers would experience. This would make the HKM theorem also potentially interesting for our context, since we used this reasoning above when describing the significance of causal structure and in Sklar's argument. However the HKM theorem is less useful from our perspective than Malament's theorem for two reasons. Firstly, it is a weaker theorem because it has stronger assumptions than Malament's theorem: it assumes the strong causality condition described above, which is stronger than the distinguishability condition assumed in Malament's theorem. Secondly, it is not a theorem that is *explicitly* about causal isomorphisms, but instead it concerns homeomorphisms in the path topology.

Remark 3.5. As a preview of our work to come, we note that a category can be seen as a generalisation of a poset. We denote an arbitrary poset as $(P, \leq)$ (we shall generally use this notation for arbitrary posets). This forms a category $\mathbf{C}$ by defining:

- *Objects*: the objects are $|\mathbf{C}| := P$.

- *Morphisms*: the morphisms are defined such that a morphism $f : A \to B$ exists in $\mathbf{C}$ if and only if $A \leq B$.
- *Composition*: for morphisms $f : A \to B$ and $g : B \to C$, composition $g \circ f : A \to C$ is just the statement of transitivity, i.e. $A \leq B$ and $B \leq C$ implies $A \leq C$.

Posets are therefore a 'minimal' type of category, in the sense that each hom-set has at most one morphism. This categorical description suggests that CQM can accommodate causal structures in a natural way. Moreover, since a category *generalises* a poset, it offers the possibility that *richer* notions of causality can be captured by CQM. This topic will be addressed in Chapter 5 and 6.

We conclude this Subsection by noting that it is sometimes argued that causality is not the essence of relativity. For example, Brown [17] and Maudlin [80] have argued that the possible existence of tachyons suggests that the essence of relativity is not causal structure, i.e. the prohibition of superluminal signalling, but instead properties such as Lorentz covariance. Their conclusion is that when considering the compatibility of (special) relativity and quantum theory, phenomena such as quantum nonlocality do not yield a *prima facie* conflict with relativity, since the latter does not prohibit signalling. However, the results above would suggest that, at least *formally*, causal structure is indeed a crucial part of the theory, since it determines much of the structure of a relativistic spacetime.

3.2 Compatibility with quantum theory

In this Section we shall discuss some previous work that is concerned with the compatibility between relativity and quantum theory from the perspective of relativistic causality.

Now, it might be argued that this is best investigated by considering the broader question of the *general* compatibility between relativity and quantum theory. For example, modern quantum field theory was developed by considering the consistency between relativity and the Schrödinger equation, and this led to e.g. the Dirac equation. Moreover, the representation theory of the Poincaré group $\mathcal{P}$ could also be seen as a type of compatibility between the two theories. That is, the unitary irreducible representations of the symmetry group of Minkowski spacetime are exactly classified by the mass and spin of elementary particles in a quantum field theory. This then leads to the idea that elementary particles are described as unitary irreducible representations of the Lie group $\mathcal{P} \times \mathcal{G}$, where $\mathcal{G}$ is the internal symmetry group of the quantum field theory in question, e.g. $\mathcal{G} = U(1) \times SU(2) \times SU(3)$ for the standard model.

Hence the compatibility between Minkowski spacetime and quantum theory is clearly at the heart of theoretical physics. But our justification for using a more abstract approach, in which we do not consider detailed physical scenarios, is that CQM has already been successful in isolating interesting phenomena in this abstract way. Therefore, we shall have a much more *narrow* focus than the very general notion of compatibility described above. In particular we are concerned with the following topics:

1. Our focus will be on causal structure specifically, in the sense of studying physical processes with background causal structure (and so we are not concerned with *consequences* of the compatibility with Minkowski spacetime such as the Dirac equation).
2. We want to formalise the relationship between the quantum theory and causal structure in an information-theoretic way (although 'informational' way might be a more appropriate description, since we will not necessarily use information theory—a distinction also made in recent reconstructions of quantum theory [23]).

3. We shall use categorical quantum mechanics to provide a way of isolating features such as information flow from detailed aspects of the underlying physics, such as *how* this information is encoded (e.g. in spin observables).

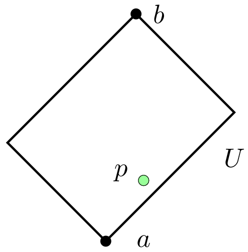
Hence we are concerned with the formal compatibility between relativistic and quantum causality from a structural or 'informational' view. In this respect, there are several pieces of work which are relevant as context. We shall discuss two approaches:

1. Algebraic quantum field theory.
2. Quantum causal histories.

1. Algebraic quantum field theory. This is the most standard axiomatisation of quantum field theory (for more details see [52]). *Prima facie*, algebraic quantum field theory (AQFT) would also seem to bear some connection to our concerns, since the idea of this approach is to assign observables to regions of spacetime. Specifically, the idea is to assign a C^*-algebra to each region of a spacetime $\mathcal{M}$. Consider the partial order $<$ on the points $p \in \mathcal{M}$ of a spacetime that represents the existence of a causal curve when $p < q$. The regions are open *double cones* of the spacetime $\mathcal{M}$, which are open sets U defined as follows. Give two points $a, b \in \mathcal{M}$ we make the definition

$$U := \{p \,|\, (a < p) \wedge (p < b)\}.$$

This can be depicted as:



The set D of all double cones U (i.e. the sets U defined for all points a, b) forms a poset $\mathcal{K} = (D, \subseteq)$, using the inclusion $U \subseteq V$ to define the partial order. AQFT then proceeds by assigning a C^*-algebra to these double cones. This assignment uses the category $\mathbf{C^*Alg}$, which is the category of C^*-algebras and $*$-homomorphisms. The C^*-algebra at the double cone θ contains the observables that can be observed at θ. Now, in Remark 3.5 we described how a poset can be viewed as a category. This then leads to the basic structure of AQFT.

Definition 3.6. For a poset of double cones $\mathcal{K}$, a *net of local algebras* is a functor

$$\mathbb{A} : \mathcal{K} \longrightarrow \mathbf{C^*Alg}.$$

Defining $\mathbb{A}$ as functor means that, for double cones $\theta_1, \theta_2 \in \mathcal{K}$, if $\theta_1 \subseteq \theta_2$ then $\mathbb{A}(U) \subseteq \mathbb{A}(V)$. This means that what can be observed in the region θ_1 can be observed in the region θ_2, since θ_1 is part of θ_2. From this assignment, one can use the machinery of operator algebras to construct a physical theory. Recall that a *directed poset* is a poset $\mathcal{P} = (P, \leq)$ such that every subset $S \subset P$ is bounded above, i.e. there exists $b \in P$ such that

$$\forall s \in S : s \leq b.$$

For Minkoswki spacetime and other Lorentzian manifolds, the poset of double cones $\mathcal{K}$ is a directed poset. This allows the standard construction of the direct limit of algebras to be used.[3] This creates a *quasi-local algebra* $\hat{\mathbb{A}}$, for which there exists an embedding $\mathbb{A}(U) \hookrightarrow \mathbb{A}$ for every double cone U. The algebra $\hat{\mathbb{A}}$ should be thought of as a 'large' algebra encompassing all the local algebras $\mathbb{A}(U)$ at each double cone U.

Let us give two examples that illustrate the importance of $\hat{\mathbb{A}}$. Firstly, it is the main tool for connecting the axioms of AQFT to field-theoretic calculations. This is done by considering representations of this algebra on the bounded operators of a Hilbert space $B(\mathcal{H})$:

$$\pi : \hat{\mathbb{A}} \longrightarrow B(\mathcal{H}) \tag{3.2}$$

Secondly, to impose causality conditions, we use $\hat{\mathbb{A}}$. Specifically, *Einstein causality* is a condition that is imposed in AQFT: this means that for doubles cones U, V, if U and V are spacelike separated, then the local algebras $\mathbb{A}(U)$ and $\mathbb{A}(V)$ pairwise commute in $\hat{\mathbb{A}}$. The definition of Einstein causality shows that the approach of algebraic quantum field theory is to build the theory using the regions of the spacetime, i.e. the double cones, whilst minimising reference to the role of points of a spacetime. There are several reasons for this, but aside from technical reasons, the main physical reason has a relation to our project. This reason is that the aim of AQFT is to build the theory entirely from observables, viz. the C^*-algebras $\mathbb{A}(\theta)$. The representation on a specific Hilbert space (i.e. Eq. 3.2) is a *subsequent* notion. The physical motivation for this is essentially operational, since AQFT originated with the idea that each element θ of $\mathcal{K}$ is a 'local laboratory', in which the observables of $\mathbb{A}(\theta)$ are available to observers. As we briefly described, the mathematical constructions for the theory *start with* the net of local algebras, and hence define the theory by using the observational data $\mathbb{A}(\theta)$ of experimenters as *primitives*.

This operational motivation would seem to make AQFT especially interesting for our aim of describing causal structure in a relativistic setting. Let us mention another reason why AQFT is interesting for us. This is one that we have encountered already. In Remark 2.32 we mentioned that one of the most interesting uses of dagger compact categories is in the Doplicher-Roberts theorem. This is a theorem in the framework of AQFT, allowing the gauge group $\mathcal{G}$ of a theory (i.e. the internal symmetry group) to be constructed from its category of representations. This is in the operational spirit of AQFT mentioned above, since it constitutes reconstructing the symmetry group of particle charges from a structure which can be traced back to the observable part of the theory, i.e. the net of local algebras. Hence AQFT would seem to satisfy three conditions that are desirable for our project: (i) it has a strong formal connection to CQM, i.e. using the same mathematics of dagger compact categories; (ii) it is a formalism based on quantum theory in spacetime; (iii) it has an operational foundation. These reasons might even suggest that AQFT is the natural source of ideas for an extension of CQM to a field-theoretic setting. An approach based on this idea this has been pursued in [8].

However, set against these reasons is the following. AQFT is *not* a formalism based primarily on causality. That is, the domain of the net of local algebras does not encode causal structure directly. Instead, it encodes the inclusion of regions. The inclusion ordering is not appropriate for representing even simple types of quantum evolution, since quantum evolution is an equation for causally related systems (e.g. a unitary $\Phi = U \circ \Psi$ between spacelike hypersurfaces with states Ψ and Φ). But since our quantum information perspective requires this type of simple evolution (e.g. a CNOT gate), AQFT would not seem to be suited to our aim of describing features such as entanglement with background causal structure. We could try to define a causal ordering for the double cones, but there exist different choices for this ordering, and each of them has deficiencies. As an example, consider the order $\sqsubseteq$ defined by Crane and Christensen [25], for the purpose of describing quantum theory in spacetime. This

[3] The direct limit of algebras is analogous to the direct family of sets. The latter is defined as follows. Let $\{A_i\}_{i\in I}$ be an indexed family of sets, such that the index set I is a directed set, and there exists a function $\phi_{ij} : A_i \to A_j$ whenever $i \leq j$. For elements $a \in A_1$ and $b \in A_2$, we define an equivalence relation $a \sim b$ if and only if there is A_k such that $\phi_{ik}(a) = \phi_{jk}(b)$. Then we define the direct limit A_∞ of $\{A_i\}_{i\in I}$ as the quotient $A/\sim$. The direct limit of algebras used in an AQFT is analogous but requires lifting the algebra structure to $\hat{\mathbb{A}}$. See [93] for a full definition of the direct limit of algebras.

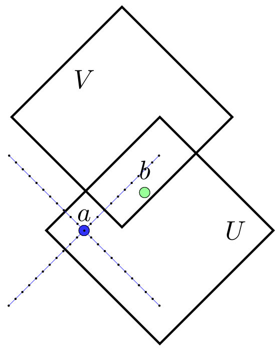


Figure 3.3: A space-time diagram showing that the pair of double cones U and V do not satisfy $U \sqsubseteq V$ using the Crane-Christensen ordering. The forward light-cone of the element $a \in U$ does not include $b \in V$, hence the Crane-Christensen condition fails here.

is defined as follows. For $U, V \in \mathcal{K}$

$$U \sqsubseteq V \quad \text{iff} \quad \forall p \in U, \forall q \in V, p < q.$$

As can be seen from Figure 3.3, the Crane-Christensen ordering on $\mathcal{K}$ is problematic, since it does not give $U \sqsubseteq V$ for a pair of double cones that provide an example of evolution from U to V. In other words, overlapping double cones U, V are not related by the order $\sqsubseteq$, despite the fact that every point $q \in V$ is in the future light cone of some point $p \in U$. Hence although AQFT has an interesting formal relationship to the setting of CQM (viz. the appearance of dagger compact categories), this relationship is apparently not useful for our particular aim of describing causality. However, we note that in [8] an order on double cones has been proposed (the *Egli-Milner* order) for which U and V in Figure 3.3 *are* causally related. However, the constructions involved have no straightforward relationship to categorical quantum mechanics, and so we shall not discuss that work further.

2. Quantum causal histories. A more relevant approach has been pursued by Markopoulou [77], called a *quantum causal history*. This also starts with a poset $\mathcal{P} = (P, \leq)$, which is assumed to represent the causal relations in a spacetime $\mathcal{M}$ (note that the poset is abstract, in the sense that it is not defined using the causal curves of a spacetime—hence we denote it with $\leq$ as before). Specifically, the spacetime data is defined to be a *causal set*, meaning that, for any two events $x, y \in P$, there exist finitely many events $z \in P$ such that $x \leq z \leq y$. Now, the role of double cones in AQFT is taken by *acausal sets* in a quantum causal history: these are antichains, i.e. subsets ξ of $\mathcal{P}$ such that[4]

$$\forall x, y \in \xi : x \not\leq y \wedge y \not\leq x.$$

We can define the analogue of certain notions in a spacetime manifold as follows. A *future-directed path* is a chain in $\mathcal{P}$, i.e. a subset $C \subseteq P$ such that

$$\forall x, y \in C : x \leq y \vee y \leq x$$

A future-directed path C is *future inextendible* if there is no event $y \in P$ such that

$$\forall x \in C : x \leq y$$

[4] We use $\vee$ and $\wedge$ for the logical operators 'or' and 'and' respectively.

and, similarly, a future-directed path is *past inextendible* if there no event $y \in P$ such that

$$\forall x \in C : y \leq x.$$

In other words, a path C is future (respectively, past) inextendible when there is no event that is to the future (respectively, past) of the entire path C, i.e. the path C is not bounded from above (respectively, below). We can then define the discrete analogue of spacelike hypersurfaces. A *complete future* of an event x is an acausal set ζ such that ζ intersects all future-inextendible future-directed paths starting at x. Similarly, a *complete past* of an event x is an acausal set ζ such that ζ intersects all past-inextendible future-directed paths ending at x. A complete future of an event x can be thought of as a hypersurface in the timelike future of x. Then we extend the notions of complete future and complete past from events to acausal sets. We say that ζ is a *complete future* of an acausal set ξ if ζ is a complete future for all events $x \in \xi$; similarly ξ is a *complete past* of an acausal set ζ if ξ is a complete past for all events $y \in \zeta$. Finally, we extend the notation $\leq$ to acausal sets: we write $\xi \leq \zeta$ to denote that ξ and ζ form a *complete pair*, meaning that ξ is the complete past of ζ and ζ is the complete future of ξ.

We can now define a quantum causal history.

Definition 3.7. A *quantum causal history (QCH)* is the following assignment:

- a Hilbert space $\mathcal{H}(x)$ to each element x of a causal set $\mathcal{C}$;
- tensor products $\mathcal{H}(\xi) := \mathcal{H}(x) \otimes \mathcal{H}(y)$ for an antichain $\xi = \{x, y\}$; and
- a unitary map between complete pairs $\xi \leq \zeta$:

  $$U(\xi, \zeta) : \mathcal{H}(\xi) \longrightarrow \mathcal{H}(\zeta).$$

  such that the multiplication of unitaries satisfies:

  $$U(\nu, \zeta)U(\xi, \nu) = U(\xi, \zeta)$$

  when both $\nu \leq \xi$ and $\xi \leq \zeta$ are complete pairs.

The idea behind unitary evolution for a complete pair is that a complete pair represents a pair of spacelike hypersurfaces, and hence the evolution should be the evolution of a closed system. QCHs have been used mainly for theoretical analysis of quantum systems with a background causal structure, for example, to provide a different kind of quantum logic [77].

However, our concern will be *information flow* in quantum protocols. With this in mind, let us consider examples of quantum causal histories in more detail. For example consider the two causal structures shown in Figure 3.4. Both involve the complete pair $\xi \leq \zeta$, for which there are Hilbert space assignments $\mathcal{H}(\xi) = \mathcal{H}(x) \otimes \mathcal{H}(y)$ and $\mathcal{H}(\zeta) = \mathcal{H}(v) \otimes \mathcal{H}(w)$. As pointed out in [58], according to the definition of a QCH, both will be assigned the same unitary:

$$U(\xi, \zeta) : \mathcal{H}(\xi) \longrightarrow \mathcal{H}(\zeta).$$

This is because a QCH does not take into account how ξ are related ζ are related, except insofar as they form a complete pair. For example, for the diagram $\mathcal{B}$ in Figure 3.4, let the initial state at ξ be

$$|\psi_x\rangle \otimes |\psi_y\rangle \in \mathcal{H}(x) \otimes \mathcal{H}(y).$$

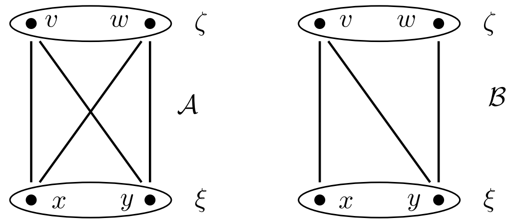


Figure 3.4: Two examples of causal structure for a QCH. Lines indicate that the events are causally related.

Then the final state at ζ is:

$$|\Phi\rangle := U(\xi, \zeta)(|\psi\rangle_x \otimes |\psi_y\rangle). \tag{3.3}$$

The marginal state at the event $w \in \zeta$ is the partial trace:

$$\rho_w := \mathrm{Tr}_{\mathcal{H}(v)} |\Phi\rangle\langle\Phi|. \tag{3.4}$$

From the causal structure of $\mathcal{B}$, the state ρ_w should not depend on ψ_x. That is, the function $f_{\mathcal{B}}$ for the causal structure $\mathcal{B}$ induced by Eq. 3.3 and Eq. 3.4 (for fixed $|\psi_y\rangle$), given by

$$f_{\mathcal{B}} :: |\psi_x\rangle \longmapsto \rho_w$$

should be constant (although the analogous function $f_{\mathcal{A}}$ for causal structure $\mathcal{A}$ need not be). But there is no restriction in the definition of a QCH that prohibits this. This is a deficiency that our approach, discussed in Chapters 5 and 6, will not share.

Chapter summary. We have discussed causal structure from two perspectives. Firstly, we discussed the reconstruction theorems that identify a relativistic spacetime from its causal structure, in particular Malament's theorem. These results illustrate the importance, from a foundational perspective, of causal structure in relativity. Secondly, we have discussed two ways of combining relativistic causal structure with quantum state spaces. We noted the attendant problems, and these provide context for our later work.

Outlook. Part of the aim of what follows is to broaden the operational or structural approaches in foundations to include a more detailed understanding of the formal compatibility between quantum theory, and possible extensions, and (special) relativity. This project has only recently been a concern of foundations of quantum theory [55, 84]. But as Hardy has discussed [56], the understanding of this compatibility is one of the great problems of physics, and it therefore should be a target of foundational work.

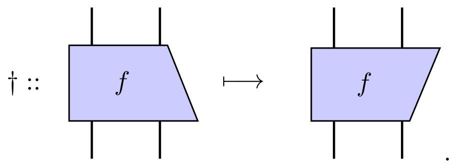
Chapter 4

Time-asymmetry and causal structure

Contents

In the previous Chapter we discussed our aim of understanding the compatibility between relativistic causality and quantum theory from a structural perspective. In this Chapter we shall consider the no-signalling conditions that physical devices must satisfy to be consistent with background relativistic causality. This will allow us to consider arbitrary 'no-signalling devices'. The abstract study of these no-signalling devices is well-suited to our aim, since we shall not need to describe how the devices are realised in detail (e.g. the spin measurements occurring inside the devices). Accordingly, in this Chapter we consider whether no-signalling devices share the properties of the spacetime in which they reside. In particular, we shall consider the property of time-reversal invariance. We will show that there exist no-signalling devices for which time-reversal turns a non-signalling device into a perfect signalling device, despite the fact that the agents may be located outside of each other's light cones. In particular, a *non-channel in one time direction* becomes a *perfect channel in the other direction*, contra the time-reversal symmetry of relativity.

CQM does not play a central role in this chapter— but we have arrived at these results by considering the dagger of a no-signalling process:



Does this exist, does it have physical meaning, and is it non-signalling? These questions are motivated by CQM, since answering them establishes whether a dagger category of no-signalling correlations can be constructed.

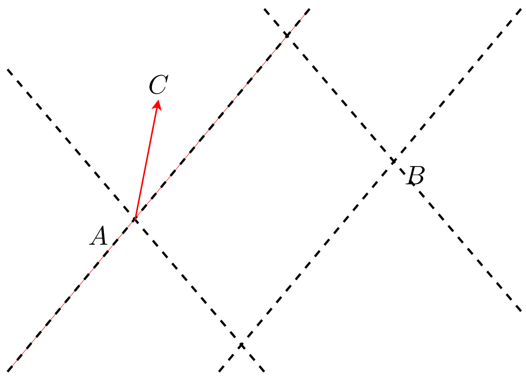
4.1 Time-reversal and signalling

4.1.1 Defining time-reversal

Time-reversal invariance is a subtle symmetry: the other known spatiotemporal symmetries, except for parity inversion, can be considered to be 'operational'. For example, whether spatial-translation invariance is a property of a theory can be checked by experiments which implement spatial translation. In contrast, time-reversal invariance cannot be implemented. However, there are two ways of understanding what time-reversal invariance is. Firstly, it means that, given some observed statistics, is the set of time-reversed statistics consistent with physical law? Equivalently: could the time-reversed statistics happen in the forward (actual) direction? Secondly, for the devices we shall consider below, could the time-reversed device be built (in the forward direction)?

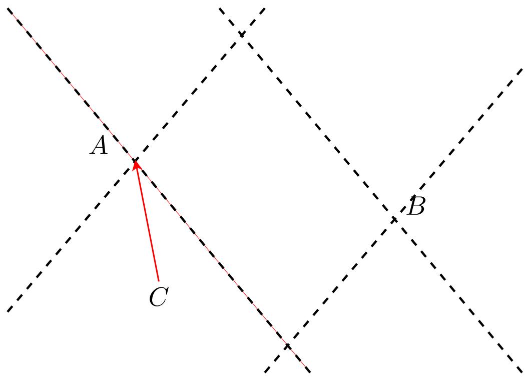
Time-reversal for a spacetime manifold

Consider a spacetime manifold $\mathcal{M}$, e.g. Minkowski spacetime $\mathbb{M}$. As before, there exists a partial order $<$ on points in $\mathcal{M}$, defined by $x < y$ iff there is causal curve from x to y. We denote the locations of agents Alice and Bob by A and B respectively[1]. Let A and B be space-like separated, i.e. $A \nless B$ and $B \nless A$. Then Alice is not able to signal to Bob, and vice versa. Consider another agent Charlie, located at C, such that A and C are time-like related: A causally precedes C and so we have $A < C$ and $C \nless A$. For Alice and Charlie, signalling is in principle possible, along the direction of time, but not backwards.



We now want to consider time-reversal of $\mathcal{M}$. In the case of Minkowski spacetime, the intuitive notion is that of 'flipping' the light-cone structure upside-down:

[1]We are assuming that the agents' locations can be described by points of $\mathcal{M}$. To be more realistic, we could instead describe their locations using regions $r_A, r_B \subset \mathcal{M}$. This would not affect the subsequent argument, except that we would now have to introduce a causal structure for regions, such as the Crane-Christensen relation $\sqsubseteq$ used above, defined as $r_1 \sqsubseteq r_2$ iff $\forall x \in r_1, \forall y \in r_2, x < y$.



That this intuitive notion is correct, and extends to general spacetimes, rests on the following two (contingent) facts about time-symmetry in general relativity[2] :

1. As Wald [105] emphasises, despite the restrictions that Einstein's equations makes on the metric, it makes no restriction on the time-orientation of the metric. Hence if a solution $(\mathcal{M}, \leq)$ exists then a solution $(\mathcal{M}, \leq^{op})$ with the opposite time-orientation exists. Indeed, if this were not the case then it would not make sense to consider the time-reversal of $\mathcal{M}$. (Although we note that the time-reversed spacetime may have very different properties, which might be unphysical, but only when considering principles additional to the core of relativity. For example, the time-reverse of a black hole is a white hole, and the latter violates the Cosmic Censorship hypothesis [85].)

2. Time-reversal preserves spacelike separation. Let us denote the partial order of $\mathcal{M}^{op}$ as $<^{op}$, and if $A <^{op} B$ is false we write $A \not<^{op} B$. Then $A < B$ if and only if $B <^{op} A$, and so $A \not< B$ if and only if $B \not<^{op} A$. So if $A \not< B$ and $B \not< A$ then $A \not<^{op} B$ and $B \not<^{op} A$, i.e. time-reversal preserves spacelike separation.

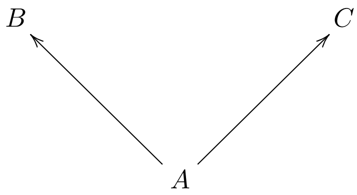
Remark 4.1. It might objected that time-reversal for a spacetime has little conceptual significance, because a spacetime is usually thought of as a 'block universe', for which each hypersurface has equal claim to existence. However, we shall take an operational perspective, and in particular we aim to establish how the causal constraints *appear* to agents in a spacetime (which is a perspective that is consistent with the block universe view). Hence our concern is how agents in the spacetime can signal to one another, and $A \leq B$ means that A can signal to B in $\mathcal{M}$, if there are agents located at A and B. So in operational terms, 'preservation of spacelike separation' means that time-reversal does not introduce the ability to signal.[3]

Remark 4.2. From a formal perspective, note that preservation of spacelike separation under time-reversal is a weaker property than a partial order $\leq$ being isomorphic to its opposite $\leq^{op}$. In the fragment of Minkowski space (consisting of the light cones of two parties) that we have considered above, the partial order $\leq$ is indeed

[2] These two facts seem to be logically independent: even if the time-reverse of a solution is guaranteed to exist in a particular theory of spacetime, it may not be that a pair of points remain spacelike separated in the time-reversed spacetime. However, a counter-example to their logical dependence will have to be pathological. For example, one could consider a variable-speed-of-light theory, such as has been considered in [73]. A crude approximation of such a theory would be to consider light cones which are 'non-linear', since the speed of light is changing with respect to some chosen time parameter. However, it is straightforward to see that even in this theory time-reversal preserves spacelike separation of a pair of points.

[3] We are therefore taking an operational view of the constraints of relativity, despite the fact that the theory makes no mention of agents or experiments. This may seem conceptually problematic, but the motivation for studying no-signalling devices is exactly such a view.

isomorphic to $\leq^{op}$. But consider the following partial order $\mathcal{P}$ (writing it is a category):



Spacelike separation is preserved under reversing the order of $\mathcal{P}$, but $\mathcal{P}$ is not isomorphic to its opposite $\mathcal{P}^{op}$, because the hom-set $\mathcal{P}(A, B)$ is non-empty, since $A \leq B$, but $\mathcal{P}^{op}(A, B)$ is not, since $A \not\leq^{op} B$.[4]

Time-reversal for I/O-boxes

Now consider a device with two inputs $a_I, b_I \in \{0, 1\}$, and two outputs $a_O, b_O \in \{0, 1\}$. We assume that Alice and Bob are spacelike separated, and Alice has access to a_I and a_O, and Bob has access to b_I and b_O. Such devices mimic the Bell-type setup, since the inputs can be thought of as measurement choices, e.g. spin measurements along two different axes, and the outputs would then be spin up or down outcomes. We call these devices *input-output (I/O) boxes*; an example is depicted in Figure 4.1.

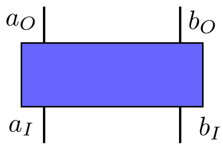
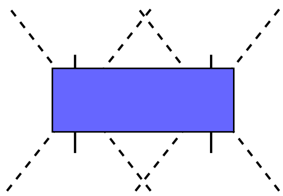


Figure 4.1: Bipartite probabilistic input-output box.

To each such device we associate a conditional probability distribution $P(a_O, b_O \mid a_I, b_I)$. Now, Alice, Bob and the I/O box are located in spacetime[5]:



Since Alice and Bob are spacelike separated whilst interacting with the device, the correlations are non-signaling. If this is the only constraint then the device could exhibit any non-signaling correlations: classical (i.e. shared randomness), quantum or super-quantum.

We have defined time-reversal for the spacetime above. We now want to consider the behaviour of I/O-boxes under time-reversal. There are two issues that present themselves:

1. *Statistical sufficiency*: In order to collect the required relative frequencies, the statistical ensemble will typically involve repeatedly using the same I/O-box. But these relative frequencies would be the result of

[4] In fact, an arbitrary category $\mathbf{C}$ is usually not isomorphic to $\mathbf{C}^{op}$. For example, even $\mathbf{FinSet}$ is not isomorphic to $\mathbf{FinSet}^{op}$: any isomorphism F must preserve limits. In particular F must preserve the terminal object, so $F(\{*\}) = \emptyset$. But then for all $A \neq \emptyset$, $|\mathbf{FinSet}(\{*\}, A)| \neq |\mathbf{FinSet}^{op}(F(\{*\}), F(A))|$, since the latter is empty whereas $|\mathbf{FinSet}(\{*\}, A)|$ has the cardinality of A.

[5] Without loss of generality we can assume that the output is received immediately after the input is supplied.

the behaviour of the I/O-box at different spacetime points, since each repetition will occur at a later time—i.e., there will be time delays between successive experiments. Since we want to test the causal structure of specific spacetime points (a single pair of points), we would instead have to provide an ensemble by using an arbitrary number of boxes at the same spacetime location rather than the same box repeatedly used. This issue that will not concern us further, but it is one which would be interesting for further consideration.

2. *Implicit temporal orientation*: Since the definition of an I/O-box is an abstraction of a Bell-type experiment, its actual operation would typically involve Alice and Bob supplying spin directions as inputs, and each receiving an up or down outcome as an output. Such an implementation seems obviously asymmetric, because the input and output have different physical forms, e.g. the former might be the choice of spin direction on a dial, whereas the latter might be an electronic display. In other words, I/O-boxes seem to already implicitly have a temporal orientation.

To address the second point, we can remove any implicit temporal orientation by making the input and output symmetric in implementation. We can do this by ensuring that the agents interact with an I/O box only through a slot for each agent. This is depicted in Figure 4.2. Then, to input a_I or b_I, Alice or Bob respectively puts a card into the slot. The output a_O or b_O occurs when the device releases a card through the slot to Alice or Bob respectively.

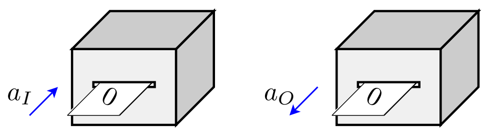


Figure 4.2: Exchange of input and output by time-reversal.

The symmetry of the situation now ensures that the time-reversal of the observed statistics also clearly exchanges the role of the inputs and outputs, since now each is the time-reverse of the other.

As we discussed above, time-reversal can be understood operationally as rewinding a videotape of the observed statistics. But what is the mathematical description of time-reversal for such a probabilistic device? We must map a conditional probability distribution with I as givens and O as conclusions to a a conditional probability distribution which has O as givens and I as conclusions. This can only be done via *Bayesian inversion*:

$$P(I|O) = \frac{P(O|I)P(I)}{P(O)}. \tag{4.1}$$

When the tape of the experiment is played backwards, the statistics will now be the same as obtained via Bayesian inversion.

Now, this requires the prior probability distribution $P(I)$. However, in a Bell-type setup this is supposed to be a free choice by the agents. Viewed another way, in such a setup the prior $P(I)$ is usually assumed to be uncorrelated to any random variables in the past light cones of the observers. Hence it is desirable to obtain results that are independent of the choice of prior. We shall do so in the next Subsection.

4.1.2 Signalling under time-reversal

Let us consider the conditional probability distribution $P(a_O, b_O \,|\, a_I, b_I)$ associated with a bipartite I/O-devices as described above. We have already assumed that $a_I, b_I, a_O, b_O \in \{0,1\}$. To these devices we can assign a 4-by-4 *(probabilistic) correlation matrix* $g = (g^{a_O,b_O}_{a_I,b_I})$ of which the entries $g^{a_O,b_O}_{a_I,b_I}$ give the probability of obtaining output pair (a_O, b_O) given input pair (a_I, b_I). Hence by a correlation matrix we just mean a stochastic matrix (i.e. entries are $[0,1]$-valued and columns sum to 1), but with the variables so defined. [6]

As discussed above, we assume that the devices are non-signalling, meaning that a_O and b_O are conditionally independent of b_I and a_I respectively, that is:

$$P(a_O|a_I, b_I) = P(a_O|a_I)$$

and

$$P(b_O|a_I, b_I) = P(b_O|b_I).$$

We shall work with both the distributions $P(a_O, b_O|a_I, b_I)$ and the matrices g so it is useful to make this explicit in the stochastic matrix, especially because we shall later consider a generalisation of g in which the entries are not probabilities.

Definition 4.3. A bipartite correlation matrix allows *(probabilistic) signalling from Alice to Bob* iff

$$\exists(b_I, b_O) : g^{0,b_O}_{0,b_I} + g^{1,b_O}_{0,b_I} \neq g^{0,b_O}_{1,b_I} + g^{1,b_O}_{1,b_I} . \tag{4.2}$$

The sums reflect that fact that the value of Alice's output is not known to Bob, and hence is marginalised. So by Alice signalling to Bob we mean that, from his input-output pairs, and after a sufficient number of rounds (i.e. in the statistical limit), Bob has obtained information about Alice's sequence of inputs. A correlation matrix is *signalling* if it allows either signalling from Alice to Bob, or signalling from Bob to Alice; for simplicity below we shall consider only signalling from Alice to Bob.

A *classical* I/O-box has a local hidden variable decomposition, i.e.

$$P(a_O, b_O|a_I, b_I) = \sum_\lambda P(a_O|a_I, \lambda)P(b_O|b_I, \lambda)P(\lambda). \tag{4.3}$$

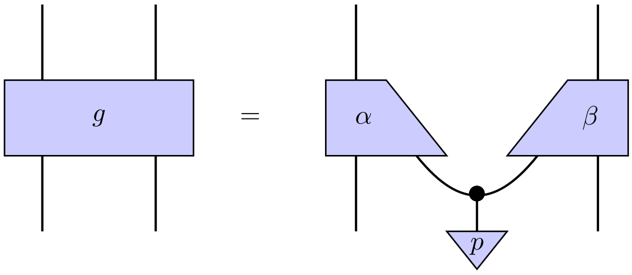
Definition 4.4. A bipartite correlation matrix g is *classical* if there exist 2-by-2 stochastic matrices $\{\alpha_\lambda\}$ and $\{\beta_\lambda\}$ and $\{p_\lambda\}$ for which $p_\lambda \in [0,1]$ and $\sum_\lambda p_\lambda = 1$, such that g decomposes as follows:

$$g = \sum_\lambda p_\lambda \alpha_\lambda \otimes \beta_\lambda \tag{4.4}$$

Remark 4.5. Since we are using stochastic matrices, no-signalling processes are morphisms in the category $\mathbf{Mat}(\mathbb{R}^+)$. This is a †-SMC for which objects are natural numbers $n \in \mathbb{N}$ and morphisms $f : n \to m$ are $n \times m$ matrices with entries in $\mathbb{R}^+$. The monoidal product is given by the Kronecker product of matrices. We can use classical structures as defined in Chapter 2 to represent the hidden-variable distribution p_i. Definition 4.4 can then

[6] Our use of the term 'correlation matrix' is possibly non-standard, since this could also be used to denote, e.g., a covariance matrix. However, we have chosen this terminology because we shall later generalise from the usual notion of probabilities valued in $[0,1]$ (hence we use the word 'correlation'), and we also want to emphasise the algebraic form of the list of probabilities ('matrix').

be depicted as:



Proposition 4.6. *Classical I/O boxes are non-signalling.*

Proof. Using the conditional probability distribution of Eq. 4.3, after summing over b_O we have:

$$P(a_O|a_I,b_I) = \sum_\lambda P(a_O|a_I,\lambda)P(\lambda) = P(a_O|a_I)$$

□

Now, given g and a prior $P(I) := P(a_I, b_I)$, we rely on Bayesian inversion to construct the time-reversed stochastic matrix $g^T_{P(I)}$. This is given explicitly as:

$$(g^T_{P(I)})^{a_I,b_I}_{a_O,b_O} = \frac{g^{a_O,b_O}_{a_I,b_I} \times P(a_I,b_I)}{P(a_O,b_O)} \tag{4.5}$$

where

$$P(a_O,b_O) = \sum_{a_I,b_I} g^{a_O,b_O}_{a_I,b_I} P(a_I,b_I)\,. \tag{4.6}$$

We shall call a distribution $P(I)$ *total* if it has full support.

Remark 4.7. In what follows, we shall restrict to prior probability distributions $P(I)$ which are total. If we did not do this, we would not be consistent with a Bell-type setup, since for a distribution $P(I)$ without full support there is a pair (a_I, b_I) which cannot occur. But then some input pairs to the device are not being used.

As described in the previous section, Bayesian inversion for our devices allows variables a_O and b_O to be treated as the inputs and the variables a_I and b_I to be treated as the outputs. By *perfect signalling* we mean that Bob receives Alice's input as his output with certainty.

Theorem 4.8. *There exist classical correlation matrices for which the time-reverse for any total prior is signalling. More specifically, each such time reverse of*

$$\tilde{g} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \tag{4.7}$$

allows perfect signalling from Alice to Bob, which is achieved when Bob has input 0*.*

Proof. First we observe that $\tilde{g}$ is indeed classical:

$$\begin{aligned}\tilde{g} \;&=\; \frac{1}{4}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{1}{4}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\\ &\quad+\frac{1}{4}\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\end{aligned}$$

and hence, by Proposition 4.6, $\tilde{g}$ is non-signalling. Using Eq. 4.5 we establish that the Bayesian inverse allows perfect signalling, since for any $P(I)$ the matrix $\tilde{g}^T_{P(I)}$ has the form:

$$\tilde{g}^T_{P(I)} = \begin{pmatrix} a & b & 0 & f\\ 0 & c & e & g\\ 1\text{-}a & d & 0 & h\\ 0 & 1\text{-}b\text{-}c\text{-}d & 1\text{-}e & 1\text{-}f\text{-}g\text{-}h \end{pmatrix} \tag{4.8}$$

Each column in Eq. 4.8 now represents a pair of inputs for the time-reversed process, and each row a pair of outputs. Now, assume that Bob's input[7] for $\tilde{g}^T_{P(I)}$ is 0. Then when Alice's input is 0 the output will be $(0,0)$ with probability a and it will be $(1,0)$ with probability $1-a$, and when Alice's input is 1 the output will be $(0,1)$ with probability e and it will be $(1,1)$ with probability $1-e$. Hence, Bob's output always perfectly matches Alice's input. □

We can make the notion of 'perfect signaling' more precise by using the channel capacity C. In calculating the channel capacity, the Alice's input is now a_O, and Bob's output is now b_I. Let us use a different notation: we define Alice's input in the reverse direction as $x := a_O$, and Bob's output as $y := b_I$. Then the channel capacity is given by

$$C = \sup_{p_x} I(x:y)$$

where the mutual information $I(x:y)$ is given in terms of entropies as

$$I(x:y) = H(x) + H(y) - H(x,y). \tag{4.9}$$

Let Alice's input distribution for x be $p(x=0)=\alpha_1$ and $p(x=1)=1-\alpha_1$. As we showed in the proof of Theorem 4.8, under time-reversal, the classical binary I/O-process g in Theorem 4.8 gives rise to the identity matrix for the Alice to Bob channel (when Bob chooses input $b_O = 0$). That is, we have the conditional probability distribution $p(y \mid x) = \delta_{x,y}$, and it then follows that $I(x:y) = 1$ for $\alpha_1 = 1/2$ using Eq. 4.9. For clarity we depict this in Figure 4.3. We emphasise that we are not claiming that there *is* a perfect channel from Alice to Bob,

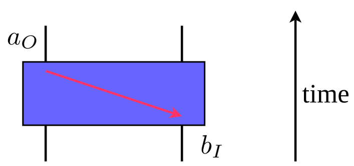


Figure 4.3: Channel from Alice to Bob, shown by the red line.

[7] Note that Bob's input here is in the time-reversed direction, and the distribution over the time-reversed inputs is not $P(I)$ which is the input in the forward direction; $P(I)$ is required to be a total distribution but not Bob's time-reversed input.

going backwards in time, but only that there appears to be such a channel, according the time-reversed stochastic matrix.

The significance of this Theorem 4.8 is two-fold:

1. *Conflict between relativity and probabilistic processes*: There is an apparent conflict between the time-symmetry of relativistic causal structure and the time-asymmetry of the no-signalling conditions. Theorem 4.8 shows that no-signalling boxes *appear* to be signalling boxes under time-reversal. That is, given the backwards statistics of an experiment with a device as in Theorem 4.8, we *infer* that the parties have a signalling device. Although signalling does not necessarily occur, the statistics imply that Alice and Bob share a signalling box in the time-reversed direction.
2. *Arrow of time*: In addition to this conflict, Theorem 4.8 reveals an *arrow of time.* That is, suppose that we are given two sequences of experimental data from a use of the device in Theorem 4.8, one being the forward direction, and the other being the backward direction. From the underlying physics we cannot infer which is the forward and which is the backward direction. However, by studying correlations of the devices above we *can* detect the backward direction of time: this is the direction in which there exist devices that potentially enable signalling between space-like separated regions.

4.2 Conditions for backwards signalling

Since Theorem 4.8 used a classical matrix, the question arises as to what other classes of no-signalling devices lead to backwards signalling, and whether this has any relation to nonlocality. To address this question we shall first consider an easier way to derive backwards signalling.

