Quantum Causality and Higher Order Processes

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Abstract

Some recent advances suggest that studying higher order processes can reveal some new features of quantum mechanics, notably regarding causality. The most famous example arose from the discovery of the quantum switch[16] supermap, which showed that causal ordering of events can, in a quantum setting, be indefinite. In particular, indefiniteness of the causal structure is deemed as a promising way of reuniting the theories of general relativity and quantum mechanics.

In most of the existing literature, processes of order at most 2 are discussed. We will in report introduce a possible way of obtaining third order processes, including some definite and indefinite examples.

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Introduction

In most of the physical theories, one always wants to assume that the cause of an observed fact precedes the observation itself: this is known as the *causality principle*[6]. This principle is closely linked with the theory of relativity[6], and hence, in the longstanding search for a unifying theory of quantum mechanics and general relativity, adopting this principle in quantum theory seems like a reasonable first step. This study of quantum causality has in turn lead to the study of more general quantum processes, namely supermaps. In particular, examples of such superoperators such as the *quantum switch*[16, 13, 38] has shown that causation itself can be quantum! However, the current state of research does seem to mainly restrict to first and second order theory: results about admissibility of higher order maps are known [31, 41], but barely any concrete example of n-order process ($n > 2$) have been established. his project aims at working towards this filling this gap, by making some preliminary observations about third order processes.

The main outcome of this project was to investigate some of the composition rules between supermaps, and hence to successfully obtain examples of third order processes. In particular, I have shown that it is possible, in third order theory, not only to act on the signalling conditions, but also on the structure of processes via newly defined operation $\sqcup$ and $\sqsubseteq$. As a consequence of these new different ways of combining operations, I have explored some of the possibilities to obtain indefinite third order processes.

However, we note that the result presented in this report consists of a collection of different types of third order operations that are allowed; not all types of operations that are possible. A possible extension of this project will then to obtain a more general theory of third order processes.

The first part acts mainly as a review of the concepts in the existing literature that motivates a study of higher order processes. The first chapter gives an overview of quantum mechanics, and in particular introduces categorical quantum mechanics and the **CPM**[**FHilb**] category. The subsequent chapter builds on the described formalism to include the causality principle; some of the consequences of including this principle in the theory is also discussed there. Chapter 3 then introduces second order processes and their link with causality.

The second part is entirely focused on the formalism of higher order processes. The work in the first chapter consists on preliminary facts and observations about general higher order processes; we, in particular, introduce the $Caus[\mathbf{CPM}]$ construction from [31], and the properties of this category. Finally, the last chapter is the core of the project, namely a discussion on concrete third order processes.

Part I

Quantum mechanics and causality

1. The framework

The aim of this chapter is to introduce the theory of quantum mechanics and the main concepts and notation that we will use throughout this project. As a side note, the approach adopted here differs from most of the textbook description of quantum theory[37]. We will in particular prone a rather operational approach of quantum theory, which, as we will see, can conveniently be abstracted using category theory.

1.1 Quantum processes

We will first review the basics of quantum mechanics, as usually described in the context of quantum information theory.

1.1.1 Quantum states and the density matrix

A quantum system is a physical object or system with observable properties which are allowed to be *non-deterministic*, e.g. a photon with a polarisation. Such a system is modelled using complex Hilbert spaces (i.e. vector space with inner product): a *quantum state* is represented as a vector, and transformations on this state are modelled by linear transformations. The standard notation is the so-called Dirac notation, where vectors are denoted as $|\psi\rangle$, and their dual vectors as $\langle\psi|$. For example, a vector in a 2-dimensional Hilbert space spanned by the basis vectors $\{|0\rangle, |1\rangle\}$ can always be written as:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{1.1}$$

with $\alpha, \beta \in \mathbb{C}$.

Remark 1. Note that we will, for the rest of this report, only consider Hilbert spaces with a finite dimension.

In addition, one might want to consider more than one system at once. Given two states $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$ for some Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$, the *joint state* of $|\psi_A\rangle$ and $|\psi_B\rangle$ is defined as:

$$|\Psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \tag{1.2}$$

with the binary operation $\otimes$ being the standard *tensor product*. Note that, in this interpretation, there is no physical difference between the states $|\psi_A\rangle \otimes |\psi_B\rangle$ and $|\psi_B\rangle \otimes |\psi_A\rangle$.

Remark 2. In matrix notation, the tensor product is given by the *Kronecker product*[21].

Now, if vectors represent properties, then we can give a probabilistic interpretation of the coefficients α, β in (1.1): the probability of *measuring* the observable represented by $|0\rangle$ (respectively $|1\rangle$) is given by $|\alpha|^2$ (respectively $|\beta|^2$). This is known as the *Born rule*[37] and takes the form:

Proposition 1 (The Born rule)**.** Given a quantum state $|\psi\rangle$, the probability of measuring the property $|\phi\rangle$ is given by:

$$|\langle\phi|\psi\rangle|^2 = |\langle\psi|\phi\rangle|^2 \tag{1.3}$$

Hence, this imposes a condition on vectors that do represent a physical state. Namely, if a Hilbert space $\mathcal{H}$ is spanned by the orthonormal basis $\{|i\rangle\}$, for any $|\psi\rangle \in \mathcal{H}$ with spectral decomposition:

$$|\psi\rangle = \sum_i a_i |i\rangle \tag{1.4}$$

$|\psi\rangle$ is the representation of a quantum state iff:

$$\sum_i |a_i|^2 = 1 \tag{1.5}$$

The density matrix Given the complex representation of a quantum state, some information is in fact redundant. For example,consider the states $|\psi\rangle$ and $|\psi'\rangle$ s.t.:

$$|\psi'\rangle = e^{i\theta} |\psi\rangle \tag{1.6}$$

where $\theta \in [0, 2\pi)$ is a global phase. Then, we clearly have $|\psi\rangle \neq |\psi'\rangle$ whenever $\theta \neq 0$. However, the Born rule applied to any state $|\phi\rangle$ is given by:

$$\left|\langle\phi|\psi'\rangle\right|^2 = \langle\phi|\psi'\rangle \langle\psi'|\phi\rangle \tag{1.7}$$

$$= e^{i\theta} \langle\phi|\psi\rangle e^{-i\theta} \langle\psi|\phi\rangle \tag{1.8}$$

$$= 1 \cdot \langle\phi|\psi\rangle \langle\psi|\phi\rangle = |\langle\phi|\psi\rangle|^2 \tag{1.9}$$

And since measurements represent what *is* observable, then those two states are physically equivalent.

An elegant way to ignore those global phases is to work with *density operators* rather than states. This object way introduced by von Neumann[36] and Landau[33] in 1927, and is defined as follows:

Definition 1 (Density matrix)**.** Given a quantum state $|\psi\rangle \in \mathcal{H}$ its corresponding *density matrix* is a linear operator $\rho_\psi : \mathcal{H} \to \mathcal{H}$ defined as:

$$\rho_\psi = |\psi\rangle \langle\psi| \tag{1.10}$$

From this definition, we indeed see that all physically distinct states will lead to different density matrices. Now, we also want to have an equivalent notion of probability. From the Born rule, one can easily see that for every states $|\phi\rangle$, $|\psi\rangle$, the probability of projecting $|\psi\rangle$ on the state ϕ is :

$$\langle\phi|\psi\rangle \langle\psi|\phi\rangle = \langle\phi| \rho_\psi |\phi\rangle \tag{1.11}$$

Motivated by this observation, we define *trace operation*:

Definition 2 (Trace)**.** We define the trace of a linear operator $\sigma : \mathcal{H} \to \mathcal{H}$ as:

$$Tr(\sigma) = \sum_i \langle i| \sigma |i\rangle \tag{1.12}$$

where the states $\{|i\rangle\}$ for an orthonormal basis of $\mathcal{H}$.

Similarly, when a state σ_{AB} is in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, we can define the *partial trace* w.r.t. to the subsystem A as :

$$tr_A(\sigma_{AB}) = \sum_{i=1}^{d_A} (\langle i| \otimes id_B)\, \sigma_{AB}\, (|i\rangle \otimes id_B) \tag{1.13}$$

Note that in the matrix representation w.r.t the same basis, the trace operator coincides with the trace defined in linear algebra (namely the sum of the terms in the diagonal).

Remark 3. For every state $|\psi\rangle$ defined as (1.4), we have:

$$\begin{aligned} Tr(\rho_\psi) &= \sum_i \langle i|\psi\rangle \langle\psi|i\rangle \\ &= \sum_i |a_i|^2 = 1 \end{aligned}$$

Mixing and superposition We have seen so far how to encode the superposition of quantum states. What if, on top of the intrinsic probabilistic nature of quantum states, we are *uncertain* about which quantum state a given system is in? More formally, how can one represent a statistical ensemble $\{(p_i, |\psi_i\rangle)\}$ of quantum states, where the $\{p_i\}$ form a probability distribution, and $\{|\psi_i\rangle\}$ is a collection of quantum states?

The solution is to use density matrices. We define the density matrix of a statistical ensemble $\{(p_i, |\psi_i\rangle)\}$ as:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| = \sum_i p_i \rho_i \tag{1.14}$$

where we use the shortcut: $\rho_i \equiv \rho_{\psi_i}$. Assuming that all the p_i's are non-vanishing, we will refer to ρ as a *mixed state* if it represents an ensemble with more than two (physically) distinct states, otherwise, ρ will be called a *pure state.* We will denote by $\mathcal{L}(\mathcal{H})$ the space of linear operators (e.g. density matrices) on $\mathcal{H}$.

Purification/Stinespring dilation There is an alternative way of obtaining mixed states via what is called *purification.*

Definition 3 (Purification[1])**.** A pure state $\widetilde{\rho} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a *purification* of the state $\rho \in \mathcal{L}(\mathcal{H}_A)$ if $\rho = tr_B(\tilde{\rho})$.

We then have the following result:

Theorem 1 (Stinespring dilation[21, 37])**.** *Every mixed state have a purification*

A proof of this result can be found in [37].

We now note that since pure states are trivially purification of themselves, then every quantum state admits a purification. This is referred to as the *purification postulate* in the more general framework of Operational Probabilistic Theories[15].

1.1.2 Quantum operations

We are now interested in a more dynamical description of quantum mechanics, and in particular how to act on quantum states.

For any ensemble $\{(p_i, |\psi_i\rangle \in \mathcal{H})\}$, the corresponding density matrix can be seen as the linear operator $\rho \in \mathcal{L}(\mathcal{H}) : |\phi\rangle \mapsto \sum_i p_i \langle\psi_i|\phi\rangle \cdot |\psi_i\rangle$. Furthermore, from the probabilistic interpretation of quantum states, we also have for every $|\phi\rangle$ a quantum state:

$$\langle\phi| (\rho |\phi\rangle) \geq 0 \tag{1.15}$$

Hence, in linear algebra terms, ρ is a *positive operator.* Going back to quantum operations, we want to impose some constraints on which operations we can or cannot do. A first naive attempt would be to preserve positivity, i.e. a quantum operation should send density matrices to density matrices. Now, recall that we are also allowed to have composite systems, e.g. $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then, if $f : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_{A'})$, then we would also want $(f \otimes id_B)(\rho_{AB})$, i.e. applying f to the system in $\mathcal{H}_A$ *locally* and leaving the system in $\mathcal{H}_B$ unchanged, should also give back a quantum state. To accommodate this situation, we define the concept of *complete positivity.*

[1]The following definition is the definition from [15] applied to the specific case of quantum theory.

Definition 4 (Complete positive maps)**.** A map $f : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is said to be *complete positive* iff for every Hilbert space $\mathcal{H}_C$, and every positive operator $\rho_{AC} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_C)$, we have:

$$(f \otimes id_C)(\rho_{AC})$$

is again a positive operator in $\mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_C)$.

Finally, we want quantum operations to preserve probabilities in the sense that given a (complete) positive operator f acting on density operators in $\mathcal{L}(\mathcal{H})$:

$$Tr(f(\rho)) = 1 = Tr(\rho) \tag{1.16}$$

for all $\rho \in \mathcal{L}(\mathcal{H})$. We can moreover extend the condition and define:

Definition 5 (Quantum channels[14])**.** A complete positive map $f : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is a *quantum channel* if it is *trace-preserving*, i.e. (1.16) is satisfied for every quantum state $\rho \in \mathcal{L}(\mathcal{H}_A)$.

To include the concept of measurements in the framework, one not only has to consider quantum channels, but also complete positive *trace non-increasing* maps[37]. However, measurements are non-deterministic processes: we will, for the rest of this project, restrict to deterministic processes and hence only consider quantum channels.

1.2 The categorical framework

As noted in the seminal papers [2, 3], the previously described process theory allows processes to be composed sequentially, via the standard composition operation $\circ$, or in parallel with the tensor product $\otimes$; hence, it can be described in terms of *monoidal categories*. In these categories, the types are the objects of the category, and the processes are the morphism.

1.2.1 Monoidal categories and the category of Hilbert spaces

We will in this section introduce the concept of monoidal categories, and how they can be used to express the quantum mechanics formalism in a more diagrammatic fashion.

Definition 6 (Monoidal category[34])**.** A *monoidal category* is a category $\mathcal{C}$ equipped with a monoidal structure $(\otimes, I, \alpha, \lambda, \rho)$ where $I \in ob\mathcal{C}$ is called the unit object, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor, α, λ and ρ are natural isomorphisms:

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \tag{1.17}$$

$$\lambda_A : I \otimes A \to A \tag{1.18}$$

$$\rho_A : A \otimes I \to A \tag{1.19}$$

s.t. the following diagrams commute:

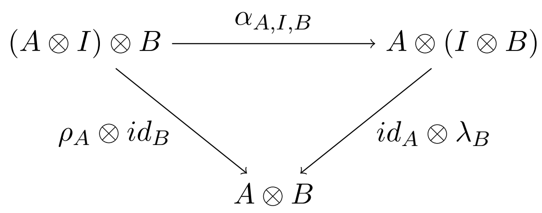


Figure 1.1: Triangle equality

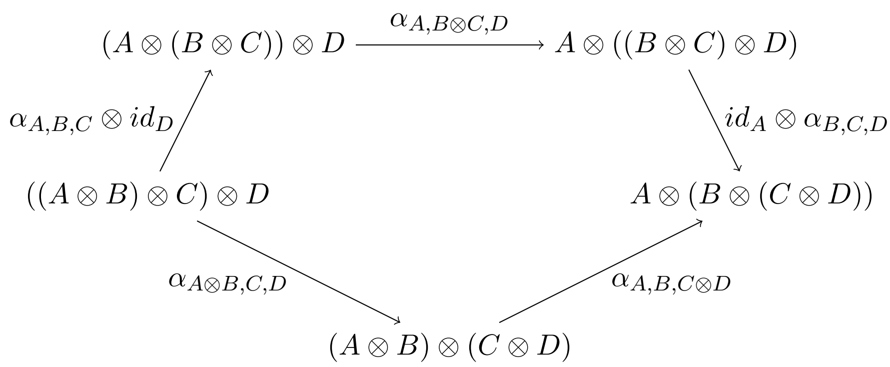


Figure 1.2: Pentagon equality

Monoidal categories also comes with a very powerful graphical language: every object is represented by a wire, and every arrow is represented by a box, and we then defined a "direction of time" (here increasing time goes upward) in order distinguish between input and outputs:

(a) Representation of $A \in ob\mathcal{C}$.

(b) Representation of $f : A \to B \in Hom_{\mathcal{C}}(A, B)$.

Figure 1.3: Graphical representation of objects and arrows.

Composition of morphism is represented by vertical composition:

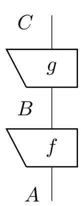


Figure 1.4: Graphical representation of the sequential composition $g \circ f$, with $f : A \to B$, $g : B \to C$.

and the tensor product is represented via horizontal juxtaposition as:

(a) Representation of $A \otimes B$. (b) Representation of $f \otimes g$, with $f : A \to B$ and $g : C \to D$.

Figure 1.5: Graphical representation of the tensor product.

The tensor unit I is on the other hand either omitted, or when necessary represented by a dashed wire:

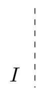
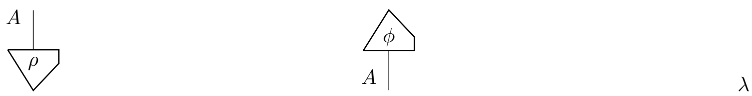


Figure 1.6: Graphical representation of I.

We then define the following families of processes:

Definition 7. In a monoidal category $\mathcal{C}$, a *state* in A, where $A \in ob\mathcal{C}$ is a process $\rho : I \to A$. An *effect* in A is a map $\pi : A \to I$. Finally, if a map is both a state and an effect, i.e. $\lambda : I \to I$, then it is called a *scalar*. These special processes have the following representation:



(a) State. (b) Effect. (c) Scalar.

The main advantage of using the graphical representation is that every (planar) isotopic diagram gives rise to the same expressions in a monoidal category up to isomorphism. This result is known as the *coherence theorem* for monoidal categories[27, 29, 46]:

Theorem 2 (Coherence theorem for monoidal categories)**.** *Isomomorphism relations between morphisms formed from the axiom of a monoidal category only holds iff their graphical representation are equal, up to planar isotopy.*

Remark 4. There several different possible graphical representation. We will in this report use the notation (direction, shape of the boxes, etc.) from [21]. We also note that it is very similar to the one adopted by Selinger[46, 45], or the one used in the context of operational-probabilistic theories[15].

Definition 8 (Strict monoidal categories[27])**.** A monoidal category $\mathcal{C}$ with monoidal structure $(\otimes, I, \alpha, \lambda, \rho)$ is said to be *strict* if the isomorphism α, λ, ρ are identities.

Remark 5. For every monoidal category, there exists an equivalence to a strict monoidal category[27].

We then define the category **FHilb**, where objects are finite dimensional Hilbert spaces, and morphisms are linear maps between Hilbert spaces. We can in addition define the binary operation $\otimes$ given by the standard tensor product, and the "unit" Hilbert space by the space of complex numbers. Then **FHilb** is a strict monoidal category.

Another property of the operation $\otimes$ in the standard quantum mechanics formulation is that swapping systems (e.g. $A \otimes B \to B \otimes A$) does not fundamentally change the meaning of a given equation. The corresponding concept is, in the language of monoidal categories, known as *symmetry*:

Definition 9 (Symmetric monoidal category[46, 27])**.** A monoidal category is symmetric if there exists a natural isomorphism σ with:

$$\sigma_{A,B} : A \otimes B \to B \otimes A \tag{1.20}$$

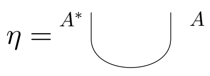
such that:

$$\sigma_{A,B}^{-1} = \sigma_{B,A} \tag{1.21}$$

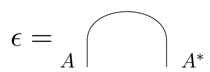
From this, we also conclude that **FHilb** is a symmetric monoidal category.

Duals We then introduce the notion of *duality* between objects. For any monoidal category, we define the (left or right) dual of an object A as:

Definition 10. A^* is said to be the right dual of A (equivalently, A is the left dual of A^*) if there exists morphisms $\eta : I \to A^* \otimes A$ and $\epsilon : A \otimes A^* \to I$ denoted by:

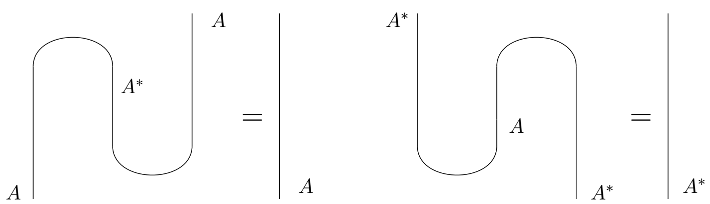


(a) Cup.



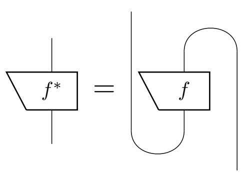
(b) Cap.

satisfying the *snake equations*:



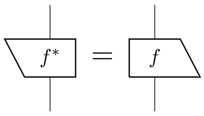
(1.22)

The dual of a morphism is moreover defined as follows:



(1.23)

We will call f^* the *transpose* of f [21], and will adopt the convenient notation of [21]:



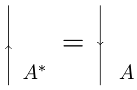
(1.24)

which is obtained by "sliding" the box along the wire.

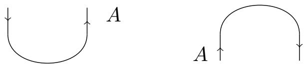
Remark 6. The dual of a given object is not unique.

It will sometimes be useful to add arrows to the wires as:

and define:



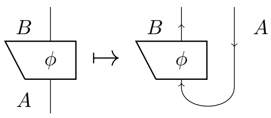
Then, for example, the cups and caps becomes:



Now, for a given Hilbert space $\mathcal{H}$, its dual space in the standard formalism is its right dual; moreover, this means that *every* object in **FHilb** has a right dual: it is *compact closed*.

Definition 11 (Compact closed category[46]). A monoidal category $\mathcal{C}$ is *complact closed* iff it is symmetric and every object has a right dual.

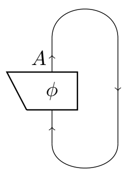
Knowing that every Hilbert space in **FHilb** admits a dual space, we can see that every map $\phi : A \to B$ is isomorphic to a state by defining:



(1.25)

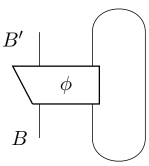
In [21], this is referred to as the *process-state duality*.

In addition, from the definition of η and ϵ, one can see that the *trace* of a given process $\phi : A \to A$ is given by:

$$tr(\phi) = \epsilon \circ \phi \otimes id_A \circ \eta(1) =$$ 

(1.26)

We can similarly define the partial trace $tr_A(\phi)$ for any $\phi : B \otimes A \to B' \otimes A$ as:

$$tr_A(\phi) =$$ 

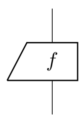
(1.27)

Note that the right dual of a process is corresponds to its (linear algebraic) transpose, not its adjoint. To deal with adjoints in terms of monoidal categories, we need to define the notions of *dagger categories*:

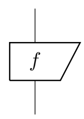
Definition 12 (Dagger monoidal categories[46]). A monoidal category $\mathcal{C}$ is a *monoidal dagger category* if there exists a contravariant functor $(\ \_\ )^\dagger : \mathcal{C} \to \mathcal{C}$ s.t.:

- For every object $A \in \mathcal{C}$, $A^\dagger = A$;
- For every $f : A \to B$ in $Hom_{\mathcal{C}}(A, B)$, $f^\dagger : B \to A$ and $f^{\dagger\dagger} = f$.
- For every maps f and g: $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- The natural isomorphism α, λ, ρ satisfies: $\alpha^\dagger = \alpha^{-1}$, $\lambda^\dagger = \lambda^{-1}$, $\rho^\dagger = \rho^{-1}$.

We then claim that **FHilb** is a dagger monoidal category, and for which the $(\ \_\ )^\dagger$ operation corresponds to taking the adjoint of a linear operator in the standard sense. Note that we can also define the *conjugate* of a map f, which coincides with the notion of conjugate for Hilbert spaces in **FHilb** as the adjoint of the transpose of f. We represent the adjoint and the conjugate of map f as[21]:



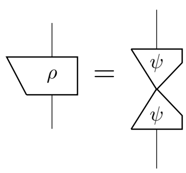
(a) Adjoint.



(b) Conjugate.

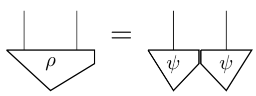
1.2.2 Density operators and the CPM construction

Recall from the previous section, using Hilbert spaces only, one can only express pure quantum states; in order to also have mixed states, one has to use density matrices. Given a pure state $|\psi\rangle$, the corresponding density operator is given by $\rho = |\psi\rangle\langle\psi|$. In term of this new diagrammatic notation, this means:



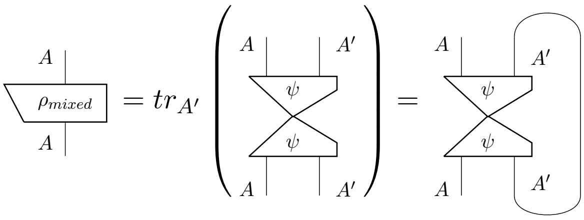
(1.28)

By applying the process-state duality (1.25) on this equation, we obtain a more convenient form:

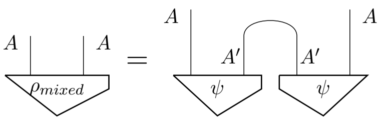


(1.29)

Now, we also know from Theorem 1 that every mixed state admits a purification, i.e.:

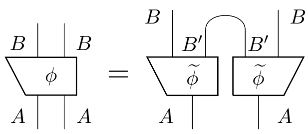


Or alternatively:



(1.30)

In addition, it has been shown in [45] that every complete positive map $\phi : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$ (acting on density operators of the form (1.30)) can be written as:



(1.31)

Motivate by these two examples, we define the *CPM*[2] *construction*[45] as follows:

Definition 13. For a dagger compact closed category $\mathcal{C}$, we define the category of complete positive maps $\mathbf{CPM}[\mathcal{C}]$ as the category for which:

- The objects are the same as the category $\mathcal{C}$
- Morphism $f : A \rightarrow B$ are morphisms in $Hom_{\mathcal{C}}(A \otimes A^*, B \otimes B^*)$ of the form (1.31).

We will denote every objects and processes as:

(a) Objects. (b) Morphisms. (c) States. (d) Effects.

Claim 1. For every dagger compact closed category $\mathcal{C}$, $\mathbf{CPM}[\mathcal{C}]$ is again a dagger compact closed category

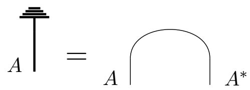
The proof of this result can be found in[45].

We then see that the category $\mathbf{CPM}[\mathbf{FHilb}]$ corresponds to the category for which objects are, as before, finite dimensional Hilbert spaces, and for which morphisms are exactly the complete positive morphism. We will, for simplicity denote $\mathbf{CPM} \equiv \mathbf{CPM}[\mathbf{FHilb}]$.

[2]From Complete Positive Map

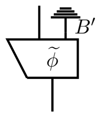
Now, from (1.30) and (1.31), we see that the caps play a special role in the **CPM** construction. We will then represent this special effect differently, and in particular, we define:

Definition 14. For every object A in a dagger compact closed category, we define the *discarding effect* as:



(1.32)

In addition, from (1.29), we define *pure states* as states of the form (1.30) s.t. $A' = I$, and *pure maps* as maps of the form (1.31) with $B' = I$. Hence, every map in **CPM** can be written as:

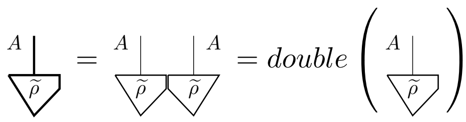


(1.33)

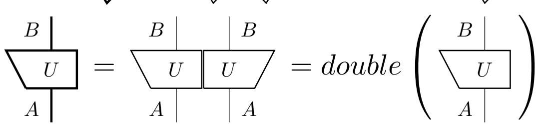
with $\widetilde{\phi}$ a pure map.

Remark 7. The discarding effect corresponds to the state representation of the trace operator; hence, discarding a system or a subsystem effectively corresponds to ignoring the said system or subsystem.

In subsequent chapters, we will sometimes adopt the notation for pure state and processes:

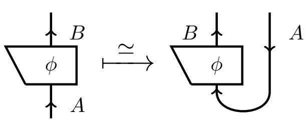


(1.34)



(1.35)

Since the category **CPM** also compact closed, and therefore has duals, one can define the analogue of the process-state duality for the **CPM** category. The mapping:



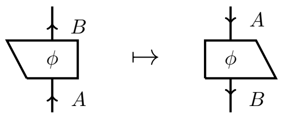
(1.36)

This isomorphism is between is known as the *Choi-Jamiołkowski isomorphism.*[17, 28].

Remark 8. Different choices of duality give rise to either the Choi[17] or Jamiołkowski [28] version of the isomorphism.

We have, in this chapter, introduced the categorical formulation of quantum theory and its associated notation.

However, in this overall description of quantum mechanics, almost no mention of time has been made. Moreover, using compact closedness of **CPM**, one can see that we can easily interchange the inputs as outputs of processes by taking the dual of the map:



In other words, in this formalism, time is symmetric. However, this is emphatically not the case in our experience of the physical world: time is always increasing. We will now investigate the consequence of this time asymmetry on our framework.

2. Causality

In the longstanding search for a unifying theory of quantum gravity, one can start by investigating cause-effect relations between quantum systems. We have, in the previous chapter, described the concepts at the foundation of quantum mechanics. In particular, we have seen that the notions of cause and effects cannot themselves be well-defined in this formalism alone. We now want to add some extra assumptions on top of our existing framework to obtain a causal theory of quantum processes.

We will start by formalising the concept of past, future and signalling relations between events. We will then discuss the notion of causal ordering and give a graph-theoretic representation of causal relations via what we will call causal graphs.

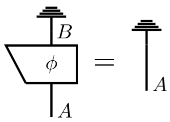
2.1 The Causality principle

In this section, we will focus on a simple question: what makes a process physically realisable? In most physical theories, one assumes what is usually referred to as *causality*[10, 4, 6]. This principle is fairly intuitive: the cause of an event is, for all possible observers, in the past of its effect. The main consequences of this assumption are that space-like trajectories are not in general allowed[6] (no signalling between space-like separated events), and *information* cannot be sent faster than light (no signalling from the future). One also notes that the causality principle does not follow from the relativity postulates[43] per sei [6]; however, it has been shown that non-causal theories that are compatible relativity are not well-behaved[4]. We then want to include the causality principle as part of our framework.

Now, in quantum theory, given a non-separable join state, it is possible to *instantly* affect a subsystem by locally acting on another, no matter how distant the two subsystems are physically separated; the most famous example of such process is the *quantum teleportation protocol* introduced in [12]. Thankfully, this type of communication is by nature not deterministic (i.e. involves quantum measurements); and hence, quantum mechanics does satisfy the causality principle.

We will now see how this translates into the framework described in the previous chapter.

Definition 15 (Admissible[1] map)**.** A map $\phi : A \to B$ is said to be *admissible* iff:



(2.1)

Let us discuss the physical interpretation of (2.1). Translated in words, it reads[21]:

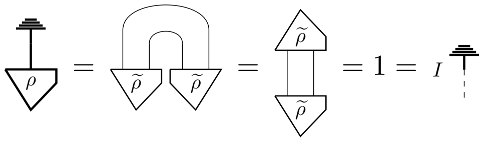
Discarding an admissible map is the same as discarding its input.

or phased differently [31]:

An admissible process can only influence its output.

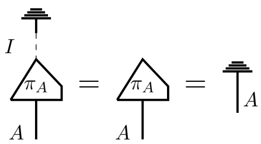
This can be interpreted further in conjunction with the causality principle as the future cannot influence the past[20].

Remark 9. From Chapter 1, we know that for any quantum state ρ, there exists a pure state $\widetilde{\rho}$ s.t.:



This has two consequences, namely that admissible states are exactly normalised states, and discarding the unit tensor I is the same as multiplying by 1, i.e. nothing happens[20].

In addition, there can only be a unique effect satisfying (2.1). To see this, assume that the exists another effect $\pi_A : A \to I$ for all finite dimensional Hilbert spaces A satisfying (2.1). Then, applying (2.1) on π_A itself gives:

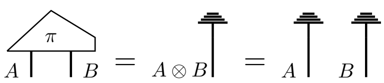


(2.2)

The discarding effect  is then the only admissible (i.e. deterministic) effect[20].

If one restrict to a process theory for which for every object there exists an effect  such that for every map f (2.1) holds, then we say that the process theory is *terminal*[2], since there is, for every object, a unique map to the tensor unit I: I is a terminal object.

Now, uniqueness of the discarding effect in turns implies that for every composite system $A \otimes B$, the discarding effect separates:



(2.3)

[1]These processes are called *causal* in [21, 31]. We will not use this terminology to avoid confusion with the concept of *causal processes* from [10]; or causal theories, e.g. [15]. Instead, this terminology is borrowed [41] and will more naturally extend when dealing with higher order process in later Chapters.

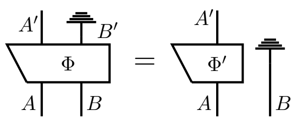
[2]We will adopt the terminology used in [20]. Such a process theory is sometimes called *causal*[15].

2.1.1 Past and causal past

We have so far question addressed the question of signalling from the future; we now want to formalise how one can characterise the other condition, namely no signalling between space-like separated parties. To do this, we will want to investigate the relations between several processes, which we will call *events*.

An event X can be seen as a local region of the space-time in which a transformation occurred, taking an initial state in X to a new state in a system X'. Adopting the notation of [31, 39], we will denote X by its ordered pair of input and output Hilbert spaces: $\mathsf{X} = (X, X')$. We can furthermore consider a global process which can include several events, with possible causal relations between them. A global process with events $\mathsf{A}_1 = (A_1, A_1'), \dots, \mathsf{A}_n = (A_n, A_n')$ can be modelled via a channel $\Phi : A_1 \otimes \dots \otimes A_n \to A_1' \otimes \dots \otimes A_n'$. By also borrowing the terminology from [31] and [39], we formally define the possible signalling correlations.

Definition 16 (One way signalling)**.** Consider the bipartite admissible process $\Phi : A \otimes B \to A' \otimes B'$, where $\mathsf{A} = (A, A')$ and $\mathsf{B} = (B, B')$ corresponds to two distinct events. We say that Φ is *one-way signalling from* A *to* B if:



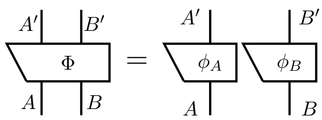
(2.4)

We then write: $\mathsf{A} \preceq \mathsf{B}$.

From this definition, one can see that $\mathsf{A} \preceq \mathsf{B}$ (and $\mathsf{A} \neq \mathsf{B}$) does not exclude the possibility of having $\mathsf{B} \preceq \mathsf{A}$. We then define:

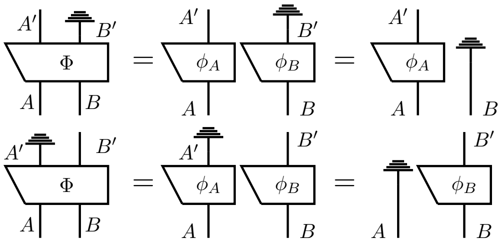
Definition 17 (No-signalling)**.** A bipartite channel Φ defined as before is *no-signalling* iff $\mathsf{A} \preceq \mathsf{B}$ and $\mathsf{B} \preceq \mathsf{A}$.

We see that if the events A and B are no-signalling, then they cannot affect one another; hence, they could be space-like separated. The most obvious examples of no-signalling channels are $\otimes$-separable channels:

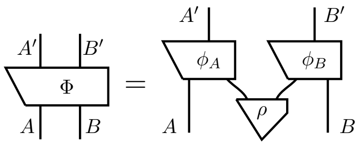


(2.5)

since:



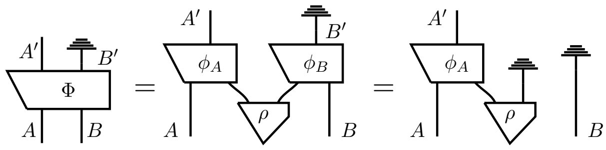
This is however not the only possibility. Suppose that the channel Φ corresponds to the situation where A and B both act independently on a local system each, and some shared state which can be either quantum[3] or classical[4]. The diagram corresponding to this situation is:



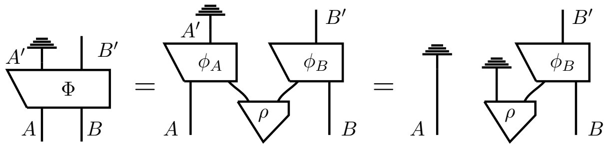
(2.6)

Remark 10. This situation is known as a Bell scenarios[48].

Now, we indeed have:

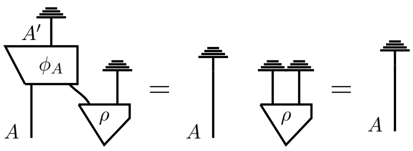


(2.7)



(2.8)

where:

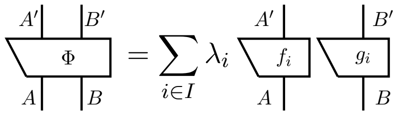


(and similarly for (2.8)). So once again, we conclude that A and B are no-signalling. There are also other admissible maps that satisfies the no-signalling condition, but which cannot be reduced to either one of (2.5) or (2.6)[10, 26]. We however do have a result from [16] (based on a previous result from [26]) regarding the form of general no-signalling channels:

Theorem 3. *Every no-signalling channel can be written as the affine combination of separable channels. In other words, for every no-signalling channel* $\Phi : A \otimes B \to A' \otimes B'$*, there*

[3]This corresponds to Local Operations with Shared Entanglement, or LOSE[26, 48].
[4]This corresponds to Local Operations with Shared Randomness, or LOSR[26, 48].

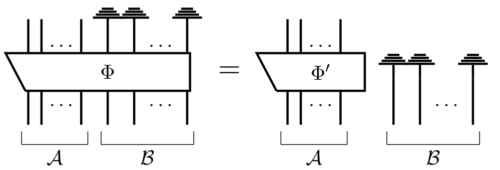
exists channels $\{f_i : A \to A'\}_{i\in I}$, $\{g_i : B \to B'\}_{i\in I}$ and $\{\lambda_i \in \mathbb{R}\}_{i\in I}$ for some indexing set I s.t.:



(2.9)

with $\sum_i \lambda_i = 1$.

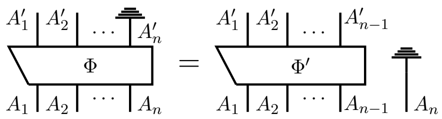
Let's now consider the case where there are more than two events. Suppose we have the set of n events $V = \{\mathsf{A_1}, \mathsf{A_2}, \dots, \mathsf{A_n}\}$, where, as before we have $\mathsf{A_i} = (A_i, A'_i)$ for each $i = 1, \dots, n$, and the joint channel $\Phi : A_1 \otimes \dots \otimes A_n \to A'_1 \otimes \dots \otimes A'_n$. We will adopt definitions from [39, 15]. For any two subsets $\mathcal{A} = \{\mathsf{A}_{a_1}, \dots, \mathsf{A}_{a_k}\}$ and $\mathcal{B} = \{\mathsf{A}_{b_1}, \dots, \mathsf{A}_{b_l}\}$ of V s.t. $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $\mathcal{A} \cup \mathcal{B} = V$, we say that $\mathcal{A} \preceq \mathcal{B}$ if there exists Φ' admissible s.t.:



(2.10)

In particular, the case where $|\mathcal{B}| = 1$ gives rise a special type of process:

Definition 18. We define the *n-partite one-way signalling processes* inductively as the processes Φ s.t.



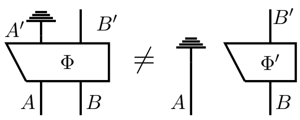
(2.11)

where Φ' is itself an $n-1$-partite one-way signalling process. A bipartite one-way signalling process is a bipartite process satisfying Definition 16.

Let's now focus on the interpretation of the Definitions 16 in more details. In the former, we have $\mathsf{A} \preceq \mathsf{B}$ means that if the output of B is not known or ignored, the input of B cannot affect the output of A: this only means that B is not in the absolute past of A[39], not that communication from A to B has happened.

We also want to consider events that do communicate information to one another. First of all, let's determine what "communicating" means. We want the system entering B to be dependent on the operation A, in other words, that discarding A has a non-trivial effect on B, specifically $\mathsf{B} \not\preceq \mathsf{A}$. We then propose the following definition:

Definition 19 (Causal past)**.** Given a bipartite channel Φ defined as usual, we say that A is in the *causal past* of B iff $\mathsf{A} \preceq \mathsf{B}$, but for all admissible maps Φ', we have:

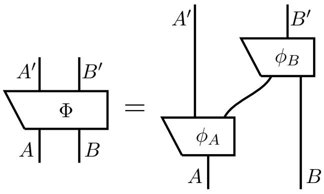


(2.12)

We write $\mathsf{A} \prec \mathsf{B}$.

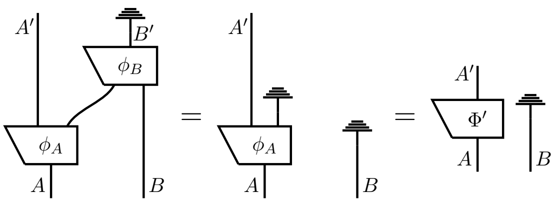
2.1.2 Causality and Localisability

Following this discussion on communication, we would expect bipartite channels satisfying $\mathsf{A} \prec \mathsf{B}$ to be of the form:



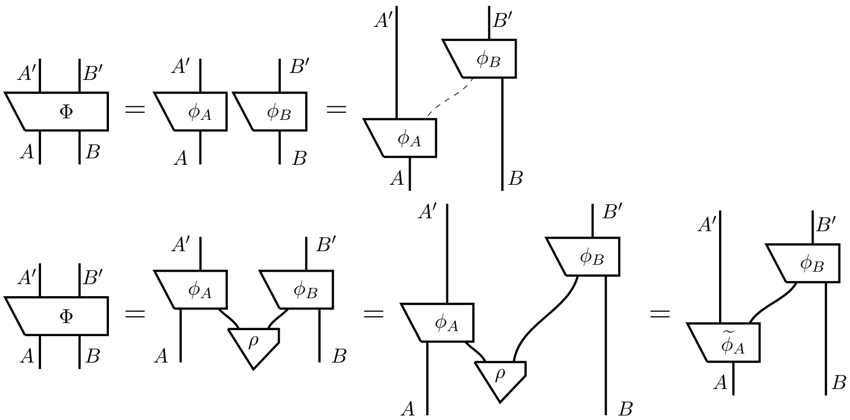
(2.13)

for some admissible maps ϕ_A, ϕ_B. Now, we see that a channel of this form only consists of local operations and communication. Hence, if a bipartite channel Φ decomposes as (2.13), we say that Φ is *semi-localisable*[10, 24]. It is easy to see that every semi-localisable channel is one-way signalling since:

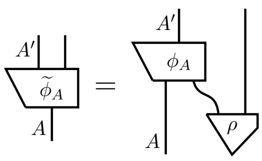


with Φ' admissible since ϕ_A is itself admissible.

Moreover, we also note that the previously discussed examples of no-signalling channels also admit such a decomposition:

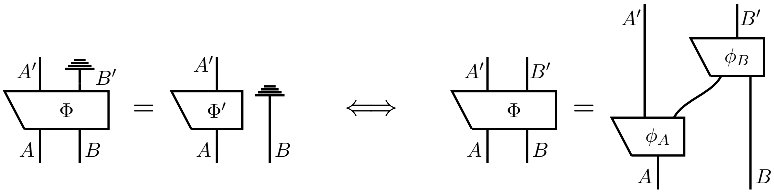


where:



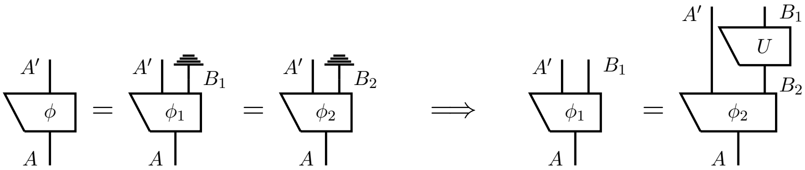
It then seems that every one way signalling channel $\mathsf{A} \preceq \mathsf{B}$ admits a decomposition of the form (2.13). And indeed the following result has been proved in [24]:

Theorem 4. *A bipartite channel is one way signalling iff it is semi-localisable, i.e.*



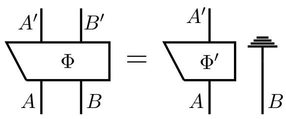
for some admissible maps $\phi_A : A \to A'$, $\phi_B : B \to B'$.