Possibilistic reasoning

Theorem 4.8 involved the Bayesian inversion of a classical probability distribution. As we mentioned, this distribution is a morphism in $\mathbf{Mat}(\mathbb{R}^+)$. Now, we defined classicality of this distribution using the diagrammatic equation in Remark 4.5. This diagrammatic equation will apply to other SMCs. Hence this suggests that we might also be able to consider the notions of classicality and no-signalling in other SMCs. Indeed, instead of $\mathbf{Mat}(\mathbb{R}^+)$, we can consider SMCs which are of the form $\mathbf{Mat}(\mathcal{R})$ for a semi-ring $\mathcal{R}$ (since any semi-ring admits a matrix calculus [51]).[8]

Let us consider the Boolean semi-ring $\mathbb{B}$ (recall also that $\mathbf{Mat}(\mathbb{B}) \simeq \mathbf{fRel}$). This means representing processes using *possibilities*, that is, for which pairs of inputs certain outputs are *possible.* A matrix of Boolean values can be considered a correlation matrix, but the entries are only 0s and 1s, standing for 'impossible' and 'possible' respectively. The interpretation here is similar to that of a probabilistic correlation matrix. The difference is that now relative frequencies do not play the same role, but only whether an outcome can occur or not (this justifies using Boolean addition, for which $1 + 1 = 1$). We could imagine that this arises because the experimenters are somehow limited in their ability to observe or record relative frequencies. There is a semi-ring homomorphism

[8] Recall that a *semi-ring* $\mathcal{R} = (R, +, \times)$ is a set R with binary operations $+$ and $\times$ which correspond to addition and multiplication respectively. Also, $(R, +)$ is a commutative monoid and $(R, \times)$ is a monoid, satisfying distributivity conditions. That is, a semi-ring is a ring without additive inverses. Hence $\mathbb{R}^+$ is a semi-ring but not a ring.

$h : \mathbb{R}^+ \to \mathbb{B}$, which induces a functor:

$$R : \mathbf{Mat}(\mathbb{R}^+) \longrightarrow \mathbf{Mat}(\mathbb{B}) \tag{4.10}$$

$$f \longmapsto \text{supp} f \tag{4.11}$$

We call this the *relational collapse functor*. It ensures every probabilistic correlation matrix M gives rise to a possibilistic correlation matrix by taking the support of the distribution. For example, for the correlation matrix in Theorem 4.8, it provides the mapping

$$R :: \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Hence we can consider R to be a coarse-graining of a probabilistic distribution to a *possibilistic distribution*. Possibilistic reasoning for nonlocality has also been studied by Abramsky in [3].

Recall that, given a prior $p(I)$, we denoted the Bayesian inversion of a probabilistic correlation matrix g as $g^T_{P(I)}$. The relational collapse of this is therefore denoted $R(g^T_{P(I)})$. The following proposition shows how the definition of time-reversal, i.e. Bayesian inversion, is simplified by relational collapse.

Proposition 4.9. *Let g be a probabilistic correlation matrix. Then for any $p(I)$ with full support, we have $R(g^T_{P(I)}) = R(g)^T$.*

Proof. From Eq. 4.5 we see that if the prior $p(I)$ has full support then g will have the same support as $g^T_{P(I)}$. □

Hence for a possibilistic correlation matrix the time-reverse is just the transpose, and there is no dependence on the prior $P(I)$. For the example of Theorem 4.8 we have:

$$R(\tilde{g}) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad R(\tilde{g})^T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{4.12}$$

The question then arises as to how signalling should be formulated. Hence by *possibilistic signalling* we shall mean Eq. 4.2, but with Boolean values instead of positive reals. This leads to the following proposition.

Proposition 4.10. *Let f be a possibilistic correlation matrix. Then if f is possibilistically signalling from Alice to Bob, then any g such that $R(g) = f$ is probabilistically signalling from Alice to Bob.*

Proof. Assume f is possibilistically signalling the for some pair (b_I, b_O) either (i)

$$f^{0,b_O}_{0,b_I} + f^{1,b_O}_{0,b_I} = 0 \qquad \text{and} \qquad f^{0,b_O}_{1,b_I} + f^{1,b_O}_{1,b_I} = 1 \tag{4.13}$$

or (ii) vice versa, i.e.

$$f^{0,b_O}_{0,b_I} + f^{1,b_O}_{0,b_I} = 1 \qquad \text{and} \qquad f^{0,b_O}_{1,b_I} + f^{1,b_O}_{1,b_I} = 0 \tag{4.14}$$

Consider case (i): suppose that Eq. 4.13 holds, i.e. $f^{0,b_O}_{0,b_I} + f^{1,b_O}_{0,b_I} = 0$. Hence both $f^{0,b_O}_{0,b_I} = 0$ and $f^{1,b_O}_{0,b_I} = 0$. Hence for any g such that $R(g) = f$, we have

$$g^{0,b_O}_{0,b_I} + g^{1,b_O}_{0,b_I} = 0.$$

On the other hand, if

$$f^{0,b_O}_{1,b_I} + f^{1,b_O}_{1,b_I} = 1$$

then either $f^{0,b_O}_{1,b_I} = 1$ or $f^{1,b_O}_{1,b_I} = 1$. Hence

$$g^{0,b_O}_{1,b_I} + g^{1,b_O}_{1,b_I} > 0,$$

and so Eq. 4.2 follows. Case (ii) proceeds similarly. □

Propositions 4.9 and 4.10 lead to the following characterisation of backwards-signalling for all priors.

Theorem 4.11. *Let g be a probabilistic correlation matrix. If $R(g)^T$ is (possibilistically) signalling, then g is backwards signalling for all priors (i.e. $g^T_{P(I)}$ is probabilistically signalling for all $P(I)$).*

Hence possibilistic collapse preserves signalling, and the conclusion of Theorem 4.8 can be obtained by considering the possibilistic collapse. This method is useful because it is an easier way to find examples of backwards-signalling matrices, and in particular, examples which are backwards-signalling for all priors. We can see from Eq. 4.12 that the relational collapse of the classical matrix of Theorem 4.8 is such an example: $R(\tilde{g})^T$ is possibilistically signalling, and $\tilde{g}^T_{P(I)}$ is signalling for all $P(I)$.

Using possibilistic signalling will be useful for answering the question of what other types of boxes lead to backwards-signalling for all priors $P(I)$. For consider the correlation matrix pr for a Popescu-Rohrlich box [89] and its time-reverse for the uniform prior $U(I)$:

$$pr = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{4.15}$$

The relational collapse and its transpose are given by

$$R(pr) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad R(pr)^T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{4.16}$$

Since $R(pr)^T$ is possibilistically signalling, we can infer from Theorem 4.11 that pr is backwards-signalling for all priors. This therefore shows that nonlocal correlations lead to the same phenomenon as Theorem 4.8.

However, we note that the PR box has a *weaker* form of backwards signalling that the example of Theorem 4.8, in the sense that the channel capacity is not maximal. The probability distributions that arise as the Bayesian inversion of the PR box can be written as

$$M = \begin{pmatrix} \beta_2 & 0 & 0 & \beta_1 \\ \beta_1 & 0 & 0 & \beta_2 \\ 1-\beta_1-\beta_2 & 0 & 0 & 1-\beta_1-\beta_2 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

for $\beta_i > 0, y_i > 0$. By symmetry we need only consider Bob's strategy of $b_I = 0$, hence we obtain a probability distribution:

$$\begin{pmatrix} \beta_1 & 0 \\ 1-\beta_1 & 1 \end{pmatrix},$$

where we have traced out Alice's outputs. We find that

$$\begin{aligned} & I(a_I : b_O) = \\ & \quad -\alpha_1 \log(\alpha_1) \\ & \quad -((1-\alpha_1) + \alpha_1(1-\beta_1)) \log((1-\alpha_1) + \alpha_1(1-\beta_1)) \\ & \quad +\alpha_1(1-\beta_1) \log(\alpha_1(1-\beta_1)). \end{aligned}$$

We obtain $I = 1$ only when $\alpha_1 = 1/2, \beta_1 = 1$, but we require $1-\beta_1-\beta_2 > 0$, and hence $\beta_1 < 1$, for M to arise as the Bayesian inversion of a PR box. Hence although PR boxes are maximally nonlocal, and backwards-signalling for all priors, the time-reverse of a PR box cannot achieve maximal signalling.

Now, what we have established is that signalling under possibilistic time-reversal is a *sufficient condition* for a correlation matrix to exhibit signalling under probabilistic time-reversal (viz. Bayesian inversion). But it may not be a necessary condition. This is relevant to quantum correlations. The correlation matrix B for a Bell-type setup using the Bell state and measurement angles ϕ_1, ϕ_2 for Alice and measurement angles θ_1, θ_2 for Bob, is:

$$B = \frac{1}{2} \begin{pmatrix} \cos^2(\theta_1-\phi_1) & \cos^2(\theta_2-\phi_1) & \cos^2(\theta_1-\phi_2) & \cos^2(\theta_2-\phi_2) \\ \sin^2(\theta_1-\phi_1) & \sin^2(\theta_2-\phi_1) & \sin^2(\theta_1-\phi_2) & \sin^2(\theta_2-\phi_2) \\ \sin^2(\theta_1-\phi_1) & \sin^2(\theta_2-\phi_1) & \sin^2(\theta_1-\phi_2) & \sin^2(\theta_2-\phi_2) \\ \cos^2(\theta_1-\phi_1) & \cos^2(\theta_2-\phi_1) & \cos^2(\theta_1-\phi_2) & \cos^2(\theta_2-\phi_2) \end{pmatrix} \tag{4.17}$$

For maximal Bell violation by a quantum state we set $\phi_1 = \frac{\pi}{4}, \phi_2 = 0$ and $\theta_1 = \frac{\pi}{8}, \theta_2 = \frac{3\pi}{8}$ and then the correlation matrix B in Eq. 4.17 has full support. Hence, maximal quantum violation cannot exhibit possibilistic backwards-signalling.

Before discussing more general conditions, we note that possibilistic time-reversal has an interesting conceptual interpretation. This is that possibilistic time-reversal (which is just taking the transpose) is analogous to time reversal with respect to a causal order: both involve transposing the order of pairs in a relation, either pairs $((x_a, t_a), (x_b, t_b))$ in the causal order relation $<$, or pairs $\big((a_I, b_I), (a_O, b_O)\big)$ which relates inputs of processes to outputs of processes, where in the latter case it can be seen as quantifying over all priors.

General conditions

If each outcome of a binary I/O-process M is possible for some input (i.e. for each output $P(O) > 0$), we say that M is *co-total*. This corresponds to each row of the correlation matrix having at least one non-zero entry. We shall call a binary I/O-process *input-independent* if it has no dependence on the inputs, i.e. each column of the correlation matrix is the same.

Theorem 4.12. *If a co-total no-signalling binary I/O process is either:*

(i) deterministic, or

(ii) input-independent

then its time-reverse is not signalling for any prior.

Proof. (i) By enumeration, there are four co-total deterministic no-signalling matrices, for example:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Because these matrices are deterministic they are closed under Bayesian inversion, for any prior, and hence not backwards-signalling. Item (ii): the Bayesian inversion of an input-independent process is again input-independent for any prior [its columns are the prior $P(I)$], which guarantees that it is non-signalling. □

An example of an input-independent matrix is the 'maximally mixed' I/O-process M, for which $P(O|I) = \frac{1}{4}$ for each entry of its correlation matrix.

Remark 4.13. Further general conditions, either necessary or sufficient, are not straightforward to obtain. This is because Bayesian inversion is a non-linear operation. However, using Mathematica we can develop further results. Firstly, we can show that the I/O box corresponding to the maximal Bell-inequality violation by quantum theory is backwards-signalling for all priors. Secondly, it might be thought that all nonlocal boxes (i.e. those violating a Bell-type inequality) are backwards-signalling for all priors. However, Mathematica yields a counterexample.

Decomposing backwards-signalling devices

Let us now consider classical correlation matrices from the point of view of the underlying processes. Based on the decomposition in Remark 4.5, we depict such a process in Figure 4.4. It is useful to consider the joint distribution

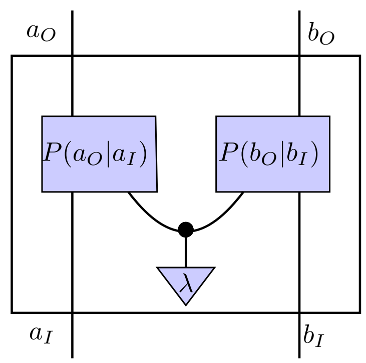


Figure 4.4: Classical process giving rise to an I/O box.

of all the random variables in Figure 4.4. This is given by:

$$P(a_O, b_O, a_I, b_I, \lambda) = P(a_O|a_I, \lambda)P(b_O|b_I, \lambda)P(a_I, b_I, \lambda) \tag{4.18}$$

In Proposition 4.6 we showed that classical processes are non-signalling, however our assumption was that $P(a_I, b_I, \lambda) = P(a_I)P(b_I)P(\lambda)$. Now, non-signalling still occurs if a_I and b_I are correlated with each other but not with λ, since the proof in Proposition 4.6 carries through. That is, classical processes are non-signalling if $P(a_I, b_I, \lambda) = P(a_I, b_I)P(\lambda)$. Hence a necessary condition for signalling in the *forward* direction is that

$$P(a_I, b_I, \lambda) \neq P(a_I, b_I)P(\lambda).$$

But this condition is well-known as a failure of λ-*independence*. This is sometimes characterised as either a lack of 'free will' for the experimenters, since their inputs are now correlated with the hidden variable and so cannot be 'freely' chosen; or as the existence of a prior common cause that correlates λ with the inputs a_I and b_I. In any case, however it is interpreted, λ-independence is an assumption in the derivation of Bell's theorem, and is therefore a standard assumption for Bell-type experiments.

Now, the question arises, how is this necessary condition for signalling in the *forward* time direction related to *backwards* signalling? We can make a connection by making the classical box internally symmetric. Consider Figure 4.5. We show a classical box for which the hidden variable λ is not only sent out to Alice and Bob at the beginning of the experiment, but is also sent to a common future of Alice and Bob after their local processes have occurred. Since the depiction in Figure 4.5 is now symmetric, when viewing the time-reversed box, the

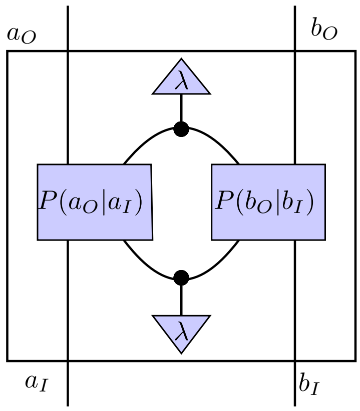


Figure 4.5: Symmetrised classical process inside an I/O box.

process will appear to involve the creation of a hidden variable in Alice and Bob's past light cone, just as for the forward time direction. In particular, we can hope to apply our reasoning for signalling in the forward direction to signalling in the backward direction, viz. a failure of λ-independence.

Now, to do so, we require the analogue of Eq. 4.18 for the backward time direction. That is, in order to view the backward direction as arising from a hidden variable process, we require the following equation to be satisfied:

$$P(a_O, b_O, a_I, b_I, \lambda) = P(a_I|a_O, \lambda)P(b_I|b_O, \lambda)P(a_O, b_O, \lambda) \tag{4.19}$$

This is obtained from Eq. 4.18 by swapping inputs and outputs, e.g. a_I and a_O. For a given classical I/O box, Eq. 4.19 is not necessarily satisfied. This is because Eq. 4.18 and Eq. 4.19 describe the same joint distribution, and so must be consistent, which implies that:

$$P(a_O|a_I,\lambda)P(b_O|b_I,\lambda)P(a_I,b_I,\lambda) = P(a_I|a_O,\lambda)P(b_I|b_O,\lambda)P(a_O,b_O,\lambda). \tag{4.20}$$

Now, this equation does follow if

$$P(a_I,b_I,\lambda) = P(a_I|\lambda)P(b_I|\lambda)P(\lambda), \tag{4.21}$$

that is, if λ screens off any correlation between the inputs (in the forward time direction). We shall make this assumption in this Subsection: in a typical Bell type experiment we would, in any case, expect that the inputs (in the forward time direction) are uncorrelated regardless of λ, i.e. $P(a_I,b_I) = P(a_I)P(b_I)$.

To show that Eq. 4.20 follows from Eq. 4.21, we first note that $P(a_I|a_O,\lambda)$ must satisfy:

$$P(a_I|a_O,\lambda) = \frac{P(a_O|a_I,\lambda)P(a_I|\lambda)}{\sum_{a_I} P(a_O|a_I,\lambda)P(a_I|\lambda)}$$

and similarly for $P(b_O|b_I,\lambda)$. Substituting these expressions and Eq. 4.21 into Eq. 4.20 yields:

$$P(a_O|a_I,\lambda)P(b_O|b_I,\lambda)P(a_I|\lambda)P(b_I|\lambda)P(\lambda) = \frac{P(a_O|a_I,\lambda)P(a_I|\lambda)}{\sum_{a_I} P(a_O|a_I,\lambda)P(a_I|\lambda)} \frac{P(b_O|b_I,\lambda)P(b_I|\lambda)}{\sum_{b_I} P(b_O|b_I,\lambda)P(b_I|\lambda)} P(a_O,b_O,\lambda).$$

After cancellation of terms we obtain:

$$P(\lambda) = \frac{P(a_O,b_O,\lambda)}{\sum_{a_I} P(a_O|a_I,\lambda)P(a_I|\lambda) \sum_{b_I} P(b_O|b_I,\lambda)P(b_I|\lambda)} \tag{4.22}$$

Using Eq. 4.18 to rewrite $P(a_O,b_O,\lambda)$, we obtain that Eq. 4.22 is an identity. Hence Eq. 4.19 is satisfied.

The significance of this is as follows. Under the assumption that the inputs (in the forward time direction) to an I/O box factorise, we can view the time-reversed I/O box as a classical process, i.e. with the local hidden variable decomposition of Eq. 4.19. But in that case, we know from Proposition 4.6, and from the remarks above, that signalling in the backward direction can only occur if the (backwards) inputs are not λ-independent. That is, backwards-signalling occurs (for classical decompositions) only if

$$P(a_O,b_O,\lambda) \neq P(a_O,b_O)P(\lambda).$$

This is interesting because it shows that in the backward direction the agents lack the free will that they have in the forward direction. In other words, when watching the time-reversed video, Alice and Bob's inputs would appear to be correlated to the hidden variable (if we could see inside the I/O box). We would infer either that there is a prior common cause to explain this correlation, or that Alice and Bob are *retrocausally* affecting the hidden variable λ. Price has developed a model that displays such retrocausality is given in [90], and a formal similarity can be shown between Price's model and backwards-signalling boxes.

However, we note that the analysis of this Subsection does not apply to nonlocal I/O boxes which are backwards-signalling. That is, by definition, nonlocal boxes lack a local hidden variable decomposition. Hence the signalling of a nonlocal box under time-reversal cannot be given the interpretation of a failure of λ-independence—there is no λ for these boxes.

4.3 Discussion

We conclude this Chapter by discussing the significance of the time-asymmetry discussed above from two perspectives. Firstly, it is interesting to place our result in the same setting as similar results concerning CTCs, and we shall describe how this can be done. Secondly, we shall describe how the change in thermodynamic entropy might be related to our notion of time-asymmetry.

1. Realising time-reversal. Let us consider how backwards-signalling and CTCs can be thought of similarly. To do so, we now show how one can effectively *realise* time-reversal and CTCs in the same way. To realise time-reversal we will require signalling resources, since the outcome may be a signalling device as we showed above. The signalling resource that we will rely on is *post-selection*, that is, conditioning on an outcome of a probabilistic process.

Now, consider the diagram of post-selected teleportation that we introduced in Chapter 2, shown in Figure 4.6.

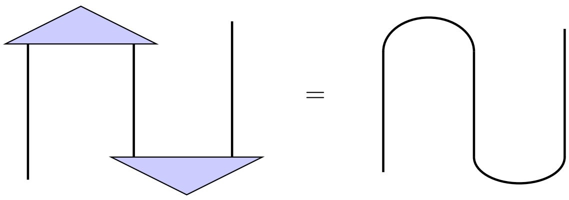


Figure 4.6: Post-selected teleportation in the graphical calculus of CQM.

Svetlichny [103], proposed post-selected quantum teleportation as a means of simulating closed timelike curves (CTCs)[9]. The diagrammatic formalism of CQM is immediately useful, since we can see how this simulation works just by 'twisting around' the Bell state in Figure 4.6, yielding the configuration of Figure 4.7.

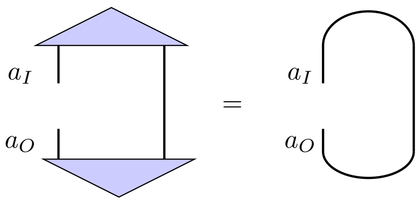


Figure 4.7: Realisation of a CTC via post-selection.

On the left of Figure 4.7, half of a Bell pair $|00\rangle + |11\rangle$ is subject to part of the entangled effect $\langle 00| + \langle 11|$: on the right the wires indeed show that this leads to an apparent flow of information backwards in time through a 'loop' (note that we have added labels for Alice's input a_I and output a_O).

Now, let us apply the same technique to I/O boxes. We have earlier described how these can be seen as morphisms in $\mathbf{Mat}(\mathbb{R}^+)$. This is a dagger compact category, with compact structure given as follows. The analogue of the Bell state is given by a morphism:

$$\tau := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

[9]Bennett and Schumacher had earlier suggested, in unpublished work, that post-selected teleportation provides a model of time travel.

and the analogue of a Bell effect is given by a morphism:

$$\upsilon := \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

Then consider the configuration of Figure 4.8.

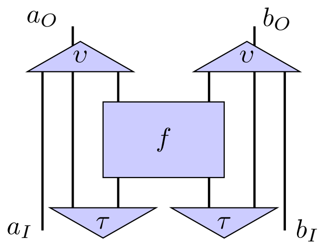


Figure 4.8: Realisation of the transpose via post-selection.

As with teleportation, we can make the apparent 'flows' of information explicit by replacing the triangles by wires:

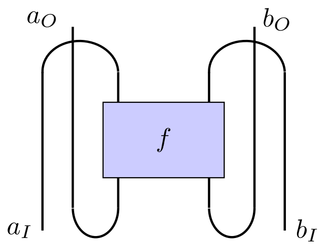


Figure 4.9: Information flow of Figure 4.8.

Figure 4.9 indeed shows that the inputs a_I and b_I are seemingly 'fed' into the outputs of the device. This leads us to propose a realisation of time-reversal as follows. It is easily calculated that the entire post-selected device of Figure 4.8 will produce the transpose of the original correlation matrix. Moreover, Coecke and Spekkens [31] have shown how the configuration of Figure 4.8 can also realise Bayesian inversion: the states and effects that are now used depend on the prior $P(I)$. This produces a realisation of time-reversal that is therefore relative to a particular input-output pair of distributions $(P(I), P(O))$.

Note that if the device of Figure 4.8 were to have pairs of qubits as inputs and outputs, and taking the states and effect respectively to be $|00\rangle + |11\rangle$ and $\langle 00| + \langle 11|$, then we immediately obtain that transpose of the quantum operation. Post-selection is hence used both in our proposal for the realisation of time-reversal and in the proposals for simulation of CTCs [103]. This proposal for realisation should be compared to the scenario of using a video tape and reversing it. In the video tape scenario, we never will observe the perfect signalling device being used to signal: we can only deduce from the statistics that it could be potentially used for that purpose, as Bob's backward inputs will typically not always be 0. In contrast, in the realisation using post-selection we *will* observe signalling for the device of Theorem 4.8: this is consistent with how Svetlichny uses post-selection to 'break' causal structure.

2. Thermodynamic entropy. Theorem 4.12 shows that probabilistic processes are needed for backwards-signaling. Hence one might think that backwards-signaling is reducible to, or a manifestation of, entropy decreasing under time-reversal. To consider this, we shall briefly investigate the entropic properties of backwards-signalling processes.

First, let us consider the change in Shannon entropy due to a stochastic binary I/O-process $\tilde{s}$. The Shannon entropy $H(X)$ of a probability distribution $p(X)$ is

$$H(X) = -\sum_i p(x_i) \log p(x_i). \tag{4.23}$$

Let the random variable $I = \{00, 01, 10, 11\}$ represent the possible inputs for the two-party input-output devices shown in Fig. 1. We denote the change in Shannon entropy of a process $\tilde{s}$, for a given input probability distribution $P(I)$, as $\Delta H := H(O) - H(I)$. Now consider the classical correlation matrix $\tilde{g}$ of Theorem 4.8. By using the probability distribution

$$P(I) = \begin{pmatrix} 0.4 \\ 0.1 \\ 0.1 \\ 0.4 \end{pmatrix},$$

we have $\Delta H = 0.09$, so in this case entropy increases in the forwards direction, which is non-signaling, and it decreases in the backwards direction, which by Theorem 4.8 is always signaling.

However for $\tilde{g}$ it is also possible to find probability distributions $P(I)$ for which entropy *decreases* in the forward direction. For the probability distribution

$$P(I) = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \\ 0.1 \end{pmatrix},$$

we have $\Delta H = -0.1$, so in this case entropy increases in the backwards direction, which is signaling. Therefore, backwards-signaling is independent of the change in Shannon entropy. Moreover, there are binary I/O processes which do not lead to backwards-signaling, for which Shannon entropy can also either increase or decrease in the forward direction. For example, the input-independent process $\tilde{f}$, each of whose columns are given by

$$f_{a_I, b_I} = \begin{pmatrix} 0.1 \\ 0.4 \\ 0.4 \\ 0.1 \end{pmatrix}^T,$$

is not backwards-signaling for any prior, but can also have positive or negative ΔH, depending on the prior used.

Now, the connection to thermodynamics is given by Landauer's principle, the generalised form of which [78] includes indeterministic operations. We expand the system to include a heat bath, and the entire system of heat bath and $\tilde{s}$ is thermodynamically closed. Then, by Landauer's principle, the change in Shannon entropy ΔH of a process $\tilde{s}$ corresponds to a change in thermodynamic entropy ΔS_{th} of the heat bath and the non-information bearing degrees of freedom of the apparatus encoding $\tilde{s}$:

$$\Delta S_{th} \geq -\Delta H k \ln 2 \tag{4.24}$$

If the Shannon entropy of $\tilde{s}$ decreases, then the thermodynamic entropy of the environment increases, i.e. the 'computer' $\tilde{s}$ dissipates heat in the heat bath. However if the Shannon entropy of $\tilde{s}$ decreases then instead heat can flow from the heat bath to the process $\tilde{s}$, and work may in principle be extracted from the process. Importantly, in both cases of increasing or decreasing Shannon entropy of $\tilde{s}$, the entire system has increasing or constant entropy. Therefore a process $\tilde{s}$ can always be considered as part of a system which respects the thermodynamical law of entropy increase. (And since the sign of ΔH is independent of whether or not the process is backwards-signaling (such as $\tilde{g}$), the sign of ΔS_{th} is also independent of this.) This would seem to suggest that backwards-signalling is independent of the thermodynamic arrow of time.

But this is not a sufficient analysis, since it treats Alice and Bob as a *single system*. That is, it does not take into account the fact that correlations are created between Alice and Bob's systems, and these correlations may well create thermodynamic irreversibility. Moreover, we know that correlations are necessary for producing backwards-signalling. The Shannon entropy calculation above is analogous to studying backwards-signalling for one system, e.g. only Alice: but then of course no backwards-signalling can occur. Hence the arrow of time that we have identified and the thermodynamic arrow of time may well be connected. This is a topic for future work.

Chapter summary. We showed that, for probabilistic theories, the time-reversed picture (cf. reversing the tape) fundamentally clashes with relativistic light cone structure. Moreover, an analysis of these probabilistic correlations allows one to detect an arrow of time. We showed that this is not a nonlocal phenomenon, but also occurs for classical devices. We derived possibilistic backwards-signalling as a sufficient condition for probabilistic backwards-signalling. We also derived some necessary conditions for backwards-signalling. We then discussed how backwards-signalling can be seen as a failure of λ-independence in the backwards direction. Finally, we discussed some general themes, which suggest possible future directions. Firstly we showed how backwards-signalling can be thought of as a similar phenomenon as closed timelike curves. Both phenomena can be realised using post-selection. Secondly, we discussed how backwards-signalling devices might be related to thermodynamics and Landauer's principle.

Chapter 5

Causal structure in SMCs

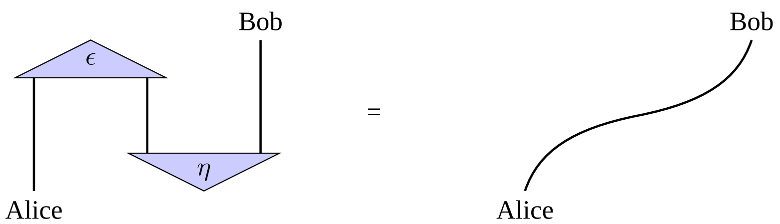
Contents

In the previous Chapter we found that the probabilistic notion of causal structure conflicts with the notion that arises from spacetime. In this and the next Chapter we shall develop a categorical structure, a *causal category*, that will incorporate both notions. We shall do so by deriving this categorical structure from physical considerations in Sections 5.1 and 5.2 below.

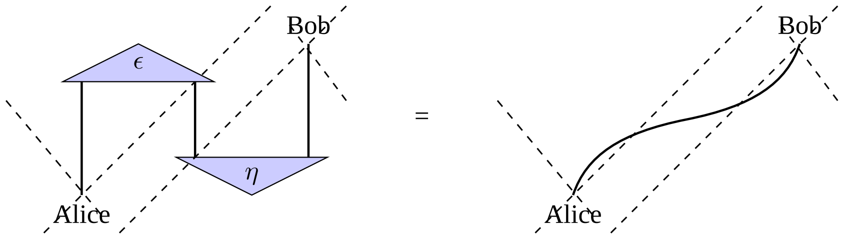
Specifically, our approach will be to *identify* the objects in a category with spacetime points (or regions), and to derive the consequences therein. Our first step is to show how, with this approach, the need for new categorical structure can be seen in the formalism of CQM itself. We shall do so by considering how CQM describes information flow.

The motivating example

In Subsection 2.1.3 we introduced compact structure, and discussed its interpretion as *post-selected quantum teleportation*, that is, quantum teleportation conditioned upon the measurement outcomes, such that no unitary correction is needed:



With the aim of treating the diagrams of CQM literally, let us assume that spacetime locations can be assigned to the agents in this protocol. We can then assign light cones to the compactness diagram, which now appears as:



(5.1)

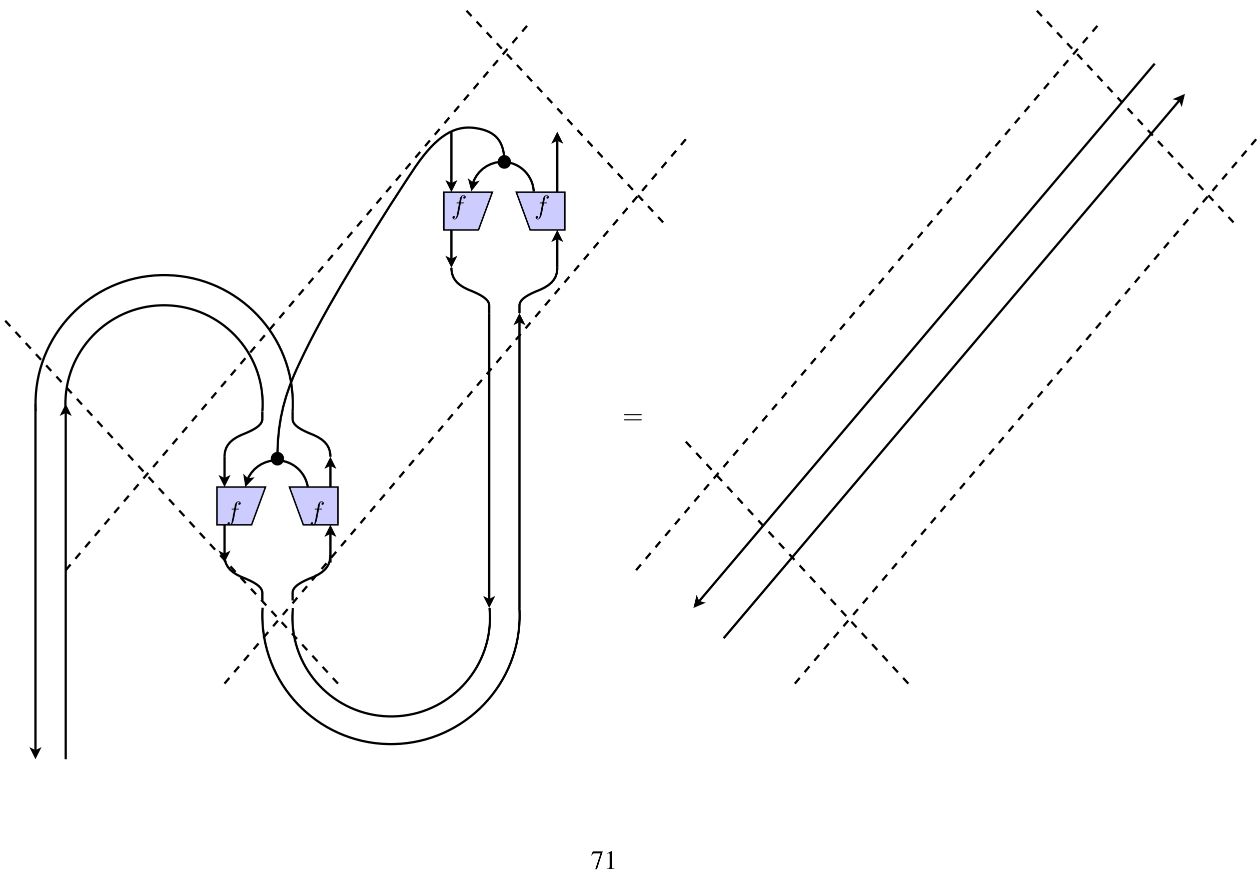
Hence Alice communicates superluminally to Bob with this assignment of causal structure, and so:

Compact structure is unphysical

This suggests that a category that encodes causal structure should not be compact.

Now, the origin of this ability for Alice to signal to Bob is post-selection. Note that we obtain signalling even for classical probability theory: for if we consider the compact structure in $\mathrm{Mat}(\mathbb{R}^+)$ we obtain classical teleportation. This corresponds to Alice and Bob sharing a classical correlated state: then if Alice post-selects x then consequently Bob will also have x. Hence Alice has signalled the bit x to Bob.

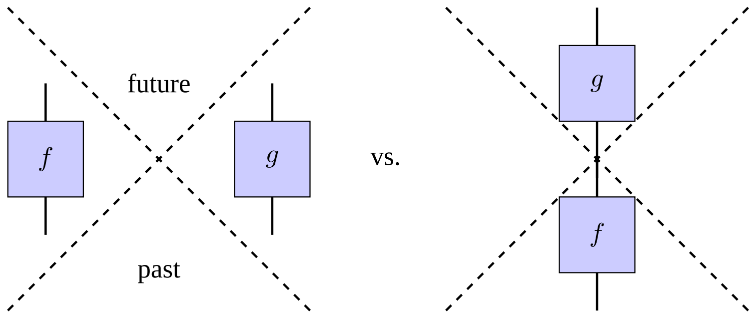
To avoid this, we must consider all possible measurement outcomes together. In the quantum teleportation protocol this requires classical communication, and this requires Bob to be in Alice's future light cone. To express this in the existing formalism of CQM requires using classical structures to specify classical communication and classical control, as we saw in Subsection 2.3. Once these various *internal* structures are used, then CQM does become consistent with the light cone structure that we would like to assign to the diagram:



The question now is whether we can identify an abstract structure, i.e. a category, which defines allowed operations 'externally', meaning that the definition of the category itself only allows operations which respect causal structure, and which are therefore physically realisable. Taking an operational view, this would correspond to excluding from the category those operations that cannot be realised in the laboratory. In this sense, we are proposing a *retreat*—from the previously defined structures for classical data, to something simpler.

In fact, we can identify two more general questions that arise from attempting to use the formalism of CQM in a literal way with respect to causality:

1. Compact structure would seem to lead to backwards-in-time information flow in Eq. 5.1. The question arises: what is the interpretation of this, and is this logically independent from the signalling that is also present in Eq. 5.1?
2. The two notions of composition in CQM, i.e. the categorical composition $\circ$ and the monoidal composition $\otimes$, seem to encode causal and acausal relationships respectively:



To what extent is this true?

The first question in particular has been a pressing question for CQM since its inception. In the structure we develop below, Eq. 5.1 will be resolved, and in doing so we shall address the two questions above.

5.1 Terminality of the monoidal unit

The aim of this Section and the next Section is to show how physical considerations allow us to derive certain properties of a category that will encode causal structure. This process will lead to us to the definition of a causal category. In this Section we shall proceed by extending the form of reasoning that we used above when discussing Eq. 5.1. That is, we shall assume that we are given an SMC, and we attempt to assign a specific causal structure to it. Then we infer the categorical properties that the SMC must have (e.g. morphisms of a certain type). Alternatively we can think of the task ahead as *building* an SMC that we want to encode a given causal structure. This is then analogous to way that we introduced compactness as teleportation: identify some physical phenomenon, and find the rules that morphisms should obey to capture the phenomenon.

We shall apply this reasoning in Subsection 5.1.1 to show how causal structure can be thought of in terms of information flow, and how connectedness in the graphical language captures this. Then in Subsection 5.1.2 we shall show that the existence of a certain type of entanglement in a physical theory leads to properties that the monoidal unit must satisfy.

5.1.1 Causality as information flow

The notion of causal structure that we have used so far is that of a partially ordered set $(P, \leq)$, as discussed in Chapter 3. But in the framework of CQM, we shall need a more refined notion of causality, which can be seen as follows.

As we emphasised previously, we want to identify objects A and B in an SMC with spacetime points. Then an initial idea for encoding causal structure in an SMC $\mathbf{C}$ would be to make a hom-set $\mathbf{C}(A, B)$ empty if the objects A and B are spacelike separated (i.e. if we they are to be assigned $A \not\leq B$ and $B \not\leq A$). This would mimic the fact that, when describing a partial order $\mathcal{P} = (P, \leq)$ as a category, the relation $A \leq B$ holds if and only if there exists a morphism $f : A \to B$, where $A, B \in P$. Hence, according to this idea, the hom-set $\mathcal{P}(A, B)$ is empty if A and B are spacelike separated.

But this will not be expressive enough for our purposes. Consider Figure 5.1, in which we depict physical scenarios of the kind we discussed above. In Figure 5.1, in addition to the teleportation example (on the left), we consider

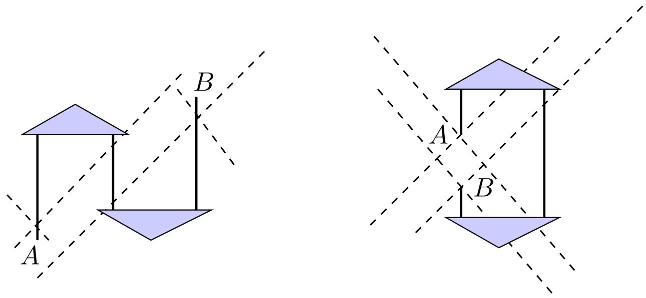


Figure 5.1: Examples of protocols with an informal assignment of spacetime points.

another causally problematic example, viz. a ‘closed time-like curve’ (on the right): in both diagrams Alice and Bob are at locations A and B respectively. As before, we have assigned light cones in the diagrams. Then, from causality, we infer that no information can flow from Alice to Bob in either the teleportation or the closed time-like curve case. Hence $\mathbf{C}$ cannot be compact. However, even though $\mathbf{C}$ is not compact, the composite process in each of these diagrams does *physically exist*. For example, suppose that ϵ is a destructive measurement (i.e. not involving post-selection). Then each part of the diagram (ϵ, η and identities) is physically realisable. Since each of these parts is a morphism in a category, the composite morphism must exist by the definition of a category. The composite process will not be the identity channel, but instead a process that *does not allow information flow*. For example, for the picture on the left, the teleportation example, we have morphisms $f : I \to C \otimes B$ and $g : A \otimes C \to I$ which yield the composition:

$$h = (g \otimes 1_B) \circ (1_A \otimes f) : A \to B\,.$$

Now, the morphism h in the teleportation example (when no classical communication occurs) is a *constant map*: Alice can input any state, but Bob always receives the maximally mixed state. But there still *exists* a morphism h between A and B.

This shows that a formalisation of causal structure within CQM should allow the existence of morphisms between acausal regions, i.e. $\mathbf{C}(A, B) \neq \emptyset$, since otherwise the composition law would have to be partially defined. Hence we make a distinction is between:

- the *existence* of a physical process, that is $\mathbf{C}(A, B) \neq \emptyset$; and,

- the *flow of information* enabled by such a process.

We will use the latter to encode a given causal structure. This means that the *possibility* of information flow, i.e. a non-constant map, shall 'witness' a causality assertion $A \leq B$. In a sense, we are reversing the usual direction of inference. Usually, from $A \leq B$ we infer that information flow can occur; instead, in our scheme, from the hom-set $\mathbf{C}(A, B)$ providing information flow, we infer that $A \leq B$. Hence the incorporation of causal structure in CQM will involve more than just expressing that there *is* a causal connection: it will involve specifying the processes that establish this causal connection from A to B, e.g. by providing a channel.

Remark 5.1 (Proof theory)**.** We have described our idea of capturing causal structure by using the *space* of physical processes between points (or regions) instead of a partial order. In fact, this mirrors a similar transition that has occurred in the development of proof theory [67]. Indeed, the proof-theoretic transition is also one from partially ordered sets to categories. In algebraic logic the object of study is whether there *is* a proof which derives proposition B from proposition A, represented by the partial order $A \leq B$. However, in categorical logic the object of study is *how* this implication can be established. This is done by giving a mathematical description of the space of proofs $\mathbf{C}(A, B)$ between propositions, so that a proof is now a morphism in a category $\mathbf{C}$ of type $A \to B$. So rather than focussing only on provability, categorical logic also takes the structure of the *space* of proofs into account. This example shows how a category-theoretic viewpoint is useful, since we have established an interesting analogy between proof theory and causal structure that could be useful[1] However, we note that our scheme for causal structure differs from proof theory in an important respect. The relationship between the poset and the category is:

$$A \leq B \iff \mathbf{C}(A, B) \neq \emptyset\,. \tag{5.2}$$

or in words, B is derivable from A if there exists a proof that does so. However, as we discussed above, in the case where regions are spacelike separated, we cannot have $\mathbf{C}(A, B) = \emptyset$. That is, in our setting there are no empty hom-sets, and so the equivalence Eq. 5.2 will fail to hold. Translating this to the proof-theoretic setting would be interesting: this would mean that between all propositions there exists a morphism, i.e. a proof, but that in some cases the existence of a morphism does not allow 'information flow' between the premises and the conclusion. This might mean, for example, that the premise is not used in the proof. But we shall not consider this idea further.

The reasoning above shows that we should capture causality in an SMC using the information flow enabled by a morphism. We stated that 'information flow' means a non-constant morphism, but let us be more precise about what this means in an SMC.

Definition 5.2. We say that a morphism $f : A \to B$ in an SMC is:

- *constant on states* if and only if, for all $\psi, \phi : I \to A$, we have $f \circ \psi = f \circ \phi$;
- *is determined by its action on states* if and only if, for all $g : A \to B$ such that $f \circ \psi = g \circ \psi$ for all $\psi : I \to A$, we have $f = g$.

The definition of a morphism f being determined by its action on states (i.e. morphisms with domain I) just means that f can be distinguished from a distinct morphism g by their action on at least one state ψ. (This condition is sometime known as *well-pointedness* in the category theory literature [7].)

Example 5.3. In $\mathbf{fHilb}$, the zero map is constant on states. Also, every morphism is determined by its action on states, which follows from $\mathbf{fHilb}$ admitting a matrix calculus. For example, consider $f \neq g : \mathbb{C}^2 \to \mathbb{C}^2$. There

[1] Indeed, such an analogy already exists formally for cartesian closed categories, a type of monoidal category, via the Curry-Howard-Lambek correspondence [100]. This is a three-way correspondence between intuitionistic logic, simply-typed lambda calculus, and cartesian closed categories. It has been useful for extending the formal techniques of each area.

exists $\psi : I \to A$ such that $f \circ \psi \neq g \circ \psi$ as follows. Since the hom-set $\mathbf{fHilb}(\mathbb{C}^2, \mathbb{C}^2)$ is a ring, we have $f \circ \psi = g \circ \psi$ iff $(f - g) \circ \psi = 0$. Then the matrix equation of $(f - g) \circ \psi = 0$ written in some basis is

$$\begin{pmatrix} f_1 - g_1 & f_2 - g_2 \\ f_3 - g_3 & f_4 - g_4 \end{pmatrix} \circ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0.$$

But if $f \neq g$ then for at least one i, there is a component of the matrix of $f - g$ which satisfies $f_i - g_i \neq 0$. Then we can always choose ψ_1 and ψ_2 in the correct ratio to ensure that

$$(f_1 - g_1) \circ \psi_1 - (f_2 - g_2) \circ \psi_2 \neq 0,$$

and hence $(f - g) \circ \psi \neq 0$ and there exists ψ which distinguishes f and g.

By *Type I information flow* we shall mean a morphism $f : A \to B$ in an SMC that is not constant on states, as in Definition 5.2. Since in this case there exists $\psi, \phi : I \to A$ with $f \circ \psi \neq f \circ \phi$, then Alice can choose to send either ψ or ϕ 'into' the process f, so that Bob receives $f \circ \psi$ or $f \circ \phi$ respectively. In other words, Type I information flow corresponds to the existence of a non-constant function

$$f \circ - : \mathbf{C}(I, A) \to \mathbf{C}(I, B) :: \psi \mapsto f \circ \psi,$$

induced by the morphism f, which we schematically depict in Figure 5.2[2]. Moreover, we shall also say that

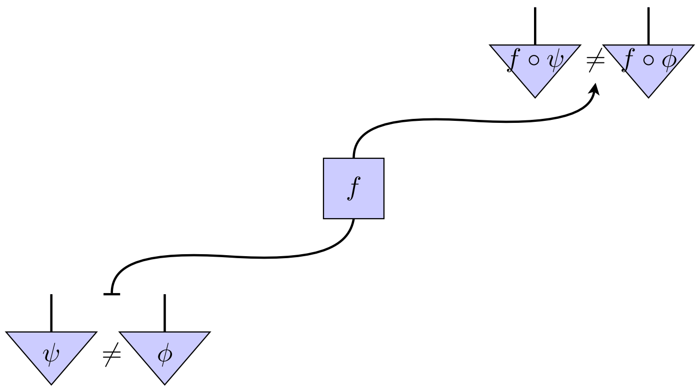


Figure 5.2: Type I information flow

objects A and B in an SMC are *causally related* iff Type I information flow can occur between A and B, meaning the existence of a morphism in $\mathbf{C}(A, B)$ that is not constant on states.

Remark 5.4. Note that for most examples of SMCs that we have introduced, every pair of objects is causally related, in the sense of allowing Type I information flow. For example, in $\mathbf{Set}$, for any two objects A and B, such that $B \neq \{*\}$, we can define a non-constant function $f : A \to B$. This shows that, as is implicit from the discussion above, the type of category we are developing to encode causal structure will be quite different in structure from the categories typically used in CQM.

Remark 5.5 (Well-pointedness)**.** Conditions of 'well-pointedness', as used in Definition 5.2, are sometimes thought to be undesirable, for both mathematical and physical reasons [59, 61]. However, our level of generality will also apply to categories with 'point-free' objects, i.e. objects which do not have states. That is, we will

[2] Note that this is *not* a formal diagram corresponding to morphisms in category. Instead Figures 5.2 and 5.3 *schematically* show the concept of Types I and II information flow respectively, by depicting the induced set-theoretic functions.

show that the notion of information flow in Definition 5.2 that refers to points can be equivalently stated purely in terms of a notion of 'connectedness's. This notion of connectedness makes no reference to points. It is this point-free characterisation of information flow that we will use subsequently.

Definition 5.6. In an SMC we shall say that:

- a morphism $f : A \to B$ is *disconnected* if it decomposes as $f = p \circ e$ for some $e : A \to I$ and $p : I \to B$;
- a morphism is *connected* if it is not disconnected;
- a *hom-set* $\mathbf{C}(A, B)$ *is disconnected* when it contains only disconnected morphisms, and then we say that the *objects* A *and* B *are disconnected.*

The most trivial example of Definition 5.6 is that of scalars $s : I \to I$, which are disconnected in any category. A simple non-trivial example is as follows.

Example 5.7. In the SMC $\mathbf{Set}$, a constant function on a two-element set $II = \{0, 1\}$, say

$$f : II \longrightarrow II :: x \longmapsto 1,$$

is a disconnected morphism: with

$$p : I \longrightarrow II :: * \longmapsto 1$$

and

$$e : II \longrightarrow I :: x \longmapsto *$$

we have $f = p \circ e$. However, the identity morphism 1_A for any set A is connected, except for $A = I$.

The justification for using disconnectedness to encode causality arises from the following proposition.

Proposition 5.8 (Equivalence of constancy and disconnectedness)**.** *Let* $\mathbf{C}$ *be an SMC. If all scalars are equal to* 1_I *and if morphisms are determined by their action on states, then the following are equivalent:*

- $f : A \to B$ *is constant on states;*
- $f : A \to B$ *is disconnected.*

Proof. Let f be constant on states and let $\phi := f \circ \psi : I \to B$ be that constant (for any $\psi : I \to A$). Then consider an arbitrary morphism $\pi : A \to I$. Now, for all $\psi : I \to A$ we have:

$$(\phi \circ \pi) \circ \psi = \phi \circ \underbrace{(\pi \circ \psi)}_{1_I} = \phi = f \circ \psi$$

Hence, since morphisms are determined by their action on states, we indeed have that f is disconnected, i.e.:

$$f = \phi \circ \pi$$

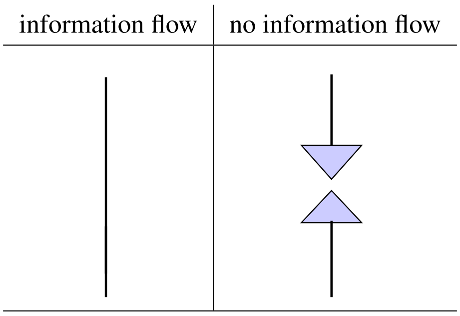
for any $\pi : A \to I$ (and therefore for some $\pi : A \to I$, as required by Definition 5.6). Conversely, if $f = \phi \circ \pi$ then for all $\psi : I \to A$:

$$f \circ \psi = (\phi \circ \pi) \circ \psi = \phi \circ \underbrace{(\pi \circ \psi)}_{1_I} = \phi$$

where we have again used uniqueness of scalars: hence f is constant on states. □

Now, Proposition 5.8 uses the assumption that the category has a unique scalar 1_I. Since the scalars represent probability amplitudes in CQM, this means such a category is a category of *deterministic processes*, i.e. non-postselected processes.

With this assumption, Proposition 5.8 establishes that information flow from A to B is captured by topological connectedness in the graphical language.



If we can justify the assumption of determinism, i.e. the unique scalar 1_I, then this will provide an elegant characterisation of information flow. This characterisation would also conform to the general methodology of CQM, since would make use of the graphical language. In the next Subsection we shall indeed justify this assumption, since we will show that the monoidal unit is required to be terminal if certain causality constraints are present.

5.1.2 Terminality of I from no-signalling

Deriving terminality

In the previous Subsection information flow was defined as a non-constant morphism $f : A \to B$. We called this *Type I information flow*. We shall now generalise this notion to include other forms of information flow. Given a bipartite state $\eta : I \to A \otimes B$, there may also be another type of information flow, which we call *Type II information flow*. Type II information flow corresponds to the existence of a non-constant function from effects to states induced by the bipartite state η:

$$(- \otimes 1_B) \circ \eta : \mathbf{C}(A, I) \longrightarrow \mathbf{C}(I, B) \quad (5.3)$$

$$\pi \longmapsto (\pi \otimes 1_B) \circ \eta. \quad (5.4)$$

Schematically, this is depicted in Figure 5.3.

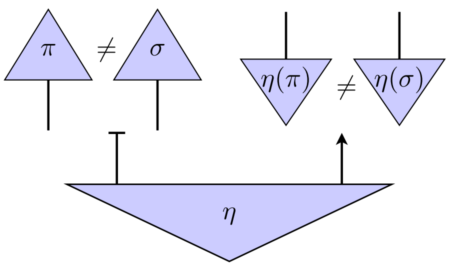


Figure 5.3: Type II information flow

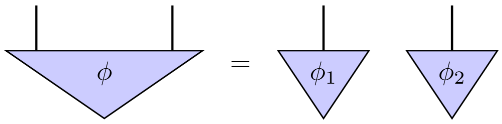
Example 5.9 (Quantum entanglement)**.** In quantum theory a bipartite state may be entangled. In that case, Type II information flow corresponds to correlations between measurement outcomes of the two parties that are *signalling*, which can only happen if we allow post-selection[3].

The following postulate imposes compatibility between Type II information flow with Type I information flow. In other words, it forbids correlation-induced signalling when systems are not causally related. If this were not the case, then Type II information flow could be used to produce Type I information flow. For example, given a state η that enables Type II information flow, Alice could select effects $\pi \neq \sigma$ corresponding to states $\psi \neq \phi$, thus producing a non-constant function of type $\mathbf{C}(I, A) \to \mathbf{C}(I, B)$, i.e. Type I information flow. This would violate causal structure.

Definition 5.10 (Causal consistency)**.** An SMC obeys *causal consistency* if, for a bipartite state $f : I \to A \otimes B$, Type II information flow cannot occur when A and B are not causally related, i.e. when Type I information flow cannot occur.

Remark 5.11. Note that we could have made the stronger requirement that entanglement-induced signalling does not occur even for causally related systems, e.g. if there exist entangled states for a bipartite system $A \otimes B$ for which A and B *are* causally related. However, since Type II information flow 'across time' does not cause any inconsistency with causal structure, we shall not impose this extra condition. We note that this type of information flow, corresponding to using entanglement 'across time', has been explored by Taylor et al. in [104].

As well as the idea of a *process* $f : A \to B$ being disconnected, we can also consider a bipartite *state* $\phi : I \to A \otimes B$ as being 'disconnected':



and if A and B are not causally related, then there will be no Type II information flow. Hence in this case Definition 5.10 is trivially satisfied. However, the kind of physical theories that we want to capture *do* have connected bipartite states, both in quantum theory (entangled states) and classical probability theory (for example, a probabilistic bipartite state that has perfect correlations). The following definition asserts the existence of states of this kind. Moreover, it states that all processes can be faithfully represented by bipartite states. In the context of quantum theory this corresponds to the Choi-Jamiołkowski isomorphism [24, 60], as described in Example 5.22 below.

Definition 5.12. For an SMC:

- A bipartite state $\zeta_A : I \to A^\star \otimes A$ is called a *CJ-state*, if for all B,

$$\tau_A : \mathbf{C}(A, B) \longrightarrow \mathbf{C}(I, A^\star \otimes B) \tag{5.5}$$

$$f \longmapsto (1_{A^\star} \otimes f) \circ \zeta_A \tag{5.6}$$

  is an injective function.

- A *CJ-universe* is an SMC for which, for all objects A:
  - (i) there exists an object $A^\star$ such that A and $A^\star$ are not causally related; and
  - (ii) there exists a CJ state $\zeta_A : I \to A^\star \otimes A$.

[3] The use of post-selection to provide signalling using bipartite entangled states is discussed in various places, but a discussion in the spirit of this Chapter can be found in [76], in the context of time-symmetric quantum mechanics, where it is also used to provide post-quantum correlations .

Note that this definition implies in particular that, for the case $B = I$, there is an injection from effects $\pi : B \to I$ to states $(1_{A^\star} \otimes \pi) \circ \zeta_A : I \to A^\star$[4].

Example 5.13. A CJ state weakens the definition η in compact structure: consider a compact category $\mathbf{C}$. Compactness provides an involution on hom-sets, which establishes a bijection between $\mathbf{C}(A, B)$ and $\mathbf{C}(I, A^\star \otimes B)$ using $\zeta_A := \eta_A$ and $A^\star := A^*$. Therefore $\mathbf{C}$ has a CJ state for each object. But since a CJ state needs only provide the *injection* τ_A of Definition 5.12, instead of the *bijection* that compactness ensures, it is a weaker notion.

Example 5.14 (Classical probability theory)**.** We define classical probability theory as a subcategory $\mathbf{Stoch}$ of the category of real matrices $\mathbf{Mat}(\mathbb{R}^+)$: morphisms are stochastic maps, i.e. finite-dimensional real matrices with entries $p_{ij} > 0$, whose columns are normalised, i.e. $\forall j\ \Sigma_i p_{ij} = 1$. Hence states are given by normalised column vectors with entries in $\mathbb{R}^+$. The monoidal product is the Kronecker product of matrices. A CJ state is then given by a perfectly correlated bipartite probability distribution. Denoting Alice's outcome index by i, and Bob's outcome index by j, this can be written as

$$\mathbf{v}_{ij} = \begin{cases} \frac{1}{n} & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

which provides an injective mapping from operations to states. We note that Abramsky and Heunen have described a scheme for interpreting $\mathbf{Mat}(R)$ as a category of physical processs, for any ring R, in [6].

Remark 5.15. Note also that although the usual examples of compact categories such as $\mathbf{fRel}$ and $\mathbf{Mat}(\mathbb{R}^+)$ have CJ states, these categories are *not* CJ universes. This is because a CJ universe requires that, for every object A, there exists at least one other object $A^\star$ that is not causally related to it, and for which there is a CJ state $\eta : I \to A^\star \otimes A$. But, as we discussed in Remark 5.4, for standard SMCs there do not exist pairs of objects which are not causally related. This is consistent with the fact that, as we shall in see Subsection 6.1.2, compactness conflicts with our notion of causal structure, but the definition of a CJ-universe does not. This raises the question of how restrictive the definition of a CJ universe is, since it requires that for every object A there is an object $A^\star$ which is not causally related to A. There are two points we can make about this: a technical one and a physical one. The physical one is that Minkowski space satisfies the property that every point is not causally related to some other point. This provides justification for the assumption, because it is the main example of a spacetime that we would hope to capture. However, there are spacetimes, such as the Schwarzschild or FLRW spacetimes, which may not satisfy this property, depending on how their causal structure is modelled. For example, the singularity for a Schwarzschild spacetime might be thought to not satisfy this property, since it is at future null infinity. Such examples are therefore beyond the scope of the work in this Chapter. We shall make the technical point shortly, in Remark 5.21 below.

Definition 5.16. By an *environment structure* for an SMC, we mean a family of effects $\top_A : A \to I$, one for each system A. We call a CJ universe with an environment structure a *CJ$\top$-universe*.

Example 5.17. Definition 5.16 is quite a weak requirement, since it only specifies the existence of an effect for each object in the category. For example, in the category $\mathbf{fRel}$, any family of relations $\{f_A\}_A$ (with types $f_A : A \to I$) for all sets $A \in |\mathbf{fRel}|$, provides an environment structure. But, as discussed in Remark 5.15, $\mathbf{fRel}$ is not a CJ universe, and so this is not a significant example of an environment structure.

Definition 5.18. A *terminal object* in a category $\mathbf{C}$ is an object A for which, for each object $B \in |\mathbf{C}|$, there is a unique morphism from B to A.

Example 5.19. In $\mathbf{Set}$ the unit object $I = \{*\}$ is terminal since there is a unique function from every set to the

[4] We have used the star notation '$\star$' instead of the asterisk '$*$' in Definition 5.12 since the object $A^\star$ may not be the dual object A^* in the definition of compact structure that is usually denoted with an asterisk.

one-element set. In $\mathbf{Vec}_{\mathbb{K}}$ the trivial vector space 0 is terminal (and initial). However, note that the unit object $I = \mathbb{K}$ is not terminal, since e.g. for $\mathbb{K} = \mathbb{R}$ there are a continuous infinity of linear maps from a vector space V to $\mathbb{R}$, since this is the number of linear functionals on V.

Note that the uniqueness of 1_I as a scalar is implied by terminality of the tensor unit.

Theorem 5.20. *A CJ$\top$-universe obeying causal consistency has a terminal tensor unit.*

Proof. Suppose for contradiction that there exist distinct effects $\pi \neq \pi' : A \to I$. Then by Definition 5.12 (with $B := I$) we have

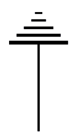
$$(1_{A^*} \otimes \pi) \circ \zeta_A \neq (1_{A^*} \otimes \pi') \circ \zeta_A \,,$$

But this contradicts Definition 5.10: it provides Type II information flow between objects A and $A^\star$ which are not causally related, i.e. between which Type I information flow cannot occur. Hence there can exist at most one effect and its existence is guaranteed by the environment structure. □

Remark 5.21. In Remark 5.15 we discussed that a CJ universe has a CJ state, which is defined to ensure that every object A is not causally related to at least one other object $A^\star$, and that the physical consequence is the potential exclusion of models such as the FRLW spacetime. Now, consider an alternative definition, in which a CJ universe has a CJ state for every pair of objects that is not causally related: let us call this a 'CJ* universe'. This is a weaker definition than the one we have used, because a CJ* universe does not impose that every object A has another object $A^\star$ which is not causally related to A. Although this would seem to encompass models such as FRLW, it would not yield Theorem 5.20, since it does not guarantee that the injectivity argument applies to *every* object in the category.

Examples and consequences

As we pointed out in Remarks 5.4 and 5.15, examples of CJ-universes, and hence CJ$\top$-universes, will not include the standard examples of SMCs. Hence the categories to which Theorem 5.20 applies will have to await Section 6.2, in which we construct examples of causal categories (and hence CJ universes). This should be as expected: we are deriving the properties necessary for a causal category: since such a project has not been previously attempted, we should not expect to have examples of such categories to hand. However, we see from Theorem 5.12 that, since an environment structure in a non-trivial CJ$\top$-universe has a unique effect for each object A, this effect should be interpreted as a process that removes the system from the scope of consideration. That is, it represents tracing out a system. We have denoted this symbolically as $\top_A$, but given this physical meaning we shall also depict it as a 'ground morphism' for each object A:

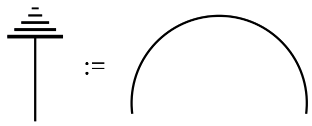


Hence, as an intermediate step to constructing CJ$\top$-universes, it is useful to consider examples of categories to which the conclusion of Theorem 5.20 applies: those with such a tracing-out operation, especially those with CJ states and terminal objects.

Example 5.22 (Quantum theory)**.** The category **FdHilb** is the motivating example of a †-SMC in CQM, and so one might attempt to define it as a (trivial) $CJ\top$-universe, using the Bell state as the CJ state. However this is problematic for the following reason. As discussed above, we want to study examples in which the environment structure provides a unique morphism $\top_A : A \to I$ for each object A, and the interpretation of this family of morphisms $\{\top_A\}_A$ is the partial trace operation (i.e. the operation which sends the system A to the environment). But consider tracing out one half of a bipartite system in quantum theory: if the bipartite state is pure and entangled then the marginal state will be mixed. Hence we must use a category of mixed operations rather than $\mathbf{FdHilb}$, whose states are always pure (see also Remark 6.7 of [30]). We do so as follows, and our example culminates in Proposition 5.23 below.

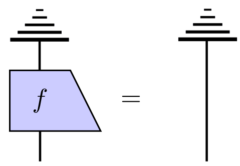
We can consider categories of mixed states in a general way by using the CPM construction described in Chapter 2. Recall that, given a †-compact category $\mathbf{C}$, this construction gives a †-compact category $CPM(\mathbf{C})$; and if $\mathbf{C} = \mathbf{fHilb}$ then we obtain the category of unnormalised density matrices and completely positive maps, which we denoted $\mathbf{Mix}$.

Let us show that if we restrict to the subcategory $CPM_\top(\mathbf{C})$ of $CPM(\mathbf{C})$ whose morphisms are completely-positive *trace-preserving* maps, then we obtain a category whose monoidal unit I is terminal. Now, we can propose an environment structure for $CPM(\mathbf{C})$ by choosing $\top_A$ to be the cap ϵ_A in compact structure (we shall justify this in Chapter 7):



(5.7)

To restrict to the subcategory of trace-preserving maps, we select the morphisms f in $CPM_\top(\mathbf{C})$ that satisfy



(5.8)

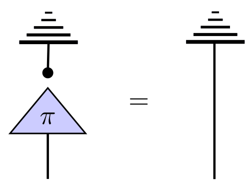
This is indeed a category, because for any two morphisms $f : A \to B$ and $g : B \to C$ in $CPM(\mathbf{C})$, if f and g both satisfy Eq. 5.8, then the composite $g \circ f$ does also:

$$\top_C \circ g \circ f = \top_B \circ f = \top_A.$$

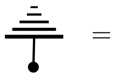
In the following proposition we make the assumption that $\top_I = 1_I$. This is physically justified in the sense that it corresponds to discarding the trivial system. But discarding the trivial system should not correspond to a physically meaningful operation, and so its probability should be 1_I.

Proposition 5.23. *Given any †-compact category $\mathbf{C}$, the category $CPM_\top(\mathbf{C})$ has a terminal unit object if $\top_I = 1_I$.*

Proof. Let f in Eq. 5.8 be an effect $\pi : A \to I$. Then we have the equation:



Hence terminality follows if $\top_I = 1_I$, i.e.



where the right-hand side is empty since it depicts 1_I. $\square$

Hence Proposition 5.23 yields examples of categories with terminal I, in particular $CPM_{\top}(\mathbf{fHilb})$.

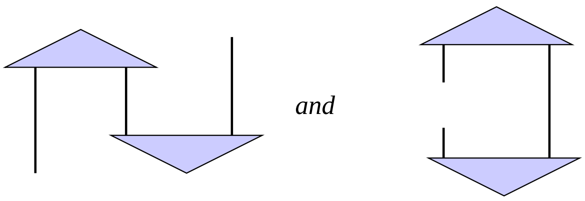
We also see from Proposition 5.23 that terminality follows from enforcing that all processes are deterministic. The converse is also true, since Theorem 5.20 has the following consequences.

Corollary 5.24. *Under the assumptions of Theorem 5.20 we have:*

- *All scalars are equal to* 1_I.
- *States are 'normalised' i.e.* $\top_A \circ \psi = 1_I$ *for all* $\psi : I \to A$.
- *For all* A, B *we have* $\top_{A \otimes B} = \top_A \otimes \top_B$.
- *All bipartite effects are disconnected.*

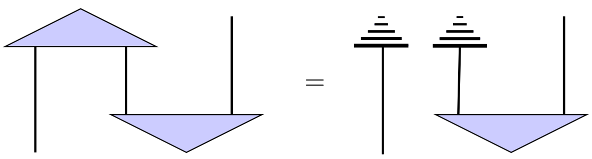
With the assumptions of Theorem 5.20, we can now show that the protocols depicted in Figure 5.1 do not lead to information flow. In particular, teleportation without classical communication cannot generate any information flow.

Corollary 5.25. *Under the assumptions of Theorem 5.20, the composite of the protocols*

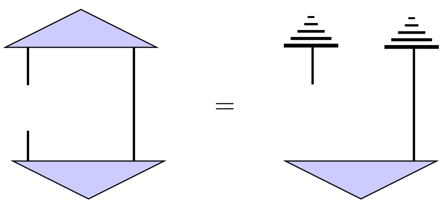


are disconnected.

Proof. From Corollary 5.24, we have that all bipartite effects are disconnected. We therefore have:



Similarly we have:



$\square$

We shall discuss this further in Section 6.1.2; in Section 6.2 we show how we can retain the full power of CQM, despite the breakdown of compactness in a causal setting.

Remark 5.26 (Time-symmetric quantum mechanics)**.** The passage from dagger compact categories to causal categories can also be seen as an abstract counterpart to the passage from time-symmetric quantum mechanics (TSQM) [9] to the usual formalism of quantum mechanics. In the formalism of TSQM, not only do we assume the existence of a pre-selected state (i.e. one which has been prepared by measurement), but we also assume that measurement outcomes have been post-selected. This corresponds to how a dagger symmetric monoidal category is used in CQM, because the dagger imposes a formal symmetry between states and effects. In TSQM, the violation of Definition 5.10 has been partially addressed by restricting the formalism to those classes of intial and final (post-selected) states which do not lead to signalling [76]; this is ad hoc, and these classes lack an elegant formal characterisation.

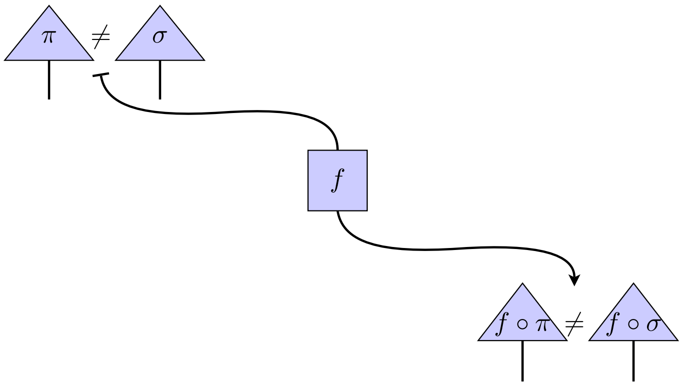


Figure 5.4: 'Dualised' Type I information flow

Remark 5.27. In addition to Type I and Type II information flow, we can identify two further kinds of information flow, which are shown schematically in Figures 5.4 and 5.5. However, these are both excluded by terminality. In the first case, shown in Figure 5.4, terminality implies that there cannot be two non-equal effects. In the second case, shown in Figure 5.5, terminality implies that the bipartite effect η must itself be $\top_A \otimes \top_B$.

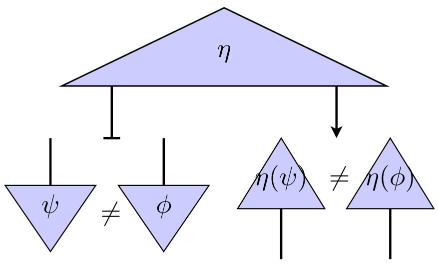


Figure 5.5: 'Dualised' Type II information flow

Remark 5.28 (Earlier work)**.** In [22], Chiribella, D'Ariano and Perimotti (CDP) use the existence of a unique deterministic effect, which they call the *causality axiom* [22, Definition 25 & Lemma 3], to derive information-theoretic features of quantum theory. In that work, much use is made of the probabilistic structure of measurements and classical outcomes. However, in our framework, we can already derive such features without assuming probabilistic structure, e.g. Corollary 5.25. By using category theory, we expose the structural—as opposed to probabilistic—aspects of information flow that follow from requiring causality. We shall discuss the connection between our work and the CDP axioms in Chapter 7.

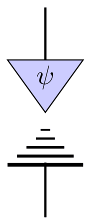
Section summary. Our progress in this Section has been as follows:

1. We started with the notion of information flow as representing the causal relation between regions, where 'information flow' means non-constant morphism. We then derived two results that concerned the categorical structure of information flow.
2. The first result was Proposition 5.8: we derived the fact that graphical disconnectedness is equivalent to being a constant morphism, assuming that there is a unique scalar in the category.
3. The second result was Theorem 5.20: we showed that the assumption of a unique scalar follows from terminality of I, which in turn we derived by prohibiting correlation-induced signalling. The category $CPM_{\top}(\mathbf{C})$ provided an example and interpretation of an SMC with terminal I.

Hence we can now characterise information flow more elegantly than as a non-constant morphism. That is, with terminality, we can now characterise information flow using the structure of an SMC: A and B *are causally related* iff a process $f : A \to B$ can take place which is not disconnected. Conversely, A and B *are not causally related* iff all processes $f : A \to B$ are disconnected (i.e. factor through the monoidal unit), that is:

$$\mathbf{C}(A, B) = \{\psi \circ \top_A \mid \psi : I \to B\}.$$

That is, all processes in $\mathbf{C}(A, B)$ are of the form:



We also noted that the existence of a unique scalar seemed to exclude the standard Hilbert space formalism for quantum mechanics, since if we take $\mathbf{FdHilb}$ to be the setting for this, then the scalars are $\mathbb{C}$. However Example 5.22 showed that the standard formalism does provide an example by instead using the category of completely positive trace-preserving maps, for which the tensor unit is terminal.

5.2 Partiality of the tensor from existence of local states

In a monoidal category $\mathbf{C}$ the monoidal product $A \otimes B$ exists for every pair of objects (A, B), i.e. the functor

$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

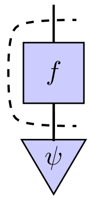
is a *total* operation. Let us continue with our intention to assign causal structure to $\mathbf{C}$. In that case, the fact that $\otimes$ is total presents some difficulties for the project of assigning causal structure to an SMC. This can be seen in two ways:

1. In an SMC, the monoidal product $A \otimes A$ exists for any system A. Then consider the hom-set $\mathbf{C}(I, A \otimes A)$, which includes morphisms of the form $\psi \otimes \phi$ for $\psi, \phi \in \mathbf{C}(I, A)$. Such morphisms correspond to state preparations, but if A denotes a single system, at a *single spacetime location*, then the meaning of $\psi \otimes \phi$

is not clear. In particular, with such a spatiotemporal interpretation, we must have $\phi = \psi$. Hence the independence of ψ and ϕ in $A \otimes A$, i.e. the fact that we could have $\phi \neq \psi$, lacks physical meaning,

2. On the other hand, consider the case of $A \neq B$ for the object $A \otimes B$. Then, as with the previous point, there is the same possibility of preparing arbitrary product states ϕ of a system A and a state ψ of system B, to form a joint state $\phi \otimes \psi$ of $A \otimes B$. But this suggests that the states ϕ and ψ should not be causally related, i.e. $A \otimes B$ should represent a 'spacelike hypersurface'.

Hence for systems that are not independent, we have to *restrict composition of the states of subsystems*. We can achieve this by restricting the monoidal composition $\otimes$ of pairs of objects, so that it is a *partial* instead of a *total* operation, and such that it only exists for causally independent systems. We will derive this feature of partiality of $\otimes$ in this Section. To do so, we shall determine which 'hypersurfaces' (i.e. object formulae such as $A_1 \otimes A_2$) we can assign states to, in particular by considering the embedding of states in arbitrary protocols. The basic idea that we will exploit is as follows. If A and B are connected, then a state of the object $A \otimes B$, i.e. a state $\phi : I \to A \otimes B$, cannot be defined for a morphism $f : A \to B$ that takes in an input state ψ. Diagrammatically this corresponds to attempting to define a state for the dashed line:



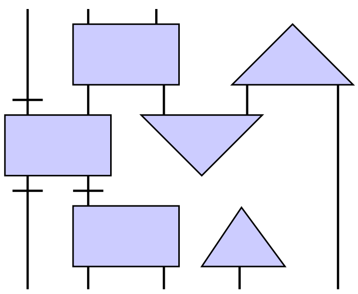
We will show that this is not possible. In this manner, the partial monoidal product that we will develop will be the analogue of *spacelike* hypersurfaces in a relativistic spacetime: we will therefore call these objects *spatial slices*.

We now proceed with the formal exposition of this idea. Recall that a protocol is defined to be a morphism formula $\mathcal{F}$ in the symbolic language of SMCs (note that our notational convention is to use calligraphic letters for morphism formulae). Recall also that we defined a protocol to be a morphism formula in the language of SMCs, such that the morphism formula contains only atomic morphisms as defined in Definition 2.30, i.e. expressions $\mathcal{F}$ that do not contain topologically disconnected components.

Definition 5.29. Let $\mathcal{F}$ be a protocol.

- A *slice* is an object formula in the symbolic language of SMCs.
- A slice $\mathcal{B}$ *is included in* $\mathcal{F}$ if the set of objects occurring in $\mathcal{B}$ is a subset of the input and output types of the atomic morphisms that are in $\mathcal{F}$.

In terms of the graphical language, a slice is included in a diagram just when it is a subset of the wires in the diagram, which we can denote by putting ticks on the wires, as follows:

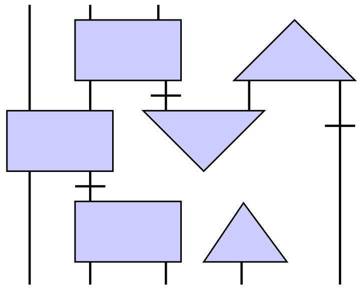


(5.9)

Recall that a disconnected hom-set $\mathbf{C}(A, B)$ contains only disconnected morphisms.

Definition 5.30. A *spatial slice* is a slice $B_1 \otimes \ldots \otimes B_n$ for which every pair of objects $(B_i, B_{j \neq i})$ that occur in it are *disconnected objects*, that is, the hom-sets $\mathbf{C}(B_i, B_j)$ and $\mathbf{C}(B_j, B_i)$ are disconnected.

While the slice in picture (5.9) cannot be spatial, since it involves objects that are explicitly connected within the diagram, the following slice may be spatial:

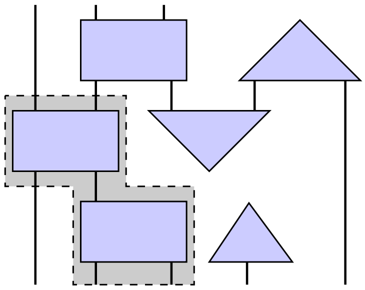


(5.10)

provided there are no protocols for which the ticked objects are connected.

Definition 5.31. Let $\mathcal{F}$ be a protocol. Another protocol $\mathcal{G}$ is a *sub-protocol* of $\mathcal{F}$, if $\mathcal{F}$ can be formed from $\mathcal{G}$, $\otimes$, $\circ$ and other morphism formulae.

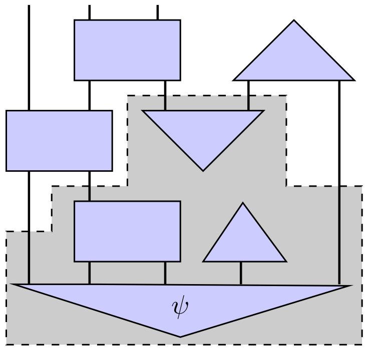
In the graphical language a sub-protocol is simply a sub-diagram of the diagram corresponding to the protocol. For example the shaded region in the following diagram is a sub-protocol of the protocol defined by picture 5.10:



The following definition states the conditions under which we can assign a state to a slice included within a protocol. Note that part of this definition is that an initial state is specified (i.e. an 'initial condition'). We denote by σ the appropriate composite of symmetry isomorphisms that realises the stated type (as defined previously in Eq. 2.8).

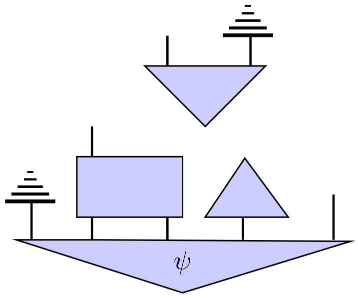
Definition 5.32. Let $\mathcal{F} : A \to C$ be a protocol, and let $\mathcal{B}$ be a slice included in it with $B := \mathcal{B}$, and let $\psi : I \to A$ be a state. The *local state at $\mathcal{B}$ relative to $\mathcal{G}$*, where $\sigma \circ \mathcal{G} : I \to B \otimes B'$ with $g := \mathcal{G}$ is a sub-protocol of $\mathcal{F} \circ \psi$, is the state $(1_B \otimes \top_{B'}) \circ \sigma \circ g$.

As an example, let $\mathcal{F}$ and $\mathcal{B}$ be as in picture (5.10). Then consider a sub-protocol $\mathcal{G}$ defined the shaded part of $\mathcal{F} \circ \psi$, for some state ψ:



(5.11)

With these assignments, the local state at $\mathcal{B}$ relative to $\mathcal{G}$ is $(1_B \otimes \top_{B'}) \circ g$, which is depicted as:



(5.12)

Remark 5.33. Note that the definition of local state requires *three* pieces of information to be supplied:

1. The protocol $\mathcal{F}$ which is to be considered.
2. The 'target' slice $\mathcal{B}$ at which the state is to be localised.
3. The sub-protocol $\mathcal{G}$ which provides both the initial state ψ and the 'total slice' $B \otimes B'$ of which $\mathcal{B}$ is a part.

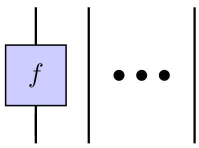
We now employ this definition to show that only spatial slice admit local states.

Theorem 5.34 (Existence of local states)**.** *Let* **C** *be an SMC with terminal unit object. A slice $\mathcal{B}$ admits a local state, relative to some sub-protocol $\mathcal{G}$, for any protocol $\mathcal{F}$ in which it is included if and only if it is a spatial slice.*

Proof. ($\Rightarrow$) First we show that if a slice is not spatial, then there exists a protocol $\mathcal{F}$ for which $\mathcal{B}$ does not admit any local state relative to any sub-protocol $\mathcal{G}$. If $\mathcal{B}$ is non-spatial then there exists B_i and $B_{j \neq i}$ for which there is a morphism $f : B_i \to B_j$ that is connected. Then we can define a protocol $\mathcal{F}$ using the connected morphism f:

$$\mathcal{F} := f \otimes 1_{\otimes_{k \neq i,j} B_k}$$

This is depicted as



However $\mathcal{F} \circ \psi$ admits no sub-protocol $\mathcal{G}$ of the type required to yield a local state at $\mathcal{B}$, since there is no sub-protocol of $\mathcal{F}$ which includes a slice $B_i \otimes B_j$.

($\Leftarrow$) The implication we need to prove is: if $\mathcal{B}$ is spatial, then for any scenario $\mathcal{F}$ which includes $\mathcal{B}$ we can define the local state at $\mathcal{B}$. Now, as we discussed in Remark 5.33, the definition of a local state requires *three* pieces of data: the implication that we want prove universally quantifies over *two* of these pieces of data, $\mathcal{F}$ and the target slice $\mathcal{B}$. Hence to prove the implication we will have to *construct* the third piece of data, i.e. the sub-protocol $\mathcal{G}$, whilst keeping $\mathcal{F}$ and $\mathcal{B}$ arbitrary. We shall construct $\mathcal{G}$ in two parts:

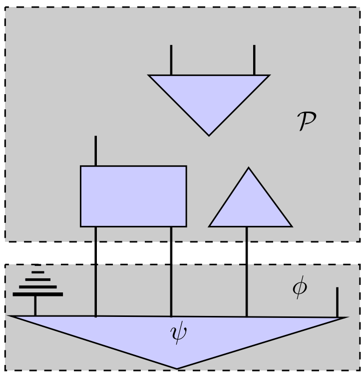
(i) We shall first construct a morphism $\mathcal{P}$, which we call the *causal past* of $\mathcal{B}$, since it will consist of all the morphisms in the 'past' of the target slice $\mathcal{B}$.

(ii) We shall then obtain $\mathcal{G}$ as the composition $\mathcal{G} := \mathcal{P} \circ \phi$, where ϕ will either be the initial state ψ, or a marginal state of ψ (up to symmetry isomorphism).

We shall achieve step (i), i.e. construct $\mathcal{P}$, by the following recursive algorithm. We first define $\mathcal{P}$ to contain all the morphisms in $\mathcal{F}$ whose output type is in $\mathcal{B}$. If an object A is both in $\mathcal{B}$ and in the input type of $\mathcal{F}$, then we include in $\mathcal{P}$ the identity morphism 1_A. Then denote by $\mathcal{D}$ the slice consisting of all the input types of the morphisms in $\mathcal{P}$: we now repeat the previous step, with $\mathcal{D}$ playing the role of $\mathcal{B}$, thus growing $\mathcal{P}$ to include the morphisms with output type in $\mathcal{D}$. We iterate this procedure until $\mathcal{D}$ becomes a slice $\mathcal{Z}$ consisting of objects wholly contained in the input type of $\mathcal{F}$ (but which may not be all of the input type of $\mathcal{F}$). We have now obtained the causal past $\mathcal{P}$: putting $Z := \mathcal{Z}$ and $B := \mathcal{B}$, then this is of type $\mathcal{P} : Z \to B \otimes B'$. We can now define the state ϕ, as

$$\phi := \sigma^{-1} \circ (1_Z \otimes \top_{Z'}) \circ \sigma \circ \psi$$

and $Z \otimes Z'$, which, up to symmetry, is equal to the input type of $\mathcal{F}$. To achieve step (ii), we then define $\mathcal{G} := \mathcal{P} \circ \phi$. For example, for the diagram (5.10), the causal past $\mathcal{P}$ and the state ϕ are



This completes the definition of $\mathcal{G}$. We then obtain the local state relative to $\mathcal{G}$ by applying $\top_E$ to any object E in the output type of $\mathcal{G}$ which is not in $\mathcal{B}$, i.e. we trace out the systems that we are not concerned with. For the example of diagram (5.10), this yields the diagram (5.12). $\square$

Theorem 5.34 shows that local states exist only for spatial slices; we now show their uniqueness.