Proof. (This is a sketch of the proof in [24]).
The proof of this result is based on the fact that given a complete positive channel in a Hilbert space, its Stinespring dilation is unique up to unitary transformation[5], i.e.:

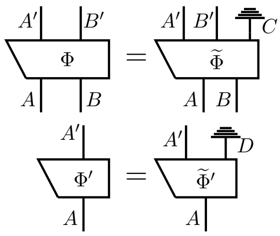


(2.14)

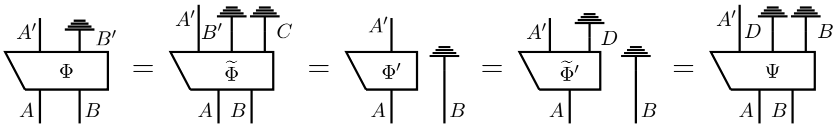
for some unitary U. Now, suppose that we have a bipartite channel Φ s.t.:



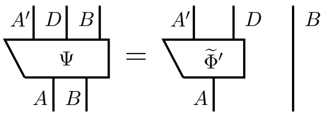
Then, both Φ and Φ' admits a Stinespring dilation:



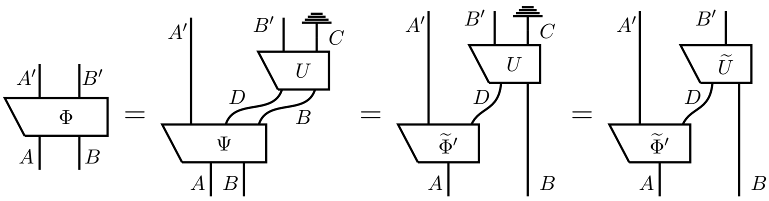
Then, the one-way signalling condition becomes:



where:



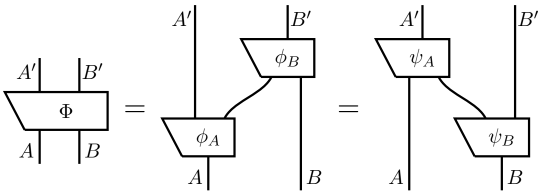
Then, by uniqueness of the Strinespring dialtion, this implies that there exists a unitary U s.t.:



[5]We will admit this result without proof. The full proof can however be found in [47].

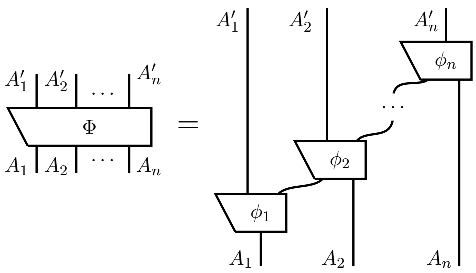
And since we have already proved that every semi-localisable channel is one-way signalling, then the result follows. □

The consequence of this result is that in some special cases, we can obtain a factorisation of processes in terms of local operations and communication, by only knowing the signalling conditions. For example, every no-signalling (bipartite) channels admit two decompositions:



(2.15)

where $\phi_A, \phi_B, \psi_A, \psi_B$ are all admissible. Another example is that every n-partite one-way signalling channel can be decomposed as:



(2.16)

where since again all the ϕ_i's are admissible.

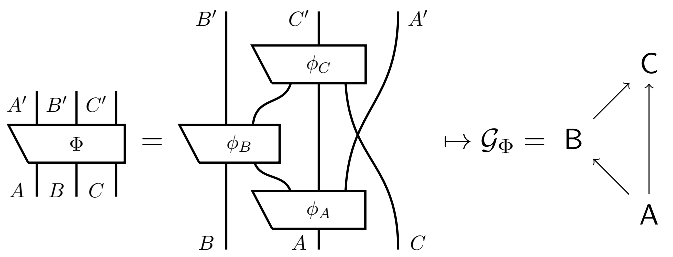
2.2 Causal orders and Causal graphs

We will see in this section how we can represent the different signalling relations in channels with more than two localisable events. We will in particular consider a very simple type of channel:

- The channel $\Phi : A_1 \otimes \ldots \otimes A_n \to A'_1 \otimes \ldots \otimes A'_n$ is associated to the set events $V = \{\mathsf{A}_1 = (A_1, A'_1), \ldots, \mathsf{A}_n = (A_n, A'_n)\}$, with $|V| = n$ is finite;
- Every event $\mathsf{A}_i = (A_i, A'_i)$ is associated to a unique operation $\phi_i : A_i \otimes K_i \to A'_i \otimes K'_i$ (and identities);
- The global channel Φ has a well-defined circuit representation, which includes the local events and communication wires only.

If this is the case, we say that Φ has a *definite causal structure.*

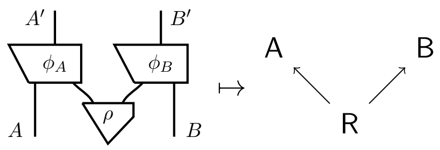
If the circuit representation is known, then we can associate the channel with a directed graph generated as follows: for every (local) event we associate a node, and we add an arrow from the node A to B if there is a wire connecting the output of A to the input of the event associated with B. For example:



(2.17)

$\mathcal{G}_\Phi$ will be referred to as the *causal graph* of Φ.

Note that the input or output Hilbert spaces of each event are allowed to be trivial. Hence, we will assume that all the internal (localised) operations are depicted by a node, e.g.:



(2.18)

where $\mathsf{R} = (I, I)$. We will in addition, for simplicity, not consider purely no-signalling operations, i.e. two events will be considered as no-signalling iff they are $\otimes-$separable or they share some randomness or entanglement. Hence, two events A and B cannot affect one another iff there are $\otimes-$separated in the circuit representation, and appear as two unconnected nodes in the causal graph.

As a directed graph, $\mathcal{G}_\Phi$ is an object of the category **Grph**. Now, we know that there exists a free functor[35]:

$$X : \mathbf{Grph} \to \mathbf{Cat} \tag{2.19}$$

then, every circuit with local events gives rise to a (small) category: $X\mathcal{G}_\Phi$ s.t.:

- Objects are exactly the set of nodes V, i.e. the set of local events in Φ;
- For every $\mathsf{A} \neq \mathsf{B}$, we have:

  $$Hom_{X\mathcal{G}_\Phi}(\mathsf{A}, \mathsf{B}) = \{\mathcal{G}_\Phi\text{-paths from } \mathsf{A} \to \mathsf{B}\}$$

  and $Hom_{\mathcal{G}_\Phi}(\mathsf{A}, \mathsf{A}) = \{id_\mathsf{A}\}$ for every object A.

Note that since we do not admit loops in circuits, then if there exists a paths $\mathsf{A} \to \mathsf{B}$ in $\mathcal{G}_\Phi$, then there is no paths $\mathsf{B} \to \mathsf{A}$, i.e.:

$$Hom_{X\mathcal{G}_\Phi}(\mathsf{A}, \mathsf{B}) \neq \emptyset \implies Hom_{X\mathcal{G}_\Phi}(\mathsf{B}, \mathsf{A}) = \emptyset \qquad \forall \mathsf{A} \neq \mathsf{B}$$

Similarly, there cannot be an edge in $\mathcal{G}_\Phi : \mathsf{A} \to \mathsf{A}$; then every edge in the causal graph does give rise to an arrow in $X\mathcal{G}_\Phi$. Now, one observes that we almost have the structure of a partial order; however, given two objects A and B, the homset $Hom_{X\mathcal{G}_\Phi}$ can contain more that one element. For example if the causal graph of a channel Φ is given by:

$$\mathcal{G}_\Phi = \begin{array}{ccccc} & & \mathsf{D} & & \\ & f_3 \nearrow & & \nwarrow f_4 & \\ \mathsf{B} & & & & \mathsf{C} \\ & f_1 \nwarrow & & \nearrow f_2 & \\ & & \mathsf{A} & & \end{array} \tag{2.20}$$

with the edges labelled, then the corresponding category $X\mathcal{G}_\Phi$ will have:

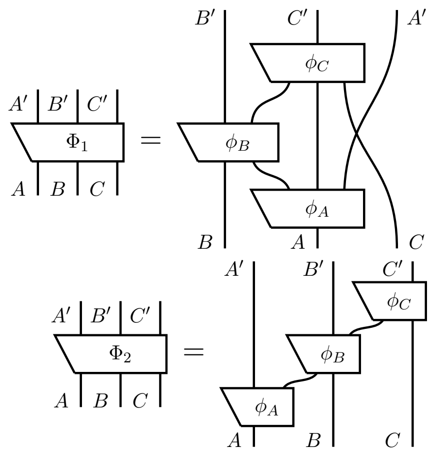
$$Hom_{X\mathcal{G}_\Phi}(\mathsf{A}, \mathsf{D}) = \{f_3 \circ f_1, f_4 \circ f_2\}$$

Following this remark, we define the quotient category[35] $P_\Phi = X\mathcal{G}_\Phi/_\sim$ with the equivalence relation $\sim$ defined as:

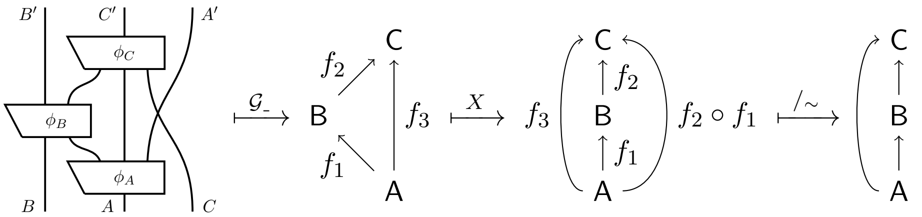
$$\forall \mathsf{A}, \mathsf{B} \text{ s.t. } Hom_{X\mathcal{G}_\Phi}(\mathsf{A}, \mathsf{B}) \neq \emptyset.\ \forall f, g \in Hom_{X\mathcal{G}_\Phi}(\mathsf{A}, \mathsf{B}) : f \sim g \tag{2.21}$$

Here, P_Φ is by construction a preorder, and moreover, from the previous observations, we have P_Φ is a partial order, i.e. $P_\Phi \in \mathbf{Pos}$. The poset P_Φ will be called here the *causal order* of the channel Φ.

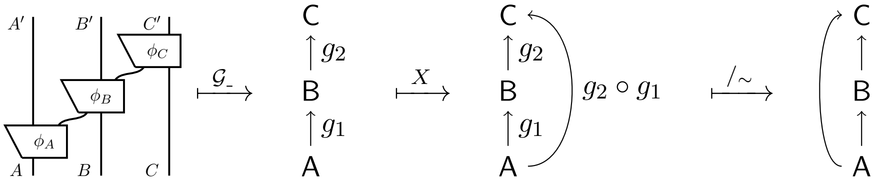
Remark 11. Different circuits can lead to the same partial order. For example, consider the two circuits:



Then, we can construct P_{Φ_1} and P_{Φ_2} as:



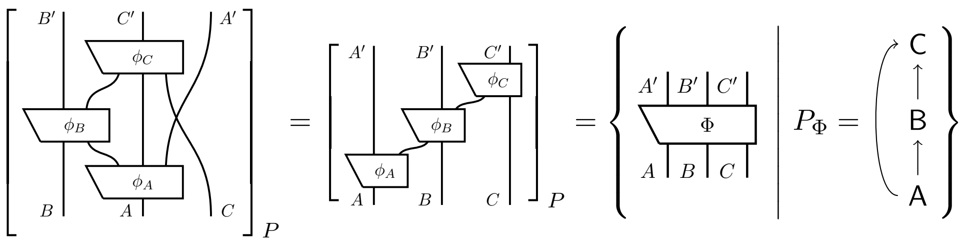
(2.22)



(2.23)

Note that in (2.22) and (2.23), the categories $X\mathcal{G}_{\Phi_{1,2}}$ and $P_{\Phi_{1,2}}$ are depicted as a arrows between objects (identities are omitted); and in particular labels are not needed for $P_{\Phi_{1,2}}$ since all morphisms are unique.

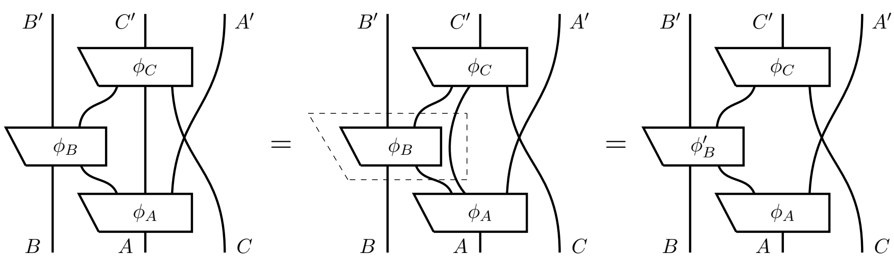
We can then define an equivalence class:



(2.24)

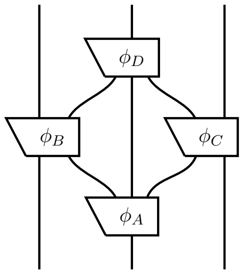
of all the circuits giving the same partial order under this construction.

Furthermore, we can show that all circuits in a given equivalence class have intrinsically the same structure. For example in the causal graph in (2.22), the arrow $f_3 : \mathsf{A} \to \mathsf{C}$ is redundant since there is already, by transitivity, signalling from A to B via the composition $f_2 \circ f_1$. Moreover, we have:



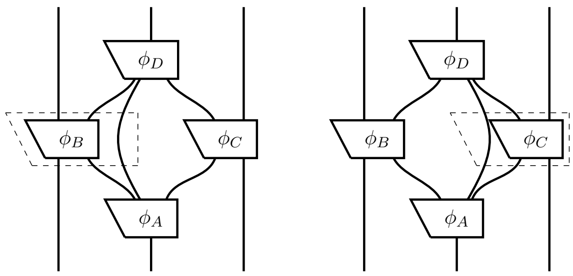
(2.25)

Hence, we can always incorporate the redundant communication wires with intermediate local operations. Note that this "merging" is not unique for example, given:



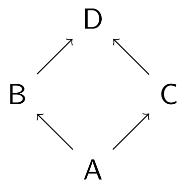
(2.26)

the following two reorganisation of the local events are distinct, but equally valid:



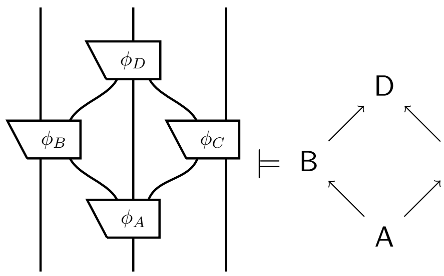
(2.27)

However, we also note that they both give rise to the same causal graph:



(2.28)

And deleting any edge on this reduced causal graph will result in a different partial order P_Φ. We will say that such a causal graph is *minimal*. We can see that all circuits admits such a minimal graph; we will write[6], e.g.:



(2.29)

Definition 20. Given a process Φ with set of events V and causal order P_Φ, we define the *causal past* of an event $\mathsf{A} \in V$ as the set:

$$\mathsf{A}_\downarrow = \{\mathsf{X} \in V \mid \exists P_\Phi\text{-arrow: } \mathsf{X} \to \mathsf{A}\} \tag{2.30}$$

The *causal future* $\mathsf{A}_\uparrow$ of A is defined the same way, but with the opposite partial order P_Φ^{op}.

[6]The notation is similar to [31]

Remark 12. The set $\mathsf{A}_{\downarrow}$ corresponds the downwards closure of $\{\mathsf{A}\}$ of P_Φ; similarly, the set $\mathsf{A}_{\uparrow}$ corresponds to the upward closure of $\{\mathsf{A}\}$.

2.2.1 Causal order and signalling conditions

We have now defined a partial order for any channel Φ which admits a causal graph. We will now investigate how we can relate this ordering with the signalling relation defined in Section 2.1.1.

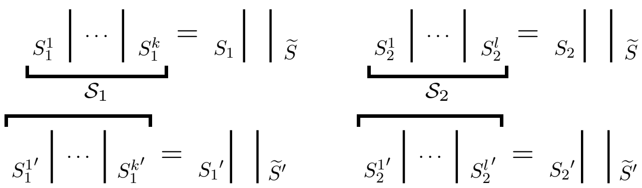
Given a channel Φ and set of events V, we define for every event $\mathsf{A} \in V$ the *future of* A:

$$\mathcal{F}(\mathsf{A}) = \bigcup_{\mathcal{S} \subseteq V - \{\mathsf{A}\}} \{\mathcal{S} \mid V - \mathcal{S} \preceq \mathcal{S}\} \tag{2.31}$$

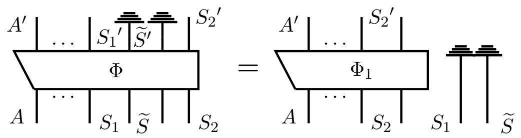
where $V - \mathcal{S} \preceq \mathcal{S}$ is defined as in (2.10).

Claim 2. If $\mathcal{S}_1 \in \mathcal{F}(\mathsf{A})$ and $\mathcal{S}_2 \in \mathcal{F}(\mathsf{A})$, then $\mathcal{S}_1 \cup \mathcal{S}_2 \in \mathcal{F}(\mathsf{A})$.

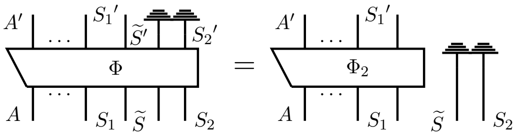
Proof. For simplicity, we will write:



with $\widetilde{\mathcal{S}} = \bigotimes\{\mathsf{B} \in V | \mathsf{B} \in \mathcal{S}_1 \cap \mathsf{S}_2\}$. Then, the assumptions $\mathcal{S}_1 \in \mathcal{F}(\mathsf{A})$ and $\mathcal{S}_2 \in \mathcal{F}(\mathsf{A})$ reads:

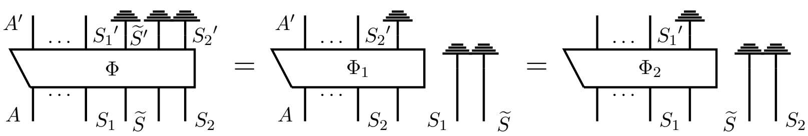


(2.32)



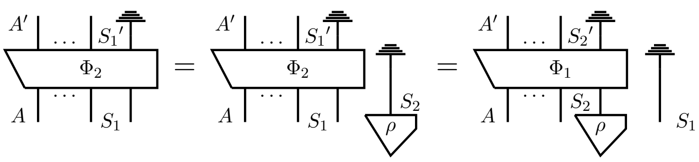
(2.33)

Then, we have:



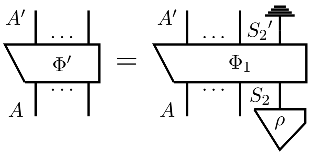
(2.34)

Now, we have for some admissible state $\rho : I \to S_2$:

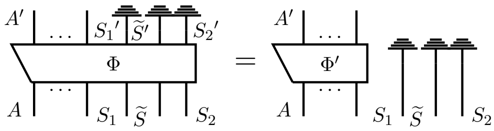


(2.35)

Let's define:



Then Φ' is admissible. Hence, by using (2.34) and (2.35), we obtain:



(2.36)

Hence, we indeed have $\mathcal{S}_1 \cup \mathcal{S}_2 \in \mathcal{F}(\mathsf{A})$. $\square$

Now, a few consequences of this claim is that $\mathcal{F}(\mathsf{A})$ does indeed satisfy $V - \mathcal{F}(\mathsf{A}) \preceq \mathcal{F}(\mathsf{A})$ and is moreover the *largest subset of* $V - \{\mathsf{A}\}$ satisfying this property.

Now, we introduce further definitions:

Definition 21. For any event A in the channel Φ, we define the set of events *no-signalling* to A:

$$\mathcal{N}(\mathsf{A}) = \{\mathsf{B} \in \mathcal{F}(\mathsf{A}) \mid \mathsf{A} \notin \mathcal{F}(\mathsf{B})\} \tag{2.37}$$

We also define the *causal future* of A as:

$$\mathcal{F}_c(\mathsf{A}) = \mathcal{F}(\mathsf{A}) - \mathcal{N}(\mathsf{A}) \tag{2.38}$$

In addition, we will introduce the notation:

$$\mathsf{A} \preceq_\Phi \mathsf{B} \iff \mathsf{B} \in \mathcal{F}(\mathsf{A}) \vee \mathsf{A} = \mathsf{B} \tag{2.39}$$

$$\mathsf{A} \prec_\Phi \mathsf{B} \iff \mathsf{B} \in \mathcal{F}_c(\mathsf{A}) \vee \mathsf{A} = \mathsf{B} \tag{2.40}$$

From this definition, we see that if $\mathsf{A} \neq \mathsf{B}$, then $\mathsf{A} \prec_\Phi \mathsf{B} \implies \mathsf{B} \not\prec_\Phi \mathsf{A}$. In addition, we will prove the following result:

Proposition 2. The operation $\prec_\Phi$ is transitive, i.e. $\mathsf{A} \prec_\Phi \mathsf{B}$ and $\mathsf{B} \prec_\Phi \mathsf{C}$ implies that $\mathsf{A} \prec_\Phi \mathsf{C}$.

Before proving this, we introduce an intermediate result:

Lemma 1. If $\mathsf{A} \prec_\Phi \mathsf{B}$, then:

$$\mathcal{F}(\mathsf{B}) \subseteq \mathcal{F}(\mathsf{A}) \tag{2.41}$$

Proof. If $\mathsf{A} = \mathsf{B}$, the statement is trivial. We will then assume $\mathsf{A} \neq \mathsf{B}$.

Since $\mathsf{A} \prec_\Phi \mathsf{B}$, then we know $\mathsf{A} \notin \mathcal{F}(\mathsf{B})$, in other words, we have $\mathcal{F}(\mathsf{B}) \subseteq V - \{\mathsf{A}\}$.

In addition, since we know that $V - \mathcal{F}(\mathsf{B}) \preceq \mathcal{F}(\mathsf{B})$, then the result follow from the definition of $\mathcal{F}(\mathsf{A})$. $\square$

We can now prove Proposition 2:

Proof. (Proposition 2.)
Note that if $\mathsf{A} = \mathsf{B}$ or $\mathsf{B} = \mathsf{C}$, then the result is once again trivial, so we will assume here $\mathsf{A} \neq \mathsf{B}$. If $\mathsf{A} \prec_\Phi \mathsf{B}$ then we know from Lemma 1 that:

$$\mathcal{F}(\mathsf{B}) \subseteq \mathcal{F}(\mathsf{A})$$

Now, since $\mathsf{B} \prec_\Phi \mathsf{C}$, then:

$$\mathsf{C} \in \mathcal{F}(\mathsf{B}) \implies \mathsf{A} \preceq_\Phi \mathsf{C}$$

In addition, since $\mathsf{B} \prec_\Phi \mathsf{C}$ we also have $\mathcal{F}(\mathsf{C}) \subseteq \mathcal{F}(\mathsf{B})$, and $\mathsf{A} \notin \mathcal{F}(\mathsf{B})$. So we get:

$$\mathsf{A} \preceq_\Phi \mathsf{C} \wedge \mathsf{A} \notin \mathcal{F}(\mathsf{C}) \iff \mathsf{A} \prec_\Phi \mathsf{C}$$

□

It then follows that $\prec_\Phi$ does define a partial order. Recall that if Φ has a "nice" form, namely that it admits a causal graph, we can define another partial order P_Φ; we then want to link those two ordering notions.