Theorem 5.35 (Uniqueness of local states)**.** *For an SMC with terminal unit object, if a slice $\mathcal{B}$ admits a local state for any protocol in which it is included, then this state does do not depend on the choice of the sub-protocol $\mathcal{G}$ of Definition 5.32.*

Proof. Consider an arbitrary protocol $\mathcal{F}$ and sub-protocol $\mathcal{G}$. Any such $\mathcal{G}$ will include all the morphisms contained in the causal past $\mathcal{P}$ that is defined in the proof of Theorem 5.34. We now proceed by induction on the number of morphisms contained in $\mathcal{P}$: Theorem 5.34 is the base case. The inductive step, which we show now, is: if $\mathcal{P}$ contains n morphisms and yields the state ψ at $\mathcal{B}$, then $\mathcal{P}'$ with $n+1$ morphisms also yields the state ψ (relative to the same slice $\mathcal{B}$). To prove the inductive step consider an arbitrary morphism $h : D \to E$. We can enlarge $\mathcal{P}$ using h in *two* ways: either by using the monoidal product $\otimes$ or categorical composition $\circ$: let us call this the *parallel enlargement* and *sequential enlargement* respectively. These respectively yield

$$h \otimes \mathcal{P} \tag{5.13}$$

and

$$(h \otimes 1_{D'}) \circ \mathcal{P} \tag{5.14}$$

where we have omitted symmetry isomorphisms. (This mirrors step (i) in the proof of the previous Theorem.) Now, recall that $\mathcal{P}$ has type $\mathcal{P} : Z \to B \otimes B'$. Then consider the sequential enlargement of $\mathcal{P}$, i.e. Eq. 5.14. For this to be a sub-protocol relative to which we can define a local state at $\mathcal{B}$, we must have that D is an object that is part of B', i.e. it is not part of the target slice B. Now, terminality of I ensures that $\top_E \circ h = \top_D$. Hence post-composing Eq. 5.13 and Eq. 5.14 with $\top_E \otimes 1_{B \otimes B'}$ and $\top_E \otimes 1_{D'}$ respectively yields

$$(\top_E \otimes 1_{B \otimes B'}) \circ (h \otimes \mathcal{P}) = \top_D \otimes \mathcal{P}$$

and

$$(\top_E \otimes 1_{D'}) \circ ((h \otimes 1_{D'}) \circ \mathcal{P}) = (\top_D \otimes 1_{D'}) \circ \mathcal{P}.$$

Now, let us define the states corresponding to ϕ. The Definition 5.32 of local state relative to $\mathcal{G}$ requires that we trace out systems that are not in $\mathcal{B}$. Hence terminality ensures that the resulting local state will be the same as for $\mathcal{G} = \mathcal{P} \circ \psi$. □

Theorem 5.34 can be understood as arising from the way in which the object and morphism languages interact. Roughly speaking, the morphism-language defines how processes can be *composed*; the interaction of the morphism-language with the object-language defines how processes or scenarios can be *decomposed* using a slice. Theorem 5.34 shows that for this latter structure to allow local states to be defined for each slice, we require a partial monoidal structure.

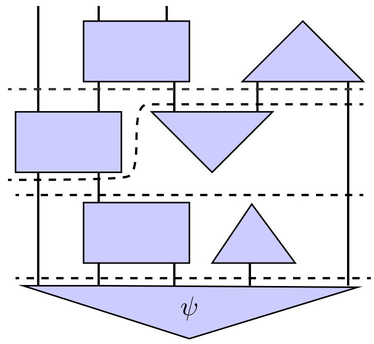
In the following two Remarks we provide some context to the proofs of Theorems 5.34 and 5.35.

Remark 5.36. Note that if we were to allow $f : B_i \to B_j$ to be a disconnected morphism in Theorem 5.34 then the ($\Rightarrow$) proof would break down, i.e. we would indeed be able to define a local state. This is possible because of our assumption that protocols (i.e. morphism formulae) are equivalent if they correspond to equivalent diagrams in the graphical language. In particular, a disconnected morphism formula $p \circ \top_{B_j}$ is equivalent to the formula $\top_{B_j} \otimes p$, i.e. this is in the same diagram equivalence class. But this would then provide a subformula $\mathcal{G} : I \to B_i \otimes B_j \otimes B'$, which ensures a local state can be given for the spatial slice $B_i \otimes B_j$. In contrast, in the connected case we were not able to 'push' B_1 next to B_2 to form $B_i \otimes B_j$—as can be done for the disconnected case—to be part of the codomain of $\mathcal{G}$.

Remark 5.37. Note that from the proof of Theorem 5.34 it also follows that in Definition 5.32 we do not always need to specify an initial state for the entire input slice $\mathcal{A}$ of the protocol $\mathcal{F}$, but only for the slice $\mathcal{Z}$ which is included in the causal past of $\mathcal{B}$.

Definition 5.38. A spatial slice $\mathcal{B}$ with $B := \mathcal{B}$ is *total* for a protocol $\mathcal{F} : A \to C$ in which it is included if $\mathcal{F}$ decomposes into two sub-protocols $\mathcal{F}_1 : A \to B$ and $\mathcal{F}_2 : B \to C$.

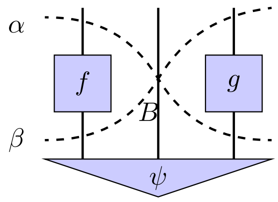
Total slices allow one to model evolution of a state through a protocol, when considering local states for 'propagating' family total slices e.g.:



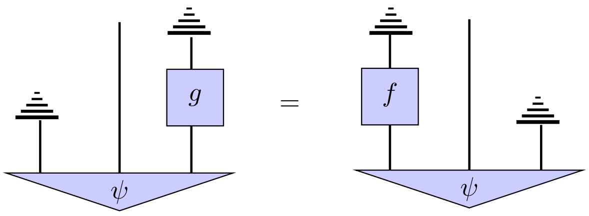
In this context, we shall take a *general covariance theorem* to mean that the state of a system does not depend on the particular choice of *foliation*, i.e. the slice it belongs to.

Corollary 5.39 (General covariance)**.** *Local states do not depend on the choice of foliation.*

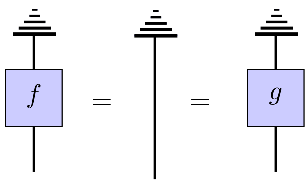
We provide a simple example: for the following protocol we calculate the state at B for the total spatial slices α and β:



Then we have:



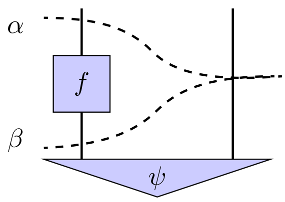
since by terminality:



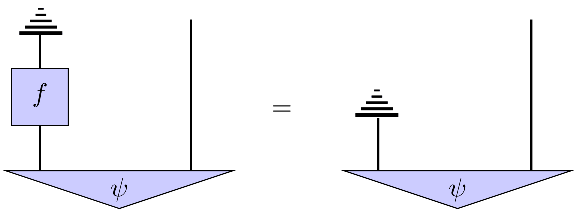
Now, above we showed that terminality of the monoidal unit implies covariance. However, in a CJ$\top$-universe the converse is not true; instead a weaker statement holds, as we now show.

Proposition 5.40. *In a CJ$\top$-universe, general covariance implies trace-preservation, i.e.* $\top_A = \top_B \circ f$ *for all morphisms* $f : A \to B$*.*

Proof. Consider the protocol $\mathcal{F} = f \otimes 1_B$, and a CJ state $\Psi : I \to A \otimes B$. There is a local state at B for each of the two sub-protocols defined by α and β:



If covariance holds then the two states are equal so:



(5.15)

However, since a CJ state provides an injective mapping from effects $\pi : A \to I$ to states $(\pi \otimes 1_B) \circ \Psi$, Eq. 5.15 implies that $\top_A = \top_B \circ f$. $\square$

Trace-preservation is weaker than terminality, since we showed in Proposition 5.23 that they are equivalent only when $\top_I = 1_I$ is assumed.

Hence, slices which are not spatial will not allow us to describe the local state on some part of the slice. Therefore, to ensure that this is always possible:

- *we will restrict the monoidal product to causally unrelated (i.e. disconnected) systems.*

Our formal definition is in the next Section. One interesting consequence of this notion of partiality is that:

- *all systems (i.e. objects) in a causal category correspond to spatial slices.*

The latter point follows from a recursive argument as follows. In a monoidal category, for any object A, either:

(i) there exists a monoidal decomposition of A into $B \otimes C$; or

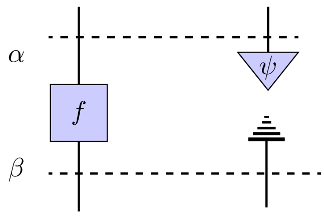
(ii) there does not.

If (ii), then A is spatial, since we can consider it to be the slice $A \otimes I$, for which A and I are disconnected (since every object is disconnected from the monoidal unit I). However if (ii), and there is such a decomposition, then B and C must be disconnected by assumption. But then we can repeat the argument using B instead A, and C instead A; this will eventually show that the object A is a spatial slice (this constitutes what is sometimes called a 'compositionality' argument, meaning that a categorical 'propagates' through the category by composition: in this case the property is that objects are spatial slices).

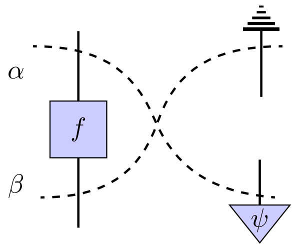
Remark 5.41 (Crossing slices)**.** Although we shall restrict tensor composition of objects we will not restrict the monoidal product of morphisms. In contrast to other work in the same vein, in particular Markopoulou [77] and Blute et al. [16], this will allow for morphisms to be defined between 'crossing' slices. For example, for slices $A \otimes B$ and $C \otimes D$, with A causally preceding C while D causally precedes B, it still makes sense to speak of processes of type $A \otimes B \to C \otimes D$, which will all be of the form

$$f \otimes (\psi \circ \top_B) = f \otimes \top_B \otimes \psi \tag{5.16}$$

with f arbitrary in $\mathbf{C}(A, C)$ and ψ arbitrary in $\mathbf{C}(I, D)$. Then the left-hand side of Eq. 5.16 is depicted as



but the right-hand side shows the 'causal' diagram:



Chapter summary. We have derived the structural properties of an SMC that are necessary to represent causal structure, when this is encoded using information flow. We showed how information flow can be represented by disconnectedness in the graphical language, provided that terminality of the monoidal unit is satisfied. We then showed that terminality can be derived by using a causal consistency property, which ensured that bipartite states cannot lead to information flow between systems which are not causally related. Finally we derived restrictions on the monoidal product by requiring that local states can be defined at a slice, for every protocol in which the slice appears. Hence the main structural properties that we have derived are terminality of the monoidal unit I, and partiality of the monoidal product $\otimes$.

Chapter 6

Causal categories

Contents

The work in the previous Chapter has led to us to the structure that we need to capture causality in CQM. In Section 6.1 we shall formally define this structure, a *causal category*, and explore its properties. In Section 6.2 we shall show how to construct examples of causal categories, and we shall also explore the formal relationship between causal categories and similar approaches.

6.1 Definition and structure of causal categories

In the previous Chapter we used physical properties to derive the mathematical structure of a monoidal category $\mathbf{C}$ that encodes causal structure, given certain assumptions. It is useful to summarise the results of the previous Section. We have done so in Table 6.1. This shows the *mathematical* structure corresponding to the *physical* properties that we aim to axiomatise.

Physical property	Mathematical structure	Assumptions
No Type I info flow	Disconnected hom-set	Terminality of monoidal unit
No Type II info flow	Terminality of tensor unit	Existence of CJ states
Unique local state for each slice	Partial monoidal structure	Terminality of monoidal unit

Table 6.1: Correspondence between physical properties, categorical structure, and necessary assumptions.

Note that the concrete models that we envisage will typically be quantum theory or classical probability theory, so the assumption of the existence of CJ states will be satisfied. Indeed, since this assumption leads to terminality (line 2 of Table 6.1), we note that there are no other assumptions necessary for deriving the other two structural features (lines 1 and 3 of Table 6.1), since they only require terminality of I. This leads to the formal definition of a causal category which we now introduce.

6.1.1 Definition and immediate properties of causal categories

This Subsection is divided into three parts:

1. We define causal categories.
2. We derive some useful technical properties of causal categories.
3. We identify some features that are relevant to their physical interpretation.

Initial definitions

In order to define causal categories we shall need to introduce some preliminary definitions. This sequence of definitions will mirror the one required for a symmetric monoidal category, since we shall define *partial* analogues of functor, bifunctor and monoidal category in turn. The meaning of 'partial' in this context is that of a partial function; i.e. as opposed to a total (ordinary) function. After this we will be in a position to define a causal category.

Definition 6.1. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{C_1}, \mathbf{C_2}, \mathbf{C_3}$ be categories.

- A *partial functor* $F : \mathbf{B} \to \mathbf{C}$ is a functor $\hat{F} : \mathbf{A} \to \mathbf{C}$, where $\mathbf{A}$ is a subcategory of $\mathbf{B}$, and $\mathbf{A}$ is called the *domain of definition* of F, written $\text{dd}(F) = \mathbf{A}$, and $\mathbf{B}$ is called the *domain* of F, written $\text{dom}(F) = \mathbf{B}$.
- A *partial bifunctor* is a partial functor G whose domain is a product category:

$$G : \mathbf{C}_1 \times \mathbf{C}_2 \longrightarrow \mathbf{C}_3.$$

Note that we use the hat notation $\hat{F}$ to denote a functor that corresponds to a partial functor F. Now, any functor F is a partial functor for which $\text{dd}(F) = \text{dom}(F)$. Hence partial functors as we have defined them generalise functors; however they are not commonly used in category theory[1]. The definition we have given is presumably not original, but no standard reference exists. However the definition and use of partial *bifunctors* has apparently not been explored elsewhere.

Example 6.2. Given any subcategory $\mathbf{A} \hookrightarrow \mathbf{B}$ and functor $\hat{G} : \mathbf{B} \to \mathbf{C}$ there is a partial functor $G : \mathbf{B} \to \mathbf{C}$. But it is often the case that a total functor might be recovered from a partial functor by redefining $\hat{G}$ with an extended domain, so that $\text{dom}(\hat{G}) = \text{dd}(G)$. For example, consider the powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$, which sends a

[1] Note that two distinct definitions of 'partial functor' have previously appeared in the literature, but both are different from our definition. Firstly, in [63] and related works, it has been used to describe the 'marginal' functor of a bifunctor. Secondly, in [18] it has been used in the context of *partial categories* for which the composition law is partial. The first definition is clearly not conceptually related to ours, but the second initially seems to be. However, it is actually quite different, since in that context a partial functor is (what we would call) a total functor, but between categories with a partially-defined composition law. This is why we emphasised in Definition 6.1 that we are concerned with *categories* not *partial categories*.

set A to its powerset $\mathcal{P}(A)$ and a function $f : A \to B$ to the function

$$\begin{aligned} \mathcal{P}(f) : \mathcal{P}(A) &\longrightarrow \mathcal{P}(B) \\ U &\longmapsto f[U] \end{aligned}$$

Since $\mathbf{Set}$ is a subcategory of $\mathbf{Rel}$, this straightforwardly defines a partial functor $\mathcal{P} : \mathbf{Rel} \to \mathbf{Set}$. However in this case $\mathcal{P}$ can be extended to a total functor $\mathcal{P} : \mathbf{Rel} \to \mathbf{Set}$. Consider a relation $R : A \to B$ in $\mathbf{Rel}$. Even though R is a relation (which may in general be multi-valued), the image of R restricted to subsets $U \subseteq \mathcal{U}$ is still a single set $V \subseteq \mathcal{P}(B)$. Hence to any relation R we can assign a function $\mathcal{P}(R)$ in $\mathbf{Set}$, defined as:

$$\begin{aligned} \mathcal{P}(R) : \mathcal{P}(A) &\longrightarrow \mathcal{P}(B) \\ U &\longmapsto V \end{aligned}$$

where $V = \{b \in B \mid \exists a \in A : aRb\}$. Note that such a function exists even if V is the empty set, since the mapping is given by $U \mapsto \emptyset$. A straightforward computation shows that functoriality of $\mathcal{P}$ is satisfied.

Remark 6.3. Partial functors have not received much attention in the literature. The reason for this is perhaps that examples of a partial functor which arise in mathematically natural situations fall into two classes, both of which are uninteresting mathematically. The first class consists of partial functors which are actually extendable to *total* functors, as in Example 6.2. The second class consists of 'contrived' partial functors, which are not extendable to total functors simply because the remaining part of the category does not carry the required structure. The following example illustrates this. Consider the category of pointed differentiable manifolds $\mathbf{Diff}$, whose objects are pairs (M, p) with M a differentiable manifold and $p \in M$, and morphisms are smooth maps. There is a tangent space functor $\hat{T} : \mathbf{Diff} \to \mathbf{Vec}_{\mathbb{R}}$ which on objects assigns a tangent space $T_p(M)$ and to a morphism f it assigns the pushforward $Tf : T_pM \to T_{f(p)}N$. Now, consider the category of topological spaces and continuous maps, $\mathbf{Top}$. Since $\mathbf{Diff} \hookrightarrow \mathbf{Top}$, $\hat{T}$ induces a partial functor $T : \mathbf{Top} \to \mathbf{Vec}_{\mathbb{K}}$. However the partial functor T is not extendable to a total functor $T_{\mathrm{tot}} : \mathbf{Top} \to \mathbf{Vec}_{\mathbb{K}}$. But this is a contrived example in the sense that $\mathbf{Top}$ contains many topological spaces S which are not smooth manifolds, and for which the tangent vector at a point $p \in S$ simply cannot be defined.

Remark 6.3 does not apply to the formalism we shall develop: firstly, we are not using partial functors to analyse a mathematically 'natural' category (such as $\mathbf{Vec}_{\mathbb{K}}$); secondly we are going to use partial *bifunctors.*

Now, recall that $\mathbf{C}$ is a *full subcategory* of $\mathbf{D}$ iff for all pairs of objects A, B in $\mathbf{C}$:

$$\mathbf{C}(A, B) = \mathbf{D}(A, B).$$

That is, for those objects A, B of $\mathbf{D}$ which are in $\mathbf{C}$, the category $\mathbf{C}$ contains *all* the morphisms in $\mathbf{D}$ (as opposed to if $\mathbf{C}$ were an arbitrary subcategory, for which only the inclusion $\mathbf{C}(A, B) \subseteq \mathbf{D}(A, B)$ holds).

Definition 6.4. A *symmetric strict partial monoidal category* is a category $\mathbf{C}$, together with a partial bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, for which dd($\otimes$) is a full subcategory of dom($\otimes$), and such that there exists a *unit object* I, which is the unit of a partial monoid $(|\mathbf{C}|, \otimes, I)$, satisfying

- Unit laws:

  (u1) $\forall A \in |\mathbf{C}|$, both $(A, I) \in |\mathrm{dd}(\otimes)|$ and $(I, A) \in |\mathrm{dd}(\otimes)|$,

  (u2) $\forall A \in |\mathbf{C}|$, $A \otimes I = A = I \otimes A$, and (u3) For all morphisms f in $\mathbf{C}$, $f \otimes 1_I = f = 1_I \otimes f$;

- Associativity laws:

(a1) $\forall A, B, C \in |\mathbf{C}|$, $(A, B), (A \otimes B, C) \in \text{dd}(\otimes)$ iff $(B, C), (A, B \otimes C) \in \text{dd}(\otimes)$,

(a2) $\forall A, B, C \in |\mathbf{C}|$, $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ when they exist, and

(a3) for any morphisms f, g, h in $\mathbf{C}$, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ when they exist;

- A symmetry law:

(s1) for all $A, B \in |\mathbf{C}|$ such that both $(A, B) \in \text{dd}(\otimes)$ and $(B, A) \in \text{dd}(\otimes)$, there exists a *symmetry isomorphism:*

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$

such that $\sigma_{A,B} \circ \sigma_{B,A} = 1_{A \otimes B}$.

Definition 6.4 introduces a type of partial monoidal category, a notion which has apparently not appeared in the literature. Partial monoids, on the other hand, have been studied in theoretical computer science [88], although no categorical generalisation seems to have been considered. Note that Definition 6.4 does not include the existence of structure isomorphisms ρ, λ, α, as the definition of an SMC does. This is because we have defined the analogue of a *strict* SMC, and so ρ, λ are replaced by the conditions (u2) and (u3), and α is replaced by the conditions (a2) and (a3).

Example 6.5. Any strict monoidal category is a strict partial monoidal category, where $\text{dd}(\otimes) = \text{dom}(\otimes)$, and any category that contains a strict monoidal category as a full subcategory is a strict partial monoidal category.

Remark 6.6 (Associativity of parallel composition for morphisms)**.** Definition 6.4 contains two existence conditions, (u1) and (a1), stating respectively that $A \otimes I$ and $I \otimes A$ always exist, and that $(A \otimes B) \otimes C$ exists iff $A \otimes (B \otimes C)$ exists. The definition of dd($\otimes$) as a *full* subcategory of dom($\otimes$) makes it unnecessary to include existence conditions for morphisms. This is because fullness implies that

$$\mathbf{CC}(A \otimes B, C \otimes D) = \mathbf{C} \times \mathbf{C}\left((A, B), (C, D)\right)$$

when $A \otimes B$ and $C \otimes D$ exist, and hence the partial monoidal product $f \otimes g$ of morphisms $f : A \to D$ and $g : B \to E$ exists if and only if $A \otimes B$ and $D \otimes E$ exist. Moreover, this means that, given a morphism $h : C \to F$, since $(A \otimes B) \otimes C$ exists iff $A \otimes (B \otimes C)$ exists, we also have that $(f \otimes g) \otimes h$ exists iff $f \otimes (g \otimes h)$ exists.

Remark 6.7 (Bifunctoriality)**.** For a partial monoidal category the bifunctoriality equation holds just as for a (full) monoidal category, i.e.

$$(h \otimes k) \circ (f \otimes g) = (h \circ f) \otimes (k \circ g). \tag{6.1}$$

This is guaranteed by the following facts. First we note that both sides of Eq. 6.1 exist, since $(h \otimes k) \circ (f \otimes g)$ and $(h \circ f) \otimes (k \circ g)$ have the same domain and codomain, and by Remark 6.6, $(h \otimes k) \circ (f \otimes g)$ exists iff $(h \circ f) \otimes (k \circ g)$ exists. Secondly, since $\otimes$ is defined to be a partial bifunctor, we have that $\otimes$ is in particular a functor $\hat{\otimes} : \mathbf{D} \to \mathbf{C}$ on a subcategory $\mathbf{D} := \text{dd}(\otimes)$. But the definition of a functor means that

$$\hat{\otimes}(p \circ q) = \hat{\otimes}(p) \circ \hat{\otimes}(q) \tag{6.2}$$

is satisfied for morphisms p, q in $\mathbf{D}$. Since $\mathbf{D}$ is a subcategory of $\mathbf{C} \times \mathbf{C}$ this means both that the morphisms p, q

in $\mathbf{D}$ are pairs $p = (h, f)$ and $q = (k, g)$, and that composition law is the same as in $\mathbf{C} \times \mathbf{C}$, i.e.:

$$(h, f) \circ (k, g) := (h \circ k, f \circ g). \tag{6.3}$$

Applying the functoriality law Eq. 6.2 to this composition law Eq. 6.3 then yields Eq. 6.1. (Note also that each side of the equation Eq. 6.2 must exist, since this is subsumed by the definition of $\hat{\otimes}$ as a functor on $\mathbf{D}$.)

A consequence of Remarks 6.6 and 6.7 is that a partial monoidal category behaves 'intuitively as expected', in the following sense. The fullness of the inclusion $\mathcal{I} : \mathrm{dd}(\otimes) \to \mathbf{C} \times \mathbf{C}$ and Remark 6.7 also means that when writing down a bifunctoriality equation such as Eq. 6.1 we can also safely do this for any *other* set of morphisms which have the same objects as domain and codomain. Indeed, Remarks 6.6 and 6.7 together informally mean that, for chosen hom-sets, the partial monoidal product behaves as the full monoidal product does: it is 'locally' a full monoidal product.

We shall now introduce the main object of study for the remainder of this Chapter. We use $[-]$ to denote pointwise application, meaning that if there is a unique morphism $\pi : A \to I$ such that

$$\mathbf{C}(A, B) = \{p \circ \pi \,|\, p : I \to B\}$$

then we write $\mathbf{C}(A, B) = [\mathbf{C}(I, B)] \circ \pi_A$ (i.e. the hom-set factors through π).

Definition 6.8. A *causal category* $\mathbf{CC}$ is a symmetric strict partial monoidal category for which

- the unit object I is terminal, i.e. for each object $A \in |\mathbf{CC}|$ there is a unique morphism $\top_A : A \to I$;
- the monoidal product, $A \otimes B$, exists iff

$$\mathbf{CC}(A, B) = [\mathbf{CC}(I, B)] \circ \top_A \quad \text{and} \quad \mathbf{CC}(B, A) = [\mathbf{CC}(I, A)] \circ \top_B\,; \tag{6.4}$$

- each object has at least one element, i.e. $\forall A \in |\mathbf{CC}| : \mathbf{CC}(I, A) \neq \emptyset$.

The requirement that every object has at least one element is the only part of Definition 6.8 that has not been hitherto motivated. The technical reason for this assumption will become clear below. However this is not a particularly restrictive assumption: for a causal category to be useful for describing a physical theory, each system should have at least one possible state.

Some immediate technical properties

The technical properties that we derive in this Subsection concern the interaction between the disconnectedness condition in a causal category, i.e. Eq. 6.4, and the partial monoidal structure.

Proposition 6.9. *In any causal category:*

(i) For all objects A, the hom-sets $\mathbf{CC}(A, I)$ and $\mathbf{CC}(I, A)$ are disconnected, and hence condition (u1) in the definition of partial monoidal category is implied by Eq. (6.4).

(ii) All morphisms $f : A \to B$ are 'normalised', i.e. $\top_B \circ f = \top_A$.

(iii) The equations $\top_I = 1_I$ and $\top_{A \otimes B} = \top_A \otimes \top_B$ are satisfied whenever $A \otimes B$ exists.

Proof. Item (i) follows from the definition of a disconnected hom-set, in particular since I is terminal we have:

$$\mathbf{CC}(I, A) = [\mathbf{CC}(I, A)] \circ 1_I = [\mathbf{CC}(I, A)] \circ \top_I,$$

and

$$\mathbf{CC}(A, I) = \{\top_A\} = \{1_I \circ \top_A\} = [\mathbf{CC}(I, I)] \circ \top_A.$$

so $I \otimes A$ and $A \otimes I$ always exist. Items (ii) and (iii) also follow straightforwardly from terminality of I. □

We shall refer to a monoidal or partial monoidal category in which the unit object is terminal, and for which each object A has at least one element, as a *normalised category.* The definition of disconnectedness for a partial monoidal category is the same as for a full monoidal category (i.e. as stated in Definition. 5.6 and Definition. 5.30), but we state it explicitly for completeness.

Definition 6.10 (Disconnectedness for a partial monoidal category)**.** In a partial monoidal category, a morphism $f : A \to B$ is *disconnected* if it decomposes as $f = p \circ e$ for some $e : A \to I$ and $p : I \to A$, and a hom-set $\mathbf{C}(A, B)$ is *disconnected* if it contains only disconnected morphisms. If both $\mathbf{C}(A, B)$ and $\mathbf{C}(B, A)$ are disconnected then we say that the objects A and B are *disconnected.*

Proposition 6.11. *For a causal category* $\mathbf{CC}$*, we have:*

(i) In Definition 6.8, Eq. (6.4) is equivalent to both $\mathbf{CC}(A, B)$ *and* $\mathbf{CC}(B, A)$ *being disconnected.*

(ii) In a causal category, $A \otimes B$ *exists iff* $B \otimes A$ *exists.*

(iii) Condition (a1) in the definition of partial monoidal category is implied by Eq. (6.4) together with the condition that if $A \otimes B$*,* $A \otimes C$*,* $B \otimes C$ *exist then also* $A \otimes (B \otimes C)$ *exists.*

Proof. Item (i) follows from the fact that, by terminality of I, any disconnected morphism $f : A \to B$ in a causal category is of the form $p \circ \top_A$. Item (ii) follows straightforwardly from the symmetry of Eq. (6.4). For item (iii): for any $f \in \mathbf{CC}(A, C)$, there exists $p_B \in \mathbf{CC}(I, B)$ such that:

$$\begin{aligned} f &= f \otimes 1_I && \text{(u3)} \\ &= f \otimes (\top_B \circ p_B) && \text{(terminality)} \\ &= (f \otimes \top_B) \circ (1_A \otimes p_B) && \text{(bifunctoriality)} \end{aligned}$$

Now, since $(A \otimes B) \otimes C$ exists, we have that $f \otimes \top_A = p_C \circ \top_{A \otimes B}$ for some $p_C \in \mathbf{C}(I, C)$, and so:

$$\begin{aligned} f &= p_C \circ (\top_A \otimes \top_B) \circ (1_A \otimes p_B) && \text{(terminality)} \\ &= p_C \circ \top_A && \text{(bifunctoriality).} \end{aligned}$$

Similarly we find that for any $g \in \mathbf{CC}(C, A)$, we have $g = p_A \circ \top_C$ for some $p_A \in \mathbf{C}(I, A)$. It follows that $A \otimes C$ exists, and by symmetry $B \otimes C$ also exists. Hence by our additional assumption $A \otimes (B \otimes C)$ exists. □

Remark 6.12. The significance of Proposition 6.11 is as follows. Item (ii) in Proposition 6.11 shows that the symmetry morphism can be consistently defined for a causal category, i.e. that Definition 6.8 is consistent. Item (iii) shows that the unit and associativity existence conditions (u1) and (a1) are implied by the disconnectedness condition of a causal category. This shows that a causal category is a somewhat 'rigid' structure: it could not have been defined without (u1) and (a1) if the disconnectedness condition is included.

Item (ii) in Proposition 6.11 is also noteworthy in relation to the definition of a partial monoidal product $\otimes$ as a bifunctor. In particular, it is relevant to the fact that, in a causal category, the domain of definition of a partial bifunctor is not necessarily a product category. For example, a natural requirement might be that since dom($\otimes$) is a product category, we should also require dd($\otimes$) to be a product category. However, our reason for not doing so is summarised in the following Proposition.

Proposition 6.13. *Let* $\mathbf{CC}$ *be a causal category for which dd($\otimes$) is a product category. Then we have that:*

(i) The object $A \otimes A$ *exists if* $A \otimes B$ *or* $B \otimes A$ *exists.*

(ii) The existence relation induced by $\otimes$ *is transitive, i.e. if* $A \otimes B$ *and* $B \otimes C$ *exist, then* $A \otimes C$ *exists.*

Proof. Item (i): let us write $\mathbf{D} := \text{dd}(\otimes)$, where $\otimes$ is the partial monoidal product for a causal category. Now, since $\mathbf{D}$ is the domain of definition for $\otimes$, we have $\mathbf{D} \hookrightarrow \mathbf{C}_1 \times \mathbf{C}_2$. But since $\mathbf{D}$ is a product category it also has the decomposition $\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2$ such that $\mathbf{D}_1 \hookrightarrow \mathbf{C}_1$ and $\mathbf{D}_2 \hookrightarrow \mathbf{C}_2$. Then the definition of $\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2$ means in particular that

$$|\mathbf{D}| = |\mathbf{D}_1| \times |\mathbf{D}_2|. \tag{6.5}$$

Then, suppose $A \otimes B$ exists, i.e. $(A, B) \in \mathbf{D}$. Now, since $\mathbf{D}$ is the domain of definition for a causal category, item (ii) in Proposition 6.11 implies that $(A, B) \in |\mathbf{D}|$ iff $(B, A) \in |\mathbf{D}|$. Hence $B \otimes A$ exists, and Eq. 6.5 implies that $B \in |\mathbf{D}_1|$ and $A \in |\mathbf{D_2}|$. Therefore we have both $A, B \in |\mathbf{D}_1|$ and $A, B \in |\mathbf{D_2}|$, and so we have $(A, A), (B, B) \in |\mathbf{D}|$, i.e. both $A \otimes A$ and $B \otimes B$ exist. Similarly we show that if $B \otimes A$ exists then $A \otimes A$ and $B \otimes B$ exist.

Item (ii): If $A \otimes B$ and $B \otimes C$ exist, then Eq. 6.5 implies that $A \in |\mathbf{D}_1|$ and $C \in |\mathbf{D}_2|$, and hence $A \otimes C$ exists. $\square$

Proposition 6.13 has more than just technical significance. It shows that the seemingly natural requirement that dd($\otimes$) is a product category is too restrictive. Concerning item (i), we have already informally argued in Section 5.2 that, in a 'spacetime' category, the object $A \otimes A$ lacks physical meaning. But we can make stronger statement, since Theorem 6.23 below shows that $A \otimes A$ cannot exist for all object A unless each object has only one state. Therefore causal categories that represent physically realistic scenarios (i.e. for which systems can have more than one possible state) cannot contain the object $A \otimes A$, and consequently dd($\otimes$) cannot be a product category. Also, requiring that $\otimes$ is transitive is too restrictive, as we shall discuss in Example 6.33 below.

Remark 6.14. Since causal categories represent processes occurring in spacetime, we can now interpret the structure morphisms in a spatiotemporally literal way. The symmetry morphism can then be seen as a 'kinematic' feature of a causal category, analogous to inversion of a spatial axis in a conventionally formulated physical theory. The isomorphism then asserts that the direction of the spatial axis does not affect the predictions of the theory. [2]

Interim summary. The technical properties that we have discussed so far in this Section are:

- The monoidal product for a partial monoidal product behaves 'locally' as a monoidal category, as shown by Remarks 6.6 and 6.7.
- Causal categories contain only normalised morphisms, shown in Proposition 6.9, and causal categories have a desirable symmetry property: $A \otimes B$ exists iff $B \otimes A$ exists, as shown in Proposition 6.11.

[2] This is also consistent with our previous comment in Remark 2.10, where we discussed the fact that a *strict* symmetry isomorphism leads to a degenerate monoidal structure. That is, although we can express an *isomorphism* between $A \otimes B$ and $B \otimes A$, we cannot *identify* them, since they are physically distinct situations, e.g. $\mathbb{C}^2$ at position x and $\mathbb{C}^4$ at y is distinct from $\mathbb{C}^4$ at position x and $\mathbb{C}^2$ at y.

- The domain of definition of $\otimes$ is an important part of defining a causal category: as shown in Proposition 6.13, requiring the domain of definition to be a product category affects the structure significantly, e.g. making slices transitive.

The last point confirms that causal categories axiomatise the phenomena in Chapter 5, where we derived terminality from post-selection.

Some basic physical properties

We shall now identify some physical properties of causal categories. These concern specific notions of causality.

Definition 6.15. In a causal category $\mathbf{CC}$:

- If both $\mathbf{CC}(A, B)$ and $\mathbf{CC}(B, A)$ are disconnected then we say that A and B are *space-like separated*.
- If $\mathbf{CC}(A, B)$ is connected but $\mathbf{CC}(B, A)$ is disconnected then A *causally precedes* B.
- If both $\mathbf{CC}(A, B)$ and $\mathbf{CC}(B, A)$ are connected then A and B are *causally intertwined*.

Proposition 6.16. *In a causal category* $\mathbf{CC}$*, every pair of objects* (A, B) *is either spacelike separated, causally intertwined, or such that either* A *causally precedes* B *or* B *causally precedes* A*.*

Proof. We first note that the hom-sets $\mathbf{CC}$ are all non-empty: in particular, each hom-set contains at least one disconnected morphism. This is because Definition 6.8 states that every object B has at least one state $\psi : I \to B$. But we also have that for any object A there is the morphism $\top_A : A \to I$. Hence for any pair of objects (A, B) the homset $\mathbf{CC}(A, B)$ contains at least the morphism $\psi \circ \top_A$. Now, Definition 6.15 classifies pairs of objects (A, B) according to the morphisms contained in the hom-sets $\mathbf{CC}(A, B)$ and $\mathbf{CC}(B, A)$. Since each hom-set contains at least one disconnected morphism, Definition 6.15 exhausts the possibilities: A and B will be intertwined unless either (i) $\mathbf{CC}(B, A)$ is disconnected (i.e. it contains *only* disconnected morphisms), in which case A causally precedes B; or (ii) both $\mathbf{CC}(A, B)$ and $\mathbf{CC}(B, A)$ are disconnected, in which case A and B are spacelike separated. $\square$

Remark 6.17. Since every object in a causal category is a spatial slice as defined in Definition 5.30, a pair of objects which is causally intertwined corresponds to the 'crossed' spatial slices that we discussed in Remark 5.41.

Example 6.18. The two simplest examples of causal categories are obtained as follows.

1. Each category induces a causal category by freely adjoining three items: a monoidal unit I, a state for each object A, and a unique morphism $\top_A : A \to I$ for all objects A. We call such a degenerate causal category *purely temporal*. The reason for this terminology is that the only disconnected hom-sets in a purely temporal causal category are, for all objects A, the hom-sets $\mathbf{CC}(A, I)$ and $\mathbf{CC}(I, A)$. However, as Proposition 6.9 shows, this is true in any causal category, and hence a purely temporal causal category is a causal category with 'minimal' spacelike structure, since it has no pairs of objects (A, B) that are spacelike separated (except $A = I$ or $B = I$).

2. Similarly, consider a monoid $(M, \odot, 1)$. We can view this as a category by taking the elements $a \in M$ to be the objects of a category $\mathbf{M}$. This induces a causal category with the monoid product $\odot$ as the monoidal product by freely adjoining, for each element $a \in M$, a morphism $\top_a : a \to 1$, and a state $p_a : 1 \to a$. This provides a disconnected morphism for each pair of objects, so that all hom-sets are of the form

$$\mathbf{M}(a, b) = \{p_b \circ \top_a\}$$

We call such a degenerate causal category *purely spatial*. A purely spatial causal category has 'maximal' spacelike structure, since every pair of objects (A, B) is spacelike separated.

Remark 6.19. In Example 6.18 we described how any category $\mathbf{D}$ induces a purely temporal causal category. But this ignored the structure of $\mathbf{D}$, e.g. whether it is already a monoidal category. We could instead attempt to define a causal category using the monoidal product of an SMC. But an arbitrary symmetric (full) monoidal category does not, by virtue of its monoidal structure, induce a causal category, in a simple way—as we have done for categories without monoidal structure in Example 6.18. (This is also a distinction between causal categories and general symmetric partial monoidal categories, since, as shown in Example 6.5, the latter are induced by monoidal categories.) There are two reasons for this.

1. A causal category needs to have a *partial* monoidal product, and hence to obtain a causal category from an arbitrary monoidal category $\mathbf{C}$, some objects $A \otimes B$ need to be removed from $\mathbf{C}$.
2. A causal category must satisfy the condition that partiality coincides with *disconnectedness* of hom-sets, i.e. Eq. 6.4 in Definition 6.8.

Moreover, a monoidal category will not in general even have disconnected hom-sets (e.g. $\mathbf{Rel}$): so even if the first point is addressed it is not immediate that Eq. 6.4 is satisfied. In contrast, although the purely spatial causal category in Example 6.18 is easily obtained from a full monoidal category, viz. a monoid, this is only because a monoid does not have any connected morphisms, as $\mathbf{Rel}$ does, which 'obstruct' the construction of a causal category. In fact, in Section 6.2 we shall address the issue of constructing causal categories from arbitrary monoidal categories.

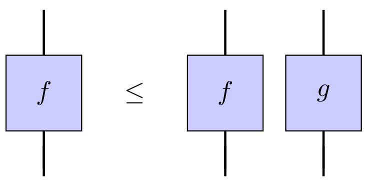
We shall shortly explore the relationship between causal categories and dagger compact categories, with a view to understanding how causal categories fit into the framework of CQM. Before doing so, it is useful to clarify the extent to which *categorical* properties are needed in the definition of a causal category. Let us consider this in relation to the domain of definition of $\otimes$. Now, we have defined a causal category to be a symmetric partial monoidal category with extra properties. This means that the partial bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a (full) functor on its domain of definition $\mathrm{dd}(\otimes)$, which is a full subcategory $\mathrm{dom}(\otimes) = \mathbf{C} \times \mathbf{C}$. However, it may be asked whether $\mathrm{dd}(\otimes)$ needs to be a sub*category* of $\mathrm{dom}(\otimes)$. In other words, could $\mathrm{dd}(\otimes)$ be another type of algebraic structure contained in $\mathrm{dom}(\otimes)$? It is instructive to consider attempting to define the domain as a subideal, since ideals are sometimes considered in spacetime posets [79].[3]

Recall that an *ideal* of a poset $(P, \leq)$ is a nonempty subset $\mathcal{I}$ of P which is both

(i) a *lower set*,meaning that if $x \in \mathcal{I}$ and $y \leq x$ then $y \in \mathcal{I}$; and

(ii) a *directed set*, meaning that if $x, y \in \mathcal{I}$ then there exists an upper bound $z \in \mathcal{I}$ such that $x \leq z$ and $y \leq z$.

Consider a strict symmetric monoidal category $\mathbf{C}$. We can define an order relation $\leq$ on objects as follows: $A \leq C$ iff there exists an object C such that $A \otimes B = C$. In other words, the order relation is defined to be inclusion of slices. For morphisms this can be defined analogously: $f \leq h$ iff there exists an object g such that $f \otimes g = h$. We depict this diagrammatically as:



[3] We thank the anonymous referee of [39] for this suggestion.

Now, consider $\mathbf{fVec}_{\mathbb{K}}$, the category of finite-dimensional vector spaces over a field $\mathbb{K}$. We denote the skeleton of $\mathbf{fVec}_{\mathbb{K}}$ as $\mathbf{S}$. Since the objects of $\mathbf{S}$ are all of the form $\mathbb{K}^{\otimes n}$, we can see that, using the order defined above, the poset $(|\mathbf{S}|, \leq)$ is in fact a total order. It is therefore an ideal, and this motivates considering defining a partial monoidal category as a subideal of $\mathbf{S}$. However, with the order defined above, the directed set condition in an ideal corresponds to the existence of a monoidal product. That is, if $|\mathbf{S}|$ is a directed set then $A, B \in \mathbf{S}$ implies that there exists $C \in \mathbf{S}$ such that $A \leq C$ and $B \leq C$. But using the order defined above this is always satisfied by using $C := A \otimes B$. Hence this shows us that a subideal will not allow us to define a notion of a partial monoidal product (which is necessary for the existence of global states, as derived in Section 5.2), since the upper bound must exist for all pairs in the subideal.

6.1.2 Relationship to dagger compact categories

We derived the structure of a causal category in Sections 5.1 and 5.2 using physical considerations within the framework of CQM. However, some basic aspects of CQM cannot be defined non-trivially for causal categories, as we show now. More specifically, we shall show that the following structures lead to a kind of degeneracy in causal categories:

(i) monoidal products $A \otimes A$;

(ii) causally related isomorphic objects;

(iii) compact structure;

(iv) dagger functor.

As we shall see, this degeneracy essentially arises from the involvement of identical or isomorphic objects (which allow us to identify systems of the same kind). However, in Subsection 6.2.4 we shall describe how this can be accounted for.

First, recall that in Section 5.2 we discussed how the monoidal product of a system with itself is arguably not meaningful in a causal setting. Now that we have defined causal categories, we can provide a formal analogue of this statement.

Lemma 6.20. *Let A be an object in a causal category $\mathbf{CC}$. If $A \otimes A$ exists then 1_A is disconnected.*

Proof. If $A \otimes A$ exists then $\mathbf{CC}(A, A) = [\mathbf{CC}(I, A)] \circ \top_A$, and 1_A is disconnected. □

Lemma 6.21. *Let A be an object in a causal category $\mathbf{CC}$. If the identity morphism 1_A is disconnected, then A has only one state.*

Proof. If 1_A is disconnected, then for some state $\psi : I \to A$ we have $1_A = \psi \circ \top_A$. Then for any state $\phi : I \to A$ we have

$$
\begin{aligned}
\phi &= 1_A \circ \phi && \text{(unit)} \\
&= \psi \circ \top_A \circ \phi && \text{(disconnectedness)} \\
&= \psi \circ 1_I && \text{(terminality)} \\
&= \psi && \text{(unit)}.
\end{aligned}
$$

□

Lemma 6.22. *Let A and B be objects in a causal category $\mathbf{CC}$. If either 1_A or 1_B is disconnected, then any morphism $f : A \to B$ is also disconnected.*

Proof. Consider a morphism $f : A \to B$ for which $1_A = \psi \circ \top_A$. Then

$$\begin{aligned} f &= f \circ 1_A && \text{(unit)} \\ &= f \circ \psi \circ \top_A && \text{(disconnectedness)} \\ &= \phi \circ \top_A \end{aligned}$$

for some morphism $\phi : I \to B$. The codomain case proceeds similarly. □

Theorem 6.23. *Let* $\mathbf{CC}$ *be a causal category. If* $A \otimes A$ *exists for all objects* A*, then all morphisms are disconnected, and each object has only one state.*

Proof. Lemmas 6.20, 6.21 and 6.22. □

Theorem 6.23 derives a property of all objects in the category $\mathbf{CC}$. We can derive similar but slightly weaker results for properties (ii)–(iv) in the list at the beginning of this Subsection. For property (ii), we now show that, for causal categories, isomorphisms cannot be used to represent the property that two systems at different spatiotemporal locations are of the same type (e.g. a qubit).

Proposition 6.24. *Given a causal category* $\mathbf{CC}$*, suppose that* A *causally precedes* B*, or that* A *and* B *are causally unrelated. If* $A \cong B$*, then both* 1_A *and* 1_B *are disconnected, and* $A \cong I \cong B$*.*

Proof. If either A causally precedes B, or A and B are space-like separated, then $\mathbf{CC}(B, A)$ is disconnected. Hence for the iso $f : A \to B$ we have, for some $\psi : I \to A$,

$$\begin{aligned} 1_A &= f^{-1} \circ f && \text{(iso)} \\ &= (\psi \circ \top_B) \circ f && \text{(disconnectedness)} \\ &= \psi \circ \top_A && \text{(terminality).} \end{aligned}$$

Since by terminality of I we also have $\top_A \circ \psi = 1_I$, we obtain $A \cong I$ and $B \cong I$ similarly. □

Hence the fact that systems at different spacetime locations are of identical types cannot be witnessed in the causal category. Instead, this can be defined in the †-SMC that will be used to construct the causal category—we shall describe this in Subsection 6.2.4.

When deriving terminality in Chapter 5, we showed that compact structure breaks down for SMCs with terminal unit object. We now consider this again, but in more detail, and specifically for a causal category (which we had not yet defined in our discussion of terminality).

Proposition 6.25. *Let* $\mathbf{CC}$ *be a causal category. If an object* A *is compact, then* 1_A *is disconnected, and morphisms between compact objects are disconnected. Hence for a compact subcategory of a causal category all morphisms are disconnected.*

Proof. If A is compact then $A \otimes A$ exists. Hence by Theorem 6.23, 1_A is disconnected, and morphisms between compact objects are disconnected. □

Hence compact structure is incompatible with the structure of a causal category. Finally, we show that a dagger functor can also not be defined for a physically meaningful causal category.

Proposition 6.26. *In a causal category with a dagger functor every object has only one state, and hence compound objects* $A \otimes B$ *only have disconnected states.*

Proof. For a given object A, a dagger functor provides a bijection

$$\mathbf{CC}(I, A) \cong \mathbf{CC}(A, I)$$

and since I is terminal this can only occur if each object has only one state. □

Interim summary. We have derived four results in this Subsubsection, corresponding to the four properties outlined at the beginning of this Subsubsection:

- Causal categories are quite different structurally from the usual categories studied in CQM: e.g. compact structure and the dagger functor are degenerate in a causal category.
- If we expect to extend CQM to a causal setting, then we need to connect causal categories to the usual structures, which we describe now.

6.2 Constructing causal categories

In this Section we shall describe methods for constructing causal categories. We shall construct them by considering the properties that causal categories satisfy, as shown in this Chapter so far. More specifically, we have seen that the full structure of a †-compact category is not compatible with causality. The question then, is:

Given a †-compact category, how do we obtain a causal category?

Let us briefly expand on the context of this question. This question assumes that a certain causal structure is to be assigned to the objects in the initial †-compact category. This is just as we described in Chapter 5, where, for example, we tried to assign causal structure to the compactness diagram—but found that it violated spacelike separation. We subsequently showed that considerations of causality lead to a highly modified type of monoidal category, i.e. a causal category, which we defined in the previous section. The two methods for constructing causal categories that we define in this Section arise from two different interpretations of these modifications.

The first step for each method consists of normalising a (†-compact) SMC $\mathbf{C}$, i.e. restricting to trace-preserving morphisms (as discussed in Example 5.22). There are then two options corresponding to each method of constructing a causal category:

1. *Carving*: Using Remark 6.19 as a starting point, we can 'carve out' an appropriate *subcategory* of $\mathbf{C}$. This will represent *discarding* the unphysical objects in the category, for example, discarding the connected morphisms between objects A and B that we want to assign spacelike separation to (i.e. for which we want $A \otimes B$ to exist).
2. *Pairing*: Alternatively, we can *combine* it with a causal structure, resulting in a partial monoidal product that exists for pairs of objects which are not causally related. However, the resulting causal category will not be a subcategory of $\mathbf{C}$.

This process is schematically depicted in Figure 6.1. In Subsection 6.2.4 we shall then describe how to reinstate the power of CQM, given that we showed above that structures such as compactness are not compatible with causal categories.

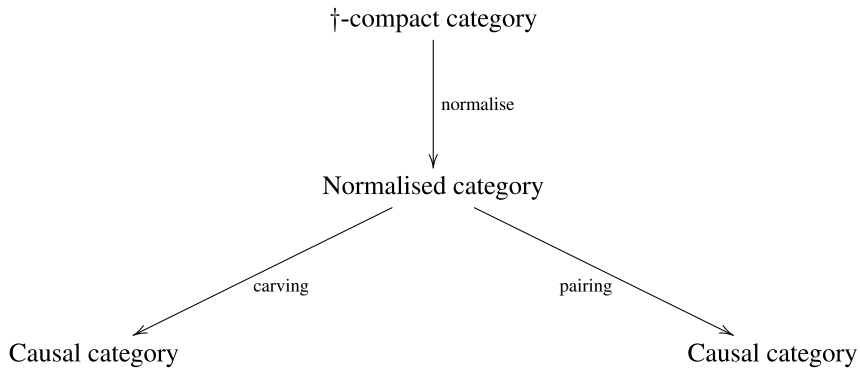
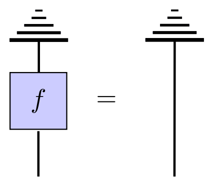


Figure 6.1: Schema of construction methods for causal categories.

6.2.1 Normalising

Recall that we constructed the category $CPM_{\top}(\mathbf{C})$ from the category $CPM(\mathbf{C})$. We defined this by restricting to morphisms f that satisfy



For $\mathbf{C} = \mathbf{fHilb}$ this gives the category of completely-positive trace-preserving maps, and the monoidal unit I is terminal. We shall mimic this method of constructing a normalised category, viz. we shall start with a †-compact category and discard the unnormalised morphisms.

Definition 6.27 (Normalisation)**.** Given a (†-compact) SMC $\mathbf{C}$ with environment structure, we define an SMC $\mathbf{C}_{\top}$, the *normalised subcategory of* $\mathbf{C}$, as having:

- the same objects as $\mathbf{C}$;
- the morphisms $f : A \to B$ are morphisms $f \in \mathbf{C}(A, B)$ that satisfy $\top_B \circ f = \top_A$, i.e. the normalised morphisms.

We repeat Proposition 5.23 for completeness.

Proposition 6.28. *A normalised subcategory* $\mathbf{C}_{\top}$ *has terminal unit object, when defining* $\top_I = 1_I$.

Recall that a *faithful functor* is a functor $F : \mathbf{C} \to \mathbf{D}$ for which, for all objects $A, B \in |\mathbf{C}|$, the function

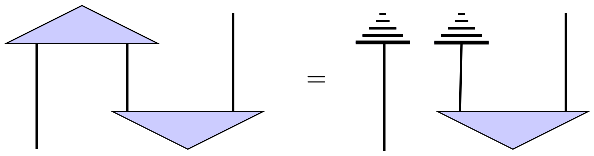
$$F_{A,B} : \mathbf{C}(A, B) \longrightarrow \mathbf{D}(FA, FB)$$

is injective. Now, note that the normalised subcategory $\mathbf{C}_{\top}$ of $\mathbf{C}$ defines an inclusion functor:

$$F_{\top} : \mathbf{C}_{\top} \hookrightarrow \mathbf{C}.$$

Since $F_{\top}$ is an inclusion functor, it is identity-on-objects (and so a strict monoidal functor), and the definition of $\mathbf{C}_{\top}$ means that this functor is faithful but not full (since it excludes unnormalised morphisms). Now, if $\mathbf{C}$

is †-compact, then $\mathbf{C}_\top$ will not be: as we have seen, terminality of I ensures that any bipartite effect becomes disconnected:



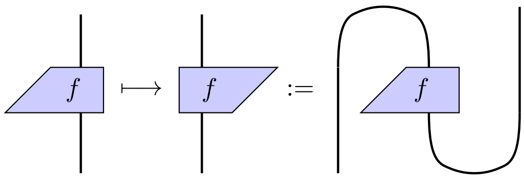
But we can retain the connection between $\mathbf{C}_\top$ and the given †-compact SMC $\mathbf{C}$ by using the strict monoidal functor $F_\top$, as we shall shortly explore.

However, for the moment let us consider what can be defined in $\mathbf{C}_\top$ *itself.* Not only can we not define compact structure, but we also cannot define a † functor, unless $\mathbf{C}$ is a preorder: this because Proposition 6.26 will apply to normalised categories $\mathbf{C}_\top$ as well as to causal categories. Hence a † functor can only be defined in $\mathbf{C}_\top$ if there is only one state for each object. However, surprisingly, we can define a *conjugate functor*, which provides complex conjugation in **fHilb**, even though this functor can be constructed from the dagger functor and the compact structure!

The construction of the conjugate functor in $\mathbf{C}$ is as follows. Firstly, given a compact category $\mathbf{C}$ recall that we can define a contravariant functor:

$$\begin{array}{rcl} (-)^* : \mathbf{C} & \longrightarrow & \mathbf{C} \\ A & \longmapsto & A^* \\ f & \longmapsto & (\epsilon_B \otimes 1_{A^*}) \circ (1_{B^*} \otimes f \otimes 1_{A^*}) \circ (1_{B^*} \otimes \eta_A)\,, \end{array}$$

that is, diagramatically:

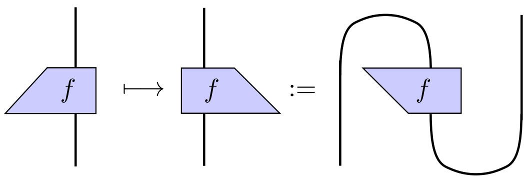


(6.6)

where we use a 180-degree rotation of the box representing the morphism to denote its *transpose*. In **fHilb**, this gives the transpose of a linear map. Moreover, if we have a dagger functor then we can define a covariant functor:

$$\begin{array}{rcl} (-)_* = ((-)^*)^\dagger = ((-)^\dagger)^* : \mathbf{C} & \longrightarrow & \mathbf{C} \\ A & \longmapsto & A^* \\ f & \longmapsto & (\epsilon_B \otimes 1_{A^*}) \circ (1_{B^*} \otimes f^\dagger \otimes 1_{A^*}) \circ (1_{B^*} \otimes \eta_A)\,, \end{array}$$

that is, in diagrams,



(6.7)

where we use reflection in the y-axis to denote the *conjugate*. In **fHilb**, the conjugate functor gives the complex conjugate of a linear map.

Let us now show that, under mild assumptions, the conjugate functor exists for $\mathbf{C}_\top$. We also show that the existence of a CJ state for every object A is retained in a $\mathbf{C}_\top$: this is the morphism η_A defined for the compact structure in $\mathbf{C}_\top$. Recall that $\bullet$ denotes the action of the scalars $\mathbf{C}(I, I)$ on an arbitrary hom-set $\mathbf{C}(A, A)$.

Theorem 6.29. *Let $\mathbf{C}$ be a dagger compact category with an environment structure for which, for all $A \in |\mathbf{C}|$:*

(cc1) *the scalar $s_A := \top_{A^*\otimes A} \circ \eta_A : I \to I$ is invertible; and,*

(cc2) $(\top_A)_* = \top_{A^*}$.

Then in $\mathbf{C}_\top$ every object has a CJ state $\eta_A^{CJ} = (s_A)^{-1} \bullet \eta_A$, and the conjugation functor $(-)_* : \mathbf{C} \to \mathbf{C}$ restricts to a conjugation functor $(-)_* : \mathbf{C}_\top \to \mathbf{C}_\top$.

Proof. For $A \in |\mathbf{C}_\top|$ the morphism $(s_A)^{-1} \bullet \eta_A \in \mathbf{C}(I, A \otimes A)$ is normalised, since

$$\begin{array}{rcll} \top_{A^*\otimes A} \circ \big((s_A)^{-1} \bullet \eta_A\big) & = & (s_A)^{-1} \bullet (\top_{A^*\otimes A} \circ \eta_A) & \text{(scalars)} \\ & = & (s_A)^{-1} \bullet s_A & \text{cc1} \\ & = & 1_I & \text{(cc1)}. \end{array}$$

Hence $(s_A)^{-1} \bullet \eta_A$ is in $\mathbf{C}_\top$. Now, the fact that this is a CJ state follows straightforwardly from compactness, since the property of being a CJ state is weakened form of compactness, as discussed in Example 5.13. Moreover, if $f : A \to B$ is normalised, then

$$\begin{array}{rcll} \top_{B^*} \circ f_* & = & (\top_B)_* \circ f_* & \text{(cc2)} \\ & = & (\top_B \circ f)_* & \text{(functoriality)} \\ & = & (\top_A)_* & \text{(normalisation)} \\ & = & \top_{A^*} & \text{(cc2)}. \end{array}$$

So conjugates of normalised morphisms are also normalised, and hence $\mathbf{C}_\top$ inherits the conjugation functor from $\mathbf{C}$. □

The importance of this Theorem is as follows:

(i) It shows that normalised subcategories of $\mathbf{C}$ can contain non-trivial categorical structure, i.e. a conjugate functor (although not compact structure). This is consistent with the idea that we can also capture causal structure in this way.

(ii) Moreover, normalised categories are 'half-way' towards a causal category, since they have a terminal unit object, and this is part of Definition 6.8.

Proposition 6.28 motivates the following terminology: by a *normalised category* we shall mean an SMC with a terminal unit object (i.e. we are extending the notion, introduced previously, of a normalised *sub*category of $\mathbf{C}$ to an arbitrary category). In Chapter 2 and Appendix A we discussed the symmetry morphism σ. In particular, we discussed the fact that strict-symmetric monoidal categories are rare, and we also gave an informal interpretation as to why this is the case. However, we also gave counterexamples to the fact that they are degenerate, i.e. a preorder. But we can show that strict-symmetric normalised categories are actually preorders.

Proposition 6.30. *Let $\mathbf{C}_\top$ be a normalised category which*

- *has at least one state $\psi : I \to A$ for every object A;*
- *is strict-symmetric, i.e. $\sigma_{A,B} = 1_{A\otimes B}$ for all $A, B \in |\mathbf{C}_\top|$.*

Then $\mathbf{C}_\top$ is a preorder: every hom-set $\mathbf{C}_\top(A, B)$ has at most one morphism.

Proof. Strict symmetry implies that, for morphisms $f : A \to B$ and $g : A \to B$:

$$f \otimes g = g \otimes f \tag{6.8}$$

Since there is at least one state using $\psi : I \to A$ for each object A, we can pre- and post-compose Eq. 6.8 with $\psi \otimes 1_A$ and $\top_B \otimes 1_B$ respectively:

$$(\top_B \otimes 1_B) \circ (f \otimes g) \circ (\psi \otimes 1_A) = (\top_B \otimes 1_B) \circ (g \otimes f) \circ (\psi \otimes 1_A).$$

But now bifunctoriality and then terminality yields:

$$(\top_B \circ f \circ \psi) \otimes g = (\top_B \circ g \circ \psi) \otimes f$$
$$f = g.$$

Since the morphisms f and g are arbitrary (including their domain and codomain), there is a single morphism in $\mathbf{C}_\top(A, B)$, for all A, B in $|\mathbf{C}|$. □

Hence strict-symmetric normalised categories are not expressive enough to model information flow in a meaningful way: e.g. each such category could only model a single protocol.

6.2.2 Causal structure and carving

We have so far considered two notions of causal structure in relation to causal categories:

1. *Global*: We defined causal structure as a poset $\mathcal{P} = (P, \leq)$ arising from a relativistic spacetime.
2. *Local*: We described bipartite causality in Chapter 6 as a non-constant morphism. We then defined causal structure in a monoidal category as arising from the existence of a connected morphism.

Now, we might have thought that the causal structure of a monoidal category is a combination of the two notions above, i.e. a partial order (point 1, the global notion) induced by the connectedness of hom-sets (point 2, the local notion). However, the connectedness of hom-sets does *not* in general induce a partial order. This is because connectedness is not transitive, as we show in Figure 6.4. Hence we shall have to expand the notion of causality

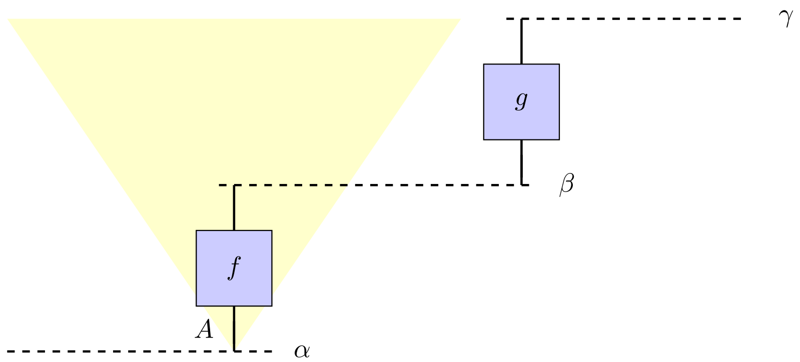


Figure 6.2: The forward light cone of the observer at A does not include the slice γ, despite the relations $a \leq b$ and $b \leq c$ holding.

once more. In a monoidal category, causal structure is defined as follows.

Definition 6.31. The *causal structure* of a partial (or full) monoidal category is a directed graph $\mathcal{G} = (G, E)$, whose vertices G are the objects of $\mathbf{CC}$, and an edge $(A, B) \in E$ exists if and only if $\mathbf{CC}(A, B)$ is connected.

Example 6.32. We can define a causal category $\mathbf{CC}$, shown in Figure 6.3, whose causal structure is the directed graph of a '3-loop'. We obtain $\mathbf{CC}$ from the graph by freely adjoining the monoidal unit. Note that pairs (A, B) for which either $A = I$ or $B = I$ are disconnected, and so $(A, B) \notin E$. However it will be useful to display such pairs with a dashed line in graph diagrams.

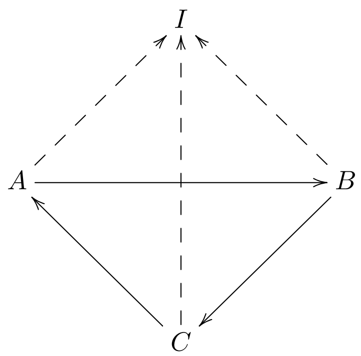


Figure 6.3: Graph corresponding to the 3-loop causal category of Example 6.32.

In this causal category, the only pairs of objects for which $\otimes$ exists are $A \otimes I$, $I \otimes A$ and $I \otimes I$. Reading off Figure 6.3, we see that the restrictions on the morphisms are as follows. Firstly, for related pairs (A, B), we have $\mathbf{CC}(A, B) \neq [\mathbf{CC}(I, B)] \circ \top_A$. Secondly, we must ensure that any pair of composable connected morphisms f, g, the composite $g \circ f$ is disconnected. This is allowed, since there is nothing in the definition of a causal category that forces the composition of connected morphisms to be a connected morphism. That is, connectedness of hom-sets is not transitive.

Example 6.33. Connectedness of hom-sets is, however, satisfied by causal categories that represent degenerate causal structure. For example, a Galilean spacetime written as a causal category $\mathbf{CC}_G$ satisfies this, since a Galilean spacetime does not have non-trivial causal structure. Moreover, the existence relation induced by $\otimes$ is transitive for $\mathbf{CC}_G$. This distinguishes $\mathbf{CC}_G$ from causal categories with non-trivial light cone structure. This can be seen from Figure 6.4, which shows points A, B, C and their light cones in Minkowski spacetime. We see that A and B are spacelike separated, and B and C are spacelike separated, but A and C are causally related. This

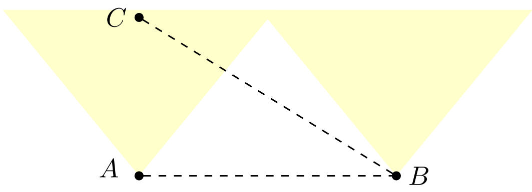


Figure 6.4: Failure of transitivity of spacelike separation in Minkowksi spacetime.

should be seen in the context of Proposition 6.13, which showed that if the domain of definition $\mathrm{dd}(\otimes)$ of a causal category is a product category then existence for $\otimes$ is transitive. Hence we now see that requiring $\mathrm{dd}(\otimes)$ to be transitive yields causal categories such as $\mathbf{CC}_G$, for which causal structure is degenerate.

Now, since every full monoidal category is also a partial monoidal category, we also can apply Definition 6.31 to full monoidal categories.

Example 6.34 (Degenerate causal structure)**.** Recall the category of mixed states and completely positive maps $CPM(\mathbf{C})$, considered in Example 5.22. This is the setting for mixed-state quantum theory. We also introduced the subcategory $CPM_{\top}(\mathbf{C}) \hookrightarrow CPM(\mathbf{C})$, which contains only morphisms that are *trace-preserving* complete-positive maps. As described above, requiring each morphism to preserve the trace implies that the unit object of this category is terminal. Hence $CPM_{\top}(\mathbf{C})$ is a normalised category, and therefore satisfies *some* of the conditions required to be a causal category. However, this category has *degenerate* causal structure: each object is connected to one another. For example, in each hom-set $CPM_{\top}(\mathbf{C})(\mathbb{C}^{\otimes m}, \mathbb{C}^{\otimes n})$ such that $m \neq n$ and $m, n > 1$, there is a morphism that does not factor through $\mathbb{C}$: for example, consider any isometry $U : \mathbb{C}^{\otimes m} \to \mathbb{C}^{\otimes n}$. Indeed, this category actually represents the 'opposite' structure of a causal category, since every pair of objects in $CPM_{\top}\mathbf{C}$ is connected but the monoidal product $A \otimes B$ also exists for every pair of objects (A, B).

The preceding Example suggests that $CPM_{\top}(\mathbf{C})$ will induce a causal category if we discard objects and morphisms according to the causal structure that we intend to encode. Hence just as we obtained a normalised category $\mathbf{C}_{\top}$ by discarding the unnormalised morphisms in $\mathbf{C}$, we can obtain a causal category $\mathbf{CC}$ as a subcategory of a normalised category by discarding the connected morphisms. To be more precise, from a normalised category we shall:

- Discard the connected morphisms between A and B when we want to assign the relation of spacelike separation to the pair (A, B);
- We will also have to ensure that, in addition to normalisation, we have: for all $A, B \in |\mathbf{C}|$, A and B are connected if and only if $A \otimes B \notin |\mathbf{CC}|$.

We now develop this idea formally.

Definition 6.35 (Carved category)**.** Let $\mathbf{C}$ be a strict SMC. A *carved causal category (of* $\mathbf{C}$*)*, denoted $\mathbf{CC}$, is a causal category that is a monoidal subcategory of $\mathbf{C}$.

We shall shorten 'carved causal category' to *carved category* from now on. Now, to obtain a causal category as a carved category, i.e. a subcategory of a category of physical processes $\mathbf{C}$, we proceed as follows. We want to extract a physical universe $\mathbf{CC}$ from $\mathbf{C}$, e.g. some systems such as qubits, and define causal relationships between them using $\mathcal{G}$. The main idea of the following construction is to *discard* objects and morphisms which could not exist in our intended 'spacetime', e.g. a connected morphism between spacelike separated points. Recall that a directed graph $\mathcal{G} = (G, E)$ is *transitive* if and only if the adjacency relation E is transitive.

Definition 6.36. Let $\mathbf{C}$ be an SMC of possible physical processes, let $\mathbf{C}_{\top}$ be its normalised subcategory, and let $\mathcal{G}$ be a transitive directed graph. The *carving construction of* $\mathbf{CC}$ is a triple $(\mathbf{C}, \mathcal{G}, \kappa : G \to |\mathbf{C}|)$, where κ is an injective mapping on the vertices of $\mathcal{G}$, that yields $\mathbf{CC}$ as a carved category of $\mathbf{C}$ as follows:

1. The objects of the category $\mathbf{CC}$ are defined as $|\mathbf{CC}| := \kappa(G)$.
2. For $A := \kappa(g_1)$ and $B := \kappa(g_2)$, we have

$$\mathbf{CC}(A, B) := \begin{cases} \mathbf{C}_{\top}(A, B) & \text{if } (g_1, g_2) \in E; \\ [\mathbf{C}_{\top}(I, B)] \circ \top_A & \text{otherwise.} \end{cases}$$

3. We define $\kappa(g_1) \otimes \kappa(g_2)$ to exist if and only if $(g_1, g_2) \notin E$.

The three parts of Definition 6.36 formalise the 'discarding' idea that we discussed above, and conceptually they correspond to the following three steps:

1. *Assign the objects*: The mapping κ identifies the objects of $\mathbf{C}$ which we want to consider in our protocol in spacetime, e.g. qubits using $\mathbb{C}^2$.

2. *Assign the hom-sets*: We define the hom-sets of $\mathbf{CC}$ to be connected, and thus identical to the hom-sets of $\mathbf{C}_\top$, only when they are causally related in $\mathcal{G}$ (i.e. only when they are causally related in the intended protocol).

3. *Define* $\otimes$: Finally, we make $\mathbf{CC}$ into a carved category of $\mathbf{C}$, i.e. a causal category, by defining $\mathbf{CC}$ to satisfy the partiality condition on the existence of the monoidal product: the partial monoidal product $\otimes$ is such that the monoidal product of a pair of objects $(\kappa(g_1), \kappa(g_2))$ exists if and only if $(g_1, g_2) \notin E$.

Remark 6.37. Note the following technical points regarding the carving construction:

(i) We required that the digraph is transitive, despite the remarks above concerning Definition 6.31 (which we explained using Figure 6.4). Now, if the digraph $\mathcal{G}$ is not transitive it is always possible to define an $E' \supset E$ such that $\mathcal{G} = (G, E)$ induces a transitive digraph $\mathcal{G}' = (G, E')$. But this transitive closure has a different physical interpretation, since it describes different causal relations, and so it not useful for representing the initial protocol. In any case, we shall explain shortly the reason for the assumption of transitivity.

(ii) We have defined the carving construction as factoring through the normalisation process. That is, first we identify the normalised subcategory $\mathbf{C}_\top$ of $\mathbf{C}$, and then we apply the steps 1–3 in Definition 6.36. Hence we have:

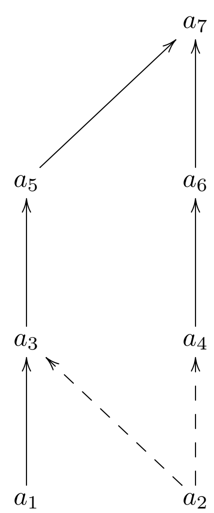
$$\mathbf{CC} \hookrightarrow \mathbf{C}_\top \hookrightarrow \mathbf{C}.$$

This explains the diagram in Figure 6.1.

(iii) Importantly, in step 2, transitivity of $\mathcal{G}$ ensures that the composition law of $\mathbf{CC}$ can be defined as the composition law of $\mathbf{C}$.

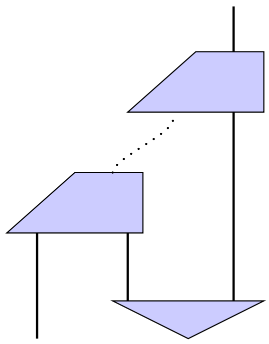
An advantage of carving is that there exists a canonical embedding into a †-SMC, i.e. the category out of which the causal category was carved. We discuss this in Subsection 6.2.4.

Example 6.38 (Teleportation protocol)**.** We can construct a causal category $\mathbf{T}$ for the teleportation (*with* classical communication) protocol using the carving construction. We define a directed graph $\mathcal{T} = (T, E)$ that represents the causal relationships[4] in the protocol:



[4] Recall that we use dashed lines for pairs (a_2, a_3) and (a_2, a_4). These pairs are not edges but the dashed lines indicate that a_2 will be assigned the monoidal unit.

This should be compared to the depiction in the graphical calculus:



where we schematically show the classical communication using the dotted line. We shall carve $\mathbf{T}$ from $CPM(\mathbf{C})$, using the carving construction $(CPM(\mathbf{C}), \mathcal{T}, \kappa : T \to |\mathbf{C}|)$ as follows:

1. We take the requisite number of isomorphic copies of $\mathbb{C}^2$, or monoidal products thereof, to be the objects of the category $\mathbf{T}$. The mapping κ is therefore defined as:

$$\kappa(a_i) \cong \begin{cases} \mathbb{C} & \text{if } i = 2 \\ \mathbb{C}^2 & \text{if } i \in \{1, 4, 6, 7\} \\ \mathbb{C}^2 \otimes \mathbb{C}^2 & \text{if } i \in \{3, 5\} \end{cases}$$

2. We define the hom-sets using the edges in $\mathcal{G}$, i.e. hom-sets are connected for related pairs, otherwise they are disconnected:

$$\mathbf{T}(\kappa(a_i), \kappa(a_j)) := \begin{cases} CPM_\top(\mathbf{C})(\kappa(a_i), \kappa(a_j)) & \text{if } (a_i, a_j) \in E; \\ [CPM_\top(\mathbf{C})(I, \kappa(a_j))] \circ \top_{\kappa(a_i)} & \text{otherwise.} \end{cases}$$

   For example, Alice's system $\kappa(a_3) \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ at a_3 has Bob's system $\kappa(a_7) \cong \mathbb{C}^2$ at a_7 in its future light-cone. Hence the hom-set $\mathbf{T}(\kappa(a_5), \kappa(a_7))$ will be connected since this hom-set is defined to be the same as for $CPM_\top(\mathbf{C})$:

$$\mathbf{T}(\kappa(a_5), \kappa(a_7)) := CPM_\top(\mathbf{C})(\kappa(a_5), \kappa(a_7))$$

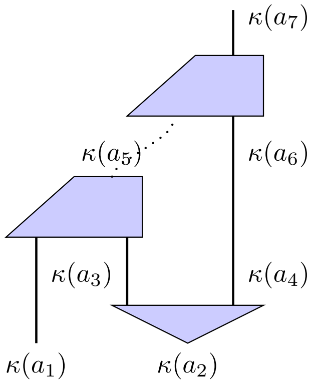
   On the other hand, when a_i and a_j are not causally related, $\mathbf{T}$ only inherits some of the morphisms from $CPM_\top(\mathbf{C})$, i.e. the disconnected morphisms:

$$\mathbf{T}(\kappa(a_i), \kappa(a_j)) \subsetneq CPM_\top(\mathbf{C})(\kappa(a_i), \kappa(a_j)).$$

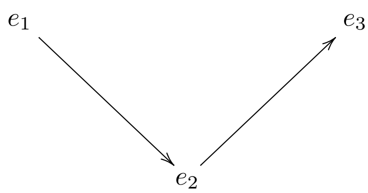
   This will define a normalised subcategory of $CPM_\top(\mathbf{C})$.

3. We define the partial monoidal product for $\mathbf{T}$ to exist only for disconnected systems.

Hence we obtain a causal category $\mathbf{T}$ which formalises both the resources and the causal relationships of available for the teleportation protocol. In particular, the teleportation protocol with classical communication is a morphism in this category, and with the objects from the carving construction annotated, this morphism is depicted as:



Now, although this seems satisfactory there are several features of this construction that are problematic. In Remark 6.37 (iii), we noted that the definition of carving required the assumption that the digraph $\mathbf{G}$ is transitive. The reason for this assumption is as follows. Suppose that we were to use the carving construction for an SMC $\mathbf{C}$ and a digraph $\mathcal{G}$ that is not transitive, for example:



Now, the carving construction will produce a category $\mathbf{CC}$ with hom-sets

$$\mathbf{CC}(\kappa(e_1), \kappa(e_2)) := \mathbf{C}(\kappa(e_1), \kappa(e_2))$$

and

$$\mathbf{CC}(\kappa(e_2), \kappa(e_3)) := \mathbf{C}(\kappa(e_2), \kappa(e_3))$$

for the edges (e_1, e_2) and (e_2, e_3) respectively. Similarly, the construction will assign a disconnected hom-set to the edge (e_1, e_3):

$$\mathbf{CC}(\kappa(e_1), \kappa(e_3)) := [\mathbf{C}(I, \kappa(e_3))] \circ \top_{\kappa(e_1)}$$

But this means that connected morphisms $f : \kappa(e_1) \to \kappa(e_2)$ and $g : \kappa(e_2) \to \kappa(e_3)$ must compose to a disconnected morphism in $\mathbf{CC}$:

$$g \circ f = \psi \circ \top_{\kappa(e_1)} \tag{6.9}$$

However, in general this composition will not hold in $\mathbf{C}$. For example, let $\mathbf{C} = \mathbf{Set}$, and let the objects defined by κ be isomorphic, i.e.:

$$\kappa(e_1) \cong \kappa(e_2) \cong \kappa(e_3).$$

Then we can choose f and g to be isomorphisms. But then $g \circ f$ will compose to an isomorphism. Such an isomorphism cannot be disconnected: suppose that $\kappa(e_1) = \{s, t\}$ and $\kappa(e_3) = \{u, v\}$. Then consider a disconnected morphism as in Eq. 6.9:

$$\begin{aligned} h : \kappa(e_1) &\longrightarrow \kappa(e_3) \\ x &\longmapsto (\psi \circ \top_{\kappa(e_1)}) \circ x = \psi \end{aligned}$$

Since h is disconnected, it is not injective; indeed h is constant. Hence h not an isomorphism in $\mathbf{Set}$. Hence with this choice of digraph $\mathcal{G}$, that is *not transitive*, Eq. 6.9 holds in $\mathbf{CC}$ but not in $\mathbf{C}$. This implies that $\mathbf{CC}$ is not a subcategory of $\mathbf{C}$, since now $\mathbf{CC}$ and $\mathbf{C}$ have different composition rules. The requirement that the digraph $\mathcal{G}$ is transitive therefore ensures that $\mathbf{CC}$ is a subcategory of $\mathbf{C}$. We could attempt to define a carving construction which does not require $\mathbf{CC}$ to be a sub*category* of $\mathbf{C}$, but then carving would not be a categorical construction, and would be outside the scope of CQM.

Now, we showed in Figure 6.4 that the causal structure of spatial slices is not transitive. Hence to capture this as a causal category, we will have to define a new construction, since carving cannot account for it. The new construction should not be based on the 'discarding' idea, but instead defines a causal category $\mathbf{CC}$ that is not a subcategory $\mathbf{C}$. Now, to do so consider the following idea. Although transitive digraphs do not capture the causal structure of slices, they do capture the causal structure of *points* of a spacetime. Equivalently, they capture the causal structure of 'atomic' objects. Hence to capture the causal structure of slices, we need a construction that builds slices from points.

6.2.3 A causal category from a causal set and an SMC

Based on the reasoning of the previous Subsubsection, we shall assume that $\mathcal{G}$ is a transitive digraph, and build slices using elements of G. Accordingly, in this Subsection we shall construct a causal category from the given transitive digraph $\mathcal{G} = (G, E)$ and an SMC $\mathbf{C}$.

Now, because $\mathcal{G}$ is transitive, we shall denote the relation E as $\leq$. We define $R(\mathcal{G}) \subseteq 2^G$ to consist of those subsets $a \subseteq G$ satisfying

$$x, y \in X \Longrightarrow (x \not\leq y \wedge y \not\leq x).$$

That is, the elements $X \in R(\mathcal{G})$ are spatial slices, or 'spacelike hypersurfaces', since no objects in X are causally related to any other.

We shall use an ordering on spatial slices in what follows. This is defined as $X \sqsubseteq Y$ if and only if:

$$\forall x \in X, \forall y \in Y : x \leq y\,. \tag{6.10}$$

Before defining our construction formally, let us give a brief description of the main idea. The idea is that the construction will *combine* both $\mathbf{C}$ and $\mathcal{G}$ into a new category $\mathbf{CC}(\mathbf{C}, \mathcal{G})$, where its notation indicates the data required to define it (cf. the fact that a carved category $\mathbf{CC}$ was defined as a *subcategory* of $\mathbf{C}$). We will do this by constructing a category in which objects contain causal data: the objects shall be pairs (A, x) where A is an object of $\mathbf{C}$, and $x \in G$. All the other data defined by the category will inherit this pairing of objects from $\mathbf{C}$ and $\mathcal{G}$. For example, when defining the monoidal product we shall keep track of the space-time point an object in $\mathbf{C}$ is assigned to. Hence objects will not just be pairs (A, x), but a set of *indexed pairs*, of the form $\{(A_i, x_i)\}_{i\in\mathcal{I}}$.

We now define this formally.

Definition 6.39 (Pairing construction)**.** The *pairing of a monoidal category* $\mathbf{C}$ *and a directed graph* $\mathcal{G}$, is a category $\mathbf{CC}(\mathbf{C}, \mathcal{G})$, defined as follows:

- Objects are either sets of pairs $\{(A_i, x_i)\}_{i\in\mathcal{I}}$ with $A_i \in |\mathbf{C}| \setminus \{I\}$ for all $i \in \mathcal{I}$ and $\{x_i\}_{i\in\mathcal{I}} \in R(\mathcal{G})$, or $(I, \emptyset)$. For $\{(A_i, x_i)\}_{i\in\mathcal{I}}$ and $\{(B_j, y_j)\}_{j\in\mathcal{J}}$ the monoidal product is the union, and exists provided that:

$$\{x_i\}_{i\in\mathcal{I}} \cap \{y_j\}_{j\in\mathcal{J}} = \emptyset \qquad \{x_i\}_{i\in\mathcal{I}} \cup \{y_j\}_{j\in\mathcal{J}} \in R(G)\,,$$

  and we set $\{(A_i, x_i)\}_{i\in\mathcal{I}} \otimes (I, \emptyset) := \{(A_i, x_i)\}_{i\in\mathcal{I}}$.

- We define the hom-sets for states as:

$$\mathbf{CC}(\mathbf{C}, \mathcal{G})\big((I, \emptyset), \{(A_i, x_i)\}_{i\in\mathcal{I}}\big) := \mathbf{C}(I, \otimes_{i\in\mathcal{I}} A_i)$$

- For general morphisms we set:

$$\mathbf{CC}(\mathbf{C}, \mathcal{G})\big(\{(A_i, x_i)\}_{i\in\mathcal{I}}, \{(B_j, y_j)\}_{j\in\mathcal{J}}\big) :=$$

$$\left\{ \sigma' \circ \Big(f \otimes (p \circ \top_{\otimes_{i\in\mathcal{I}\setminus\mathcal{I}'} A_i})\Big) \circ \sigma \;\middle|\; \begin{array}{c} \mathcal{I}' \subseteq \mathcal{I}, \mathcal{J}' \subseteq \mathcal{J} \\ \{x_i\}_{i\in\mathcal{I}'} \sqsubseteq \{y_j\}_{j\in\mathcal{J}'} \\ p \in \mathbf{C}(I, \otimes_{j\in\mathcal{J}\setminus\mathcal{J}'} B_j) \\ f \in \mathbf{C}(\otimes_{i\in\mathcal{I}} A_i, \otimes_{j\in\mathcal{J}} B_j) \end{array} \right\}$$

  where σ and σ' are the unique symmetry isomorphisms that re-order the objects of $\mathbf{C}$ to match the ordering of $\mathcal{I}$ and $\mathcal{J}$, and $X \sqsubseteq Y$ is defined according to Eq. 6.10. Finally, we close under tensoring, that is, for all $\{(A_i, x_i)\}_{i\in\mathcal{I}}$ and $\{(A'_i, x'_i)\}_{i\in\mathcal{I}'}$ for which the tensor exists, and all $\{(B_j, y_j)\}_{j\in\mathcal{J}}$ and all $\{(B'_j, y'_j)\}_{j\in\mathcal{J}'}$ for which the tensor exists, if

$$f \in \mathbf{CC}(\{(A_i, x_i)\}_{i\in\mathcal{I}}, \{(B_j, y_j)\}_{j\in\mathcal{J}})$$

  and

$$f' \in \mathbf{CC}(\{(A'_i, x'_i)\}_{i\in\mathcal{I}'}, \{(B'_j, y'_j)\}_{j\in\mathcal{J}'})$$

  then

$$f \otimes f' \in \mathbf{CC}(\{(A_i, x_i)\}_{i\in\mathcal{I}} \cup \{(A'_i, x'_i)\}_{i\in\mathcal{I}'}, \{(B_j, y_j)\}_{j\in\mathcal{J}} \cup \{(B'_j, y'_j)\}_{j\in\mathcal{J}'}).$$

The definition of the pairing construction is notationally demanding because we want to keep track of the points $x \in X$ within each slice X. This is essentially bookkeeping, however, and is not of mathematical significance. We shall shortly give a diagrammatic intuition of the definitions of the hom-sets.