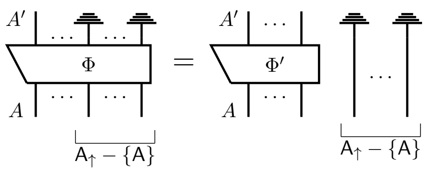
Theorem 5. *If a channel Φ admits a causal graph, then the two previously discussed partial orders coincide.*

Proof. We want to show:

$$\mathsf{A} \prec_\Phi \mathsf{B} \iff Hom_{P_\Phi}(\mathsf{A}, \mathsf{B}) \neq \emptyset \tag{2.42}$$

If $\mathsf{A} = \mathsf{B}$, then the above statement follows from the definition of the two partial orders.

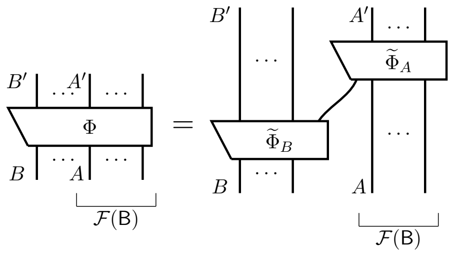
Let's now assume that $\mathsf{A} \neq \mathsf{B}$. We first show that if there is a path in $\mathcal{G}_\Phi$ from $A \to B$, then $\mathsf{A} \prec_\Phi \mathsf{B}$. It is quite easy to see that if a channel Φ admits a circuit representation with associated causal order P_Φ, then every event in $\mathsf{A}_\uparrow$ cannot affect the process associated with A. Indeed, adopting an argument similar as in [30], if $\mathsf{B} \in \mathsf{A}_\uparrow$, then there exists a path in $\mathcal{G}_\Phi$ from A to B, which characterises a sequence of processes connecting the output of A with the input of B. Now, since $\mathsf{A}_\uparrow$ is moreover the upward closure of A w.r.t P_Φ, this means that $\mathsf{A}_\uparrow$ corresponds to all the processes for which the output of A has an effect on. In particular, this means:



(2.43)

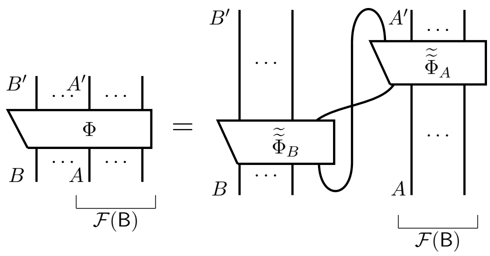
for some admissible process Φ'. Hence $\mathsf{A}_{\uparrow} - \{\mathsf{A}\} \subseteq \mathcal{F}(\mathsf{A})$.

In addition, suppose that there is a P_Φ arrow $\mathsf{A} \to \mathsf{B}$, but $\mathsf{A} \in \mathcal{F}(\mathsf{B})$. Since $\mathsf{A} \in \mathcal{F}(\mathsf{B})$, then the process Φ fatorises as follows:



(2.44)

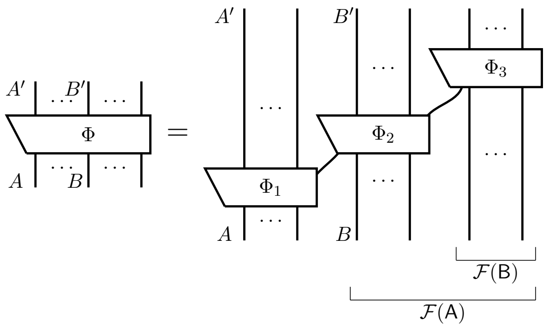
Now, using the assumption that there is indeed a path from A to B, this implies that Φ has to decompose further as:



(2.45)

which is not an admissible process. Hence, we need to have $\mathsf{A} \notin \mathcal{F}(\mathsf{B})$. We can then conclude that if there is an arrow in P_Φ from $\mathsf{A} \to \mathsf{B}$, this implies that $\mathsf{A} \prec_\Phi \mathsf{B}$.

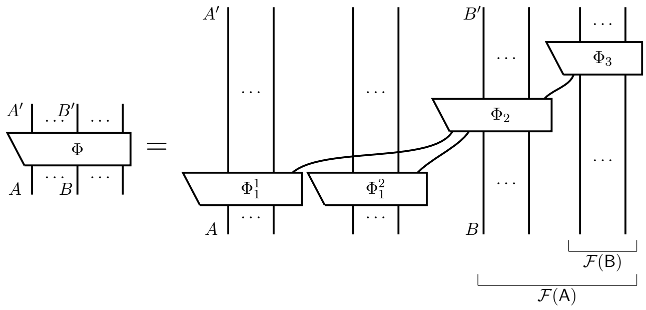
Now, if we instead suppose that $\mathsf{A} \prec_\Phi \mathsf{B}$, we want to show that there exists a arrow in P_Φ from $\mathsf{A} \to \mathsf{B}$, assuming once again the case $\mathsf{A} \neq \mathsf{B}$. From Lemma 1, we know that $\mathcal{F}(\mathsf{B}) \subseteq \mathcal{F}(\mathsf{A})$. Hence, Φ decomposes as:



(2.46)

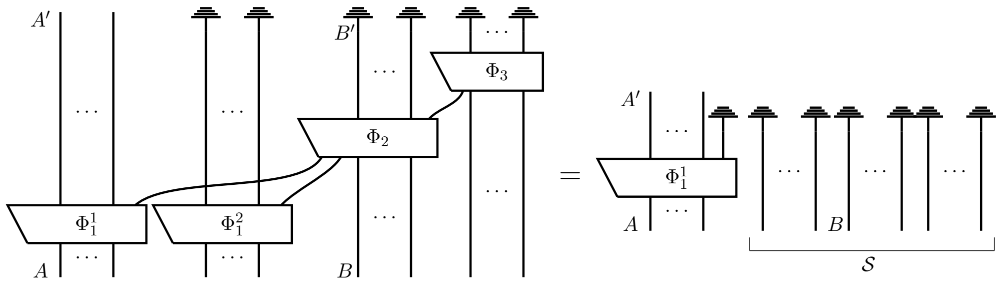
and in particular, there cannot be a path in $\mathcal{G}_\Phi$ from B to A. Now, if there is no arrow in P_Φ from A to B, this then implies that A and B are $\otimes$-separated. Hence, the process Φ_1

above has to separate as follows:



(2.47)

Then:



(2.48)

Which means that there exists a set $\mathcal{F}(\mathsf{A}) \subseteq \mathcal{S} \subseteq V - \{\mathsf{A}\}$ s.t. $V - \mathcal{S} \preceq \mathcal{S}$. So by definition of $\mathcal{F}(\mathsf{A})$, this implies: $\mathcal{F}(\mathsf{A}) \subseteq \mathcal{S} \subseteq \mathcal{F}(\mathsf{A})$. Hence, $\mathcal{S} = \mathcal{F}(\mathsf{A})$, and Φ_1^2 is trivial (i.e. does not act on any event in V).Note that this is true even if the outputs of Φ_1^2 are trivial. Hence, we have $\mathsf{A} \prec_\Phi \mathsf{B}$ implies that there is an arrow in P_Φ from $\mathsf{A} \to \mathsf{B}$. Which concludes the proof. □

We will then says that $P_\Phi = (V, \prec_\Phi)$. Besides, when no confusion can be made between the respective relations $\preceq$ and $\prec$ and $\preceq_\Phi$ and $\prec_\Phi$ respectively, we will replace the former by the latter for simplicity.

We can also extend the definitions of the future and past w.r.t. the relation $\prec_\Phi$ for any subset $\mathcal{A} \subseteq V$:

$$\mathcal{A}_\downarrow = \{\mathsf{X} \in V | \exists \mathsf{A} \in \mathcal{A}.\ \mathsf{X} \prec_\Phi \mathsf{A}\} \tag{2.49}$$

$$\mathcal{A}_\uparrow = \{\mathsf{X} \in V | \exists \mathsf{A} \in \mathcal{A}.\ \mathsf{A} \prec_\Phi \mathsf{X}\} \tag{2.50}$$

We have now formalised the causality principle and studied the main repercussions on imposing it to our description of quantum theory. We have in addition introduced a way of representing the causal relations between localised events in a directed graph. Moreover, since we are not allowing causal loops, the graph $\mathcal{G}_\Phi$ has the structure of a Directed Acyclic Graph (DAG).

This type of graphs is also used in Pearl's celebrated formalism of causal models described in [40]. Some work has been done in extending causal models to *quantum causal models*. Examples of such models can be found in [9, 23, 5].

3. Second order processes and causal ordering

We will keep focusing on channels with localisable events. However, some recent advances have shown that one can not only express causal relations in terms of second order operations, but using supermaps also gives rise to a more exotic type of channels. We will notably describe the example of the quantum switch which has raised the interest of the quantum community since the release of its concept in [16].

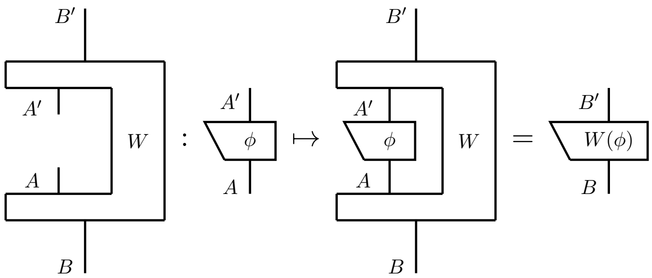
3.1 Supermaps

We have been, so far, been dealing with quantum maps, which we recall are sending quantum states to quantum states. Hence, if we view quantum states as first order systems, then quantum maps (first order processes) are second order systems. We can then extend our theory to include second order processes by defining:

Definition 22. A *supermap*[16] (or *superoperator*) is an operation that sends admissible processes to admissible processes.

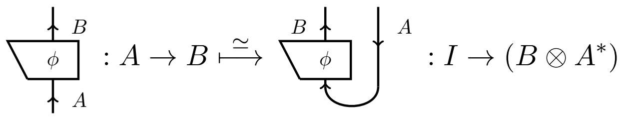
3.1.1 Picturing supermaps

As before, we are interested in the graphical notation of processes. We will then denote the superoperators as:



(3.1)

Now, we have seen in Section 1.2 that the **CPM** category do have duals, and in particular, every map is isomorphic to a state via the Choi-Jamiołkowski isomorphism:



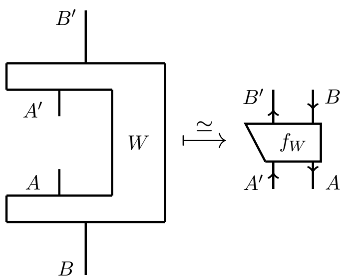
Then, if we denote by $(A \to B)$ the set of all maps (not taking admissibility into account) from the Hilbert space A to the Hilbert space B, then we have:

$$(A \to B) \simeq B \otimes A^* \simeq A^* \otimes B \tag{3.2}$$

and a superoperator W defined as in (3.1) lies in the space $(A \to A') \to (B \to B')$. Then, there are two obvious ways of using (3.2) to "collapse" a second order process into a first order process. If we were to apply (3.2) on the inner brackets, we obtain:

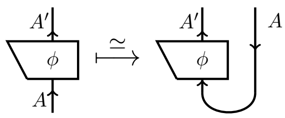
$$(A \to A') \to (B \to B') \simeq (A' \otimes A^*) \to (B' \otimes B^*) \tag{3.3}$$

which will give the following graphical representation:



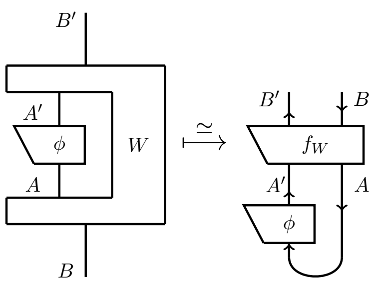
(3.4)

The interpretation of this process is fairly intuitive, given a map $\phi : A \to A'$ with corresponding Choi-Jamiołkowski representation:



(3.5)

The image of ϕ under W will then have the Choi-Jamiołkowski representation:



(3.6)

The main drawback of this representation is that the arrows and order of the wires are critical to the representation.

We can on the other hand use (3.2) on the "middle arrow" as:

$$(A \to A') \to (B \to B') \simeq (B \to B') \otimes (A \to A')^* \tag{3.7}$$

Then there are a few things to note;. Using compactness of **CPM**:

$$\begin{aligned}(A \to A')^* \simeq & (A' \otimes A^*)^* \\ \simeq & A \otimes A'^* \\ \simeq & (A' \to A)\end{aligned} \tag{3.8}$$

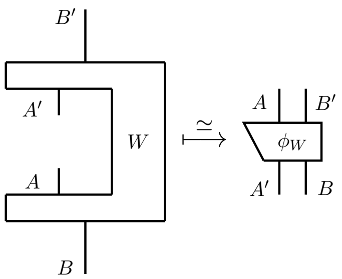
So by associativity of the monoidal tensor$\otimes$, and symmetry of **CPM**:

$$\begin{aligned}(A \to B) \otimes (C \to D) \simeq & B \otimes A^* \otimes D \otimes C^* \\ \simeq & B \otimes D \otimes A^* \otimes C^* \\ \simeq & (A \otimes C) \to (B \otimes D)\end{aligned} \tag{3.9}$$

Hence, (3.7) leads to:

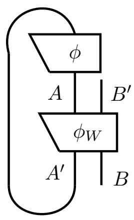
$$(A \to A') \to (B \to B') \simeq (A' \otimes B) \to (A \otimes B') \tag{3.10}$$

In diagrams:



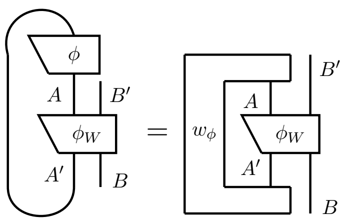
(3.11)

We now note that we do not need arrow in this representation, and the image of ϕ under W is obtained via:



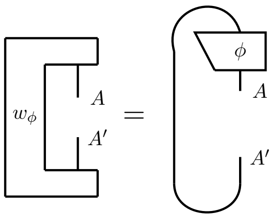
(3.12)

Remark 13. This last diagram can be seen as:



(3.13)

where:

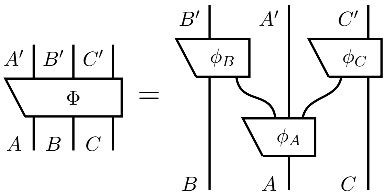


(3.14)

is a supermap that is *not* admissible. Hence, this type of diagram deformation invert the shapes of processes and supermaps in **CPM**.

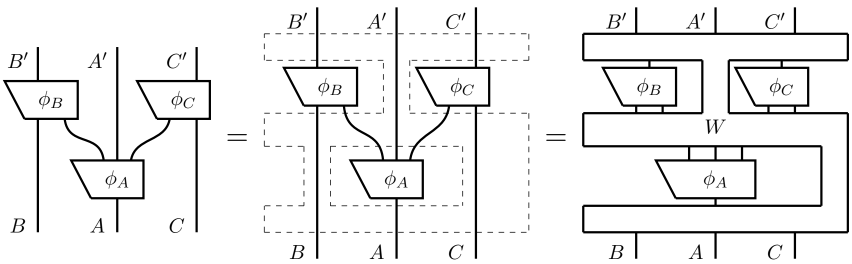
3.1.2 Causal orders and second order processes

We will now investigate how supermaps can be used to study causal relations between events. Consider once again a n-partite channel which admits a causal graph, for example:



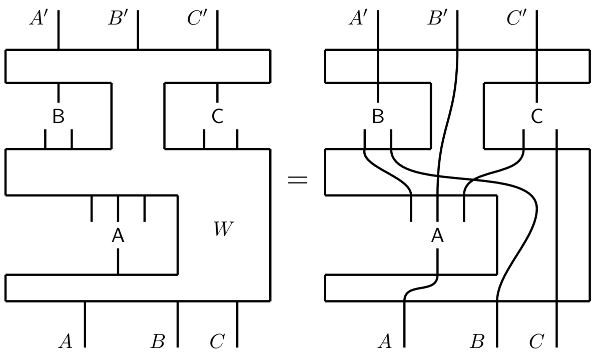
(3.15)

Then, this can be seen as a the output of a second order process acting on the local events as:



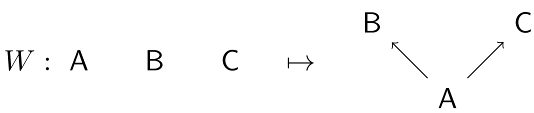
(3.16)

where:



(3.17)

We see that W defined as above sends three no-signalling processes to a process of the required causal graph:

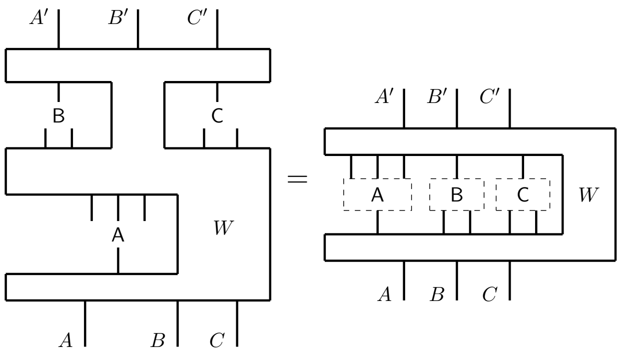


(3.18)

Remark 14. The above causal graph will be referred to as the *external causal graph* in Chapter 4 onward.

In particular, if the superoperator W gives rise a minimal causal graph $\mathcal{G}_{\Phi,min}$ given any admissible input, then the set of outputs of W corresponds to the equivalence class $[\mathcal{G}_{\Phi,min}]_P$. We then see that such a supermap stores all the information about the causal relations between events.

Remark 15. Regarding the notation for second order processes, we will, depending on which diagram is more readable either "separate" the different local operations to emphasise the no-signalling condition between the inputs and their resulting causal order if known (as in (3.17)), or denote the conditions regarding the input processes as dashed boxes, e.g.:



(3.19)

3.2 Indefinite causal structure

The second order processes that we have previously seen do not seem to add anything to the standard first order quantum theory. The reason why is that the previously described supermaps admits a circuit representation, and for that matter, does not even depend on the quantum nature of the involved maps. However, it has recently been shown, notably in [16, 13, 38], that not all channels with localised events can be represented as a circuit. If it is the case, we say that the channel has an *indefinite causal structure*. The main example in the literature arises from the *quantum switch* supermap that we will describe in the subsequent section.

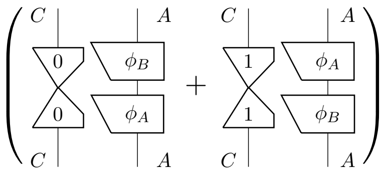
3.2.1 The quantum switch

Recall that given two separate quantum maps $\phi_A : A \to A'$ and $\phi_B : B \to B'$, there is only two ways of connecting them that doesn't lead to a no-signalling channel, namely by imposing $\mathsf{A} \prec \mathsf{B}$ or $\mathsf{B} \prec \mathsf{A}$ (where $\mathsf{A} = (A, A')$, $\mathsf{B} = (B, B')$). Besides, those two constraints cannot be imposed at the same time, assuming no time travel or Closed Timelike Curves [25].

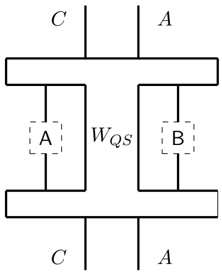
Besides, let's consider the $CNOT : C \otimes A \to C \otimes A$ map. Both the systems C and A are two dimensional Hilbert spaces spanned by the basis states $\{|0\rangle, |1\rangle\}$, with the former being called the *control system*. This map acts as follows: if the control system is in the pure state $|0\rangle\langle 0|$, then the id_A is applied to the input subsystem in A; if the control system is in the state $|1\rangle\langle 1|$, the qubit in A is flipped, i.e. $NOT : A \to A$ is applied on the input subsystem in A. The "quantumness" of the $CNOT$ gate lies in the facts that superposition of states is allowed for the control system in C. Hence, if the control state is in neither basis state, the resulting subsystem in A is neither identical nor "completely flipped".

Using those two remarks, we can extend the concept of the $CNOT$ gate by replacing the first order processes in its definition by *second order processes* associated with the imposed constraints $\mathsf{A} \prec \mathsf{B}$ and $\mathsf{B} \prec \mathsf{A}$. Formally we will define the (coherent[1]) *quantum switch* as:

Definition 23 (The quantum switch)**.** The *quantum switch* is a superoperator taking two no-signalling channels $\phi_A : A \to A$ and $\phi_B : A \to A$, and a control state in a two dimensional Hilbert space spanned by $\{|0\rangle, |1\rangle\}$, and outputs the following process:

double  $\in \mathbf{CPM}(C \otimes A, C \otimes A)$ (3.20)

We will denote the supermap as:

 (3.21)

Remark 16. This definition differs slightly from the one usually present in the literature since the dependence of the control qubit is made explicit in the previous definition, and it also includes both classically controlled (via mixed states) and quantumly controlled (via entangled states) switch. The principle of the switch is however essentially the same as the one described in [13, 39].

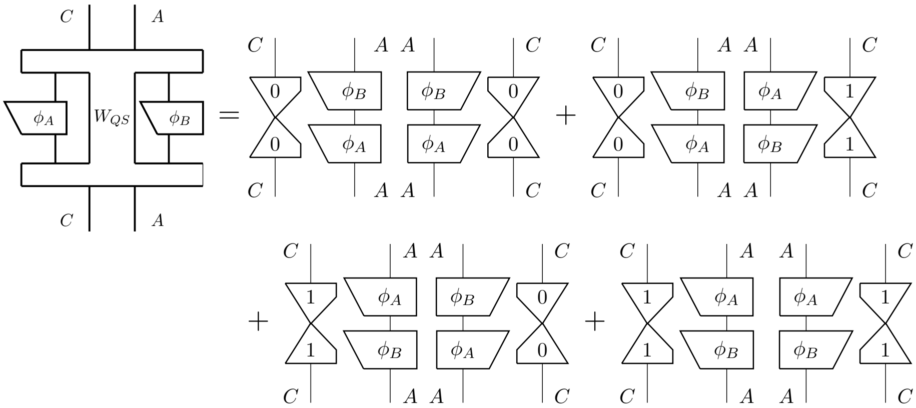
[1]The switch defined in [16] is *classically controlled*, whereas we will here also allow superpositions of causal orders, as in [13, 39].

We can see from this definition that if the control qubit is in a quantum superposition of the basis states $|0\rangle$ and $|1\rangle$ (e.g. if the control state is in the pure state $|+\rangle\langle+|$), then the causal order of events A and B is in a *quantum superposition of* $\mathsf{A} \prec \mathsf{B}$ *and* $\mathsf{B} \prec \mathsf{A}$. Similarly, if the control state is in a mixed state (e.g. in the state $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$), then the causal order is in the statistical mixture of the two possible orders. In other words, the causal graph and causal order inherit from the uncertainty (type and "amplitude") of the control state.

Since, for any admissible maps ϕ_A, ϕ_B, the resulting process is a pure process, then trivially, it is indeed a process in **CPM**. We now want to check that the quantum switch is indeed admissible.

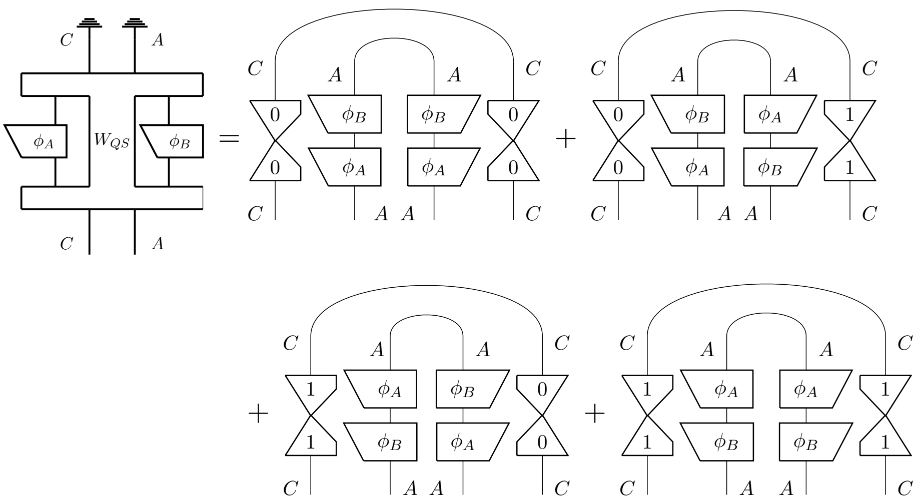
Claim 3. The quantum switch from Definition 23 is second order admissible.

Proof. For any admissible maps $\phi_A, \phi_B : A \to A$, the full decomposition of the resulting process is given by:



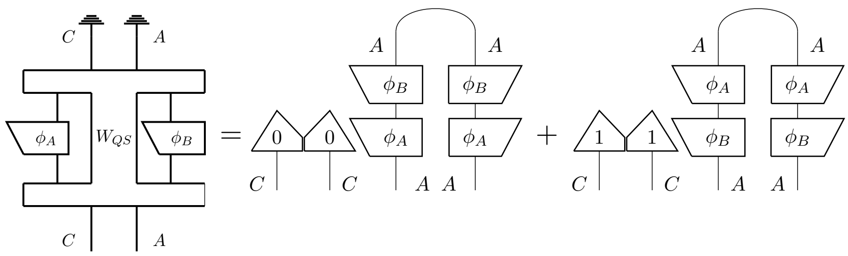
(3.22)

Then, discarding the output gives:

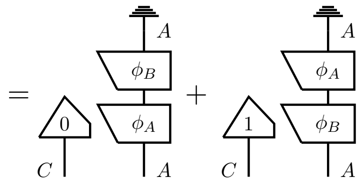


(3.23)

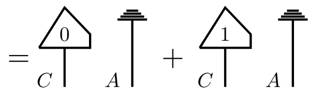
Now, by othronormality of the basis states $|0\rangle$ and $|1\rangle$ of the control system, this reduces to:



(3.24)



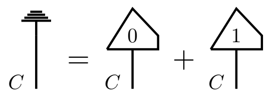
(3.25)



(3.26)

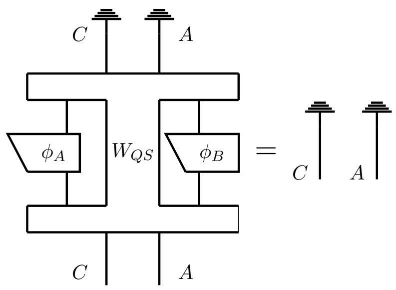
since both ϕ_A and ϕ_B are admissible.

Finally, for the 2-dimensional Hilbert space C, one can show that[21]:



(3.27)

Hence, we indeed have:



(3.28)

for all admissible processes ϕ_A, ϕ_B. □

We have now given an overview of the background notions of a causal theory of quantum mechanics. We are moreover gently moving towards a higher order generalisation of quantum processes, which will be the core of the second part of this project.

In addition, I would like to acknowledge other active areas of research in quantum causality. I have already briefly mentioned quantum causal models (e.g. [9, 23, 5]). Those models are heavily based on concepts coined process operator or process matrices, which can be seen as the generalisation of density matrices for second order processes (see e.g. [38, 23]

for formal definitions and overview of their main properties). On top of that, processes with indefinite causal structures entails some concepts of their own such as causal (non-)separability [39] (which can be seen as the analogue of the distinction between mixing and superposition, but with causal orders), or causal inequalities and the causal polytope [39, 1] (which can similarly be seen as the causality-analogue of Bell inequalities[11] and the local polytope[42] respectively).

Part II

Towards a theory of Higher order quantum processes

4. Higher order causality

Before introducing third order processes, we first want a way to formalise the admissibility conditions that need to be imposed on higher order processes. We will in this chapter describe the type system introduced in [31], and what are the main results of this construction related to causality. We will moreover extend the previously defined notion of causal graphs to encompass the higher order nature of supermaps.

Although most of the results of this chapter are indeed taken from [31], we do provide here a different proof of the *-autonomous nature of $Caus[\mathbf{CPM}]$, and an explicit derivation of the embedding $\mathbf{A} \otimes \mathbf{B} \hookrightarrow \mathbf{A} \parr \mathbf{B}$.

4.1 The type system

We have seen in the previous chapter that in **CPM**, all the higher order processes are isomorphic to a first order system via the Choi-Jamiołkowski isomorphism. However, we have also seen that processes in higher order, to be physical, will need to satisfy higher versions of the causality principle, namely:

- 1st order processes need to be admissible;
- 2nd order processes need to preserve admissibility
- 3rd order processes preserve "preservation of admissibility"
- etc.

Hence, Hilbert spaces alone are not enough to distinguish higher order quantum processes. For example, consider a second order process $W : (A \to A') \to (B \to B')$, which is isomorphic to a first order process in in $(A' \otimes B) \to (A \otimes B')$ (see (3.11)), and a first order process $\phi : (A' \otimes B) \to (A \otimes B')$; we obtain two different admissibility conditions:

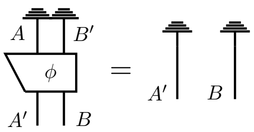
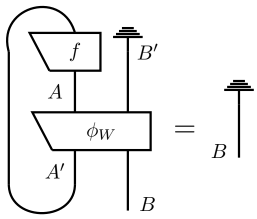
First order process	Second order process
	 $\forall f : A \to A'$ first order admissible.

Table 4.1: First and second order causality principle.

Hence, we need extra structure to have a complete physical theory of higher order processes. We will now describe the $Caus[\mathcal{C}]$ construction introduced in [31]. Note that the idea behind this formalism is essentially the same as the one adopted in [41]. Besides, there have been several other ways of dealing with the causality principle (see [22] for example of constructions of *causal categories*); but which does not include higher order processes.

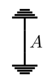
4.1.1 The $Caus[\mathcal{C}]$ construction

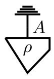
The whole of this section is based on the work and notation of Aleks Kissinger and Sander Uijen in [31].

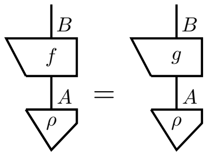
The basic idea of this construction is to associate with each "type" of element (e.g. $\mathbf{A} \multimap \mathbf{B} \equiv$ maps from $\mathbf{A}$ to $\mathbf{B}$) a set of elements satisfying an appropriate version of the causality principle (e.g. admissible maps (first order) sends admissible states to admissible states, supermaps (second order) sends admissible maps to admissible maps, third order processes sends supermaps from admissible maps to admissible maps to supermaps from admissible maps to admissible maps, etc.). We will altogether say that a process (of any order) is admissible if it satisfies its required admissibility condition.

The $Caus[\mathcal{C}]$ construction is defined over a special type category $\mathcal{C}$ called a *precausal category.*

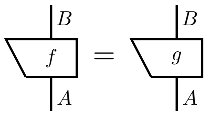
Definition 24 (Precausal category)**.** A category $\mathcal{C}$ is said to be *precausal* if:

- $\mathcal{C}$ is a symmetric compact closed category;
- There exists a discarding morphism  for each object $A \in ob\mathcal{C}$;
- For every object $A \in ob\mathcal{C}$, the scalar  $\neq 0$;

- $\mathcal{C}$ has enough admissible states. In other words, if for every state $\rho : I \to A$ s.t.  $= 1$ we have:

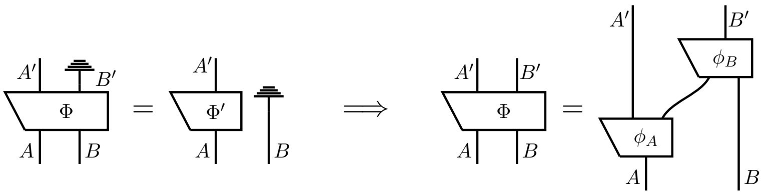
  

  Then:

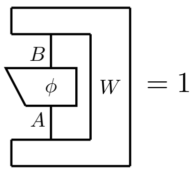
  This condition ensures states that maps are completely determined from their effect on admissible states;

- Every one way signalling process $\mathsf{A} \preceq \mathsf{B}$ factorises as:

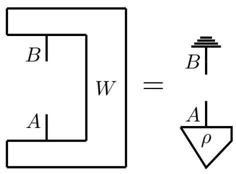
  

  (4.1)

- If for every $\phi : A \to B$ admissible we have:

  Then there exists a admissible state $\rho : I \to A$ s.t.:

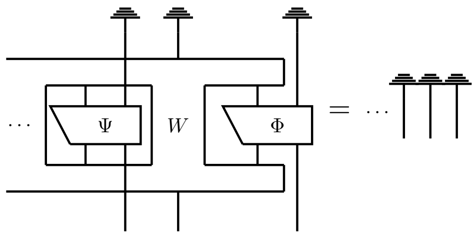
  

The condition of having enough admissible states can, in general, be quite tricky to check. However, for every compact closed category, we have the following result:

Proposition 3. A compact closed category $\mathcal{C}$ has enough admissible state iff it has enough separable admissible states.

The proof of this fact can be found in [32, 31].

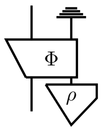
Corollary 1 ([32][1])**.** If **CPM** has enough admissible states, then every admissible second order map is "completely" admissible, i.e.:



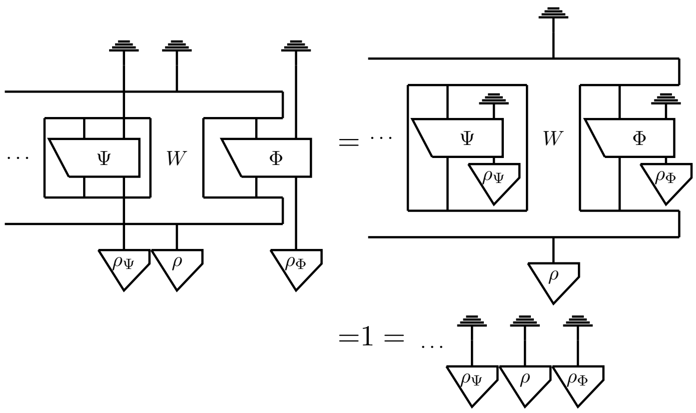
(4.2)

for all $\Phi, \Psi, \ldots$ admissible.

Proof. For every Φ admissible, then the following map is also admissible:



for any ρ admissible. We then indeed have:



The results follows if **CPM** has enough admissible states. □

In order to use this construction to describe quantum processes, we want to have:

Claim 4. The category **CPM** is a precausal category

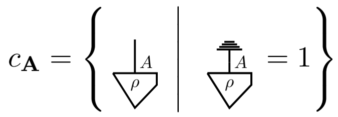
Assuming that **CPM** does have enough admissible states, we have already shown that **CPM** already satisfies most of these conditions; the others are proved in [31].

We will hence focus in the rest of this report on the $Caus[\mathbf{CPM}]$ category. An object $\mathbf{A}$ in $Caus[\mathbf{CPM}]$ will be referred to as a type, and corresponds to a pair $(A, c_{\mathbf{A}})$ where $A \in ob\mathbf{CPM}$ and $c_{\mathbf{A}} \subseteq Hom_{\mathbf{CPM}}(I, A)$ is the subset of admissible states of type $\mathbf{A}$. A morphism $f : \mathbf{A} = (A, c_{\mathbf{A}}) \to \mathbf{B} = (B, c_{\mathbf{B}})$ in $Caus[\mathbf{CPM}]$ is a morphism $f : A \to B$ in **CPM** which preserves the admissible states, i.e. for every $\rho \in c_{\mathbf{A}}$, the composite $f \circ \rho \in c_{\mathbf{B}}$.

[1]The result in [32] for bipartite processes can easily be extended to multipartite processes.

Constructing types What we have right now is just a broad idea of what types represent; let's now define them more concretely. We start by redefining quantum states in terms of types, and then describe how to construct types compatible with higher order admissibility conditions.

Definition 25. For every Hilbert space $A \in ob\mathbf{CPM}$ we define the *first order type* $\mathbf{A} = (A, c_{\mathbf{A}})$ where:

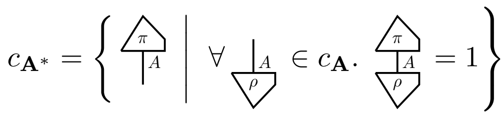


(4.3)

These corresponds exactly to the quantum states in the Hilbert space A.

Now, recall that the issue with the standard formalism arises when taking duals; then, the next obvious question is how do we define duals in the $Caus[\mathbf{CPM}]$ category?

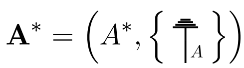
Definition 26. Given a type $\mathbf{A} = (A, c_{\mathbf{A}})$ in $Caus[\mathbf{CPM}]$, we define its dual as $\mathbf{A}^* = (A^*, c_{\mathbf{A}^*})$, where:



(4.4)

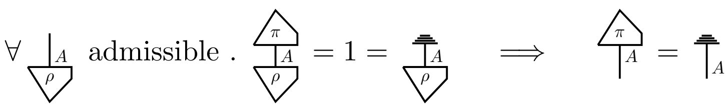
Remark 17. The notion of dual here does not correspond to the notion of (right) duality defined in Definition 10. In particular, we will show later on that $Caus[\mathbf{CPM}]$ is a $*$-autonomous category; hence the notion of right and left duals themselves are defined with two different tensor products[7].

There are a few things that one can note. For example, for the first order type $\mathbf{A}$, we have:



(4.5)

which comes from the assumption that **CPM** has enough states, and hence:



Now, we also want the category $Caus[\mathbf{CPM}]$ to inherit from the "nice" properties of **CPM**; in particular, we want $Caus[\mathbf{CPM}]$ to be a symmetric monoidal category. We first need to define a tensor product:

Definition 27. For two types $\mathbf{X} = (X, c_{\mathbf{X}}), \mathbf{Y} = (Y, c_{\mathbf{Y}}) \in obCaus[\mathbf{CPM}]$, we define the tensor product $\mathbf{X} \otimes \mathbf{Y}$ as:

$$\mathbf{X} \otimes \mathbf{Y} = (X \otimes Y, c_{\mathbf{X}\otimes\mathbf{Y}}) \tag{4.6}$$

with:

$$c_{\mathbf{X}\otimes\mathbf{Y}} = (c_{\mathbf{X}} \otimes c_{\mathbf{Y}})^{**} \tag{4.7}$$

We have shown in Appendix A.1 that for first order systems $\mathbf{A}$ and $\mathbf{B}$, the set $c_{\mathbf{A}\otimes\mathbf{B}}$ indeed corresponds to the set of quantum states in $\mathbf{A} \otimes \mathbf{B}$.

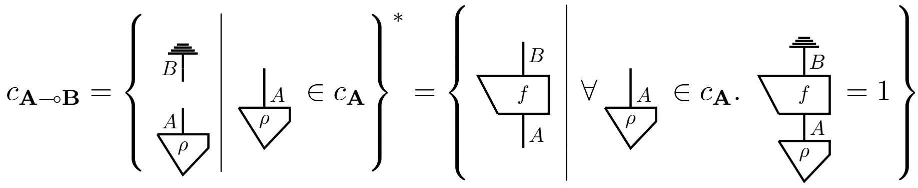
We have so far set up the ground rules of types applied to first order systems. We now move on to higher order types by defining the type $\mathbf{A} \multimap \mathbf{B}$ of processes from a type $\mathbf{A}$ to a type $\mathbf{B}$ as:

$$\mathbf{A} \multimap \mathbf{B} \equiv (\mathbf{A} \otimes \mathbf{B}^*)^* = (A \to B, (c_{\mathbf{A}} \otimes c^*_{\mathbf{B}})^*) = (A^* \otimes B, (c_{\mathbf{A}} \otimes c^*_{\mathbf{B}})^*) \tag{4.8}$$

Remark 18. The fact that this type is indeed a valid type is shown in Appendix A.2. In addition, we have also shown there that for every type $\mathbf{X}$, $c^{**}_{\mathbf{X}} = c_{\mathbf{X}}$. In particular, this implies:

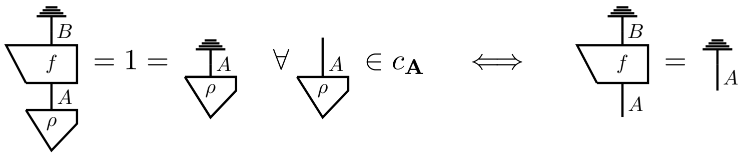
$$\mathbf{X}^{**} = \mathbf{X} \tag{4.9}$$

Let's see what (4.8) means if both $\mathbf{A}$ and $\mathbf{B}$ are first order types. The set of admissible processes sending states in A to states in B are:

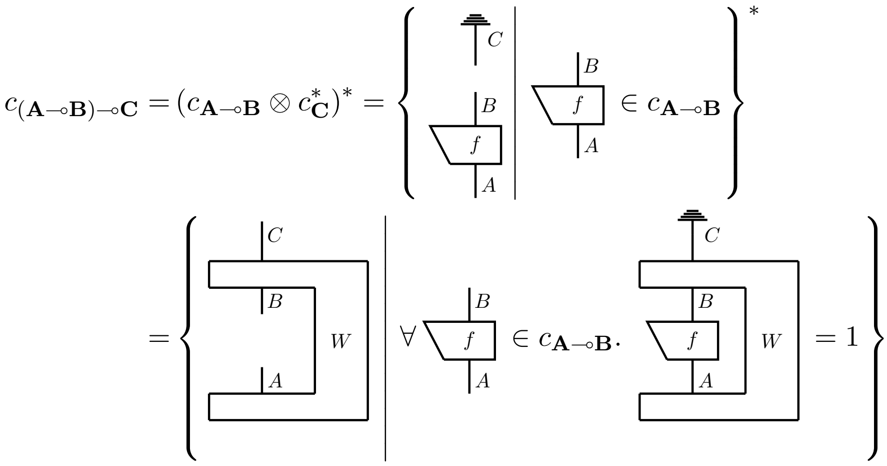


(4.10)

In other words, we recover the causality principle since **CPM** has enough admissible states. Therefore:



Now, since this definition is valid for any arbitrary type (not only first order), we can define types of arbitrary order by using (4.8) recursively. For example let's look at the type $(\mathbf{A} \multimap \mathbf{B}) \multimap \mathbf{C}$. The corresponding Hilbert space would be $A \otimes B^* \otimes C$, and the admissible states are:



Claim 5. The category *Caus*[**CPM**] forms a symmetric monoidal category with the tensor product defined above, and with unit $\mathbf{I} = (I, \{1\})$.

This has been proved in [31].

Remark 19. As in [31], we will adopt the following convention:

- $\multimap$ associates to the right, i.e. $\mathbf{A} \multimap \mathbf{B} \multimap \mathbf{C} \equiv \mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C})$;
- $\otimes$ has priority over $\multimap$, i.e. $\mathbf{A} \otimes \mathbf{B} \multimap \mathbf{C} \equiv (\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C}$.

4.1.2 Properties of the category *Caus*[CPM]

As expected, the category *Caus*[**CPM**] inherits from some of the nice properties of **CPM**. In particular, it has been shown in [31] that it is still a symmetric monoidal category. It is, however, not compact closed, but has the properties of a **-autonomous category.* We will start by reviewing the basic properties of such categories, and how they translate in the context of *Caus*[**CPM**].

***-autonomous categories** The notion of *-autonomous categories first appeared in the context of linear logic[8, 44]. Before defining formally what a *-autonomous category is, we need to first introduce a few concepts.