Remark 6.40. The causal structure (from Definition 6.31) of the causal category $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ in Definition 6.39 is simply the set $R(\mathcal{G})$.

Remark 6.41. Now, when defining the hom-sets in Definition 6.39, we ensured that we kept track of the space-time points with which objects are associated. To see why this is necessary, consider the following 'naive' approach to constructing a causal category, which we shall denote with an asterisk as $\mathbf{CC}^*(\mathbf{C}, \mathcal{G})$. We set $a \leq b$ for $a, b \in R(\mathcal{G})$ when there exist $x \in a$ and $y \in b$ such that $x \leq y$. We now attempt to define a causal category $\mathbf{CC}^*(\mathbf{C}, \mathcal{G})$ as follows:

- Objects are either pairs (A, a) with $A \in |\mathbf{C}| \setminus \{I\}$ and $a \in R(P) \setminus \emptyset$, or $(I, \emptyset)$.
- Morphisms are:

$$\mathbf{CC}^*(\mathbf{C}, \mathcal{G})((A,a),(B,b)) := \begin{cases} \mathbf{C}(A,B) & a \leq b \\ \mathbf{C}(I,B) \circ \top_A & a \not\leq b \end{cases}$$

- The monoidal product of (A, a) and (B, b) exists if and only if both $a \not\leq b$ and $b \not\leq a$, or if a or b (or both) is the empty set, which implies that a and b are disjoint and that $a \cup b \in R(\mathcal{G})$, so we can set:

$$(A, a) \otimes (B, b) := (A \otimes B, a \cup b)\,.$$

Consider the scenario of Figure 6.4. Let us contrast how Definition 6.39 and the naive approach each describe this scenario. We have $a \leq b$ and $b \leq c$ but $a \not\leq c$. For example, consider the spatial slices $a = \{x_1, x_2\}$, $b = \{y_1, y_2\}$ and $c = \{z_1, z_2\}$, with the relations $x_2 \leq y_1$ and $y_2 \leq z_1$, but with all other elements of the spatial slices a, b, c spacelike separated.

First we use the construction of Definition 6.39. The objects we consider in the causal category are $\{(A_1, x_1), (A_2, x_2)\}$, $\{(B_1, y_1), (B_2, y_2)\}$ and $\{(C_1, z_1), (C_2, z_2)\}$. Suppose that in $\mathbf{C}$ we have connected morphisms $f : A_2 \to B_1$ and $g : B_2 \to C_1$. Ignoring symmetry isomorphisms, in the causal category $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ we have the morphisms

$$f' = f \otimes (p_{y_2} \circ \top_{A_1}) \in \mathbf{CC}(\{(A_i, x_i)\}_{i \in \mathcal{I}}, \{(B_j, y_j)\}_{j \in \mathcal{J}})$$

and

$$g' = (p_{z_2} \circ \top_{B_1}) \otimes g \in \mathbf{CC}(\{(B_j, y_j)\}_{j \in \mathcal{J}}, \{(C_k, z_k)\}_{k \in \mathcal{K}})$$

These morphisms are depicted in Figure 6.5. The composite is

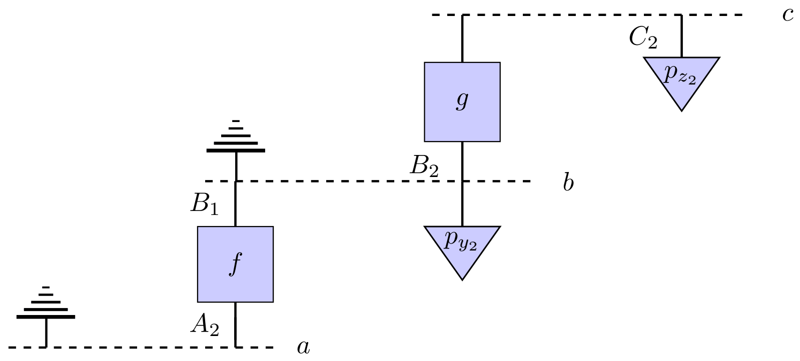
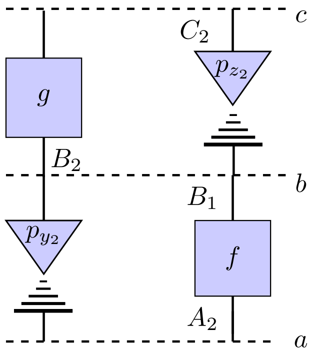


Figure 6.5: The morphisms of Remark 6.41

$$g' \circ f' = (p_{z_2} \circ \top_{B_1} \circ f) \otimes (g \circ p_{y_2} \circ \top_{A_1}) = (p_{z_2} \otimes (g \circ p_{y_2})) \circ \top_{A_1 \otimes B_1}$$

which is disconnected, as required since $a \not\sqsubseteq c$. We can depict the composition as:



We can see from this that the composition in this construction 'folds' Figure 6.5 into the usual composition of a category, but using disconnected hom-sets to encode spacelike separation.

But consider these morphisms in the naive approach: let $A := A_1 \otimes A_2$, $B := B_1 \otimes B_2$ and $C := C_1 \otimes C_2$. Then we have

$$f' \in \mathbf{CC}^*(\mathbf{C}, \mathcal{G})((A, a), (B, b))$$

and

$$g' \in \mathbf{CC}^*(\mathbf{C}, \mathcal{G})((B, b), (C, c)).$$

Then f' and g' can be connected morphisms, and as a morphism in $\mathbf{C}$, the composite $g' \circ f'$ may also be connected. But the hom-set of which $g' \circ f'$ is an element, i.e. the hom-set

$$\mathbf{CC}^*(\mathbf{C}, \mathcal{G})((A, a), (C, c))$$

is disconnected! Hence, because the naive approach ignores how the domain and codomain of morphisms are assigned in space-time, the composition law in the naive construction $\mathbf{CC}^*(\mathbf{C}, \mathcal{G})$ of a causal category fails to be consistent with the composition law of $\mathbf{C}$. This is similar to the problem that we encountered in the carving construction, where $g' \circ f'$ was forced to be disconnected in the analogous situation.

We now investigate the conditions on $\mathbf{C}$ and $\mathcal{G}$ under which $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ in Definition 6.39 is a causal category. We first provide some useful examples.

Remark 6.42. In Definition 6.39, we stipulated that the objects of $\mathbf{CC}(\mathbf{C}, \mathcal{G})$, which are sets of pairs, cannot consist of a pair (I, a) unless $a = \emptyset$. This excludes the following degenerate case. Let $\mathbf{C}_0$ be the *trivial monoidal category*, i.e. $|\mathbf{C}_0| = \{I\}$ and the only morphism is the identity morphism 1_I. Since $\mathbf{C}_0$ has no connected morphisms, the only way to form $\mathbf{CC}(\mathbf{C}_0, \mathcal{G})$ is for the graph $\mathcal{G}$ to be trivial, i.e. the graph relation satisfies $x \leq y$ iff $x = y$. This follows from the fact that Definition 6.39 requires the hom-sets of causally related objects, i.e. where slices $a, b \in R(\mathcal{G})$ satisfy $a \sqsubseteq b$, to be connected.

Example 6.43. Let $\mathbf{C}_1$ be the monoidal category with only one non-trivial object $A \neq I$, and only one connected morphism $f : A \to A$. Let $\mathcal{G}$ be a directed graph with elements $G = \{x, y\}$, with the order $x \leq y$. Then $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ has one connected morphism $f : (A, x) \to (A, y)$. This also shows that the category $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ does not exist for an arbitrary pair of monoidal category $\mathbf{C}$ (even if non-trivial) and a graph $\mathcal{G}$, since if f is disconnected in this example then $\mathcal{G}$ must be the trivial graph as in Remark 6.42.

This leads to the following proposition.

Proposition 6.44. *For any digraph $\mathcal{G}$, the category $\mathbf{CC}(\mathbf{C}_1, \mathcal{G})$ is a causal category.*

This suggests that we can identify the conditions under which $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ is a causal category.

Lemma 6.45. *The pairing $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ of an SMC $\mathbf{C}$ and a digraph $\mathcal{G}$ is a normalised category if and only if $\mathbf{C}$ is a normalised category.*

Proof. We need to show that the unit object of $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ is terminal if and only if the unit object of $\mathbf{C}$ is terminal. By the definition of the pairing construction the unit object in $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ is $(I, \emptyset)$. For any object $(\{(A_i, x_i)\}_{i\in\mathcal{I}}$, the morphisms to the unit object are defined as

$$\mathbf{CC}(\mathbf{C}, \mathcal{G})\big(\{(A_i, x_i)\}_{i\in\mathcal{I}}, (I, \emptyset)\big) := \mathbf{C}(\otimes_{i\in\mathcal{I}} A_i, I).$$

and hence the cardinality of the hom-sets is equal:

$$\left|\mathbf{CC}(\mathbf{C}, \mathcal{G})\big(\{(A_i, x_i)\}_{i\in\mathcal{I}}, (I, \emptyset)\big)\right| = \left|\mathbf{C}(\otimes_{i\in\mathcal{I}} A_i, I)\right|.$$

Hence the unit I in $\mathbf{C}$ is terminal iff the unit $(I, \emptyset)$ in $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ is. □

Remark 6.42 shows that using a trivial monoidal category in the pairing construction does not yield a causal category unless $\mathcal{G}$ is also trivial, i.e. $\mathcal{G}$ has no edges. Hence in the following theorem we provide an equivalence for non-strict monoidal categories.

Theorem 6.46. *The pairing $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ of a non-trivial SMC $\mathbf{C}$ and a digraph $\mathcal{G}$ is a causal category if and only if $\mathbf{C}$ is normalised and $\mathbf{C}(I, A)$ is non-empty for all $A \in |\mathbf{C}|$.*

Proof. ($\Rightarrow$) If $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ is a causal category then by Lemma 6.45, the category $\mathbf{C}$ is normalised. Also, by Definition 6.39, each object in $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ has at least one state only if each object in $\mathbf{C}$ has at least one state. ($\Leftarrow$) A normalised category is a causal category if it has a partial tensor whose existence is defined by disconnected hom-sets, and if each object has at least one state. By Lemma 6.45, if $\mathbf{C}$ is normalised then the pairing $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ is normalised. Also, Definition 6.39 ensures both that the monoidal product of two objects in $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ exists only when they are disconnected, and that each object in $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ has at least one state if each object in $\mathbf{C}$ has at least one state. □

Hence Theorem 6.46 establishes that the pairing construction is a method of constructing causal categories.

6.2.4 Recovering CQM

The results of Subsection 6.1.2 showed that important structures in CQM, such as compactness and the dagger functor, cannot be retained in causal categories. These structures are used in CQM to formalise notions such as measurement and dynamics. However, we can still *indirectly* represent the structures of CQM in a causal category. We shall do so by providing a precise description of the connection between a causal category and the dagger compact category that is typically used to construct it. Indeed, each of the constructions described above ensure the existence of a functor from the causal category $\mathbf{CC}$ to the dagger compact category of processes $\mathbf{C}$.

In the first case, a carved category $\mathbf{CC}$ is a subcategory of a given †-SMC $\mathbf{C}$, and so has an embedding functor

$$\iota : \mathbf{CC} \longrightarrow \mathbf{C}.$$

In the case of the second construction, a paired category $\mathbf{CC}(\mathbf{C}, \mathcal{G})$ has a projection functor

$$\pi : \mathbf{CC}(\mathbf{C}, \mathcal{G}) \longrightarrow \mathbf{C}.$$

We can now show how these functors can be exploited to recover CQM.

Example 6.47. We can identify unitary operations in a causal category as follows. Consider a carved category $\mathbf{CC}$ of a †-SMC $\mathbf{C}$. We have the inclusion functor $\iota : \mathbf{CC} \to \mathbf{C}$, from which we can define *causal unitary* operations in $\mathbf{CC}$ using conditions on the image of ι. Since $\iota(U) = U$, $\iota(A) = A$ and $\iota(B) = B$, causal unitary morphisms $U : A \to B$ in $\mathbf{CC}$ are those which are unitary in $\mathbf{C}$, i.e. those which satisfy

$$U^\dagger \circ U = 1_A \ \ \& \ \ U \circ U^\dagger = 1_B.$$

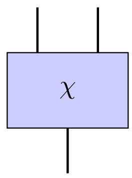
in $\mathbf{C}$. A general (i.e. not necessarily dagger-) isomorphism is similarly associated with a morphism in $\mathbf{CC}$, by witnessing it in $\mathbf{C}$.

Now, we showed in Proposition 6.24 that the fact that a system remains of the same type (e.g. a qubit) during an evolution process cannot be defined in the causal category itself. However the above example shows that this can instead be defined using the functor ι and data in $\mathbf{C}$.

Example 6.48. Classical data can be identified in a causal category in a similar way to unitaries, although some care is needed due to the same issue of isomorphisms. A morphism $v : A \to B \otimes C$ in $\mathbf{CC}$ is a candidate for representing the comonoid, i.e. the copying morphism, in a classical structure, since its codomain can represent the two copies B and C of the input data A. However, classical data in $\mathbf{C}$ is usually defined with *identical* types for the input and output state spaces, i.e. it has the form $\delta : A \to A \otimes A$. We account for this by requiring instead that the input and output types are isomorphic, where, as discussed in the previous example, we define this isomorphism in $\mathbf{C}$. So one of the conditions for a morphism $v : A \to B \otimes C$ to be considered a measurement in $\mathbf{CC}$ will be that $B \cong A$ holds in $\mathbf{C}$.

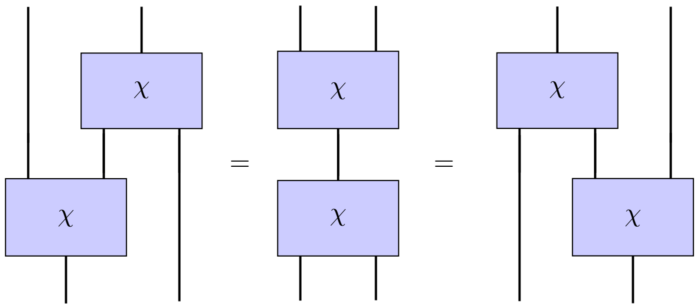
Consequently, for a carved category $\iota : \mathbf{CC} \to \mathbf{C}$, a *causal classical structure* is a morphism $x : A \to B \otimes C$ in $\mathbf{CC}$ such that

- there are isomorphisms $i_1 : B \cong A$ and $i_2 : C \cong A$ in $\mathbf{C}$;
- denoting the morphism $\chi := (i_1 \otimes i_2) \circ x : A \to A \otimes A$ as



  the morphism χ satisfies the classical structure axioms defined for morphisms in a †-compact category in

Chapter 2, for example the Frobenius law:



Note in particular that although $\chi^\dagger$ does not exist in $\mathbf{CC}$, it does exist in $\mathbf{C}$, and so using the functor ι to identify χ, we can make use of its properties.

Remark 6.49. To develop quantum field theory one requires the state spaces to be Fock spaces

$$\mathcal{F}(\mathcal{H}) = \oplus_{n=1}^{\infty} \mathcal{H}^{\otimes n} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \ldots,$$

to ensure that particles can be created or annihilated. This construction necessarily involves using the tensor product of identical Hilbert spaces. Therefore the Fock space construction cannot take place in a causal category, but will instead be defined in the process category. This requires defining a dagger compact category for which the processes can now involve pair creation and annihilation. In [66], we have defined such a category: morphisms are Feynman propagators, and compactness in this category represents pair creation and pair annihilation instead of the post-selected quantum teleportation protocol. The exploration of this in the context of causal categories is left for future work.

6.2.5 Connection to other approaches

In this Subsubsection we shall briefly compare causal categories to quantum causal histories, the approach to combining causal structure and quantum processes that we discussed in Chapter 3.

Let us recall the definition of a quantum causal history (QCH) from Chapter 3. This assigns Hilbert spaces to the anti-chains ξ of a partially ordered set $\mathcal{P} = (P, \leq)$, i.e. the subsets $\xi \subset \mathcal{P}$ such that $\forall x, y \in \xi : x \not\leq y \wedge y \not\leq x$. We have called these *spatial slices* in the context of causal categories. Moreover, a QCH assigns a unitary between spatial slices ξ and ζ:

$$U(\xi, \zeta) : \mathcal{H}(\xi) \longrightarrow \mathcal{H}(\zeta) \tag{6.11}$$

when $\xi \leq \zeta$, which is defined to hold when ξ and ζ are a complete pair, as defined in Chapter 3. The unitary assignments satisfy

$$U(\zeta, \gamma) \circ U(\xi, \zeta) = U(\xi, \gamma) \tag{6.12}$$

In other words, a QCH can be thought of as a causal set with Hilbert spaces attached to the nodes. This structure is similar to the pairing construction for causal categories, since a QCH 'pairs' a causal set with a quantum evolution U, but as a mapping from the former to the latter. We can make a closer comparison by rewriting a QCH categorically. In particular, Eq. 6.11 and Eq. 6.12 imply that a QCH can be more concisely defined as a functor. We define a category $R(\mathcal{P})$ of spatial slices of $\mathcal{P}$, where a morphism $f : \xi \to \zeta$ exists if and only if $\xi \leq \zeta$. Then we can write a QCH as functor:

$$U : R(\mathcal{P}) \longrightarrow \mathbf{fHilb},$$

Note that we earlier defined a set of spatial slices $R(\mathcal{G})$ for a directed graph. The context of this definition is that of causal structure for a causal category. Specifically, $R(\mathcal{G})$ is part of the pairing construction. In particular, the similarity between the pairing construction and a QCH would seem to be formal, in the following sense. Given elements ξ, ζ of $R(\mathcal{P})$, a QCH assigns a *single* element of $\mathbf{fHilb}$, viz. $U(\xi, \zeta)$, to $\xi \leq \zeta$. This is because a QCH is a *functor*, and not a *category*. On the other hand, the pairing construction for a causal category $\mathbf{CC}(\mathbf{fHilb}, \mathcal{P})$ provides a *set* of morphisms from $\mathbf{fHilb}$ between ξ and ζ, viz. the hom-set $\mathbf{C}(\otimes_{i\in\mathcal{I}} A_i, \otimes_{j\in\mathcal{J}} B_j)$, where the index sets $\mathcal{I}, \mathcal{J}$ label the elements of ξ, ζ respectively (i.e. $\xi := \{x_i\}_{i\in\mathcal{I}}$ and $\zeta := \{y_j\}_{j\in\mathcal{J}}$).

This shows that a QCH is conceptually quite different to a causal category. A QCH is a *single* sequence of events, but a causal category is represents *all possible sequences* of events.

With this in mind, there are several ways in which a causal category is a generalisation, and improvement, of a QCH.

1. *Functor vs. category*: The fact that a QCH is a functor means that it is not suitable for describing information flow, at least in the way we outlined at the beginning of Chapter 6. Recall that we wanted to formalise the information flow in protocols, in particular features such as the lack of information flow that arises in quantum teleportation without classical communication. Now, this requires a formal notion of 'lack of information flow', and in this example it should be assigned to the 'lack of causal relation' between Alice and Bob. In a causal category, this corresponds to the fact that only disconnected morphisms exist for spacelike separated regions. But this same feaure *cannot be described with a QCH*. This is because a functor cannot assign a morphism in $\mathbf{fHilb}$, e.g. a disconnected morphism, to 'the lack of a morphism' in $R(\mathcal{G})$. That is, a QCH defines spacelike separation as both $x \not\leq y$ and $y \not\leq x$, so the hom-sets $\mathrm{Hom}(x, y)$ and $\mathrm{Hom}(y, x)$ are both empty. But then, purely by the definition of a functor, the QCH functor cannot assign any morphism in $\mathbf{fHilb}$ to the *absence* of a morphism in $\mathcal{P}$. In other words, it cannot assign the property of 'no information flow' to 'spacelike separation', because the latter is represented as the *lack* of a morphism.

2. *Detecting causal structure*: Recall from Chapter 3 that the assignment of unitaries between antichains ξ and ζ does not take into account the causal structure between elements $x \in \xi$ and $w \in \zeta$. Hence a unitary $U : \mathcal{H}(\xi) \to \mathcal{H}(\zeta)$ for which the state at w is not a constant function of the state at x is allowed *even if x and w are not causally related*. For example, as we discussed previously, the two diagrams in Figure 6.6 can both be assigned the same unitary. Therefore, the context of quantum protocols that we are exploring, a QCH cannot enforce the fact that teleportation without classical communication cannot provide information flow. In contrast, since the morphisms in a causal category *define* the causal structure, the two diagrams in Figure 6.6 are forced to have different unitaries in a causal category.

Both these points demonstrate that the way we defined causal categories, by encoding causal structure using the morphisms *themselves*, offers more expressiveness than previous approaches such as QCH.

Chapter summary. We defined causal categories, the properties of which are motivated by the results in the previous Chapter. We explored technical properties of a causal category such as requiring that the domain of definition is a product category, and also various ways in which a causal category exhibit the features discussed in the previous Chapter, e.g. containing only normalised morphisms. We then showed how causal categories are

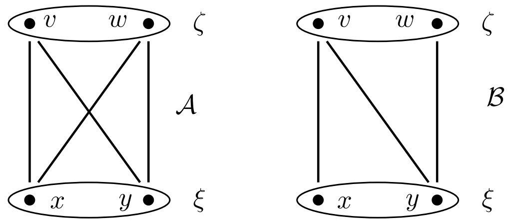


Figure 6.6: Two examples of causal structure which can both be assigned the same unitary U in a QCH. A line between $u \in \xi$ and $v \in \zeta$ indicates that $u \leq v$

different in structure to the usual categories studied in categorical quantum mechanics. For example, we showed that compact structure and a dagger functor cannot be defined for a causal category, unless the causal category is trivial. We then provided constructions for causal categories, and gave examples such as a causal category that encodes teleportation with classical communication. We showed how these constructions allow a connection to a dagger compact category to be retained. Of the two constructions, the more comprehensive one—the pairing $\mathbf{CC}(\mathbf{C}, \mathcal{G})$—is quite intricate. But perhaps this demonstrates the success of the definition of a causal category, since it shows that the definition captures a complicated construction in a simple definition.

Chapter 7

Reconstruction axioms and CQM

Contents

In previous Chapters we have frequently considered the property of terminality of the monoidal unit I in an SMC. We derived this from considerations of causality in Chapter 5. This led us to develop a framework for causal processes in Chapter 6, for which terminality is part of the definition of a causal category. In this Chapter we shall show that terminality is also connected to a particular reconstruction of quantum theory, viz. the work by Chiribella, D'Ariano and Perinotti, which is based on the use of a purification axiom. In fact, in addition to terminality, we shall show that other parts of this reconstruction are connected to CQM. In particular, the purification axiom has a formal connection to CQM via the CPM construction that we have encountered several times. Our aim will not just be to expose such connections; we also want to establish whether the mathematical framework of CQM is *useful* for reconstructions. By the criterion of being 'useful', we mean mathematical simplicity: the recent wave of reconstructions all involve a lengthy sequence of results, and the crucial aspects are not always apparent.[1] Simplifying such reconstructions is not work that we shall complete here, but in this spirit we shall sometimes make assumptions that are not operationally justified, e.g. the existence of a dagger functor.

7.1 Operational vs. CQM axioms

This Section provides mostly background material on a particular reconstruction of quantum theory (but also a redescription thereof). We first present the set of axioms used by Chiribella, D'Ariano and Perinotti (CDP) to reconstruct quantum theory [22, 23], identifying the particular aspects that we are concerned with. We shall then present an axiomatisation of the CPM construction by Coecke in [32]. In the next Section we shall show how the two sets of axioms are connected.

[1] This is perhaps a subjective notion, but this notion of simplicity could be made more concrete by the following steps. CQM has been shown to be useful for incorporating various toy models in a simple way, e.g. the Spekkens toy theory [36]. If a connection could be made between CQM and reconstructions, then it might be possible to 'provide' partial reconstructions of a range of theories, e.g. quantum theory but also e.g. real quantum mechanics, and various toy theories. This would be interesting since it would reveal how the reconstruction axioms correspond to different models, and perhaps suggest how new models can be developed.

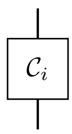
7.1.1 CDP axioms

We shall only sketch the framework and axioms of the CDP reconstruction in this Subsection, since our purpose is to describe just the axioms that we capture categorically in Section 7.2. A more detailed description is given by CDP in Sections 2 and 3 of [23].

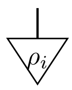
The CDP reconstruction of quantum theory uses six operational axioms to reconstruct quantum theory. 'Reconstruction' means that the only physical theory satisfying these axioms (and background assumptions) is mixed-state quantum theory, i.e. density matrices and completely positive maps over complex Hilbert spaces. We take 'operational axiom' to mean an axiom that concerns only the *macroscopic* operations that are available to experimenters. More specifically, the *primitive* terms of the language in which the axioms are formulated are to be interpreted as referring to experimental devices, such as a Stern-Gerlach device, or observations and manipulations of these devices. To build a physical theory with this language, the axioms must be stated in a particular mathematical framework. CDP's use of such a framework to reconstruct quantum theory builds on the work of Hardy [53][2]. CDP's framework is stated in slightly more abstract terms than Hardy's original presentation. This level of abstraction will allow us to show, even before considering the axioms themselves, that we can view the framework *itself* as being implicitly categorical.

CDP's framework begins by describing a collection of named systems $A, B, C \ldots$, including a *trivial* system I. Systems are the inputs and outputs of *tests* $\{\mathcal{C}_i\}_{i \in X}$, which represent a single use of some physical device, e.g. a beam-splitter. The input and output systems need not be identical. The elements of tests, $\mathcal{C}_i$, are referred to as *events*, and are indexed by outcomes $i \in X$. An event has the same input and output types as the test $\{\mathcal{C}_i\}_{i \in X}$ of which it is a member. Note that tests are 'deterministic' in the sense that they represent the use of a physically realisable device, without e.g. post-selection on the outcomes. An example of a test in quantum theory is a quantum instrument, i.e. a collection $\{\mathcal{C}_i\}_{i \in X}$ of completely-positive trace non-increasing maps $\mathcal{C}_i$, such that $\sum_{i \in X} \mathcal{C}_i$ is trace-preserving. Note that tests are not just a sum of events $\sum_{i \in X} \mathcal{C}_i$; they also contain the information about the classical indexing $\{\mathcal{C}_i\}_{i \in X}$.

CDP introduce a graphical calculus for their framework. We shall use colourless boxes for CDP diagrams, since *a priori* they appear to be a new type of diagram as compared to our previous diagrams. Hence an event with unspecified input and output systems is depicted as:



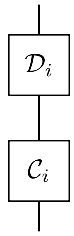
If the input of an event is the trivial system then it is depicted as



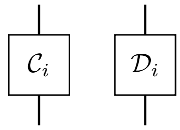
and referred to as a *preparation-event. Observation-events* are the dual notion, for which the output is the trivial system. Events $\mathcal{C}_i$ and $\mathcal{D}_i$ can be composed *in sequence* when the output system of $\mathcal{C}_i$ is the same as the input

[2] The problem of providing an axiomatic account of quantum theory was posed by Mackey [71, 72]; the subsequent work by Ludwig [70] is cognate to Hardy's approach.

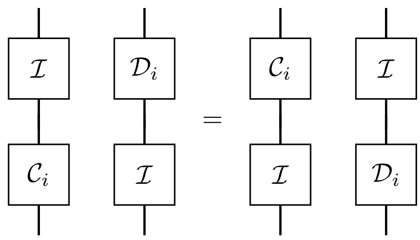
system of $\mathcal{D}_i$, depicted as



or the events can be composed *in parallel:*



Each type of composition yields another event, i.e. events are closed (in the algebraic sense) under parallel and sequential composition. In the case of parallel composition, if $\mathcal{C}_i$ has A and B as input and output types respectively, and if $\mathcal{D}_i$ has C and D as input and output types respectively, then the parallel composition of $\mathcal{C}_i$ and $\mathcal{D}_i$ has the *composite systems*, AC and BD, as the input and output types respectively. 'Composite system' is a primitive notion for CDP, and so it is not defined with respect to any other mathematical structure. The framework includes an identity test $\mathcal{I}$, which is subject to the rule:



(7.1)

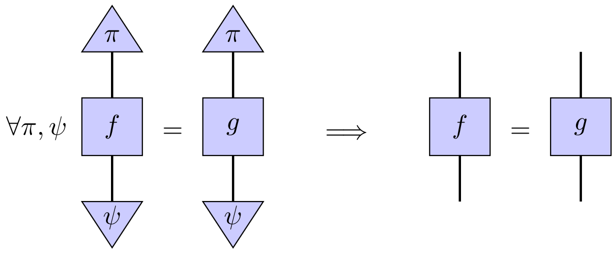
Now, what we have described of the CDP framework so far is actually just part of:

Defining an SMC.

So CDP are *implicitly* using SMCs. For example, as we described in Chapter 2, Eq. 7.1 is just a consequence of bifunctoriality in an SMC. Other graphical rules are presented by CDP which capture the remaining axioms for an SMC, and this amounts to defining an SMC for events, and another SMC for tests. Note that CDP define an SMC using a graphical language, instead of using the symbolic language of category theory. However, as we discussed in Chapter 2, such a language suffices to express the content of an SMC. Since SMCs have useful mathematical properties, part of our aim in this Chapter will be to establish the extent to which these properties are connected to the CDP reconstruction. For example, we can now see that a choice of an SMC specifies a particular operational theory. The probabilistic structure is an *additional* layer: it arises from assuming that the sequential composition $\sigma \circ \psi$ of a preparation-test ψ and an observation-test σ is a probability, i.e. a scalar $s \in [0, 1]$.

The introduction of the scalars leads to an important background assumption, which is that indistinguishable events are collected into equivalence classes, called *transformations.* Two events $\mathcal{C}_i$ and $\mathcal{D}_i$ are indistinguishable if and only if they produce the same probabilities for all pairs of a preparation-test and and an observation-test. Since we have established that the CDP framework is one of an SMC, we can now revert to our usual graphical notation and symbolic nomenclature. Then we can define the indistinguishability assumption more precisely as follows.

Definition 7.1. An SMC **C** satisfies *behavioural equivalence* iff, for any two morphisms f, g in **C**:



(7.2)

where the scope of the quantifier is only the antecedent.

Hence if f and g are not identical then they can be distinguished by some preparation-observation pair. In particular, CDP define states to be equivalence classes of preparation-events, and effects are defined as equivalence classes of observation-events; this is just applying behavioural equivalence when either the input or output type of the event $\mathcal{C}_i$ is the trivial system.

Two more notions need to be introduced. Firstly, CDP's initial notion of a *deterministic* test is a test $\{\mathcal{C}_i\}_{i \in X}$ for which X is a singleton, i.e. $\mathcal{C}$ is a test with a single outcome. With further notions CDP show that this leads to a more familiar description, e.g. deterministic states are normalised states, and deterministic operations are trace-preserving (both of which can be stated only once the trace operation has been identified using the axioms below). Secondly, CDP use the outcome sets X in the tests $\{\mathcal{C}_i\}_{i \in X}$ to define pure tests and pure transformations.

This is defined by considering the notion that outcomes $x \in X$ can be joined together. For example, consider a transformation $\mathcal{C}_0 \in \{\mathcal{C}_i\}_{i \in X}$ and a test $\{\mathcal{D}_j\}_{j \in Y}$. If there exists a subset $Y_0 \subset Y$ such that

$$\mathcal{C}_0 = \sum_{j \in Y_0} \mathcal{D}_j$$

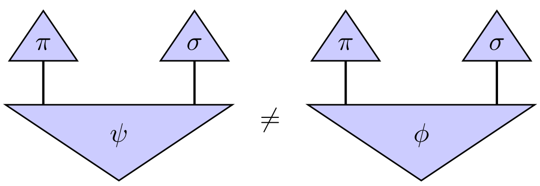
then we say that each $\mathcal{D}_j$ is a *refinement* of $\mathcal{C}_0$. Then we say a transformation is *pure* if it has only trivial refinements, meaning that if $\mathcal{D}_j$ is a refinement of $\mathcal{C}_0$ then $\mathcal{D}_j = s \bullet \mathcal{C}_0$ (recall that '$\bullet$' denotes scalar multiplication in an SMC). For example, in quantum theory an effect P (i.e. in CDP's language, a transformation with trivial output) is atomic if P is a rank-one projector, since $P = \sum_i Q_i$ only if Q_j are each proportional to P.

With the minimal elements of the framework in place, we can state the CDP reconstruction axioms.

1. *Causality*: Every system A has a unique deterministic effect.

2. *Pure composition*: The sequential composition of two pure events is a pure event.

3. *Local distinguishability*: If two bipartite states are different, then they give different probabilities for at least one product experiment. That is, for states $\phi \neq \psi$ there exist effects π and σ such that



4. *Purification*: Every state has a purification. For a specific purifying system B, every two purifications $\psi : I \to A \otimes B$ and $\phi : I \to A \otimes B$ of the same state ρ are connected by a reversible transformation on the purifying system $R : B \to B$.

We shall not discuss Axioms 5 and 6, but we state them for the sake of completeness (without defining the new terms that they use).

5 *Ideal compression*: For every state there exists an ideal compression scheme.

6 *Perfect distinguishability*: Every state that is not completely mixed can be perfectly distinguished from some other state.

We note that, in one sense, the purification axiom is the most important axiom: both classical probability theory and quantum theory satisfy the other five axioms, but only quantum theory also satisfies purification.

7.1.2 Axiomatisation of the CPM construction

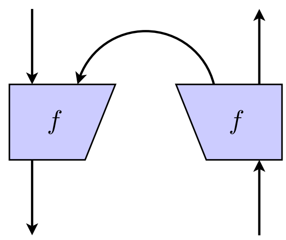
We introduced the CPM construction in Chapter 2. Recall that this yields the category of mixed states and completely positive maps $\mathbf{Mix}$ from the category of pure states $\mathbf{fHilb}$, i.e. $CPM(\mathbf{fHilb}) \cong \mathbf{Mix}$. Now, since this is a construction, the following question arises: given two categories $\mathbf{C}$ and $\mathbf{D}$, can conditions be given which are satisfied if and only if $CPM(\mathbf{C}) \cong \mathbf{D}$? In other words, the question is: does there exist an *axiomatisation* of the CPM construction?

Dilation structures

Coecke showed that there does exist such an axiomatisation [32], and in fact its form anticipates the connection to the CDP axioms. This is because the CPM axiomatisation is implicitly based on the idea of purification. For example, recall that a morphism $g : A \to B$ in $CPM(\mathbf{C})$ is defined as a morphism

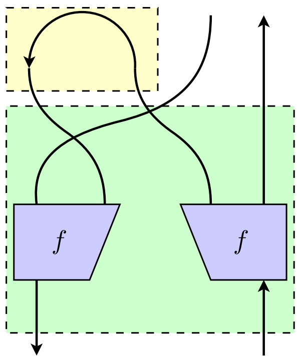
$$g := (1_B \otimes \epsilon_C \otimes 1_{B*}) \circ (f \otimes f_*) : A \otimes A^* \to B \otimes B^* \tag{7.3}$$

in $\mathbf{C}$ depicted as:



Note that this diagram contains a kind of redundancy, because the same morphism $f : A \to B$ is depicted twice.

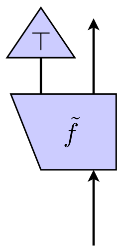
Using the graphical calculus to rearrange g we obtain



Let us make the object definitions $\tilde{A} := A \otimes A^*$, $\tilde{B} := B \otimes B^*$ and $\tilde{C} := C \otimes C^*$. Similarly, for morphisms we write $\tilde{f} := \sigma \circ (f \otimes f_*)$ and $\tilde{\top}_C := \epsilon_C$, so that $\tilde{f}$ and $\tilde{\top}$ correspond to the green and yellow areas of the diagram respectively. Eq. 7.3 then becomes

$$g = (\top \otimes 1_{\tilde{B}}) \circ \tilde{f} : \tilde{A} \to \tilde{B} \tag{7.4}$$

and corresponds to the diagram:



The hint of purification arises as follows. The morphism $\tilde{f}$ is defined in $\mathbf{C}$, and so we can consider it to be pure (e.g. when $\mathbf{C} = \mathbf{fHilb}$). However, g is mixed since it is a morphism in $CPM(\mathbf{C})$. Hence the *mixed* morphism g in $CPM(\mathbf{C})$ is obtained from a *pure* morphism $\tilde{f}$ by 'tracing out' the system $\tilde{C}$, as specified by Eq. 7.4. This is therefore similar to the notion of purification. As described in Remark 2.42, we have the inclusion $\mathcal{I}(\mathbf{C}) \hookrightarrow CPM(\mathbf{C})$ of a pure category into a mixed category, and the morphisms of the mixed category can always be obtained from a 'larger' pure morphism in $\mathcal{I}[\mathbf{C}]$ using the trace $\top$. This is the main idea behind the axiomatisation of the CPM construction, which we now present.

Definition 7.2. A *dilation structure* for a dagger SMC $\mathbf{C}$ is a pair $(\mathbf{C}_E, \{\top_A\}_A)$, where:

(i) $\mathbf{C}_E$ is a dagger SMC such that $\mathbf{C}$ is a subcategory of $\mathbf{C}_E$, where the inclusion functor

$$\mathcal{E} : \mathbf{C} \longrightarrow \mathbf{C}_E$$

is surjective on objects and preserves the dagger functor and the monoidal structure;

(ii) $\{\top_A\}_A$ is a family of morphisms $\top_A : A \to I$ for each object A in $\mathbf{C}_E$, satisfying the following axioms:

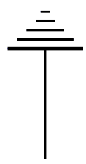
D1. For all effects $e : A \to I$ in $\mathbf{C}_E$ there exists a morphism $f : A \to B$ in $\mathbf{C}$ such that $\top_B \circ f = e$;

D2. For all pairs of morphisms $f, g : A \to B$ in $\mathbf{C}$:

$$f^\dagger \circ f = g^\dagger \circ g \Longleftrightarrow \top \circ f = \top \circ g$$

D3. For all objects $A, B \in \mathbf{C}$, we have $\top_{A \otimes B} = \top_A \otimes \top_B$ and $\top_I = 1_I$.

Note that we have previously used the symbol '$\top$'. This was used to denote the *environment structure* for an SMC, which we defined as the existence of a chosen family of morphisms $\{\top_A\}_A$ for each object A in an SMC. The existence of this family of morphisms is essentially Definition 7.2 without conditions D1–D3. Hence a dilation structure is an environment structure subject to the extra axioms of Definition 7.2. Now, for an environment structure, the morphisms $\{\top_A\}_A$ represent tracing out a system, i.e. considering the system as part of the environment. As we shall see in Example 7.3, the morphisms $\{\top_A\}_A$ for a dilation structure still represent the trace. Accordingly, we shall also use the same graphical notation as before: we depict the morphism $\top$ in a dilation structure using the ground symbol that we used previously:



In the next Section we shall consider both environment structures and dilation structures: it should be clear from the context when $\top$ refers to an environment structure or a dilation structure. (The reason for overloading $\top$ is that we shall sometimes want to prove that an environment structure satisfies the further conditions of Definition 7.2.)

Example 7.3. Based on the reasoning above, any category of mixed states $CPM(\mathbf{C})$ for a dagger compact category should provide an example of a dilation structure. We shall shortly comment on this, but let us first give a concrete example of a dilation structure. For standard quantum theory we can use the category of completely positive maps $\mathbf{Mix}$ defined in Chapter 2. Recall that the objects of $\mathbf{Mix}$ are finite-dimensional Hilbert spaces $\mathcal{H}$, and the morphisms are completely positive maps $f : L(\mathcal{H}_1) \to L(\mathcal{H}_2)$. In $\mathbf{Mix}$ we define $\top_{\mathcal{H}}$ to be the trace:

$$\begin{aligned} \top_{\mathcal{H}} : L(\mathcal{H}) &\longrightarrow \mathbb{C} \\ \rho &\longmapsto Tr(\rho) \end{aligned}$$

There is a subcategory of pure processes $\mathbf{Pure} \hookrightarrow \mathbf{Mix}$, defined using Kraus decomposition. $\mathbf{Pure}$ contains the morphisms f in $\mathbf{Mix}$ which, for some linear map L, can be defined as:

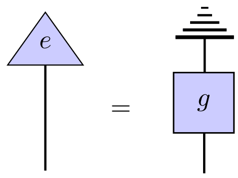
$$\begin{aligned} f : L(\mathcal{H}_1) &\longrightarrow L(\mathcal{H}_2) \\ \rho &\longmapsto L\rho L^\dagger \end{aligned}$$

With these definitons, $(\mathbf{Mix}, \{\top_{\mathcal{H}}\}_{\mathcal{H}})$ is a dilation structure for $\mathbf{Pure}$, since it is easily shown that Axioms D1–D3 are satisfied.

Remark 7.4. It is useful to explain the significance of axioms D1–D3:

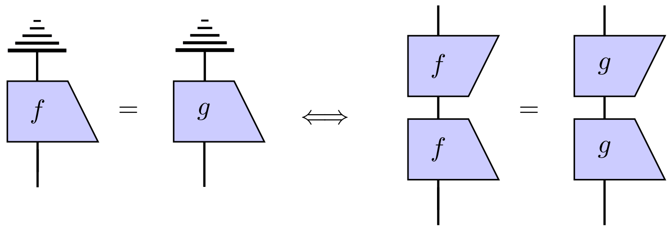
- *Axiom D1*: This provides a connection between $\mathbf{C}_E$ and $\mathbf{C}$, since it imposes that effects $e : A \to I$ in $\mathbf{C}_E$

are obtained from morphisms g in $\mathbf{C}$ via:



This justifies the name 'dilation structure', since it constrains the mixed effects in the category $\mathbf{C}_E$ to be generated using pure morphisms g in $\mathbf{C}$ using the tracing operation $\top$.