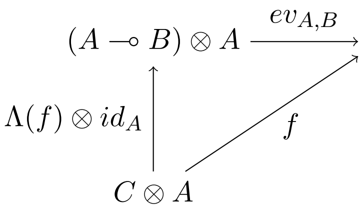
Definition 28 (Closed category[35, 8])**.** A symmetric monoidal category $\mathcal{C}$ is said to be *closed* iff for every object A in $\mathcal{C}$, the following functor:

$$\_\_ \otimes A : \mathcal{C} \to \mathcal{C} \tag{4.11}$$

has a right adjoint:

$$A \multimap \_\_ : \mathcal{C} \to \mathcal{C} \tag{4.12}$$

That is, for all $B \in ob\mathcal{C}$, there exists a map $ev_{A,B} : (A \multimap B) \otimes A \to B$ s.t. for every object C, and map $f : C \otimes A \to B$, there exists a unique map $\Lambda(f) : C \to A \multimap B$ making the following diagram commute:



(4.13)

The maps $ev_{A,B}$ is called an *evaluation map.*

An intuitive example is the category of sets **Set** for which an object A is a set, arrows are functions. It is moreover a strict symmetric monoidal category, with the tensor product being the cartesian product $\times$, and unit $I = \{\bullet\}$ the singleton. We then note that given two sets A, B, the homset $Hom(A, B)$ is once again a set. If we now set $A \multimap B = Hom(A, B)$ with the corresponding evaluation map $ev_{A,B} :: (g \in Hom(A, B), a) \mapsto g(a)$, then one can construct, for every map $f : C \times A \to B :: (c, a) \mapsto b$ as:

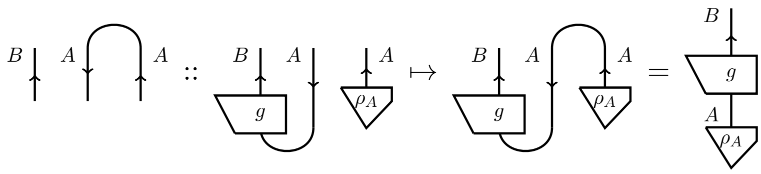
$$\Lambda(f)(c) = (g_c : A \to B) \quad \text{s.t.} \quad g_c(a) = b$$

By construction, this makes (4.13) commute, and every other map $\widetilde{\Lambda}(f)$ making (4.13) commute will, once again by construction, be equal to $\Lambda(f)$.

Now, moving back to the $Caus[\mathbf{CPM}]$ category, we want to have the operation $\_ \multimap \_ : Caus[\mathbf{CPM}] \times Caus[\mathbf{CPM}] \to Caus[\mathbf{CPM}]$ to satisfy for every $\mathbf{A} \in Caus[\mathbf{CPM}]$:

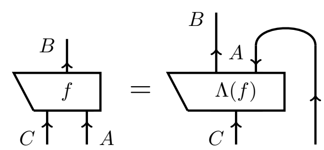
$$\_ \otimes \mathbf{A} \dashv \mathbf{A} \multimap \_ \tag{4.14}$$

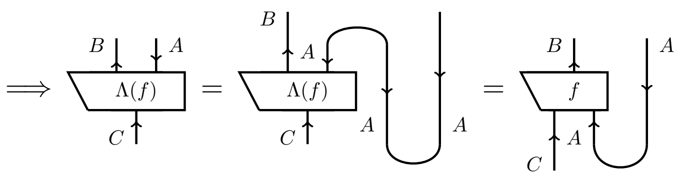
Given any other object $\mathbf{B}$, we claim that the map in **CPM**:

$ev_{\mathbf{A},\mathbf{B}} =$ 

(4.15)

Then we can, as for **Set**, construct the morphism $\Lambda(f) : \mathbf{C} \multimap (\mathbf{A} \multimap \mathbf{B})$ for every map $f : \mathbf{C} \otimes \mathbf{A} \to \mathbf{B}$ as follows:

$f = ev_{\mathbf{A},\mathbf{B}} \circ \Lambda(f) \otimes id_{\mathbf{A}} \iff$ 

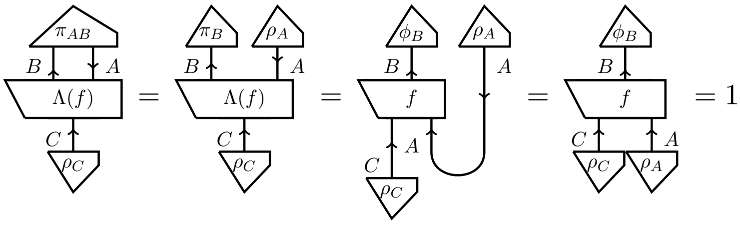


(4.16)

Then, by strictness of **CPM** (and coherence of the graphical calculus), $\Lambda(f)$ as defined above is the only map making (4.13) commute. We hence only need to check that $\Lambda(f)$ is an valid morphism $\mathbf{C} \to (\mathbf{A} \multimap \mathbf{B})$, i.e. an admissible morphism in $\mathbf{C} \multimap (\mathbf{A} \multimap \mathbf{B})$. First, we note that:

$$c^*_{\mathbf{A} \multimap \mathbf{B}} = (c_{\mathbf{A}} \otimes c^*_{\mathbf{B}})^{**} = (c_{\mathbf{A}} \otimes c^*_{\mathbf{B}})$$

hence, for every effect in $(\mathbf{A} \multimap \mathbf{B})^*$ separates, and for every $\pi_{AB} \in c^*_{\mathbf{A} \multimap \mathbf{B}}$, $\rho_C \in c_{\mathbf{C}}$, we have:



(4.17)

since f is by assumption a valid map $\mathbf{C} \otimes \mathbf{A} \to \mathbf{B}$. Hence, $\Lambda(f)$ is an admissible morphism, and $Caus[\mathbf{CPM}]$ is closed.

Furthermore, a standard result about adjunctions is[35]:

$$F \dashv G \quad \iff \quad Hom(F(C), D) \xrightarrow[\iota]{\simeq} Hom(C, G(D))$$

with ι a natural isomorphism, which given the adjuction $\_ \otimes A \dashv A \multimap \_$ in closed categories gives for every object $B, C \in ob\mathcal{C}$:

$$Hom(B \otimes A, C) \xrightarrow[\iota]{\simeq} Hom(B, A \multimap C) \tag{4.18}$$

Definition 29 (Dualising object[8, 7])**.** Given a symmetric closed monoidal category $\mathcal{C}$, a *dualising object* $\bot$ is an object in $\mathcal{C}$ such that the canonical morphism:

$$\gamma : A \to (A \multimap \bot) \multimap \bot \tag{4.19}$$

derived from the identity morphism $id_{A \multimap \bot} : A \multimap \bot \to A \multimap \bot$ as:

$$\begin{aligned}(id_{A \multimap \bot} : A \multimap \bot \to A \multimap \bot) &\xrightarrow{\iota^{-1}} (\gamma_1 : A \otimes (A \multimap \bot) \to \bot) \\ &\xrightarrow{\sigma_{A,(A \multimap \bot))}} (\gamma_2 : (A \multimap \bot) \otimes A \to \bot) \\ &\xrightarrow{\iota} (\gamma : A \to (A \multimap \bot) \multimap \bot)\end{aligned} \tag{4.20}$$

is an isomorphism for all object A of $\mathcal{C}$.

Claim 6. $Caus[\mathbf{CPM}]$ has a dualising object $\bot \equiv \mathbf{I}$.

Proof. We first note that $\mathbf{A} \multimap \mathbf{I} = (A \to I, c_{\mathbf{A}\multimap\mathbf{I}})$ where:

$$A \to I = A^*$$

and:

$$\begin{aligned} c_{\mathbf{A}\multimap\mathbf{I}} =& (c_{\mathbf{A}} \otimes c_{\mathbf{I}}^*)^* \\ =& c_{\mathbf{A}}^* \end{aligned}$$

since $c_{\mathbf{I}} = c_{\mathbf{I}}^* = \{1\}$. Hence, we conclude: $\mathbf{A} \multimap \mathbf{I} \equiv \mathbf{A}^*$. Now, since (4.20) is formed only from isomorphisms and identities, then by the coherence theorem for the graphical calculus (for symmetric monoidal categories), the result follows since $(\mathbf{A} \multimap \mathbf{I}) \multimap \mathbf{I} = \mathbf{A}^{**} = \mathbf{A}$, and the only morphism from the symmetric monoidal structure only is the identity $id_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$. □

We then finally define:

Definition 30 (*-autonomous category[7, 8])**.** A **-autonomous category* is a closed symmetric monoidal category with a dualising object.

Hence, $Caus[\mathbf{CPM}]$ is indeed a *-autonomous category.

Remark 20. This proof of the *-autonomous nature of $Caus[\mathbf{CPM}]$ is different from the one provided in [31].

Tensor products $\otimes$ and $\parr$ In a *-autonomous category with dualising object $\perp$, one defines for all objects A[8, 7]:

$$A^\perp = A \multimap \perp$$

and the binary operation:

$$A \parr B = \left(A^\perp \otimes B^\perp\right)^\perp \tag{4.21}$$

So in the case of the $Caus[\mathbf{CPM}]$ category, this will give:

$$\mathbf{A} \parr \mathbf{B} = (\mathbf{A}^* \otimes \mathbf{B}^*)^* \tag{4.22}$$

It can be shown for an arbitrary *-autonomous category that $\parr$ defines another monoidal structure with unit $\perp$[7]. In the case of $Caus[\mathbf{CPM}]$ one can see that:

$$(\mathbf{A} \parr \mathbf{B}) \parr \mathbf{C} = [(\mathbf{A}^* \otimes \mathbf{B}^*)^{**} \otimes \mathbf{C}^*]^* = (\mathbf{A}^* \otimes \mathbf{B}^* \otimes \mathbf{C}^*) = \mathbf{A} \parr (\mathbf{B} \parr \mathbf{C}) \tag{4.23}$$

and:

$$\mathbf{A} \parr \mathbf{I} = (\mathbf{A}^* \otimes \mathbf{I}^*)^* = \mathbf{A}^{**} = \mathbf{A} = \mathbf{I} \parr \mathbf{A} \tag{4.24}$$

Hence, the pentagon and triangle equalities (Figures 1.2 and 1.1 respectively) are trivially satisfied.

Moreover, as noted in [8], from the definition of a dualising object, $\perp$ does not have to satisfy $\perp \simeq I$. When this is the case, e.g. $Caus[\mathbf{CPM}]$, the category is an *isoMIX* category[19]. In particular it follows from [19] and [18] that for every types $\mathbf{A}$, $\mathbf{B}$, there exists a canonical map:

$$\varphi : \mathbf{A} \otimes \mathbf{B} \to \mathbf{A} \parr \mathbf{B} \tag{4.25}$$

Proof. (Sketch)
We know from [18] that every *-autonomous category is *weakly distributive*[18], or equivalently *linearly distributive*[19]. We have in addition shown that in $Caus[\mathbf{CPM}]$, the unit tensor $\mathbf{I}$ and (the cotensor) $\perp$ coincides, so in particular $\mathbf{I} \simeq \perp$. Hence, it follows from Lemma 6.6 of [19] that $Caus[\mathbf{CPM}]$ is an isoMIX category. Now, by definition (once again from [19]), an isoMIX category is a MIX category for which we here have the mix morphism: $m : \mathbf{I} \xrightarrow{id_{\mathbf{I}}} \mathbf{I}$. Hence, by definition of a MIX category, the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{A} \otimes \mathbf{B} & \xrightarrow{id_{\mathbf{A}} \otimes \rho^{-1}_{\parr,\mathbf{B}}} & \mathbf{A} \otimes (\mathbf{I} \parr \mathbf{B}) \\
\downarrow {\scriptstyle \lambda^{-1}_{\parr,\mathbf{A}}} & & \downarrow {\scriptstyle \delta^{L}_{L}} \\
(\mathbf{A} \parr \mathbf{I}) \otimes \mathbf{B} & & (\mathbf{A} \otimes \mathbf{I}) \parr \mathbf{B} \\
\downarrow {\scriptstyle \delta^{R}_{R}} & & \downarrow {\scriptstyle \lambda_{\otimes,\mathbf{A}} \parr id_{\mathbf{B}}} \\
\mathbf{A} \parr (\mathbf{I} \otimes \mathbf{B}) & \xrightarrow[id_{\mathbf{A}} \parr \rho_{\otimes,\mathbf{B}}]{} & \mathbf{A} \parr \mathbf{B}
\end{array} \tag{4.26}$$

i.e. $\varphi = \lambda_{\otimes,\mathbf{A}} \parr id_{\mathbf{B}} \circ \delta^{L}_{L} \circ id_{\mathbf{A}} \otimes \rho^{-1}_{\parr,\mathbf{B}} = id_{\mathbf{A}} \parr \rho_{\otimes,\mathbf{B}} \circ \delta^{R}_{R} \circ \lambda^{-1}_{\parr,\mathbf{A}}$, where the natural isomorphisms $\lambda_{\otimes,\parr}$, $\rho_{\otimes,\parr}$ are the isomorphisms from the monoidal structure defined w.r.t $\otimes$ and $\parr$ respectively. □

Now, in **CPM**, we know that:

$$(A^{*} \otimes B^{*})^{*} = A^{**} \otimes B^{**} = A \otimes B \tag{4.27}$$

hence, the difference between the types $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A} \parr \mathbf{B}$ lies in the set of admissible states. In particular this means:

$$\forall (\rho : I \to A \otimes B) \in c_{\mathbf{A}\otimes\mathbf{B}}.\ (\varphi(\rho) : I \to A \otimes B) \in c_{\mathbf{A}\parr\mathbf{B}} \tag{4.28}$$

i.e. the function φ is an embedding $\varphi : \mathbf{A} \otimes \mathbf{B} \hookrightarrow \mathbf{A} \parr \mathbf{B}$.

4.2 n-partite definite causal order from nth-order processes

We will introduce here the notion of *combs* as the first example of a higher-order process. We will, in particular, see how this particular type of process can be used to describe second order processes with a definite causal structure.

We will start with a remark that will be useful in the rest of this discussion.

Remark 21. For any types $\mathbf{A}, \mathbf{B}, \mathbf{X}$, we have:

$$\mathbf{A} \multimap \mathbf{X} \multimap \mathbf{B} \simeq \mathbf{X} \multimap (\mathbf{A} \multimap \mathbf{B}) \tag{4.29}$$

Proof. This is shown by unfolding the definitions of types:

$$\begin{aligned}\mathbf{X} \multimap (\mathbf{A} \multimap \mathbf{B}) =& \mathbf{X}(\mathbf{A} \otimes \mathbf{B}^*)^* \\ =& (\mathbf{X} \otimes (\mathbf{A} \otimes \mathbf{B}^*)^{**})^* = (\mathbf{X} \otimes \mathbf{A} \otimes \mathbf{B}^*)^* \\ \simeq& (\mathbf{A} \otimes \mathbf{X} \otimes \mathbf{B}^*)^* = (\mathbf{A} \otimes (\mathbf{X} \otimes \mathbf{B}^*)^{**})^* \\ =& (\mathbf{A} \otimes (\mathbf{X} \multimap \mathbf{B})^*)^* \\ =& \mathbf{A} \multimap (\mathbf{X} \multimap \mathbf{B}) \equiv \mathbf{A} \multimap \mathbf{X} \multimap \mathbf{B}\end{aligned}$$

□

We will now focus on a special family of higher order processes referred to as *combs*[41, 31].

Definition 31. We define the comb types recursively as:

$$\begin{cases} \mathbf{C}^{(0)} & = \mathbf{I} \\ \mathbf{C}^{(n+1)} & = \mathbf{A}_{-(n+1)} \multimap \mathbf{C}^{(n)} \multimap \mathbf{A}_{n+1} \end{cases} \tag{4.30}$$

where $\mathbf{A}_{\pm n}$ are first order types for all $n \geq 1$

From the previous remark, we see that the type of an n-comb can be rewritten as:

$$\mathbf{C}^{(n)} = \mathbf{A}_{-n} \multimap \mathbf{C}^{(n-1)} \multimap \mathbf{A}_n = \mathbf{C}^{(n-1)} \multimap (\mathbf{A}_{-n} \multimap \mathbf{A}_n)$$

i.e. takes a $n-1$-comb as an input and output a first order map. Note that any first order process can be seen as an nth order process as:

$$\mathbf{A} \multimap \mathbf{B} \simeq (\mathbf{I} \multimap \mathbf{I}) \multimap (\mathbf{A} \multimap \mathbf{B}) \simeq [(\mathbf{I} \multimap \mathbf{I}) \multimap (\mathbf{I} \multimap \mathbf{I})] \multimap [(\mathbf{I} \multimap \mathbf{I}) \multimap (\mathbf{A} \multimap \mathbf{B})] \simeq \dots$$

Hence, we we observe that:

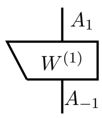
- $\mathbf{C}^{(0)}$ is a 0th order process;

- $\mathbf{C}^{(1)} = \mathbf{A}_{-1} \multimap \mathbf{A}_1$ is a 1st order process;
- $\mathbf{C}^{(2)} = (\mathbf{A}_{-1} \multimap \mathbf{A}_1) \multimap (\mathbf{A}_{-2} \multimap \mathbf{A}_2)$ is a 2nd order process;
- $\mathbf{C}^{(3)} = [(\mathbf{A}_{-1} \multimap \mathbf{A}_1) \multimap (\mathbf{A}_{-2} \multimap \mathbf{A}_2)] \multimap (\mathbf{A}_{-3} \multimap \mathbf{A}_3)$ is a 3rd order process;

$\vdots$

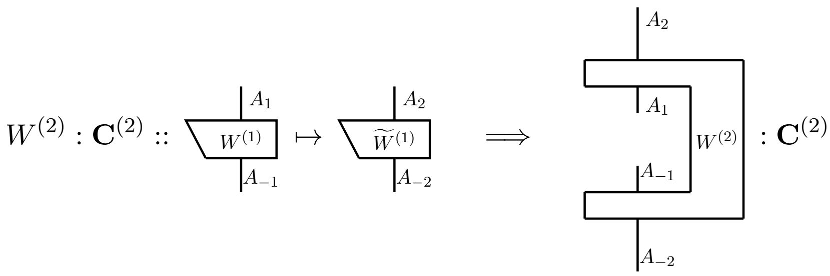
- $\mathbf{C}^{(n)}$ is a nth order process.

What are the processes of types $\mathbf{C}^{(n)}$, and how do we represent them? The 0-combs are empty diagrams in **CPM**, so the first representable combs are the 1-combs, which are as we recall standard quantum processes:



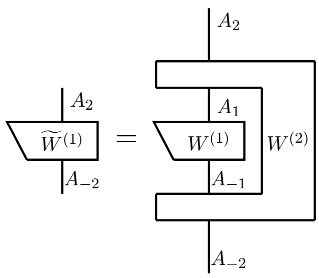
$: \mathbf{C}^{(1)}$ (4.31)

Now, 2-combs takes quantum maps from A_{-1} to A_1 to maps from A_{-2} to A_2, i.e. we have:

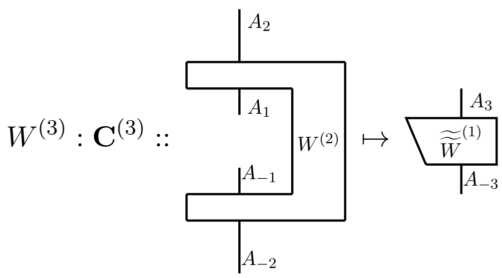


(4.32)

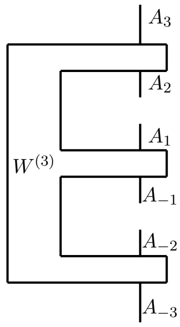
where:



Similarly, we know that:

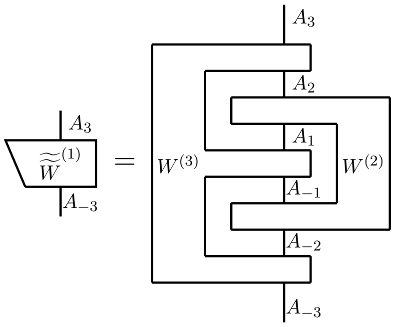


Hence, we can represent 3-combs as:

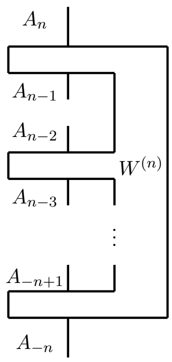


$$: \mathbf{C}^{(3)} \tag{4.33}$$

and:



One can see from the first few iterations, and from the definition of the types $\mathbf{C}^{(n)}$, that the general form of an n-comb can be represented as:



$$: \mathbf{C}^{(n)} \tag{4.34}$$

4.2.1 Combs and causal graphs

Now that we have defined combs, we can see how they are related to causal ordering. One of the main results of [41, 31] was to show that there is a one-to-one correspondance between n-partite one way signalling channels and nth order combs. In other words:

Theorem 6. *A n-partite channel $A_1 \otimes A_2 \otimes \ldots \otimes A_n \to A'_1 \otimes A'_2 \otimes \ldots \otimes A'_n$ admits the minimal causal graph:*

$$\begin{array}{c} \mathsf{A}_n \\ \uparrow \\ \vdots \\ \uparrow \\ \mathsf{A}_2 \\ \uparrow \\ \mathsf{A}_1 \end{array}$$

then it can be deformed into an n-comb of type:

$$\mathbf{A}_1 \multimap (\mathbf{A}'_1 \multimap (\ldots) \multimap \mathbf{A}_n) \multimap \mathbf{A}'_n$$

The proof is in the Appendix A.3 and is entirely based on [31].

Remark 22. From linear distributivity of $Caus[\mathbf{CPM}]$, we know that there exists a morphism[18, 31, 19]:

$$\vartheta : (\mathbf{A} \parr \mathbf{B}) \otimes \to \mathbf{A} \parr (\mathbf{B} \otimes \mathbf{C}) \tag{4.35}$$

Then it has been shown in [31] that every no-signalling channels can be transformed into combs, i.e. there exists an embedding:

$$(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}') \hookrightarrow \mathbf{A} \multimap (\mathbf{A}' \multimap \mathbf{B}) \multimap \mathbf{B}' \tag{4.36}$$

Intersection of types Recall that one-way signalling processes give rise to a very restricted class of processes with definite causal order. Is it then possible to obtain any causal order from higher order processes? The answer to this question follows from the definition of no-signalling channels, in particular that, if A and B are no-signalling, then $\mathsf{A} \preceq \mathsf{B}$ and $\mathsf{B} \preceq \mathsf{A}$. So every process that has a definite causal structure is compatible to at least one *total order*, i.e. causal order of the form $\mathsf{A}_1 \prec \mathsf{A}_2 \prec \ldots \prec \mathsf{A}_n$. To see this, recall that a channel with definite causal structure gives rise to a DAG in which the events are the nodes, and each edge (A, B) corresponds to a strictly one-way signalling relation, i.e. $\mathsf{A} \prec \mathsf{B}$. Now, one can construct a total order for a given DAG as follows: start by choosing a pair of no-signalling nodes, and add an edge with a chosen direction between them; then we obtain a new DAG. One can then apply the same process on the newly obtained DAG, and so on until there are no no-signalling nodes anymore. We see by construction that this does not induce any loop, and is completely connected. Hence, it can be reduced to a minimal graph of the form:

$$\begin{array}{c} \mathsf{A}_n \\ \uparrow \\ \vdots \\ \uparrow \\ \mathsf{A}_2 \\ \uparrow \\ \mathsf{A}_1 \end{array}$$

We then call the resulting linear causal order induced from this causal graph a *totalisation* of the original causal order of the channel. Hence, it has been shown in [31] that a channel with definite causal structure has a causal graph given by a certain DAG iff it is compatible with *all* its totalisations.

Now, we know that a channel has a total causal ordering if it is (up to isomorphism) an admissible state of a comb type. So a channel that is compatible with a set of totalisations lies within the intersection of all admissible sets of the corresponding combs.

We want to formalise this intuition about the intersection of types. To do this, we need to first show that taking such an intersection indeed leads to a well-defined type. We start by showing the following:

Proposition 4. [31] For every type $\mathbf{X}$, there exists an embedding:

$$e : \mathbf{X} \hookrightarrow \mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_n \multimap \mathbf{A}'_1 \otimes \ldots \otimes \mathbf{A}'_m \tag{4.37}$$

for some $\{\mathbf{A}_i\}_{i=1}^n$, $\{\mathbf{A}'_i\}_{i=1}^m$ first order systems.

The proof is shown in Appendix A.4.

We can then formally define the intersection of types as follows:

Definition 32 ([31])**.** Given two types $\mathbf{X}, \mathbf{X}'$ (not necessary combs), both of which can be embedded into the same first order process type:

$$\mathbf{Y} = (\mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_n) \multimap \left(\mathbf{A}'_1 \otimes \ldots \otimes \mathbf{A}'_m\right)$$

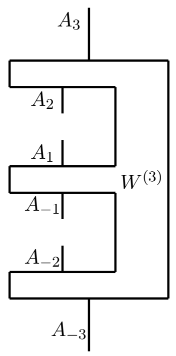
we define the intersection type $\mathbf{X} \cap \mathbf{X}'$ as the type in $Caus[\mathbf{CPM}]$ making the following diagram a pullback:

$$\begin{array}{ccc} \mathbf{X} \cap \mathbf{X}' & \xrightarrow{e_1^{-1}} & \mathbf{X} \\ {\scriptstyle e_2^{-1}}\downarrow & & \downarrow{\scriptstyle e_1} \\ \mathbf{X}' & \underset{e_2}{\hookrightarrow} & \mathbf{Y} \end{array} \tag{4.38}$$

It has been shown in [31] that this does indeed give rise a well-defined pullback. More explicitly, we have $\mathbf{X} \cap \mathbf{X}' = (Y, c_{\mathbf{X} \cap \mathbf{X}'})$ with Y being the same underlying Hilbert space as $\mathbf{X}$ and $\mathbf{X}'$, and $c_{\mathbf{X} \cap \mathbf{X}'}$ being the intersection of $e_1(c_{\mathbf{X}}) \cap e_2(c_{\mathbf{X}'})$.

4.2.2 External and Internal causal graphs

There are two obvious ways of associating a causal graph to a given process W. We have seen, for example for $W^{(3)} : \mathbf{C}^{(3)}$:

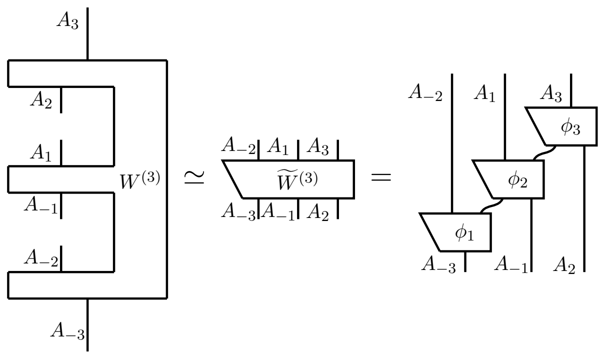


$$\models \quad \begin{array}{c} \mathsf{W}_3 \\ \uparrow \\ \mathsf{W}_2 \\ \uparrow \\ \mathsf{W}_1 \end{array} \tag{4.39}$$

where:

$$\begin{cases} \mathsf{W}_1 = & \mathbf{A}_{-3} \multimap \mathbf{A}_{-2} \\ \mathsf{W}_2 = & \mathbf{A}_{-1} \multimap \mathbf{A}_{1} \\ \mathsf{W}_3 = & \mathbf{A}_{2} \multimap \mathbf{A}_{3} \end{cases} \tag{4.40}$$

and this is because the (3rd order) process $W^{(3)}$ can be deformed into the channel:



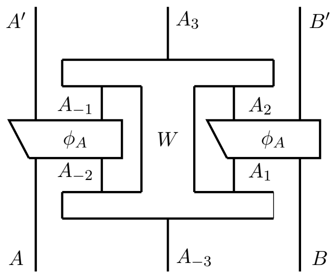
(4.41)

An alternative way of characterising the causal order is the following:

We start by define the events $\mathsf{A} = (A, A')$ and $\mathsf{B} = (B, B')$, associated respectively with admissible maps $\phi_A : A \otimes A_{-2} \to A' \otimes A_{-1}$ and $\phi_B : B \otimes A_1 \to B' \otimes A_2$. Then suppose we have a supermap W of type:

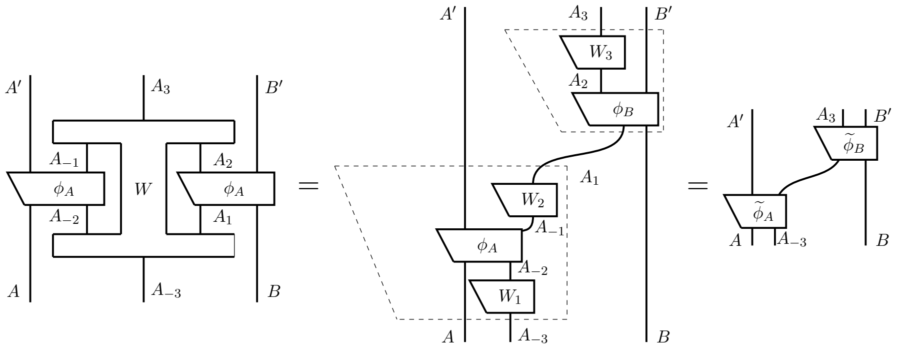
$$W : [(\mathbf{A}_{-2} \multimap \mathbf{A}_{-1}) \otimes (\mathbf{A}_1 \multimap \mathbf{A}_2)] \multimap (\mathbf{A}_{-3} \multimap \mathbf{A}_3)$$

such that:



(4.42)

gives a channel where there exists a process connecting the global input in A_{-3} and the input A_{-2} of ϕ_A, the output system A_{-2} of ϕ_A is connected to the input of ϕ_B in A_1 (via some possible unknown operation internal to W), and finally the output of ϕ_B in the system A_2 is connected to the global output in A_3. Diagrammatically, this means:

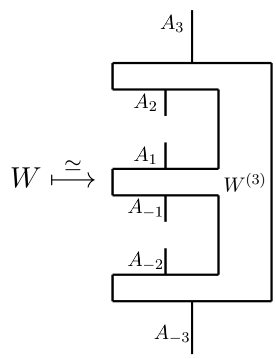


(4.43)

Now, we see that it is similar to the situation described in Section 2.2, up to a change of input system in A and output system in B. Hence, we would expect the causal graph to be of the form:

$$\mathsf{B} \uparrow \mathsf{A}$$

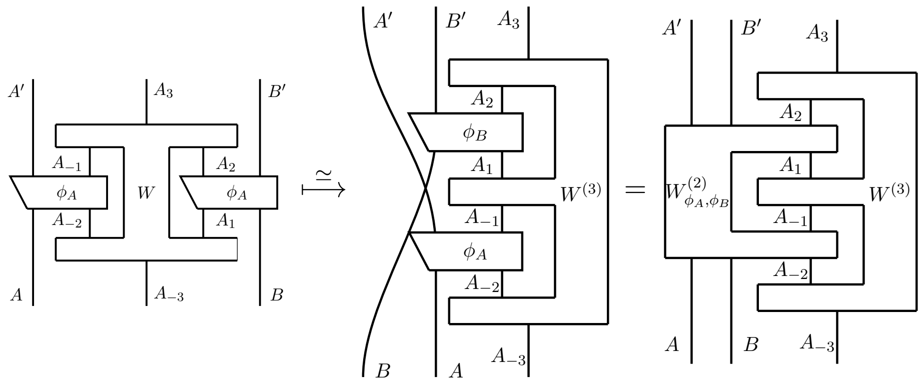
However, we can also model this process using the 3-comb:



since this process can clearly be extended to a process in:

$$\mathbf{A}_{-3} \multimap (\mathbf{A}_{-2} \multimap (\mathbf{A}_{-1} \multimap \mathbf{A}_1) \multimap \mathbf{A}_2) \multimap \mathbf{A}_3$$

Indeed, inputting a strictly one-way signalling process $\mathsf{A} \prec \mathsf{B}$ will not induce any causal loop; and:



(4.44)

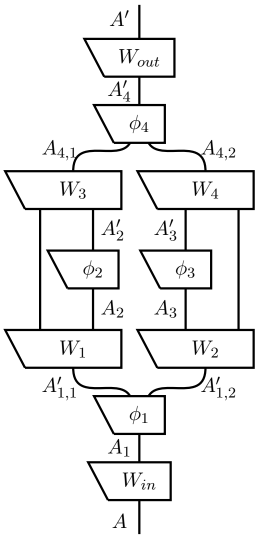
We then see that even in the simple example of 3-combs, two possible causal graphs emerges:

$$\begin{array}{ccc} \mathsf{W}_3 & & \mathsf{B} \\ \uparrow & & \Big\uparrow \\ \mathsf{W}_2 & \text{and} & \Big| \\ \uparrow & & \Big| \\ \mathsf{W}_1 & & \mathsf{A} \end{array}$$

Now, we also see from (4.43) that these two causal graphs are not incompatible, but rather complementary - although they do address different questions. We will therefore propose an extension of the previously defined causal graphs to include this extra information.

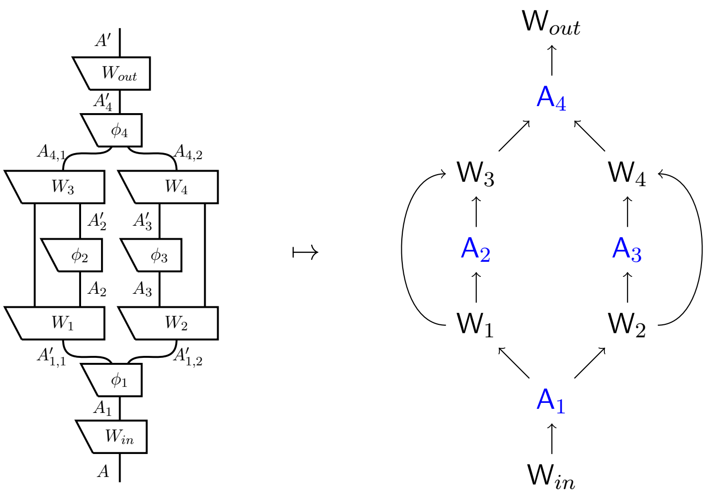
The first thing to notice is that the causal order relating the W's only depends on the internal structure of the process W, while the second kind of causal order only makes sense when the two events A and B have been fixed. Hence, given a process W, we will refer to its associated causal graph of the first kind (in the previous example $\mathsf{W}_1 \to \mathsf{W}_2 \to \mathsf{W}_3$) as the *internal causal graph*, and the second type of graph (in the previous example $\mathsf{A} \to \mathsf{B}$) as the *external causal graph.*

Augmented causal graph We then want to be able to condense all the information given from both causal graphs within a single graph, whenever the causal structure is definite. For example, let $n = 4$ and $W(\phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4)$ is given by:



(4.45)

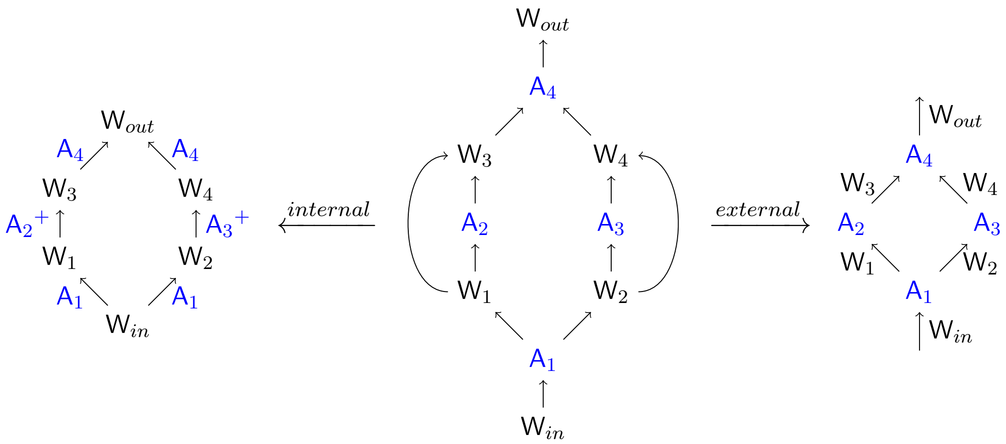
where $A_1' = A_{1,1}' \otimes A_{1,2}'$ and $A_4 = A_{4,1} \otimes A_{4,2}$, and the maps labelled with W's are processes internal to W. Now, the overall structure is the same for every input $\phi_{1,2,3,4}$, which are allowed to vary, while $W_{1,2,3,4,in,out}$ are fixed. Hence, we can generate a causal graph with a node for every process as before, in which the external events (input) with a different colour. For example, (4.45) will translate as:



(4.46)

Some features of this type of causal graph can be made more specific to second order processes. For instance, the arrows that do not factor through blue nodes (external events) are representative of the memory channels within the supermap. Moreover, the processes with no descendant (in the previous example W_{out}) correspond to the global outputs of the supermap, while the nodes with no ancestors (e.g. W_{in}) and the overall inputs; we will see later that those special nodes correspond to the "boundaries" of the processes, i.e. the processes that allow the supermap to be composed with other supermaps.

We also note that it is easy to recover both the internal and external causal orders by abstracting respectively the external and internal nodes by arrows. Following the previous example this will give:



(4.47)

where the + superscript in the internal causal order emphasizes the presence of a memory channel.

4.3 Indefinite causal orders and extendability

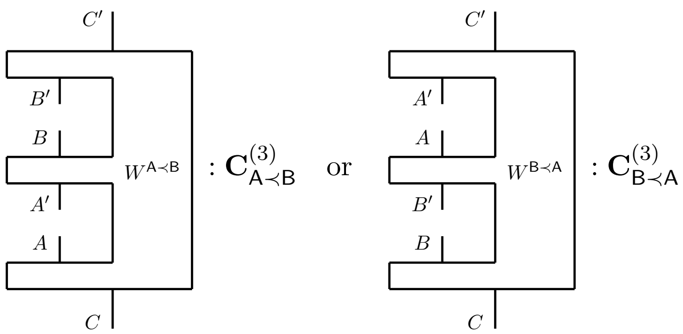
Moving on from the case when the process admits a causal order, let us see what happens when the causal structure is indefinite. In a bipartite superoperator:

$$W : (\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}') \multimap \mathbf{C} \multimap \mathbf{C}' \tag{4.48}$$

we only have two external events, so it has a definite causal structure iff it reduces to a comb:

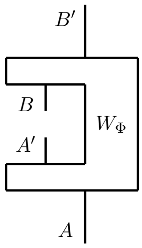
$$W : \mathbf{C} \multimap (\mathbf{A} \multimap (\mathbf{A}' \multimap \mathbf{B}) \multimap \mathbf{B}') \multimap \mathbf{C}' \quad \text{or} \quad W : \mathbf{C} \multimap (\mathbf{B} \multimap (\mathbf{B}' \multimap \mathbf{A}) \multimap \mathbf{A}') \multimap \mathbf{C}'$$

(including both the case when A and B are in no-signalling in the resulting process.) In diagrams:

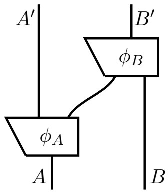


(4.49)

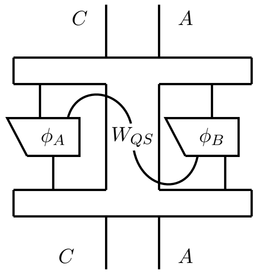
Now, as expected, this is not possible in the case of the quantum switch since plugging any 2-comb, say:



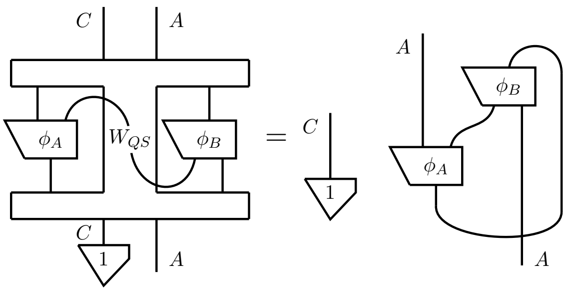
$$: \mathbf{A} \multimap (\mathbf{A}' \multimap \mathbf{B}) \multimap \mathbf{B}' \overset{\simeq}{\longmapsto} \tag{4.50}$$



in the switch gives:



Now, if we impose the control state to be in the pure state $|1\rangle\langle 1|$, then, according to (23), this will give:



(4.51)

which is not a admissible (unless $A \simeq I$). We also reach the same conclusion by interchanging $\mathbf{C}^{(2)}_{\mathsf{A}\prec\mathsf{B}}$ with $\mathbf{C}^{(2)}_{\mathsf{B}\prec\mathsf{A}}$, and the control state from the basis state $|1\rangle$ to the basis state $|0\rangle$.

From Remark 22, we known that there is a canonical embedding from no-signalling channels $(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')$ to the two combs $\mathbf{C}^{(2)}_{\mathsf{A}\prec\mathsf{B}}$ and $\mathbf{C}^{(2)}_{\mathsf{B}\prec\mathsf{A}}$. I.e.:

$$(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}') \hookrightarrow \mathbf{A} \multimap (\mathbf{A}' \multimap \mathbf{B}) \multimap \mathbf{B}'$$
$$(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}') \hookrightarrow \mathbf{B} \multimap (\mathbf{B}' \multimap \mathbf{A}) \multimap \mathbf{A}' \tag{4.52}$$

However, the example of the switch shows that:

$$c_{(\mathbf{A}\multimap\mathbf{A}')\otimes(\mathbf{B}\multimap\mathbf{B}')\multimap\mathbf{C}\multimap\mathbf{C}'} \not\subset c_{(\mathbf{A}\multimap(\mathbf{A}'\multimap\mathbf{B})\multimap\mathbf{B}')\multimap\mathbf{C}\multimap\mathbf{C}'} \tag{4.53}$$

This can be seen as the analogue of functions not extendable to another function acting on a bigger domain (e.g. $f : \mathbb{R} - \{0\} \to \mathbb{R} :: x \mapsto x^{-1}$ is not extendable to a function $\tilde{f} : \mathbb{R} \to \mathbb{R}$). Hence, we will say that the quantum switch is *not extendable* to a comb.

Remark 23 (On extendability and augmented causal graphs)*.* When the causal structure is definite, it is possible to read the extendability conditions from the augmented causal graph of the supermap: whenever adding "blue arrows" (i.e. external signalling relations) does not alter the acyclicity of the graph, this gives a possible way of extending the supermap to a more general type.

Now that we have provided a formal way of dealing with the causality principle (i.e. admissibility conditions) for processes of any order. However, even though the constraints that need to be applied for higher order processes are known, there are very few occurrences of concrete higher order processes in the literature; the exception being the combs. The aim of the final chapter is then to investigate examples of third order processes.

5. Third order processes

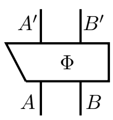
As quantum theory currently stands, it seems like processes of different order give rise to a hierarchy of features of quantum systems. For example, from first order processes such as the teleportation protocol, or the Bell scenarios, scientists have exposed the non-local nature of quantum mechanics; from second order processes, examples such as the quantum switch have shown that quantum causality is allowed to be indefinite (i.e. non-deterministic). What then? Very little is currently known about higher order processes in their generality. We will then, in this last chapter investigate third order processes, and describe some concrete examples.

Before introducing these third order processes, we will start with a discussion on a way to describe localisable events, which will be useful throughout the whole of this chapter. In particular, we define the notion of a generalised local event via the $\sqcup$ operation, and show some of its properties. This notion of a local event in particular comes with another interesting operation, denoted as $\sqsubseteq$, which gives, as we will see, another "dimension" to act upon when composing superoperators.

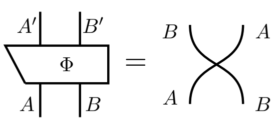
This chapter is entirely original.

5.1 On the definition of an event

We have so far assumed that events, e.g. $\mathsf{A} = (A, A')$, were predefined subset of $\{input\} \times \{output\}$ Hilbert spaces, and concepts such as localisability (see Section 2.1.2) or even signalling correlations (see Section 2.1.1) are defined with respect to this assumption. Now, let's consider the swap channel:

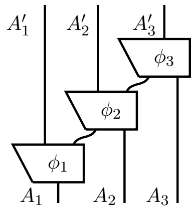


with $A' \equiv B$, $B' \equiv A$ and:



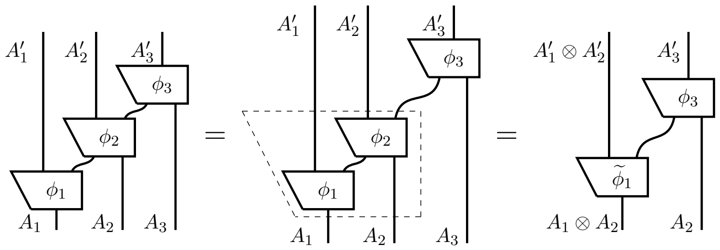
(5.1)

Then $\mathsf{A} = (A, A') = (A, B), \mathsf{B} = (B, B') = (B, A)$ are not suitable events since $\mathsf{A} \not\preceq \mathsf{B}$ and $\mathsf{B} \not\preceq \mathsf{A}$, and hence neither A nor B are localisable. Another example is the tripartite total order:



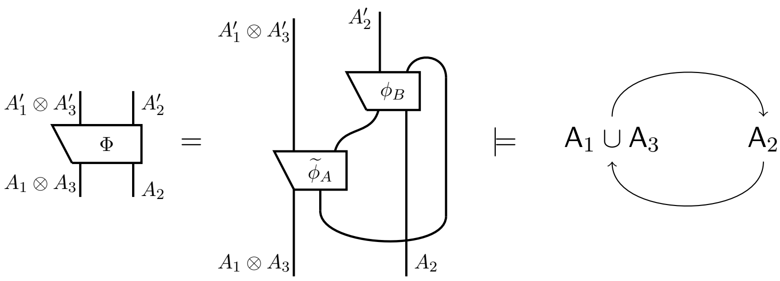
(5.2)

Here, all the three events $\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3$ are well-defined and localisable[1]. In addition, we can also view this channel as a bipartite channel, for example with events $\widetilde{\mathsf{A}}_1, \mathsf{A}_3$, where we redefine $\widetilde{\mathsf{A}}_1 = (A_1 \otimes A_2, A_1' \otimes A_2')$:



(5.3)

However, we cannot "group" the events A_1 and A_3 together since this will result in a causal loop:



(5.4)

Now, if we assume that we are in a flat space-time, in particular if we exclude CTCs[25], this causal graph is not a valid causal graph.