- *Axiom D2*: For a morphism $f : A \to B$, the condition in D2 is graphically depicted as:



(7.5)

This axiom is quite powerful, as we shall see in Proposition 7.6 below. Various useful features of a dilation structure have been derived in [30], where they have been used to describe classical data in quantum protocols, providing an extension of the formalisation of classical data using Frobenius algebras presented in Chapter 2. We shall use this axiom in the next Section where we shall derive its connection to purification; hence we refer to it as *CPM purification.*

- *Axiom D3*: This states just that $\top$ is compatible with the monoidal product $\otimes$, and also that the trace of I is 1_I. Note that we encountered these conditions earlier. In Chapter 5 we assumed the second condition, and then derived the first condition from the requirement of trace-preservation, i.e. $\top_B \circ f = \top_A$ for all f, since then trace-preservation leads to terminality. However, neither $\mathbf{C}_E$ nor $\mathbf{C}$ is required to have terminal I in Definition 7.2.

Recall that an *isometry* in a †-SMC is a morphism $f : A \to B$ such that $f^\dagger \circ f = 1_A$.

Definition 7.5. Let $\mathbf{C}$ be a †-SMC $\mathbf{C}$ with an environment structure $\{\top_A\}_A$. We say that $\mathbf{C}$ satisfies *trace-preservation of isometries* if isometries preserve the morphism $\top$, i.e. for any morphism $f : A \to B$, if $f^\dagger \circ f = 1_A$ then $\top_B \circ f = \top_A$.

The †-SMC $\mathbf{Mix}$ concretely satisfies trace-preservation of isometries, but in general this need not be the case.

Proposition 7.6. *If* $\mathbf{C}$ *has a dilation structure then* $\mathbf{C}$ *satisfies trace-preservation of isometries.*

Proof. Setting $g = 1_A : A \to A$ in Eq. 7.5, we have

$$\top_A \circ f = \top_B \Longleftrightarrow f^\dagger \circ f = 1_A$$

for any morphism f in $\mathbf{C}$. That is, f is an isometry if and only if it preserves the trace. □

Axiomatisation of the CPM construction

We motivated the definition of a dilation structure by making reference to how the CPM construction informally encodes purification. We shall now make the connection more precise. To demonstrate that Definition 7.2 axiomatises the CPM construction, we need to provide a logical equivalence (potentially with side conditions) between categories providing dilation structures $\mathbf{C}_E$ and categories of the form $CPM(\mathbf{C})$. One side of this equivalence is as follows.

Proposition 7.7 ([32])**.** *Let* $\mathbf{C}_E$ *be a dilation structure for* $\mathbf{C}$*. Then* $CPM(\mathbf{C}) \cong \mathbf{C}_E$*.*

Proof. (Sketch) Since $\mathbf{C}_E$ is a dilation structure for $\mathbf{C}$, we have $|\mathbf{C}_E| = |\mathbf{C}|$ from condition (i) in Definition 7.2, as required for isomorphic categories. The isomorphism $F : CPM(\mathbf{C}) \cong \mathbf{C}_E$ is then obtained by defining

$$F : CPM(\mathbf{C}) \longrightarrow \mathbf{C}_E$$
$$(1_{B^*} \otimes \epsilon_C \otimes 1_B) \circ (f_* \otimes f) \longmapsto (\top_C \otimes 1_B) \circ f$$

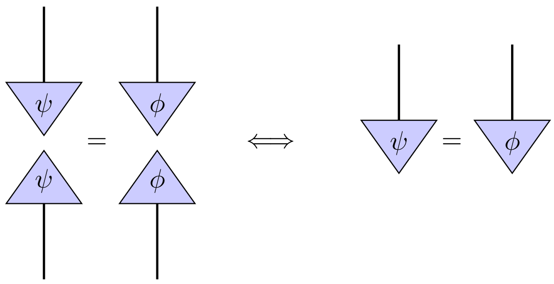
as described in Eq. 7.4. □

The other side of the equivalence requires some technical assumptions, which we now explain.

Definition 7.8. A †-SMC $\mathbf{C}$ satisfies *state-preparation agreement* if, for all objects A, and all states $\psi, \phi : I \to A$, we have

$$\psi^\dagger \circ \psi = \phi^\dagger \circ \phi \iff \psi = \phi,$$

which graphically is depicted as:



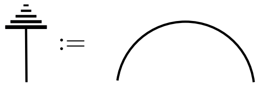
This is not satisfied in $\mathbf{fHilb}$: for projectors we have $|\psi\rangle\langle\psi| = |\phi\rangle\langle\phi|$ if $|\psi\rangle = |\phi\rangle$, but the converse is not true, since if we have $|\psi\rangle = e^{i\theta}|\phi\rangle$ then $|\psi\rangle\langle\psi| = |\phi\rangle\langle\phi|$ but $|\psi\rangle \neq |\phi\rangle$. However state-preparation agreement is satisfied in $\mathbf{Pure}$ because the points $\psi : I \to L(\mathcal{H})$ are projectors $|\psi\rangle\langle\psi|$ and the morphism $\psi^\dagger : L(\mathcal{H}) \to I$ is defined as $\text{Tr}(|\psi\rangle\langle\psi|-)$: i.e. complex phases have been quotiented out. Hence the example of a pure category that we have in mind in what follows is $\mathbf{Pure}$, not $\mathbf{fHilb}$. This just amounts to a change of formalism for pure processes, from Dirac notation to density matrices. Earlier we made the connection between the two categories by defining a functor:

$$\mathcal{I} : \mathbf{C} \longrightarrow \mathbf{C}$$
$$f \longmapsto f \otimes f_*$$

such that $\mathcal{I}[\mathbf{fHilb}] = \mathbf{Pure}$.

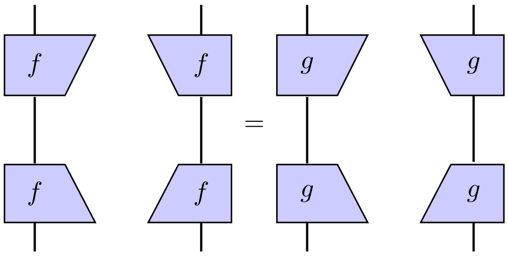
Proposition 7.9 ([32]). *Let* $\mathbf{C}$ *be a dagger compact category such that* $\mathcal{I}(\mathbf{C})$ *satisfies state-preparation agreement. Then* $CPM(\mathbf{C})$ *is a dilation structure for* $\mathcal{I}[\mathbf{C}]$.

Proof. (Sketch) We define $\top_A := \epsilon_A$, which graphically means:

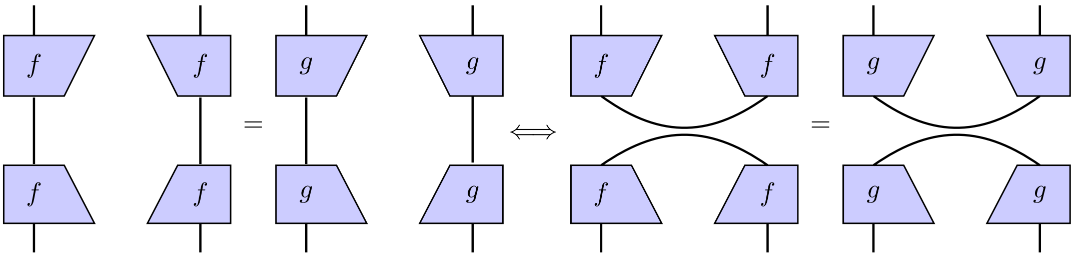


(7.6)

Straightforward calculation then shows that conditions D1–D3 are satisfied. In particular, D2 is satisfied using the condition of state-preparation agreement: the right-hand side of the conditional in Eq. 7.5 is depicted in $\mathcal{I}[C]$ as

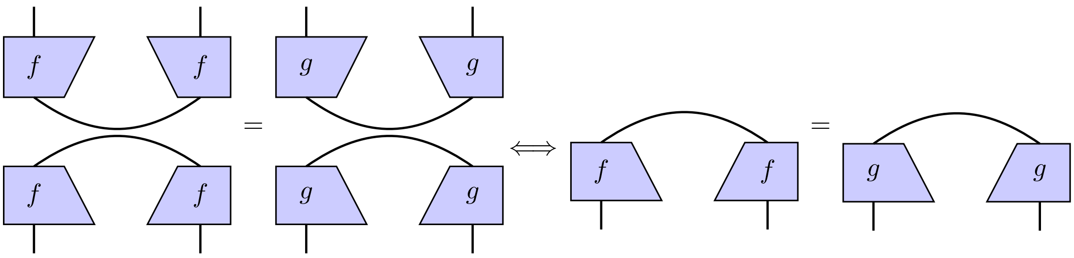


and graphical manipulation yields the equivalence:



(7.7)

On the other hand, state-preparation agreement for $\mathcal{I}[\mathbf{C}]$ is the equivalence:



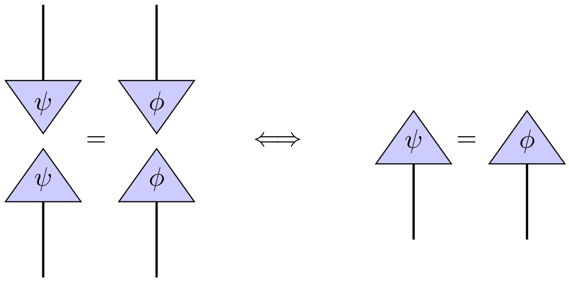
(7.8)

Combining the equivalences of Eq. 7.7 and Eq. 7.8, and employing Eq. 7.6, yields condition D2 of a dilation structure. □

Since state-preparation agreement is required for Proposition 7.9, which is one half of our axiomatisation, let us check that dilation structures have the property of state-preparation agreement.

Lemma 7.10. *Let* $\mathbf{C}$ *have a dilation structure* $\mathbf{C}_E$. *Then* $\mathbf{C}$ *satisfies state-preparation agreement.*

Proof. Let $B = I$ in Eq. 7.5. Then using axiom D3 we have $\top_I = 1_I$ and so for morphisms $\psi, \phi : I \to A$ in $\mathbf{C}$ we have



(7.9)

But since $\mathbf{C}$ is a † category, we have $\psi = \phi$ if and only if $\psi^\dagger = \phi^\dagger$, which we apply to the right-hand side of the equivalence Eq. 7.9, which yields state-preparation agreement. □

We can now combine Propositions 7.7 and 7.9 into a single statemen using Lemma 7.1.2 as well we have:

Theorem 7.11. *Let $\mathbf{C}_E$ and $\mathbf{C}$ be dagger compact categories. Then:*

(i) If $\mathbf{C}_E$ is a dilation structure for $\mathbf{C}$ then $CPM(\mathbf{C}) \cong \mathbf{C}_E$ and $\mathbf{C}$ satisfies state-preparation agreement.

(ii) If $\mathcal{I}[\mathbf{C}]$ satisfies state-preparation agreement, then $CPM(\mathbf{C})$ is a dilation structure for $\mathcal{I}[\mathbf{C}]$.

The significance of Theorem 7.11 is that it shows that the definition of dilation structure axiomatises the CPM construction, as was our stated aim[3]. This means that a dilation structure provides the conditions for a pair of categories $(\mathbf{C}, \mathbf{D})$ to be interpreted, for a particular physical theory, as a category of pure states $\mathbf{C}$ and a category of mixed states $\mathbf{D} = CPM(\mathbf{C})$ respectively (assuming state-preparation agreement holds).

7.2 Categorical description of CDP axioms

We shall give a categorical version of four of the six CDP axioms. Three of these can be translated straightforwardly. Purification will be less straightforward.

First we make some comments on our strategy for translating the framework of the CDP axioms. Our translation of the framework will *not* be a one-to-one translation of every term in the CDP language. As we emphasised in the introduction to this Chapter, we are seeking to understand how the CDP axioms might be *formally* related to CQM. So we shall assume the existence of certain formal structures at the expense of an operational justification. In particular, we shall assume that certain categories, to be defined below, are dagger compact categories, although these have no counterpart in the CDP framework or axioms. This will allow us to interpret the CDP language according to how they might most naturally fit into the formalism of CQM.

Now, the first four axioms crucially use the following terms: 'test', 'event', 'pure', 'deterministic', and 'purification'. We must explain how we are going to interpret these terms.

- We shall not include outcome sets in our framework. Hence our translation will be at the level of events: our categories should not be interpreted as having tests $\{\mathcal{C}_i\}$ as morphisms, but instead they have events $\mathcal{C}_{i \in X}$

[3] This Theorem is stated slightly awkwardly, in the sense that we did not state it as an equivalence of the form "A iff B". To be an equivalence, part (ii) should state: if $CPM(\mathbf{C}) \cong \mathbf{C}$ and $\mathbf{C}$ satisfies state-preparation agreement, then $CPM(\mathbf{C})$ is a dilation structure for $\mathbf{C}$. An equivalence of this kind can be stated, but would require introducing further side conditions: these constitute conditions on the functor providing the isomorphism $CPM(\mathbf{C}) \cong \mathbf{C}_E$, and a stronger condition than state-preparation agreement. To minimise technical assumptions we did not introduce these extra conditions. However, with these assumptions, a cleaner statement follows: $CPM(\mathbf{C}) \cong \mathbf{C}_E$ if and only if $\mathbf{C}_E$ is a dilation structure for $\mathbf{C}$.

as morphisms (which we shall also refer to as 'operations'). To describe tests would involve formalising the classical data associated with a test, viz. the set of outcomes X. However, we shall retain the notion that events form an SMC. As we have seen, the operational theory of CDP amounts to defining an SMC for all events, which we denote $\mathbf{C}_{\text{mix}}$.

- Since we do not include outcome sets X in our framework, we cannot retain the 'refinement' notion of purity that CDP use. Instead we shall assume that the pure operations are given. Our connection to purification will be via dilation structures, and since dilation structures axiomatise the CPM construction, we can always use the CPM construction to obtain mixed operations. Hence in principle we need not assume that the mixed operations are given to us, even if the pure operations are. This shows how our assumptions are not particularly operationally justified, since usually the mixed processes would be considered primary, since in general we would assume that, in practice, mixed states are what we prepare and observe in the lab.
- We assume that $\mathbf{C}_{\text{mix}}$ has an environment structure, meaning a chosen family of morphisms $\{\top_A\}_A$ which are interpreted as discarding a system. Unlike the previous two points, this is certainly operationally justifiable. By a *deterministic operation*, we shall mean an operation which preserves the discarding morphism $\top$, i.e. an operation $f : A \to B$ such that $\top_B \circ f = \top_A$. This is the definition we have used previously in Chapters 5 and 6.

We shall explain our notion of purification after the first part of the translation, which we turn to now.

Translation of axioms 1–3

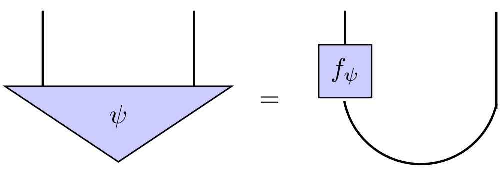
The first two axioms can be immediately interpreted in our framework as *defining* certain categories.

1. *Causality*: For CDP, this means that every system there is a unique deterministic effect.
   - In our framework, this means that there exists an SMC of deterministic processes $\mathbf{C}_{\text{det}} \hookrightarrow \mathbf{C}_{\text{mix}}$ which is defined as containing the morphisms $f : A \to B$ in $\mathbf{C}_{\text{mix}}$ satisfying $\top_B \circ f = \top_A$. In our framework, the existence of a unique deterministic effect is then Proposition 5.23, since this showed that the monoidal unit I is terminal in $\mathbf{C}_{\text{det}}$ (providing that $\top_I = 1_I$).
2. *Pure composition*: For CDP, this stated that the sequential composition of two pure operations is a pure operation.
   - In our framework, this means that there exists an SMC of pure processes $\mathbf{C}_{\text{pure}}$.

Hence we have captured two of the CDP axioms by defining the existence of the categories $\mathbf{C}_{\text{det}}$ and $\mathbf{C}_{\text{pure}}$. To translate the third axiom, local distinguishability, we shall use the following proposition.

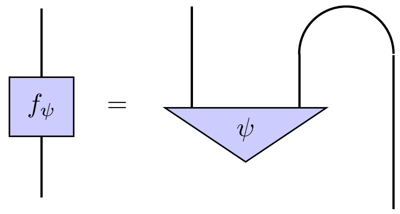
Proposition 7.12. *Let* $\mathbf{C}$ *be a dagger compact category.* $\mathbf{C}$ *satisfies behavioural equivalence if and only if it satisfies local distinguishability.*

Proof. Since $\mathbf{C}$ is compact category, for any state $\psi : I \to A$ there is a morphism f_ψ such that



(7.10)

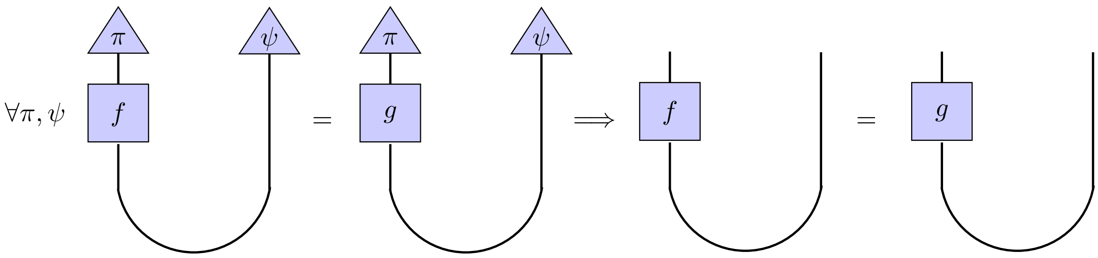
since we can always define f_ψ as:


(7.11)

and compactness then ensures that Eq. 7.10 is satisfied. Conversely, any morphism f defines a state

$$\psi_f := (f \otimes 1_A) \circ \eta.$$

Now, using compact structure, behavioural equivalence (i.e. Eq. 7.2) holds if and only if


(7.12)

Hence, applying map-state duality, defined by Eq. 7.10 and Eq. 7.11, to Eq. 7.12, we have local distinguishability if and only if behavioural equivalence holds. □

The significance of this proposition is that, given the assumption of behavioural equivalence, compactness makes the assumption of local distinguishability unnecessary. Since compactness is a mathematically natural assumption in a categorical framework, we shall use this instead of local distinguishability. Moreover, we can assume that both $\mathbf{C}_{\text{pure}}$ and $\mathbf{C}_{\text{mix}}$ are dagger compact, and that $\mathbf{C}_{\text{mix}}$ has an environment structure. We summarise our translation so far in Table 7.1.

CDP axiom	CQM axiom
Causality	$\exists$ category $\mathbf{C}_{\text{det}}$ with environment structure
Pure composition	$\exists$ category $\mathbf{C}_{\text{pure}}$
Local distinguishability	dagger compact structure

Table 7.1: Correspondence between CDP axioms and our axioms.

To explain the connection to purification, it is useful to formalise this translation scheme, which we do as follows.

Definition 7.13. A *reconstruction scheme* is a pair of dagger compact categories $(\mathbf{C}_{\text{mix}}, \mathbf{C}_{\text{pure}})$ such that:

(i) $\mathbf{C}_{\text{pure}}$ is a sub-dagger compact category of $\mathbf{C}_{\text{mix}}$, such that $|\mathbf{C}_{\text{pure}}| = |\mathbf{C}_{\text{mix}}|$;

(ii) $\mathbf{C}_{\text{mix}}$ has an environment structure.

Note that a reconstruction scheme is 'half-way between' an environment structure and a dilation structure: it has *more* conditions than an environment structure, because it has an environment structure for $\mathbf{C}_{\text{mix}}$ but it *also* requires the existence of a subcategory $\mathbf{C}_{\text{pure}}$. However, it stipulates *fewer* conditions than a dilation structure (except for the assumption of dagger compactness), because the axioms of a dilation structure, i.e. conditions D1–D3 in Definition 7.2, are *not* required to hold. Since $|\mathbf{C}_{\text{mix}}| = |\mathbf{C}_{\text{pure}}| = |\mathbf{C}_{\text{det}}|$, we depict the morphism inclusions of these categories in Figure 7.1.

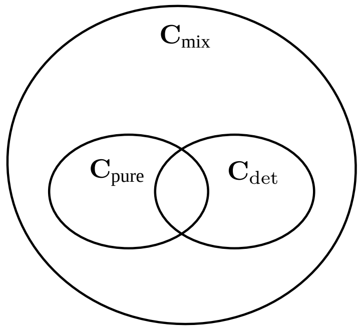
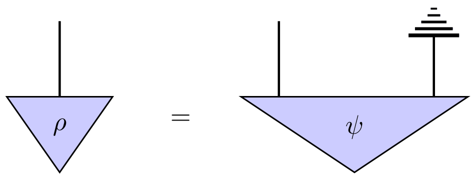


Figure 7.1: Categories defining the CDP framework in a reconstruction scheme

The purification axiom

So far we have shown that the framework and the first two axioms of CDP are captured by the existence of the categories shown in Figure 7.1. Now, consider the CDP purification axiom. This states that, for all mixed states $\rho : I \to A$, there exists a pure bipartite state $\psi : I \to A \otimes B$ in $\mathbf{C}_{\text{pure}}$ such that



and moreover, there exists a reversible transformation $R : B \to B$ between any two purifications $\psi, \phi : I \to A \otimes B$, i.e. $\psi = (1 \otimes R) \circ \phi$. In the spirit of CQM discussed above, i.e. that we want to exploit the structures of CQM, we are going to assume that a reversible transformation R is †-reversible, meaning that R is an isometry. Moreover, let us assume that an isometry connects any two systems B and C which purify ρ, so that R now has input and output types $R : B \to C$, such that B and C can be distinct.

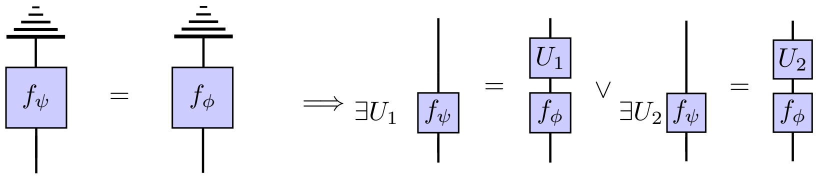
Hence considering two such purifications of ρ, say $\psi :\to A \otimes C$ and $\phi : I \to A \otimes B$, we have either

$$\psi = (1 \otimes U_1) \circ \phi \quad \text{or} \quad \phi = (1 \otimes U_2) \circ \psi \tag{7.13}$$

for an isometry U_1 or an isometry U_2. Let us use Eq. 7.10 to define morphisms f_ψ and f_ϕ for the two purifications ψ and ϕ respectively. Then, using compactness (i.e. map-state duality), we have

$$(1_A \otimes \top_C) \circ \psi = (1_A \otimes \top_B) \circ \phi \Longleftrightarrow \top_B \circ f_\psi = \top_C \circ f_\phi \tag{7.14}$$

Eq. 7.13 and Eq. 7.14 then imply that



(7.15)

where $f_\psi : A \to C$ and $f_\phi : A \to B$ are morphisms in $\mathbf{C}_{\text{pure}}$, and $U_1 : B \to C$ and $U_2 : C \to B$ are isometries in $\mathbf{C}_{\text{pure}}$.

We call this *weak CDP purification*. In an SMC which satisfies trace-preservation of isometries, the converse of Eq. 7.15 is also true. We shall take the resulting equivalence, which we now define, to capture the purification of CDP: this will be our target for CPM purification[4].

Definition 7.14. Let $(\mathbf{C}_{\text{mix}}, \mathbf{C}_{\text{pure}})$ be a reconstruction scheme. Then $\mathbf{C}_{\text{mix}}$ satisfies *(strong) CDP purification* if, for all morphisms $f : A \to C$ and $g : A \to B$ in $\mathbf{C}_{\text{pure}}$,

$$\top_C \circ f = \top_B \circ g \iff ((\exists U_1 : f = U_1 \circ g) \vee (\exists U_2 : g = U_2 \circ f))$$

for isometries $U_1 : B \to C$ and $U_2 : C \to B$ in $\mathbf{C}_{\text{pure}}$.

It is useful to make precise the relationship between weak and strong CDP purification.

Proposition 7.15. *Let $(\mathbf{C}_{mix}, \mathbf{C}_{pure})$ be a reconstruction scheme. Then $\mathbf{C}_{\text{mix}}$ satisfies strong CDP purification if and only if it satisfies both weak CDP purification and trace-preservation of isometries.*

Proof. ($\Rightarrow$) Let $A = B$, and $g = 1_B$ in Definition 7.14. Then we have, for any isometry $U : B \to C$:

$$\top_C \circ U = \top_B$$

($\Leftarrow$) Immediate. $\square$

To show the connection between the CPM construction and the CDP purification principle, we shall need to introduce some conditions relating to the † functor.

Definition 7.16. A †-SMC satisfies *polar decomposition (PD)* if, for any morphisms $f : A \to C$ and $g : A \to B$

$$f^\dagger \circ f = g^\dagger \circ g \Longrightarrow ((\exists U_1 : f = U_1 \circ g) \vee (\exists U_2 : g = U_2 \circ f))$$

where $U_1 : B \to C$ and $U_2 : C \to B$ are isometries.

In words, polar decomposition means that if the self-adjoint part of two morphisms agree, then they differ by an isometry. This is true in **fHilb**, and is sometimes known as the Douglas lemma [48]. The reason for using the name 'polar decomposition' is that the Douglas lemma is often used to show the existence of a polar decomposition for complex linear maps.

[4] Since we want to show that CPM purification implies CDP purification, strengthening the latter is not an extra assumption for CPM purification. Instead, if we can find a statement of equivalence between CPM purification and CDP purification, this would only show the strength of CPM purification, since it would then imply a stronger version of CDP purification than defined by CDP themselves.

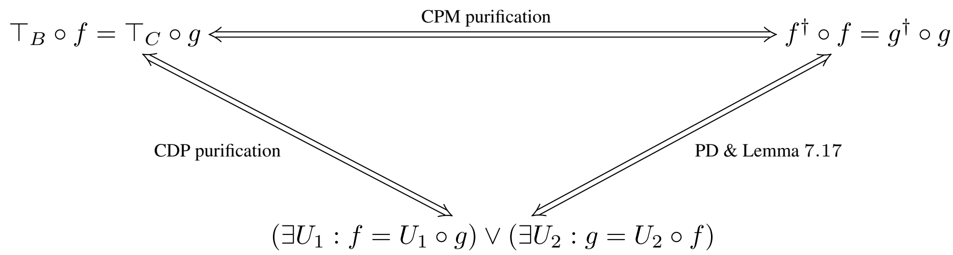
Lemma 7.17. *Let $f : A \to C$ and $g : A \to B$ be arbitrary morphisms in a †-SMC, and let $U : B \to C$ be an isometry such that $g = U \circ f$. Then $f^\dagger \circ f = g^\dagger \circ g$.*

Lemma 7.17 means that the implication in the statement of polar decomposition is actually an equivalence, informally: morphisms f and g differ by an isometry if and only if their self-adjoint parts agree.

Recall that by 'CPM purification' we mean Axiom D2 in Definition 7.2.

Theorem 7.18. *Let $(\mathbf{C}_{mix}, \mathbf{C}_{pure})$ be a reconstruction scheme. CDP purification is logically equivalent to CPM purification for $\mathbf{C}_{\mathrm{mix}}$ if and only if $\mathbf{C}_{\mathrm{pure}}$ satisfies polar decomposition.*

Proof. Lemma 7.17 completes a triangle of equivalences: for all morphisms $f : A \to C$ and $g : A \to B$ in $\mathbf{C}_{\mathrm{pure}}$:



where $U_1 : B \to C$ and $U_2 : C \to B$ are isometries. □

Note that there are *two* levels of logical equivalence in Theorem 7.18. The 'outer' level is: polar decomposition is logically equivalent to a statement about purification. The 'inner' level is this same statement about purification, viz. that CDP purification is logically equivalent to CPM purification. The reason for formulating the theorem in this way is as follows. The right-to-left (outer) implication establishes a sufficient condition for CPM to capture CDP purification (or rather, a strengthened version of it, as shown by Proposition 7.15). This was our stated aim at the beginning of this Chapter. However, the left-to-right (outer) implication establishes that this condition is also *necessary*. Hence no weaker condition can be found which ensures that CPM purification is equivalent to CDP purification.

The upshot of Theorem 7.18 is as follows. As discussed above, the most important CDP axiom for deriving quantum theory is the purification axiom. We have rephrased this in the language of dagger compact categories, which we called CDP purification. We have now shown that CDP purification is equivalent to one of the axioms for the CPM construction, as long as polar decomposition is assumed (and indeed, polar decomposition follows from assuming their equivalence). The significance of this is that a reconstruction scheme encapsulates four of the six CDP axioms if we impose that $\mathbf{C}_{\mathrm{mix}}$ is a dilation structure for $\mathbf{C}_{\mathrm{pure}}$.

Chapter summary. We have made the connection between the CDP reconstruction and CQM explicit. We showed how the graphical calculus of the CDP construction is the same as the graphical calculus of an SMC. We then showed how three of the six CDP axioms can be straightforwardly translated into a categorical scheme. Finally we showed how the purification axiom is related to the CPM construction. To do so we made of use of an axiomatisation of the CPM construction, and found that the CDP purification postulate is 'almost' the same as the CPM purification property, the difference being the 'polar decomposition' condition. Hence four of the six CDP axioms are naturally expressed in a categorical framework. We depict this in Figure 7.2, in which we also show that given a category of pure states the CPM construction gives C_E.

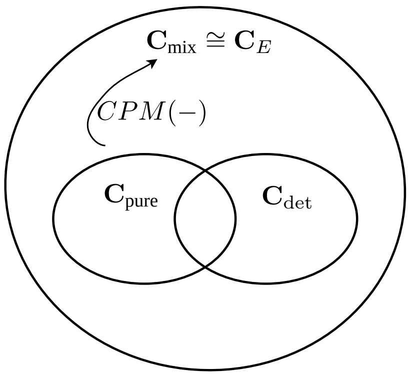


Figure 7.2: Reconstruction scheme with dilation structure

Chapter 8

Coda: the philosophy of categorical quantum mechanics

Contents

Since CQM is a relatively new formalism, there is still much to understand about its conceptual significance. In particular, although there has been a recent wave of operational approaches in the foundations of quantum theory, there has been little discussion of whether CQM is connected to such approaches. Alternatively, could CQM instead provide a formalism for a defensible version of realism?[1]

In this Chapter we shall address this question, and in doing so we shall make some progress towards a philosophy of CQM. We shall develop our view in a conservative way. This will be apparent in two ways. Firstly, our claim will be a moderate one: the claim will be that CQM does not suggest realism but instead a kind of operationalism. Indeed, this may be the intuitive interpretation that presents itself in the previous chapters. Secondly, since CQM is a new formalism, it is not obvious where to start with this project. Hence we shall take our lead from previous philosophical work on the analysis of category theory in physics.

The previous work that we shall focus on is a recent argument by Bain [12]. This states that the successful use of category theory in physics supports a certain type of scientific realism. More specifically, Bain has proposed that category theory has particular significance for a type of scientific realism known as *structural realism* [109]. Since we have been using a particular category-theoretic formulation of physical theories, the work in the preceding chapters puts us into a position to evaluate this claim. Such a proposal is particularly interesting in the context of our work for three reasons.

1. Bain argues for his claim by using examples of categories such as **fHilb** that we have been considering. He also considers category-theoretic formulations of physics such as Baez's work on n-categorical physics,

[1] Depending on how they are construed, operationalism and realism are not necessarily mutually exclusive. As Hardy and Spekkens have argued [54], we might consider these positions to be different methodologies for constructing physical theories rather than ontological statements.

which are very similar to CQM.

2. Bain's proposal is one of the few published philosophical studies of the use of category theory in physics. Moreover, it is also one which describes a *prima facie* attractive position, in the sense that it builds on a seemingly natural idea.

3. The philosophy that CQM suggests—if any—has been largely unexplored; an exception is [33]. Therefore, any proposal that connects category-theoretic formulations of physics to specific philosophical ideas is relevant, since it may shed light on the interpretation of CQM. This is especially so if it is claimed that a category-theoretic description is *crucial* to such an intepretation, as is the case with Bain's proposal. Moreover, structural realism is a significant position in the philosophy of science, since it is currently a much-discussed type of realism, particularly from the perspective of the philosophy of physics.

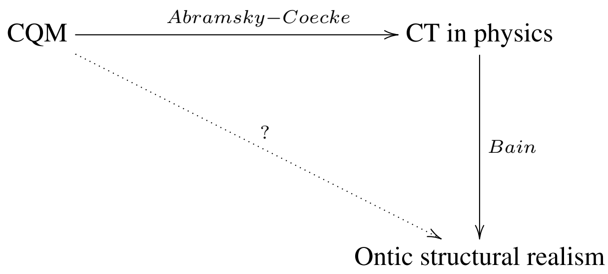
Hence we shall evaluate Bain's proposal from the perspective of CQM. Bain argument is based on the use of certain examples to support this view, and so we shall study these examples, and extend them. Our conclusion will differ from Bain's: our aim will be to show that, *contra* Bain, category-theoretic formulations of physics are neutral towards structural realism, or indeed any kind of scientific realism. On the other hand, the question then arises as to whether CQM suggests operationalism. That is, the development of CQM that we have discussed in previous Chapters might suggest that category-theoretic formulations of physics are better suited to describing information flow in physical theories, in a macroscopic way. Because its usefulness lies in this macroscopic description of protocols, it might be argued that CQM is indeed not helpful for structural realism, but instead in making precise and apparent the assumptions of operationalism. But this will not be our conclusion: by considering the work of the previous Chapter, we shall contend that CQM also seems ambiguous towards operationalism.

8.1 Category theory and structural realism

Our starting point—the aforementioned proposal by Bain—concerns a topic, structural realism, that has been previously considered in the context of the foundations of quantum theory, although in quite narrow ways (e.g. see the work of Cao [19] and Saunders [94] on structural realism and quantum field theory). However, it might still seem surprising that this topic should have any connection to the foundations of quantum theory: after all, as we shall see, it is a topic that concerns *any* physical theory, including both classical and quantum. Hence the relevance of such a general topic to the interpretation of a particular formalisation of quantum theory such as CQM may be questioned.

But we believe that it is relevant in at least two specific ways. Firstly, its generality with respect to different theories dovetails with the fact that, as we have seen, CQM is a framework for incorporating different physical theories, and it treats classical and quantum theory similarly (one might say that the 'Q' in 'CQM' is more a starting point than a restriction to quantum theory). Secondly, the claim of structural realism is, roughly speaking, that we should be committed to the existence of *structural* features of the world, and not to object-like features. This is a clear *prima facie* connection to category theory: e.g. it is often said that, in the definition of a category, 'the morphisms are important, not the objects'. We shall discuss this idea in Remark 8.2 below.

Accordingly, our concern will not be the merits of structural realism *per se* (and so we shall not review it in detail), but instead we are interested in its possible connection to category-theoretic formulations of physics—and how we can use our previous work to evaluate this. The way that these are ideas are related is illustrated by the following informal diagram, where the arrows mean ‘leads to’:



In words: the top arrow is the inference that since CQM uses category theory, it leads to the question of how to interpret the use of category theory in physics. On the other hand, the right vertical arrow is Bain’s claim that category theory supports a variant of structural realism known as ‘ontic’ structural realism (to be explained below). The question that we want to address—the dotted line—is whether CQM leads to structural realism. Our claim will be that it is does not, which we shall support by denying Bain’s inference, i.e. we deny the right vertical arrow.

8.1.1 Structural realism

Although our concern is a quite specific type of structural realism, it will still be useful to understand some general reasons in its favour, if only to make clear its plausibility. Hence we shall now explain a particular argument that is often used to support it. This is the most commonly used argument, but not the only one that is important for our purposes, and we shall mention another that is interesting in the context of category theory later.

The realist dilemma

The context of this argument is *scientific realism* concerning fundamental physical theories. This holds that we should interpret the terms of our fundamental theories as referring to a mind-independent world. In other words, we should interpret our fundamental theories *literally*[2]. Let us note two aspects of this claim. Firstly, scientific realism holds in particular that that the terms of our fundamental theories that refer to *unobservable* entities should be interpreted literally. We shall call such terms *unobservables.* For example, in electroweak theory, ‘neutrino’ is such a term. Secondly, scientific realism means that, in a fundamental theory, the true sentences that contain unobservable terms owe their truth to the existence of mind-independent entities. The property of ‘mind-independence’ means that these entities have an existence that is *independent* of our observations: i.e. their existence is independent of the experimental data that they generate, and independent of the observers processing such experimental data. To see how these two aspects are combined, suppose for example that electromagnetism is a fundamental theory. Then, according to scientific realism we account for its empirical success as follows. The sentence ‘an electron has charge 1.6×10^{-19} C’ is true because ‘electron’ refers to a mind-independent unobservable entity, and this entity bears properties such as electric charge. The electron’s property of charge is in turn responsible for our macroscopic observations on an oscilloscope. Hence part of the realist position is a causal claim, from the properties of unobservable entities to the ‘structure of appearances’, that is, the empirical data.

There are various arguments for *structural* realism, but we shall focus on the argument which requires the least amount of philosophical background. This argument was first formulated by Poincaré [87], The argument arises from considering the following two conflicting claims.

[2] The sense of ‘literal’ that is used here is just that our ordinary-language use of scientific terms has the connotation of this mind-independent existence.

ES *Empirical success implies independent existence*:

(a) The best explanation of the empirical success of a given theory is realism: at least some of the unobservables in the theory should be interpreted as referring to mind-independent unobservable entities.

∴ At least some of the unobservables in our current fundamental theories refer to unobservable entities.

TR *Theory replacement implies no independent existence*:

(a) There have been empirically-successful fundamental physical theories in the past which have been replaced. According to current theories, the unobservable entities of these discarded theories no longer exist.

(b) Our current empirically-successful fundamental physical theories are of the same kind as those in the past: they are also likely to be replaced eventually, and in particular the unobservable terms will be replaced.

∴ The unobservables in our current fundamental theories do not refer to unobservable entities.

We shall shortly explain ES and TR further; first we note that ES and TR are in conflict: ES gives us reason to favour realism concerning current theories, and TR gives us reason to doubt it. In other words, the conflict presents the following problem: if a given current theory such as quantum electrodynamics is likely to be replaced, then—despite its empirical success—how can we believe that it *currently* describes the correct ontology? Before discussing the conflict between ES and TR further, let us now briefly discuss each claim.

ES is a widely-used justification for scientific realism—often thought to be the most convincing in its favour [21]—and is sometimes called the *no-miracles argument.* This name signifies the intuition that, were realism not to hold, then the success of our scientific theories would be a remarkable coincidence: as Putnam describes [91, p. 73]:

> The positive argument for realism is that it is the only philosophy that doesn't make the success of science a miracle.

The sense of 'miracle' that is being used here is what we might expect intuitively. To elaborate: our fundamental scientific theories all consist of a set of axioms, and mathematical rules to derive consequences from the axioms. Such a formalism is essentially a single set of rules for calculating the results of a *variety* of experiments. [3] For example, the theory of classical electromagnetism, i.e. Maxwell's and supplementary equations, can be used in the same way for a huge variety of experiments involving electric charge. Now, for a successful theory, these calculations are the same as those displayed by experiment. But this success seems mysterious without a common factor to all these experiments. To state this slightly differently, the success of the formalism is a kind of 'correlation' between experiments, and a 'common cause' would constitute an explanation. One could say that the common cause is the formalism of electromagnetism itself, but this would seem circular, since the successful application of this formalism is exactly what we are trying to explain. Instead, Putnam's argument is that the best explanation of such correlations is that the unobservable terms in the formalism pick out something real, and this is a common cause *in the world*. In the case of the theory of electromagnetism, this common cause is the existence of an unobservable entity corresponding to the term 'electric field', which behaves according to the theory's formalism. The

[3] By 'formalism' we just mean the physical axioms and the mathematical rules for manipulating the physical axioms (which are phrased mathematically). In addition to the subsequent example, the formalism of quantum theory is the von-Neumann-Dirac axioms and the rules of complex Hilbert spaces. Note that philosophers sometimes restrict 'scientific theory' to mean specifically the axioms of the formalism, e.g. the von-Neumann-Dirac axioms, but not the mathematical rules. We have not emphasised this meaning, since it will not play a role in what follows.

electric field appears in each of the different experimental setups, and explains why each of them 'miraculously' obeys Maxwell's equations.

TR makes an *inductive* claim: that our current empirically-successful fundamental physical theories will be replaced in the same way that previous successful theories have been. Hence empirical success is not a guarantee of a theory's finality. Moreover, empirical success is also not a guarantee that the theory's ontology will survive. There are many examples that can be used as evidence, but let us mention Worrall's original example since this will be useful later [109]. Consider Fresnel's theory of light, which attributed optical phenomena to the wave behaviour of an aether. This theory successfully accounted for the empirical phenomena that were available when proposed by Fresnel, viz. reflection and refraction of visible light. For example, one of Fresnel's equations describe the reflection amplitude A_r, parallel to a plane of incidence, in terms of in the incoming parallel amplitude A_i as

$$A_r = \frac{-\sin(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)} A_i \tag{8.1}$$

where θ_i is the angle of incidence and θ_r is the angle of reflection. The basis of TR is that the observation that the empirical success of these equations did not guarantee the correctness of its ontology: the *correct* equations were derived from the *wrong* object, viz. the existence of an elastic aether ('wrong' in the sense that its existence was an assumption which was subsequently dropped). That is, Fresnel's theory was superseded by Maxwell's equations, which of course do not have any terms such as 'aether', but instead have terms such as 'electromagnetic field'. Now, in a sense, this example seems to be a trivial observation about the progress of physical theories. But the key point of TR is not just that physical theories are replaced, but that this replacement is accompanied by *ontological discontinuity*, which consists of the change from an aether to a field in the Fresnel-Maxwell example. This makes conventional 'entity' realism difficult to sustain. In fact, this problem was already implicit in our earlier description of scientific realism above, in which we made the *assumption* that classical electromagnetism is a fundamental theory.

Remark 8.1. Both ES and TR can be criticised on quite general grounds, especially as regards to their probabilistic content. That is, both make a claim about *likelihood*: in the case of ES, it is the likelihood of a 'miracle' as described above; in the case of TR it is the likelihood that a theory will be replaced. But it is not clear that probabilistic reasoning is appropriate (or indeed meaningful) in these cases, for example the latter case is essentially using a statistical analysis of theories in the history of science. Nevertheless, the probabilistic content of each clearly has strong intuitive appeal, as the examples above demonstrate. However, from our perspective, a more interesting objection is the question of the legitimacy of inferring metaphysics from a formalisation of physics. This criticism does not necessarily lead to a form of antirealism such as instrumentalism. For example, it could be that metaphysics is entirely independent of the formalism of a theory. But investigating the details of this criticism, or others which are similarly general, is not a topic that we shall pursue. Instead we shall take it as given that it is indeed legitimate to infer metaphysics from a formalism. In any case, inferring metaphysics (in particular, ontology) from the formalism of a physical theory is assumed in much of the contemporary literature of foundations of quantum theory—if it were not then hidden variables would not be so popular a topic.

Structural realism

So, given this commitment, how should a realist respond to the conflict between ES and TR? One way of responding to this is to impose conditions which a theory must satisfy before we should accept its implicit ontology. In more detail: we have described ES as the claim that empirical success to be the reason for accepting realism. We then explained that TR claims that empirical success is not enough to save ontology, since theory-replacement is highly likely to occur. The response could then be: perhaps ES should be augmented with further conditions, i.e. the realist also needs to take into account, for example, a theory's maturity before accepting its ontology. Then if a current fundamental theory such as quantum electrodynamics satisfies a given definition of maturity (e.g. some 'number' of successful experimenal tests), we will accept it, and its ontology. But making a test of maturity precise and meaningful is clearly difficult.

The dominant response in recent work in the philosophy of science has been *structural realism*, a view which essentially originated with Poincaré [87], but was revived more recently by Worrall in [109]. The response of the structural realist is to *weaken* the scope of ontological commitment. Instead of committing to the existence of entities denoted by terms such as 'electron', structural realism claims that commitment should only lie with the *structure* that such entities enter into. The term 'structure' here means the relational properties of entities such as electrons or fields. This takes its cue from examples of theory-change such as the transition from Fresnel to Maxwell described above, in which the Fresnel equations can be seen to describe the structure of light. In Worrall's words [109, p. 117]:

> [I]t is no miracle that his theory enjoyed the empirical predictive success it did; it is no miracle because Fresnel's theory, as science later saw, attributed to light the right structure.

So Fresnel's theory attributed to light the correct structure, *in spite* of attributing it to the wrong object, i.e. the elastic aether instead of the electromagnetic field. But Worrall's thought is that, by the principle ES, we should still attribute the success of Fresnel's theory to some ontology in the world. For the stuctural realist, this ontology is the *structure* described by the theory: in this case, 'structure' means Fresnel's equations, one of which is Eq. 8.1. Moreover, this structure is taken to be existing *in the world*, i.e. it is not just mathematical structure as part of our scientific knowledge, but it has a mind-independent existence.The key point is that this version of scientific realism navigates between the opposing forces of ES and TR:

(i) It satisfies ES because it identifies a property in the world that is responsible for the success of the theory.

(ii) It satisfies TR because the property it identifies is much more commonly *preserved* by subsequent theories: Fresnel's equations are derivable from Maxwell's equations.

There are two subtleties to the position of structural realism. These relate to the distinction between epistemology and ontology. Firstly, note that we have presented the position of structural realism as arising partly from considering how our knowledge changes as a theory is replaced (the claim TR). But structural realism is not *only* a claim about epistemology. That is, it is not only the claim that physical theories gives us only structural knowledge. Instead, it is the *stronger* claim that this structural knowledge corresponds to 'something in the world', i.e. it corresponds to ontology. This stronger claim is our focus because we are investigating the connection between category-theoretic formulations of physics and *realism*, which is an ontological claim. Hence a weaker, *epistemological* claim, i.e. one that only concerns the character of our knowledge, is not relevant. Secondly, structural realism states not only that what we should commit to about the ontology of theory is structural, but that we should *only* commit to the structural part. If we accept structural realism, then what we should infer from a physical theory is ontology *constrained* by epistemology. We can put this in the context of the realist dilemma that we presented earlier: roughly speaking, TR is an epistemological claim (that our theoretical knowledge will change), and ES is an ontological claim (the source of success of the theory is the existence of unobservable structure). Structural realism can be thought of as ES modified by TR.

Now, this form of motivating structural realism is vulnerable to a similar criticism to Remark 8.1. For, even if we have decided to 'read metaphysics off the formalism', it is not necessarily clear which part of the mathematical formalism should, in general, count as 'structure'. In the Fresnel-Maxwell example, it seems intuitively obvious that the Fresnel equations are what count as structure, since these directly describe the behaviour of the putative objects in each case: this is the aether in the case of Fresnel's theory, and the electromagnetic field in the case of Maxwell's theory.

However, other examples of structure-preservation have been proposed that are not just preservation of equations, in particular examples based on symmetry groups. For example, the most clear occurrence of structure-preservation in the transition from classical electromagnetism to quantum electrodynamics is often taken to be the gauge group $U(1)$, rather than Maxwell's equations. Hence a more general, abstract, definition of structure is needed. This leads to the following conventional notion of structure in a physical theory. Consider the Fresnel equation that we referred to above, Eq. 8.1. This refers to amplitudes $A_i, A_r \in \mathbb{R}$. Then, for fixed θ_i, θ_r, we can consider the set

$$R = \left\{ (A_i, A_r) \mid A_r = \frac{-\sin(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)} A_i \right\}$$

Hence the Eq. 8.1 identifies the relation R on the domain $\mathbb{R}^2$, and similar reasoning shows that e.g. $U(1)$ symmetry group induces a relation on a set. We can now generalise the idea of structure from this example using a basic set-theoretic definition as follows. A *structure* is a pair $(S, \mathcal{R})$, where is S is a non-empty set and $\mathcal{R}$ is a non-empty set of n-ary relations on S, e.g.

$$\mathcal{R} = \{R_1, R_1', R_2, \ldots, R_n\} \tag{8.2}$$

where R_n denotes a relation $R_n \subseteq S^n$ (in Eq. 8.2 we display two relations of arity 1, i.e. R_1 and R_1', to indicate that $\mathcal{R}$ may have more than one relation for a particular arity). For this relational notion of structure, the elements of the set S are called *relata.*

Now, even if the notion of structure has been made precise, a further objection is that although it is clear that the *individuals* (e.g. the aether) are often not preserved under theory change, it may also not be clear that the *structure* is always preserved. For example, as Redhead points out [92], in the transition from classical to quantum theory, observables shift from forming a commutative to a non-commutative algebra. This certainly seems to be a less smooth transition than in the Fresnel-Maxwell case, in which the Fresnel equations are preserved under theory change.

But note that both criticisms above concern the *universality* of both the notion of structure and its preservation under theory change: i.e. whether both structure and structure-preservation can be identified in *every* case of theory change. But we are only seeking to establish the plausibility of structural realism, and the fact that there exist examples in which structure and structure-preservation can be easily identified suffices for our purposes.

Remark 8.2. Our discussion of the notion of 'structure' would already seem to offer an interesting connection to category theory, before we have even considered Bain's proposal. We describe two possible connections. First, consider the relational definition of structure just given using Eq. 8.2. Category theory can provide various possible generalisations of this definition as follows. We note that the definition of structure above is a tuple $\mathcal{R}$ of *relations* on the set S, and so we can also view these as morphisms in $\mathbf{Rel}$. Then a natural generalisation would be to define structure as a tuple of endomorphisms $\mathcal{M}$ on an object A in an arbitrary category $\mathbf{C}$, i.e. as $(f_1, f_2, \ldots, f_n)$ where the types are $f_i : A \to A \otimes A$. Secondly, consider the second objection above, that of identifying structure-preservation. This is also related to a categorical viewpoint: in fact, category theory is used exactly for making precise the notion of structure preservation. Hence the concern, mentioned above, that the notion of structure-preservation used in the literature is not general enough to capture examples such as the transition from a commutative to a non-commutative algebra, could be addressed by using category theory. Indeed, part of the methodology of CQM is exactly to identify mathematical continuity. For example, in CQM we could describe the classical-to-quantum transition by considering the transition from a cartesian category $\mathbf{C}$ to a general monoidal category $\mathbf{D}$ (in lieu of the transition from a commutative to a non-commutative algebra). This could be formalised as a functor

$$F : \mathbf{C} \longrightarrow \mathbf{D}$$

preserving monoidal structure, which describes the inclusion of classical objects into a category of quantum objects. More generally, the transition from $\mathbf{fHilb}$ to arbitrary dagger compact categories that we have described in previous chapters is an example of continuity of mathematical structure. Moreover, this transition does indeed identify the structure of *physical* processes that the structural realist seeks: for quantum teleportation this is compact structure. Now, the two possible connections to category theory that we have described involve using the fact that category theory allows generalisations from relations on a set. Moreover, the generality of category theory means that it encapsulates general facts about mathematical representation, such as the fact that isomorphic objects 'behave the same': if $A \cong B$ then A and B both have the same dimension for example. This is related to another argument that has been used in favour of structural realism. This is the argument for structural realism based on the nature of mathematical representation; as van Fraassen writes [44, p. 522]:

> Within mathematics, isomorphic objects are not relevantly different; so it is especially appropriate to refer to mathematical objects as "structures" [...] therefore, scientific theoretical descriptions are structural; they do not cut through isomorphism.

This statement is essentially the same as the Principle of Isomorphism discussed in Chapter 2: van Fraassen concludes from it that our theories can only capture the structure of scientific phenomena. Hence structural realism would seem to be naturally related to category-theoretic formulations of physics, since the Principle of Isomorphism is an important meta-theoretical aspect of category theory.

The upshot of the discussion so far is that structural realism is a naturally-motivated position. As we mentioned in Remark 8.2, it is one which has an interesting connection to category theory. It has held sway in the philosophy of science partly because of its ability to navigate between ES and TR. To summarise its content: it is a modification of traditional scientific realism, in which our ontological commitment should be to the structure that unobservable entities enter into, rather than the entities themselves.

8.1.2 Bain's proposal

As we noted above, since structural realism is a form of realism, it is a certainly a claim about ontology. However, we also emphasised that the way in which we have presented it had an epistemic flavour: in particular, from TR we inferred the *epistemic* constraint on realism, viz. the restriction to structure. In other words, relations and relata both exist as ontology, but we only know the relations—we cannot commit to the existence of relata such as electrons. A recent idea in this debate has been to try to close the gap between epistemology and ontology. Could it be that, not only are relata are unknowable, but moreover that relata do not exist?

This has led to a form of structural realism known as *ontic structural realism (OSR)*, which is the focus of Bain's particular proposal. The various forms of OSR each hold that the relations in a structure are *ontologically basic,* and at least as fundamental as the relata, i.e. the elements of S in the structure $(S, \mathcal{R})$. By elaborating on this claim in various ways, we obtain the variants of OSR. There are two variants that concern us:

OSR1 Relata do not exist.

OSR2 The relations in a structure are ontologically prior to relata, and the latter depend on the former for their existence.

OSR1 is stronger a claim than OSR2. Bain's proposal concerns OSR1, but we shall return to OSR2 briefly later. OSR1 can be explained in more detail as follows. The definition of structure given above is the pair $(S, \mathcal{R})$. OSR1 holds that the relations $\mathcal{R}$ can exist independently of the set of objects S. It holds, for example, that the monadic relation of charge can exist without the object, the electron, that carries it.

The proponents of OSR1 favour it for the following reason. Occam's Razor (or some form of it) is the principle that we should reduce the number of different ontological types whenever possible. Therefore if relations can exist without relata, then Occam's Razor should lead to us to accept OSR1.

But the problem with OSR1 is that, using the definition of structure given above, it is not clear how to make *formal sense* of the notion of relations without relata. In the definition of structure above, the set S is a necessary part of the pair $(S, \mathcal{R})$, since e.g. a relation R_2 is defined as a subset of $S \times S$. Bain's argument begins by noting that such problems would seem to be inherent in any set-theoretic definition of structure. That is, if '$\in$' is a primitive symbol, and we interpret it as 'membership', then there would seem to be no way of getting around the need for elements when defining structure. But recall Remark 8.2. We noted that category theory offers a natural generalisation of the conventional notion of structure in the discussion of structural realism. In particular, it would seem to offer a way of defining structure without using elements of a set. For example, in a monoidal category, an element $x \in A$ could instead be described as a morphism $\tilde{x} : I \to A$ as we have done previously. This is Bain's starting point, since if this is achieved, then it suggests that structure, viz. relations, can be defined without relata.

The argument that Bain subsequently develops comes in two parts. The first part concerns generalising the notion of relata:

B1 Set-theoretic structure as defined above (i.e. Eq. 8.2) makes ineliminable reference to relata, viz. elements of a set.

B2 The same set-theoretic structure can be defined in the category $\mathbf{Set}$ using categorical definitions such as universal properties, e.g. the product. This then constitutes a new, categorical, definition of 'structure' for the ontic structural realist.

B3 These categorical definitions of structure do not depend on the elements of a set.

$\therefore$ The categorical definition of structure does not depend on relata.

The argument is valid, but we shall make the following objections:

O1 For the conclusion of the argument to have force for OSR1, Bain needs to provide examples of the definition of categorical structure in B2 which are *physically relevant* and defined in a way that is not just a one-to-one mapping from set-theoretic terms, such as 'element $x \in X$', to category-theoretic terms, such as 'morphism $\tilde{x} : I \to X$. We don't believe Bain's examples satisfy this.

O2 Even if O1 is satisfied, Bain makes the assumption in B1 that 'relata' must always mean 'element of a set'. But there is no reason why 'relata' should not correspond to a *different* mathematical notion when 'structure' is redefined categorically.

Bain recognises the potential objection O1. He attempts to address it by the following steps: (i) define what it means to just provide a 'one-to-one translation' as stated in O1; (ii) provide examples of categorical definitions of *physical* structure which evade such a one-to-one correspondence.

To achieve step (i), Bain uses the notion of a 'category of structured sets'. This can be thought of as the location of the original set-theoretic definition of structure. A 'category of structured sets' means a category whose objects are sets with extra structure (these are also called *concrete categories*). Examples of categories of structured sets include $\mathbf{Set}$, $\mathbf{Rel}$, and **fHilb**: the latter because the objects are vector spaces, and so in particular are sets subject to the axioms of a vector space. An example of category which is not a category of structured sets is a monoid viewed as a category: given a monoid $(M, \bullet, 1)$, we can describe it using a 1-object category in the usual way [68]; in doing so the object is 'formal' in the sense that it need not be a set of specified elements, and so in particular it need not be a structured set.

Now, consider the category $\mathbf{Rel}$. The original definition of structure uses morphisms in this category; using this observation, we would hope to define a new notion of structure using category theory. So as we have done before, we can use morphisms $\hat{x} : I \to X$ to represent elements $x \in X$. For example, to represent a particular relation $R \subseteq X \times X$, we use the morphism $f : X \to X$ in $\mathbf{Rel}$ that satisfies $f \circ \hat{x} = \hat{y}$ if and only if $(x, y) \in R$, where the morphisms $\hat{x}$ and $\hat{y}$ correspond to the elements $x \in X$ and $y \in Y$ respectively. Hence we seem to have got rid of relata, as proponents of OSR1 would hope, since the category-theoretic definition of R, as the morphism f, is not defined using the elements of X, but instead using other morphisms in the category. But this way of translating set-theoretic structure just seems to be a change of language: instead of referring to elements $x \in X$, we will refer to morphisms $\hat{x} : I \to X$. This constitutes the one-to-one translation that Bain needs to avoid.

The way to evade this objection is to provide physically relevant examples of categories in which objects do not always have points, i.e. objects X without morphisms of type $I \to X$. That is, the response needs to provide physically relevant examples of categories which are *not* categories of structured sets. We shall discuss why these examples fail in the next Subsection, but for now let us discuss O2: which is that even if his examples were to succeed, there is a further objection that Bain is vulnerable to.

For, it now seems that there is a new candidate for the role of relata: the objects X of the category themselves! Bain might want to respond by claiming that this does not capture the original notion of relata. For example, Bain might claim that objects in a category such as **fHilb** usually represent state spaces rather object-like entities such as 'electron'. But Bain is defining a new notion of structure for the structural realist; after all the relations $\mathcal{R}$ are no longer being defined set-theoretically but instead categorically. Hence there is no reason why the definition of relata should not change *as well* the formal definition of relations. Moreover, recall the content of the claim TR, which led to the position of structural realism, was indeed that *relata change* as theories become more refined. Hence we can simply invoke TR to explain why relata are now objects in a category. This objection applies even if O1 is addressed so that the original definition of relata are not present in a non-'structured sets' category.

There is a possible counter-response to our use of TR to claim that the objects in a category are relata. TR states that relata change under *theory-replacement*. This is arguably not the same as Bain's case: he is using a different *definition* of structure, rather than *replacing* a theory. The difference is that theory-replacement corresponds to the Fresnel-to-Maxwell transition, and Maxwell's theory has distinct empirical content to Fresnel's equations (in the sense that Maxwell's theory accounts for a broader set of phenomena). On the other hand, Bain's move is to define structure $(S, \mathcal{R})$ using a category-theoretic instead of a set-theoretic definition. This is seemingly a meta-theoretic move, not one that concerns particular physical theories. But the problem with this response is lies with Bain's evidence for responding to O1. His evidence consists of examples of the successful use of category theory in physics—'successful' in the sense that category theory identifies crucial physical structure. This is certainly not a meta-theoretic claim about the use of category theory to identify structure, since, as we shall see, it concerns *specific* physical theories. In that case, it is less clear that category-theoretic reformulation of physics are not cases of theory-replacement, since category-theory is being used to *directly* define parts of the theory.

Having explained the objections O1 and O2, we conclude that for Bain's move to category theory to be successful, he must respond as follows:

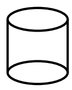
- Provide examples of defining physically relevant structure in a purely category-theoretic way, which can be used in categories which are *not* structured sets. That is, the definitions should not just be a straightforward translation of each set-theoretic term to a category-theoretic term.
- Explain why the *objects* of a category are not relata, since they seem to be candidates for such role. Note that this objection holds even if O1 is satisfied.

The first point leads to the second part of Bain's argument, which concerns the extent to which such categorical definitions can be used in a physically relevant way, but without relying on categories of structured sets.

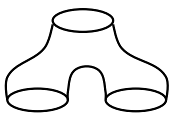
8.2 Bain's examples and CQM

Let us focus on two of Bain's examples. The first example is $\mathbf{Hilb}$, which we are familiar with. The second example uses the category $\mathbf{nCob}$: this has objects which are $(n-1)$-dimensional compact oriented Riemannian manifolds, and morphisms which are n-dimensional oriented cobordisms between such manifolds. For example, the objects of the category $2\mathbf{Cob}$ are 2-dimensional compact manifolds. An object A is depicted as:

The morphisms are essentially 3-dimensional compact manifolds which have the objects as boundaries. A morphism $f : A \to A$ is depicted as:



But $2\mathbf{Cob}$ is also a monoidal category, and hence it also has morphisms of type $g : A \otimes A \to A$, which are depicted as:



Now, the examples of $\mathbf{Hilb}$ and $\mathbf{nCob}$ are attractive for Bain's position, and are physically relevant, in the following sense. The category $\mathbf{nCob}$ is used in mathematical formulations of quantum field theory, in particular topological quantum field theory. The idea of this theory is that the objects of $\mathbf{nCob}$ represent space, and the morphisms represent spacetime: this can be seen intuitively in the diagrammatic representation. To form a type of 'quantum field theory', we define an assignment of state spaces in $\mathbf{Hilb}$ to the objects, and linear maps to the cobordisms, which formalises the idea that cobordisms represent evolution in time. Hence a *topological quantum field theory (TQFT)* is defined as a functor:

$$T : \mathbf{nCob} \longrightarrow \mathbf{Hilb}.$$

In order to respond to objection O1 above, one might argue that there is much physically relevant structure here, and moreover that it is expressed category-theoretically. For example, as Bain points out, both $\mathbf{nCob}$ and $\mathbf{Hilb}$ are $\dagger$-monoidal categories. Moreover, the fact that both categories share categorical properties such as allowing a $\dagger$ functor is seen by many as a deep feature of TQFTs. For example, Baez argues that this exposes a formal analogy between spacetime and quantum theory that should be exploited in order to obtain a theory of quantum gravity [11]. This would seem to strengthen the case for Bain, since it promises great physical relevance to these categories. But note that TQFTs here are quite unphysical in many ways. For example, the objects of 2-dimensional manifolds in $2Cob$ are the 'spacelike hypersurfaces', but they are *Riemannian* manifolds. Since they are not *pseudo-Riemannian* manifolds, there is no notion of light cones and causal structure in the category. Hence its relevance to relativity is not clear (which casts doubt on its usefulness for quantum gravity).

However, there is also an immediate objection to the usefulness of TQFTs for Bain's specific claim: both $\mathbf{nCob}$ and $\mathbf{Hilb}$ are actually categories of structured sets! That is to say, the objects in both categories are sets with extra structure: we explained this above in the case of $\mathbf{Hilb}$; in the case of $\mathbf{nCob}$, the objects are manifolds, and so are in particular topological spaces, i.e. sets with an identified topology.

Now, a counter-response might be that perhaps we can identify some useful categorical properties in $\mathbf{nCob}$ or $\mathbf{Hilb}$ that are important physically, and which can be defined in some as-yet unknown non-structured-set categories. These latter categories would then support Bain's OSR1 claim. Let us then consider the most useful mathematical result in the study of 2-dimensional TQFTs [65]. This arises by considering the category of 2-dimensional TQFTs, denoted $\mathbf{2TQFT}$: its objects are TQFTs, i.e. the functors defined above, and its morphisms are monoidal natural transformations. There is then an algebraic classification of TQFTs as follows. Consider the category $\mathbf{cFA}$ of commutative Frobenius algebras and Frobenius algebra homomorphisms. This category classifies 2-dimensional TQFTs, since we have the equivalence:

$$\mathbf{2TQFT} \simeq \mathbf{cFA}$$

This would seem to be exactly the kind of category-theoretic structure that Bain would like as evidence for OSR1. This is because it is an example both of (i) category-theoretic structure which is physically relevant (notwithstanding the criticisms of TQFTs above), since it classifies TQFTs, and (ii) also one which does not rely on a category of structured sets. That is, point (ii) means that a commutative Frobenius algebra is a purely categorical notion, which can be defined in non-concrete categories.

This bodes well, but in fact there is a problem with point (i): the idea that commutative Frobenius algebras represent physically relevant structures (i.e. TQFTs). For, we have seen exactly this kind of equivalence before, viz. an equivalence between a category $\mathbf{C}$ and a category of Frobenius algebras. Recall that in Chapter 2 we noted that (†-special) commutative Frobenius algebras classify orthonormal bases in $\mathbf{fHilb}$. For the broader class of commutative Frobenius algebras (i.e. which are not necessarily †-special), there is a bijection between commutative Frobenius algebras ($\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \epsilon : \mathcal{H} \to \mathbb{C}$) and the bases of the Hilbert space $\mathcal{H}$. In other words, commutative Frobenius algebras—seemingly the most powerful piece of categorical structure for TQFTs—classify both TQFTs *and* bases of a Hilbert space. This forms the basis of an objection to Bain's example, which we shall collect along with the previous one concerning physical relevance, as follows.

1. *TQFTs do not model actual physics*: The definition of a TQFT is too simple to expose 'physical structure'. For example, there is no causal structure (since the manifolds in $\mathbf{nCob}$ are Riemannian), or anything approaching particle physics in the mathematics of TQFTs. Therefore it is hard to see how TQFTs model the physics of our actual world.

2. *cFAs are ambiguous with respect to the physical structure they identify*: cFAs classify both TQFTs and bases in $\mathbf{fHilb}$. Because of this, we have the following situation: the most sophisticated example of categorical structure in TQFTs—Bain's (well-chosen) example of category-theoretic physics—also identifies classical data in $\mathbf{fHilb}$. Hence this categorical structure seems to be too 'coarse-grained' to identify TQFTs uniquely.

This raises doubt as to whether the categorical structure is physically relevant at all. But this objection can also be seen as the technical aspect of the previous point. That is to say, the reason that 2-dimensional TQFTs are elegantly classified by cFAs is because the definition of TQFTs is very simple, and so cFAs also classify bases in **fHilb**; and this simplicity means TQFTs cannot present much physics. Hence cFAs cannot be said to articulate 'all physically relevant structure' in a purely categorical way, as Bain's argument needs.

These two points expand on our objection O1 above, and hence complement our objection O2 concerning the idea of objects as relata. To briefly paraphrase our argument in this Subsection: our contention is that, once we look more closely the *practice* of using category theory in physics, we see that it is too general a formalism to provide an *entirely* category-theoretic formulation of physics. But this is what Bain needs, over and above a set of useful mathematical tools.

Now, we have focussed on the claim that category-theoretic formulations of physics support OSR1, following Bain's line of reasoning. Given the doubts we have cast over his claim, the question arises as to whether category theory could support weaker positions than OSR1, in particular OSR2 as explained above.

We consider this briefly now. This issue can be split into two parts:

1. *Could category theory help with other forms of ontic structural realism?* For example, could category theory help with OSR2? This position is weakening of OSR1, since it holds that relations are ontologically prior to relata (but relata still exist). This is sometimes taken to mean that relata can be defined *in terms of* relations. Prima facie, category theory could formalise this notion. Let us take morphisms to stand for relations (as before), and objects to stand for relata. Then consider the definition of n-categories. For example, in a 2-category there are morphisms between morphisms, i.e. a 2-morphism $\alpha : f \Rightarrow g$, which has as its domain and codomain the morphisms f and g. Hence for a 2-morphism, the 'relata', i.e. the domain and codomain, are actually *themselves* relations, i.e. morphisms. This is a natural analogy, but there is an immediate problem. For example, note that OSR2 would require 'relations all the way down'. This means that for every morphism $f : \alpha \Rightarrow \beta$ at level n, the domain α and codomain β of f are always morphism themselves (at level $n-1$). This might suggest the use of ∞-categories, but even an ∞-category must use the notion of objects in its definition. This would seem to be a severe problem for this idea of describing OSR2.

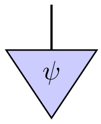
2. *Could category theory help with 'epistemic' structural realism?* In Remark 8.2 we noted some analogies between structural realism and category theory. This described a *prima facie* attractive connection, but we saw that Bain's attempt to develop such a connection for OSR specifically faced several problems. But we might then ask, can category theory play a useful role for the weaker notion of structural realism, in which relata are still allowed to exist? Note that in the case of OSR, category theory seemed useful because there was a *technical challenge* that faced OSR, i.e. the problem of formally articulating the notion of 'relations without relata'. On the other hand, for ESR, there is not a corresponding challenge. It may well be that category-theoretic formulations of physics most cleanly express the structural content of a physical theory. But this is a much weaker claim than the ontological one we have been considering, since this is merely indicating that category theory is useful 'linguistically'. As we indicated earlier, this is less interesting to us, because our aim is to understand the extent to which category-theoretic formulations of physics (in particular CQM) support realism, in the hope of understanding their ontological content.

What we infer from the work of this chapter is that category-theoretic formulations of physics do not provide any *more* reason to support structural realism than the standard arguments. Since structural realism has been argued as specifically relevant to category-theoretic formulations of physics, we conclude that these formulations of physics are agnostic towards structural realism (at least, by virtue of their *category-theoretic* formalisation).

This suggests that CQM does not suggest a (structural) realist ontology and whatever metaphysics we interpret for quantum theory will therefore be uninfluenced by using CQM. In other words, it seems hard to argue that category-theoretic formulations of physics, and in particular CQM, have significant ontological implications.

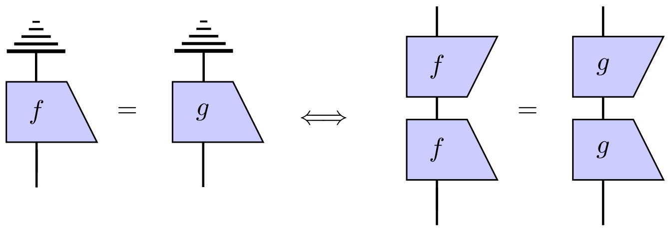
Before concluding this Chapter, let us consider how the work in previous Chapters might be related to this point. Now, the kind of success that we have described in previous Chapters instead suggest that it is a useful *methodology*. In particular, if, as we have argued in this Chapter, CQM does not especially suggest a type of realism, does it support operationalism? This question is especially interesting in the light of the what we have discussed the previous Chapter: our work there explicitly considered the connection between CQM and operational reconstructions of quantum theory.

Let us now consider this connection from a philosophical perspective. We showed that some of the CDP axioms can be reformulated in a categorical way. More specifically, we showed that CQM contains four of the six CDP axioms. But the CQM translation of this was not obviously operational. This is contrast to the prima facie view of CQM. For example, one might subscribe to the following chain of reasoning. CQM is based on a graphical calculus. This graphical calculus represents various features that are apparently operational. An example is the representation of a state:



We can view this as state preparation, since it is a morphism of type $I \to A$, and we have usually interpreted the monoidal unit I as the environment. Then, as we described in Subsection 7.1.1 on the CDP axioms, by considering the sequential and parallel composition of processes, we naturally arrive at the definition of an SMC. Hence SMCs would seem to be highly suited to formalising operational theories.

However, in our translation of the CDP axioms we had to assume the existence of certain formal structures at the *expense* of an operational justification, e.g. dagger structure. For example, our connection to the purification axiom was through the axiom D2 (which we called CPM purification):



(8.3)

This was part of the definition of a dilation structure for a pair of categories $(\mathbf{C}, \mathbf{C}_E)$. But to define a dilation structure the categories $\mathbf{C}$ and $\mathbf{C}_E$ need to be *dagger* categories: this is evident in the diagram of Eq. 8.3, since the right-hand side of the equivalence explicitly involves using the dagger functor (i.e. flipping the boxes upside-down). It is difficult to see how this assumption can be operationally motivated. Indeed, in CDP's reconstruction, the existence of a bijection between states and effects is an intermediate theorem.

Chapter summary. We discussed an existing proposal by Bain for connecting category-theoretic formulations of physics to a particular position in the philosophy of science known as structure realism. We discussed some of the motivation for structural realism. We then discussed Bain's position. We argued that his examples do not support the idea that category theory can provide coherence for ontic structural realism. We also argued that the objects in a category can also be considered to be the relata for the structural realist. Finally we argued that the practice of using category theory in physics shows that category theory identifies a very coarse-grained type of 'physically-relevant structure'. Hence it is difficult for the structural realist to claim, using current examples, that category-theoretic formulations of physics capture fundamental physics in a way that is primarily categorical.

In relation to CQM, our philosophical analysis has left us in an uncertain position. On the one hand, the most suitable form of realism is not given any more weight by CQM. On the other hand, the structures of CQM are not purely operational, and even an initial reformulation of operational axioms seems to need non-operational assumptions. The philosophy of CQM would therefore seem to require much further analysis.

Chapter 9

Outlook

Our results suggest several avenues for further work.

Time-asymmetry and causal structure

There are two specific ways to extend our work on time-asymmetry:

1. *Quantum Bayesian inference*: Leifer and Spekkens have recently proposed a quantum formalism that mimics classical Bayesian probability theory [69]. In the Leifer-Spekkens formalism, conditional density operators $\rho_{B|A}$ can be defined. The Bayesian inverse $\rho_{A|B}$ can also be defined. It would be interesting to apply this to study quantum processes which are backwards-signalling. Specifically: we showed that λ-independence fails for backwards-signalling classical processes, i.e. those with a hidden variable decomposition. If there is no hidden variable decomposition (as is the case for quantum non-local boxes) then λ-independence cannot be defined, since there is no variable λ. But since the Leifer-Spekkens formalism aims to provide a formal analogue of classical probability theory, it may provide a quantum analogue of the hidden variable λ, and so of λ-independence. The analysis of the failure of λ-independence in the classical case may then carry over to the nonlocal case in some form.
2. *Thermodynamics*: It would be interesting to establish a precise connection between backwards-signalling and the thermodynamic arrow of time. For example, it might be possible to establish a relationship between the amount of entropy increase and the type of backwards-signalling channel (e.g. the amount of entropy increase and the channel capacity of the backwards-signalling box might be proportional). This would involve the careful use of Landauer's principle: in [78] Landauer's principle has been extended to stochastic computation and also to spatially separated systems. This form of Landauer's principle will therefore be useful for further study of time-asymmetry and causal structure.

Causal categories

We can identify two ways in which causal categories can be extended:

1. *Profunctors*: Our framework is based on partial monoidal categories. Recently, *profunctors* have been used as a 'partial' analogue of functors. Moreover, profunctors have also been used to describe concurrency for *event structures*, which have been used in denotational semantics [20]. This application is similar to

ours, since concurrency is conceptually similar to our use of causal structure. It would be interesting to use profunctors to reformulate causal categories, and thence make connection to this recent work in theoretical computer science.

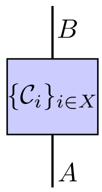
2. *Indefinite causal structure*: Hardy has argued that a theory of quantum gravity would need to incorporate indefinite causal structure [55]. Causal categories encode *definite* causal structure. However, it may be possible to generalise causal categories to describe indefinite causal structure. For example, we might use a functor $F : \mathbf{J} \to \mathbf{CC}$ to vary a causal category $\mathbf{CC}$ over the index set J; this could represent 'superposition' of causal structure. Such an idea would help to formalise a notion, viz. 'indefinite' causal structure, which has yet to be precisely analysed.

Operational axioms

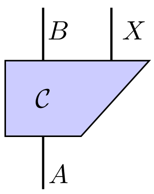
There are three specific ways to extend the work we have discussed.

1. *Axioms*: We reformulated Axioms 1–4 of the Pavia scheme categorically. It would be interesting to also reformulate Axioms 5 and 6 categorically. If this can be done, then we could potentially provide a reconstruction framework which does not assume that scalars are $x \in [0, 1]$. This would allow us to describe e.g. the Spekkens toy theory in a reconstruction framework, and therefore establish which axioms it fails to satisfy. We can then ask the question, which axiom is responsible for the nonlocality of quantum theory?

2. *Classical indexing*: We discussed that our reformulation of the CDP framework did not take into account the relationship between events and tests. It would be useful to represent this classical indexing in a categorical way. For example, instead of using an explicit indexing over events such as



it would be more elegant to internalise the classical outcomes as an object X in the category:



Philosophy of CQM

We objected to an existing argument that is relevant to understanding the philosophy of CQM. It would be interesting to develop a positive proposal for the philosophy of CQM. As suggested in [33], a *process philosophy* might be the most attractive position.

Appendix A

Appendix

In this Appendix we show two results:

1. The symmetry morphism of a non-strict-symmetric monoidal category, i.e. a symmetric monoidal category for which $\sigma_{A,B} \neq 1_{A\otimes B}$ for all A, B, cannot be strictified.
2. Strict-symmetric monoidal categories are not necessarily degenerate (in the sense of 'degenerate' to be described below).

1. Strictification of σ

As explained in Remark 2.10, to strictify the symmetry morphism σ for a category $\mathbf{C}$, we need to show the existence of a symmetric strong monoidal functor $F : \mathbf{C} \to \mathbf{D}$ that is also an equivalence, such that the symmetry morphism in $\mathbf{D}$ is the identity. Recall that a symmetric strong monoidal functor is a strong monoidal functor $F : \mathbf{C} \to \mathbf{D}$, such that:

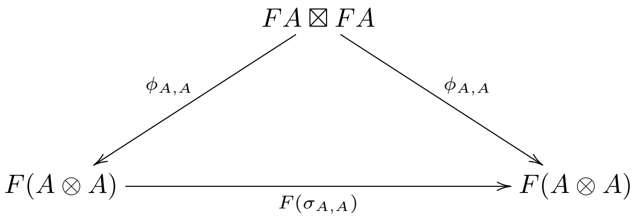
$$\begin{array}{ccc} FA \boxtimes FB & \xrightarrow{\gamma_{FA,FA}} & FB \boxtimes FA \\ \Big\downarrow{\scriptstyle \phi_{A,B}} & & \Big\downarrow{\scriptstyle \phi_{B,A}} \\ F(A \otimes B) & \xrightarrow[F(\sigma_{A,B})]{} & F(B \otimes A) \end{array} \tag{A.1}$$

where $\otimes$ and $\boxtimes$ are the monoidal products for $\mathbf{C}$ and $\mathbf{D}$ respectively; and σ and γ are the symmetry morphisms in $\mathbf{C}$ and $\mathbf{D}$ respectively.

Proposition A.1. *Let $\mathbf{C}$ be a non-strict-symmetric monoidal category, and let $\mathbf{D}$ be a strict-symmetric monoidal category. Then there does not exist a symmetric strong monoidal functor $F : \mathbf{C} \to \mathbf{D}$ such that F is an equivalence.*

Proof. Since is F is assumed to be a symmetric strong monoidal functor, it must satisfy the commutative diagram given by Eq. A.1. Now, let $B = A$ in Eq. A.1. Since $\mathbf{D}$ is assumed to be strict-symmetric, we have $\gamma_{A,A} = 1_{A\times A}$.

Hence Eq. A.1 becomes



Hence we have the equation

$$F(\sigma_{A,A}) \circ \phi_{A,A} = \phi_{A,A} \tag{A.2}$$

However, since F is a strong monoidal functor, ϕ is a natural isomorphism, and so pre-composing Eq. A.2 with $\phi^{-1}_{A,A}$ yields:

$$F(\sigma_{A,A}) = 1_{F(A \otimes A)} \tag{A.3}$$

But since F is assumed to be an equivalence, it must be faithful: for all objects A, B, the functor F induces function $F_{A,B}$

$$F_{A,B} : \mathbf{C}(A, B) \longrightarrow \mathbf{D}(FA, FB)$$

which must be injective. However, since F is a functor we have $F(1_{A\otimes A}) = 1_{F(A\otimes A)}$. Together with Eq. A.3 and the fact that $\sigma_{A,A} \neq 1_{A\otimes A}$ in $\mathbf{C}$ (since it is not strict by assumption), implies that the function $F_{A,A}$ is not injective. Hence F cannot be an equivalence. □

It is interesting to note why Proposition A.1 does not hold for the other structure morphisms. This is precisely because of the coherence theorem for monoidal categories: this states that all diagrams involving the structure morphisms and the identity morphism commute. Intuitively, this would indicate that the difference between σ and the other structure morphisms is that the latter morphisms are purely syntactical, whereas symmetry is a genuine mathematical property.

2. Non-degeneracy of strict-symmetric monoidal categories

Given a category $\mathbf{C}$, we shall consider two forms of degeneracy:

- $\mathbf{C}$ is a *preorder*, meaning that for all A, B, $|\mathbf{C}(A, B)| \leq 1$, i.e. every hom-set has at most one morphism;
- $\mathbf{C}$ is *one-dimensional*, meaning that for all $A, B, f : A \to B$, there exists a unique morphism $v : A \to B$ and a scalar $s : I \to I$ such that $f = s \bullet v$. In words, every morphism is a scalar multiple of a morphism v.

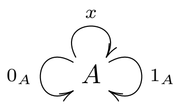
From a physical perspective, the reason for considering these properties to be 'degenerate' is that, for example, the notion of dynamics makes little sense in a category which is a preorder, since only one type of evolution can occur between any two objects in the category.

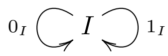
To show that strict-symmetric monoidal categories are not degenerate in either of these senses, we provide a counterexample.

Our counterexample for the preorder property is the category $\mathbf{Nat}_2$. The basic structure of this category is defined as:

- *Objects*: there are two objects, the monoidal unit I and another object A

- *Morphisms*: the morphisms are defined by the diagrams:





- *Composition*:
  - For all objects X, and for all morphisms f of appropriate type,

$$f \circ 0_X = 0_X \circ f = 0_X.$$

  - We define

$$x \circ x = 1_A$$

  The remaining composition rules are forced by the identity law, e.g. $x \circ 1_A = x$.

We define the monoidal structure of $\mathbf{Nat}_2$ as follows:

- *Objects*: the monoidal product of two objects is given by $\mathbb{Z}_2$:

$\otimes$	I	A
I	I	A
A	A	I

- *Morphisms*: The monoidal product $v_X \otimes w_Y$ is defined for three cases:
  (i) If $X = I$ and $Y = I$ then $\otimes$ is the identical to $\circ$.
  (ii) If $X = A$ and $Y = I$ (or vice versa, since $\otimes$ is symmetric) then:

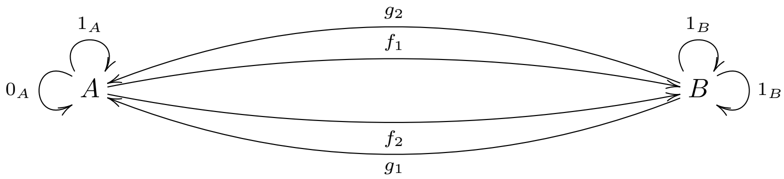
$\otimes$	0_A	1_A	x_A
0_I	0_A	0_A	0_A
1_I	0_A	1_A	x_A

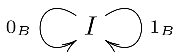
  (iii) If $X = A$ and $Y = A$ then:

$\otimes$	0_A	1_A	x_A
0_A	0_I	0_I	0_I
1_A	0_I	1_I	1_I
x_A	0_I	1_I	1_I

This completes the definition of the strict-symmetric monoidal category $\mathbf{Nat}_2$. It is not a preorder since, for example, $|\mathbf{Nat}_2(A, A)| = 3$. This category is also not one-dimensional, because $\mathbf{Nat}_2(A, A)$ is not one-dimensional: there does not exist a morphism y and a scalar s such that, for all morphisms $f : A \to A$, we have $f = s \bullet y$. This is most easily seen in part (ii) of the definition of $\otimes$ on morphisms above: there is no column which has all morphisms in $\mathbf{Nat}_2(A, A)$.

Now, it might be thought that this example is still degenerate in the sense of containing only one object $X \neq I$, viz. the object A. However, this can be extended to the following category $\mathbf{Nat}_3$, now defining $\otimes$ on objects as the cyclic group $\mathbb{Z}_3$.





It would seem that this definition can be generalised to categories with n objects by using the cyclic group $\mathbb{Z}_n$. We leave a proof of this for further work.

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