We then want to define a more general notion of event. We start by defining the pseudo-union of two events as:

Definition 33. Given a causal graph in flat space-time, for two events $\mathsf{A}, \mathsf{B} \in V$, we define $\mathsf{A} \sqcup \mathsf{B}$ as:

$$\mathsf{A} \sqcup \mathsf{B} = \{\mathsf{A}, \mathsf{B}\} \cup (\mathsf{A}_\downarrow \cap \mathsf{B}_\uparrow) \cup (\mathsf{B}_\downarrow \cap \mathsf{A}_\uparrow) \tag{5.5}$$

Now we also note that since $\mathsf{A}_{\uparrow,\downarrow}$ are defined from a partial order, we have:

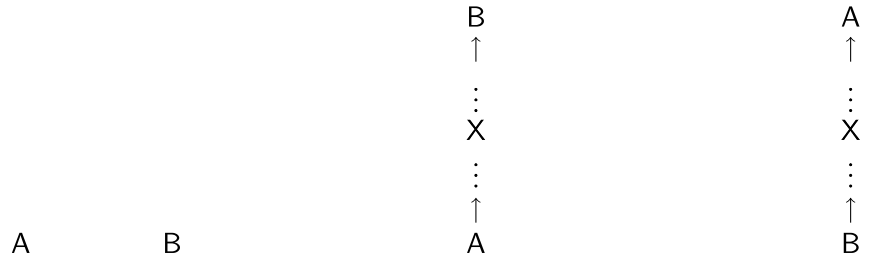
$$\{\mathsf{A}\} = \mathsf{A}_\downarrow \cap \mathsf{A}_\uparrow \tag{5.6}$$

[1]Note that we will avoid to use the terminology "point-like", since it is not generally the case; on the other hand, the term *contractible*, as in topology, would be more accurate.

(similarly for B). Then, (5.5) becomes:

$$\mathsf{A} \sqcup \mathsf{B} = (\mathsf{A}_{\downarrow} \cup \mathsf{B}_{\downarrow}) \cap (\mathsf{A}_{\uparrow} \cup \mathsf{B}_{\uparrow}) \tag{5.7}$$

How do we understand this definition? By analysing (5.5), we see that the first term ensures that both A and B are inside the pseudo-union, the second term is more interesting and includes all the nodes on all paths (if any exists) from A to B, similarly, the last term includes nodes on paths from B to A.



(a) A, B no-signalling. (b) $\mathsf{A} \prec \mathsf{B}$ (c) $\mathsf{B} \prec \mathsf{A}$

Figure 5.1: Three different cases for $\mathsf{A} \sqcup \mathsf{B}$

One may wonder whether we can simply extend (5.5) for any arbitrary set of events; however, allowing for arbitrary subsets of events implies that (5.5) and (5.7) are no longer equivalent, and in particular, for a general $X \subseteq V$:

$$X \sqcup X \neq X \tag{5.8}$$

Instead, we want to show that $\sqcup$ can be applied recursively. For any subsets $\mathcal{A}$ and $\mathcal{B}$ of V, we define:

$$\mathcal{A} \sqcup \mathcal{B} = \mathcal{A} \cup \mathcal{B} \cup (\mathcal{A}_{\downarrow} \cap \mathcal{B}_{\uparrow}) \cup (\mathcal{B}_{\downarrow} \cap \mathcal{A}_{\uparrow}) \tag{5.9}$$

Let's then start with the simplest case, namely $(\mathsf{A} \sqcup \mathsf{B}) \sqcup \mathsf{C}$, where $\mathsf{A}, \mathsf{B}, \mathsf{C}$ are (atomic) elements in V. Now, we first remark that types do not quite check in this expression, we will then assume the short hand $\{\mathsf{C}\} \equiv \mathsf{C}$, and hence, only consider subsets of events.

Proposition 5. The pseudo-union, as defined in (5.9), is associative for atomic events, i.e.:

$$(\mathsf{A} \sqcup \mathsf{B}) \sqcup \mathsf{C} \equiv (\mathsf{A} \sqcup \mathsf{B}) \sqcup \{\mathsf{C}\} = \mathsf{A} \sqcup (\mathsf{B} \sqcup \mathsf{C}) \equiv \{\mathsf{A}\} \sqcup (\mathsf{B} \sqcup \mathsf{C}) \tag{5.10}$$

for $\mathsf{A}, \mathsf{B}, \mathsf{C} \in V$.

Proof. First note that by the symmetry of the definition of $\mathcal{A} \sqcup \mathcal{B}$, we have:

$$\mathcal{A} \sqcup \mathcal{B} = \mathcal{B} \sqcup \mathcal{A} \tag{5.11}$$

for any set of events $\mathcal{A}, \mathcal{B}$.

Now, unfolding the definitions, we have:

$$(A \sqcup B) \sqcup C = \{Y \in V | Y \in A \sqcup B \ \vee \ Y = C \ \vee \ C \prec Y \prec A \sqcup B \ \vee \ A \sqcup B \prec Y \prec C\} \quad (5.12)$$

$$A \sqcup (B \sqcup C) = \{Y \in V | Y = A \ \vee \ Y \in B \sqcup C \ \vee \ A \prec Y \prec B \sqcup C \ \vee \ B \sqcup C \prec Y \prec A\} \quad (5.13)$$

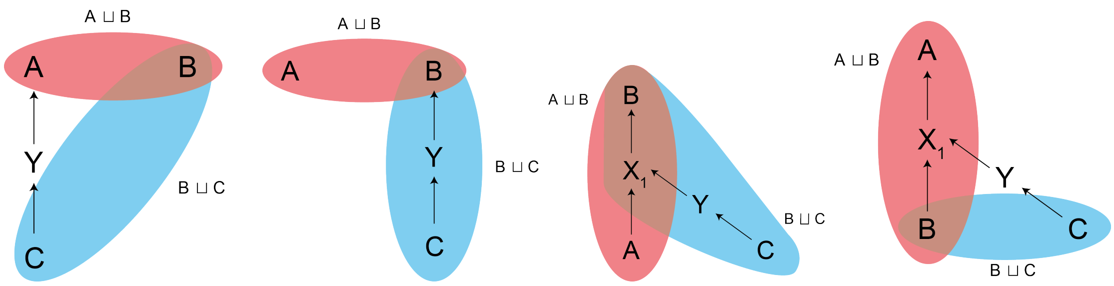
We first show: $(A \sqcup B) \sqcup C \subseteq A \sqcup (B \sqcup C)$. Suppose that $Y \in (A \sqcup B) \sqcup C$:

- If $Y \in \{A, B, C\}$, then automatically, $Y \in A \sqcup (B \sqcup C)$;
- Excluding the previous cases, suppose $Y \in A \sqcup B$, then either $A \prec Y \prec B$ or $B \prec Y \prec A$. For the former, we note that $B \in B \sqcup C$, hence:

  $$A \prec Y \prec B \implies A \prec Y \prec B \sqcup C$$

  so $Y \in A \sqcup (B \sqcup C)$. Similarly, if $B \prec Y \prec A$, then $B \sqcup C \prec Y \prec A$ and once again, we can conclude $Y \in A \sqcup (B \sqcup C)$;
- If $C \prec Y \prec A \sqcup B$, we know that $\exists X_1 \in A \sqcup B \, . \, C \prec Y \prec X_1$. We then know that either:

  1. $X_1 = A$. We then have $C \prec Y \prec A$, where as before, we have $C \in B \sqcup C$. So we conclude $B \sqcup C \prec Y \prec A \implies Y \in A \sqcup (B \sqcup C)$.
  2. $X_1 = B$. This means $C \prec Y \prec B$, hence $Y \in B \sqcup C$. So by extension, $Y \in A \sqcup (B \sqcup C)$.
  3. $A \prec X_1 \prec B$. In that case, we have by transitivity $C \prec Y \prec B$, so this gives $Y \in A \sqcup (B \sqcup C)$.
  4. $B \prec X_1 \prec A$. By transitivity again, we have $C \prec Y \prec A$ and $C \in C \sqcup B$. So: $C \sqcup B \prec Y \prec A$ and hence $Y \in A \sqcup (C \sqcup B)$



(a) Case 1. (b) Case 2. (c) Case 3. (d) Case 4.

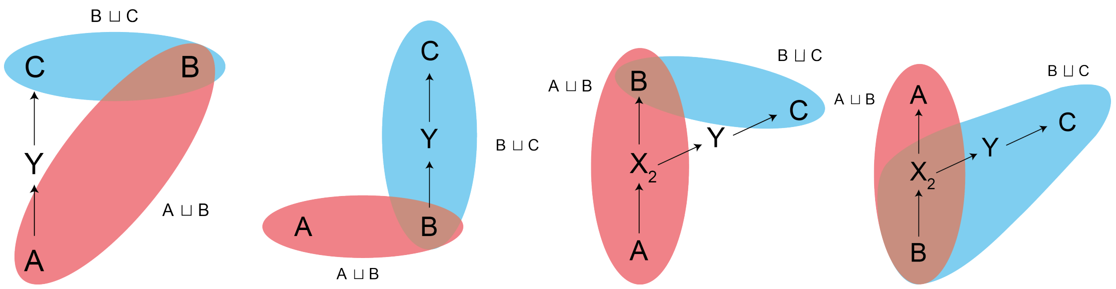
- If $A \sqcup B \prec Y \prec C$, we have $\exists X_2 \in A \sqcup B \, . \, X_2 \prec Y \prec C$. We can decompose in cases the same way:

  1. $X_2 = A$. In this case, we have $A \prec Y \prec C \implies A \prec Y \prec B \sqcup C$.

2. $X_2 = B$. This gives $B \prec Y \prec C$. Then $Y \in B \sqcup C$, and $Y \in A \sqcup (B \sqcup C)$.

3. $A \prec X_2 \prec B$. Here we will have $A \prec Y \prec C \implies A \prec Y \prec B \sqcup C$.

4. $B \prec X_2 \prec A$. Then: $B \prec Y \prec C \implies Y \in B \sqcup C$.



(a) Case 1. (b) Case 2. (c) Case 3. (d) Case 4.

Hence, we have proved:

$$(A \sqcup B) \sqcup C = C \sqcup (A \sqcup B) \subseteq A \sqcup (B \sqcup C) \tag{5.14}$$

By permuting the labels, we then obtain:

$$A \sqcup (B \sqcup C) \subseteq B \sqcup (C \sqcup A) \subseteq C \sqcup (A \sqcup B) = (A \sqcup B) \sqcup C \tag{5.15}$$

From there, we conclude: $(A \sqcup B) \sqcup C = A \sqcup (B \sqcup C) = A \sqcup B \sqcup C$.

□

From this result follows that we can define arbitrary pseudo-unions recursively from atomic events:

$$\mathcal{A} = \bigsqcup_{i \in I} A_i \qquad A_i \in V \ \forall i \in I \tag{5.16}$$

Given a process Φ with causal order $P_\Phi = (V, \prec)$, we can then define its set of *(localisable) events* $\mathcal{E}_\Phi(V)$ as the set of all subsets of V which have a decomposition as in (5.16). It is clear that for any W we have:

$$\mathcal{E}_\Phi(V) \subseteq \mathcal{P}(V)$$

where equality is achieved for example when all the events are no-signalling.

5.1.1 Indefinite causal structure

For a channel Φ with definite causal structure, we have seen that it is possible to obtain a causal graph, and a causal order from Φ; we have so far not attempted to obtain a similar representation for processes with indefinite causal structure.

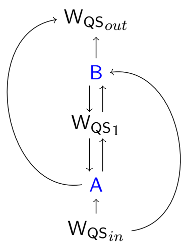
The difference between 2nd order processes inducing definite and indefinite causal structures is that in the case of indefinite causal structure, some of the wiring, i.e. the causal past and causal future of a given subprocess, is uncertain. For example, in the case of the quantum switch, we will have the two possible causal orders:

$$\begin{array}{c} \mathsf{W_{QS}}_{out} \\ \uparrow \\ \mathsf{B} \\ \uparrow \\ \mathsf{W_{QS}}_1 \\ \uparrow \\ \mathsf{A} \\ \uparrow \\ \mathsf{W_{QS}}_{in} \end{array} \quad \text{or} \quad \begin{array}{c} \mathsf{W_{QS}}_{out} \\ \uparrow \\ \mathsf{A} \\ \uparrow \\ \mathsf{W_{QS}}_1 \\ \uparrow \\ \mathsf{B} \\ \uparrow \\ \mathsf{W_{QS}}_{in} \end{array}$$

where $\mathsf{W_{QS}}_{in,out,1} = id_{C\otimes A}$. Then we see that in the general case, B is *to some extent* in both the causal future and the causal past of A (similarly, A is in the causal future and past of B). Hence, according to the previous definition, we would have:

$$\mathsf{A} \sqcup \mathsf{A} \equiv \{\mathsf{A}, \mathsf{B}, \mathsf{W_{QS}}_1\} \neq \{\mathsf{A}\} \tag{5.17}$$

(similarly, $\mathsf{B} \sqcup \mathsf{B} \equiv \{\mathsf{A}, \mathsf{B}, \mathsf{W_{QS}}_1\} \neq \{\mathsf{B}\}$) even though $\{\mathsf{A}\}$ (respectively $\{\mathsf{B}\}$) is atomic. More importantly, if the causal relations are written in the naive way, then the corresponding graph:



(5.18)

does not give rise to a partial order (namely, $\mathsf{X} \prec \mathsf{Y} \wedge \mathsf{Y} \prec \mathsf{X} \not\Rightarrow \mathsf{X} = \mathsf{Y}$).

Now, we also note that this is different from the Closed Timelike Curve (CTC)[25] situation, where B will genuinely be in both the causal past and future of A; whereas here measuring the control qubit will "collapse" the causal graph to one of the above. Hence, instead of having B being in both the causal past and future of A, we more accurately have B is either in the causal past of A *or* in its causal future, but not both, at least in a deterministic way. We can therefore denote the causal graph associated with the switch as:

$$\begin{array}{ccc} \mathsf{W_{QS}}_{out} & & \mathsf{W_{QS}}_{out} \\ \uparrow & & \uparrow \\ \mathsf{B} & & \mathsf{A} \\ \uparrow & & \uparrow \\ \mathsf{W_{QS}}_1 & \vee & \mathsf{W_{QS}}_1 \\ \uparrow & & \uparrow \\ \mathsf{A} & & \mathsf{B} \\ \uparrow & & \uparrow \\ \mathsf{W_{QS}}_{in} & & \mathsf{W_{QS}}_{in} \end{array} \tag{5.19}$$

which is not so convenient since both A and B appear twice. So, for lack of a better notation, we will "split" each atomic event according to the branching which it belongs to, and adopt the following notation for the switch:

$$\begin{array}{ccc} \mathsf{W_{QS}}_{out}^{\mathsf{A}\prec\mathsf{B}} & & \mathsf{W_{QS}}_{out}^{\mathsf{B}\prec\mathsf{A}} \\ \uparrow & & \uparrow \\ \mathsf{B}^{\mathsf{A}\prec\mathsf{B}} & & \mathsf{A}^{\mathsf{B}\prec\mathsf{A}} \\ \uparrow & & \uparrow \\ \mathsf{W_{QS}}_1^{\mathsf{A}\prec\mathsf{B}} & \vee & \mathsf{W_{QS}}_1^{\mathsf{B}\prec\mathsf{A}} \\ \uparrow & & \uparrow \\ \mathsf{A}^{\mathsf{A}\prec\mathsf{B}} & & \mathsf{B}^{\mathsf{B}\prec\mathsf{A}} \\ \uparrow & & \uparrow \\ \mathsf{W_{QS}}_{in}^{\mathsf{A}\prec\mathsf{B}} & & \mathsf{W_{QS}}_{in}^{\mathsf{B}\prec\mathsf{A}} \end{array} \tag{5.20}$$

We then obtain the "signalling[2] conditions":

$$\begin{aligned} &\mathsf{A}^{\mathsf{A}\prec\mathsf{B}} \prec \mathsf{B}^{\mathsf{A}\prec\mathsf{B}} \\ &\mathsf{B}^{\mathsf{B}\prec\mathsf{A}} \prec \mathsf{A}^{\mathsf{B}\prec\mathsf{A}} \\ &\mathsf{A}^{\mathsf{A}\prec\mathsf{B}} \not\prec\not\succ \mathsf{A}^{\mathsf{B}\prec\mathsf{A}} \\ &\mathsf{A}^{\mathsf{A}\prec\mathsf{B}} \not\prec\not\succ \mathsf{B}^{\mathsf{B}\prec\mathsf{A}} \\ &\quad\ldots \end{aligned}$$

and will refer to the global events as:

$$\mathsf{A} = \mathsf{A}^{\mathsf{A}\prec\mathsf{B}} \sqcup \mathsf{A}^{\mathsf{B}\prec\mathsf{A}} \equiv \left\{\mathsf{A}^{\mathsf{A}\prec\mathsf{B}}, \mathsf{A}^{\mathsf{B}\prec\mathsf{A}}\right\} \tag{5.21}$$

$$\mathsf{B} = \mathsf{B}^{\mathsf{A}\prec\mathsf{B}} \sqcup \mathsf{B}^{\mathsf{B}\prec\mathsf{A}} \equiv \left\{\mathsf{B}^{\mathsf{A}\prec\mathsf{B}}, \mathsf{B}^{\mathsf{B}\prec\mathsf{A}}\right\} \tag{5.22}$$

(and similarly for the internal events).

[2]There are not actually signalling conditions, but rather a practical way of representing the fact that the two branching are incompatible, i.e. cannot happen at the same time deterministically.

Remark 24. Using this way of dealing with indefinite structures, the different branches behave as if they were representing no-signalling channels. This implies that we obtain, as one would expect:

$$\mathsf{A} \sqcup \mathsf{A} = \mathsf{A} \tag{5.23}$$

and

$$\mathsf{A} \sqcup \mathsf{B} = \mathsf{A} \cup \mathsf{B} \cup \mathsf{W}_{\mathsf{QS}_1} \tag{5.24}$$

for the quantum switch, where $\cup$ is the standard set theoretic union.

This splitting is however fundamentally different from taking two no-signalling channels they cannot deterministically happen at the same time.

From this remark, one can see that we can naturally extend this definition for any process with indefinite causal structure: if a process induces n possible causal orders associated with the graphs $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$, we define the global causal graph as:

$$\bigvee_{i=1}^{n} \mathcal{G}_i \tag{5.25}$$

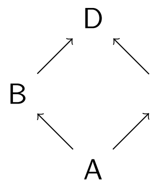
and each event is split into n subevents, labelled by the corresponding branching; and a global event is defined as:

$$\mathsf{X} = \bigsqcup_{i=1}^{n} \mathsf{X}^{\mathcal{G}_i} \tag{5.26}$$

5.1.2 Inclusion and non-separability of events

We have seen in the previous example that, in order to avoid causal loops in a causal diagram, some events cannot be made distinct. Here is a collection of other similar examples:

- In the initial example of the swapping channel, the events (A, B) and $(B.A)$ cannot be considered separately;
- For the causal graph:

  (5.27)

  the event B cannot separated from the generalised event $\mathcal{A} = \mathsf{A} \sqcup \mathsf{D}$;
- In the case of the switch, the split events $\mathsf{A}^{\mathsf{A}\prec\mathsf{B}}$ and $\mathsf{A}^{\mathsf{B}\prec\mathsf{A}}$ are not separable from the global event A, since they cannot "exist" independently.

In all of these cases, we see that the *non-separability* comes from the fact that:

- Neither of the singletons $\{(A, B)\}$ and $\{(B, A)\}$ in the swap channel are in the set of events $\mathcal{E}_\Phi$, but are somewhat included in the event $\{(A, A)\} \sqcup \{(B, B)\}$, which is valid (i.e. localisable);

- From the DAG (5.27), we see that the event $\mathsf{B} \in \mathsf{A} \sqcup \mathsf{D}$ as seen as sets of localisable events;

- By definition, $\mathsf{A}^{\mathsf{A} \prec \mathsf{B}}$ and $\mathsf{A}^{\mathsf{B} \prec \mathsf{A}}$ are included in the set of events A.

Hence, we see that, in addition to the $\to$ relation between events, which represent the existence of a signalling correlation, we have, within this formalism an additional relation that we will denote $\sqsubseteq$ which represent the *inclusion relation*. In the previous examples (excluding the swap channel), we have:

- $\mathsf{B} \sqsubseteq \mathcal{A}$;

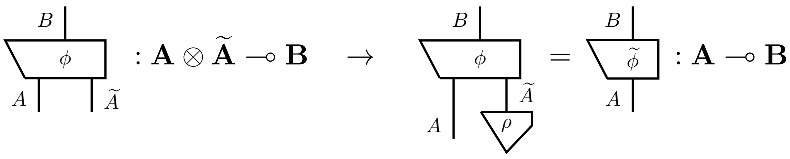
- $\mathsf{A}^{\mathsf{A} \prec \mathsf{B}}, \mathsf{A}^{\mathsf{B} \prec \mathsf{A}} \sqsubseteq \mathsf{A}$.

5.2 Bipartite 3rd order processes with definite causal structure

We now introduce the core result of this project, namely how can one construct third order processes. We will start by a fairly obvious way to act on supermaps, and then carry on to investigate more interesting composition methods.

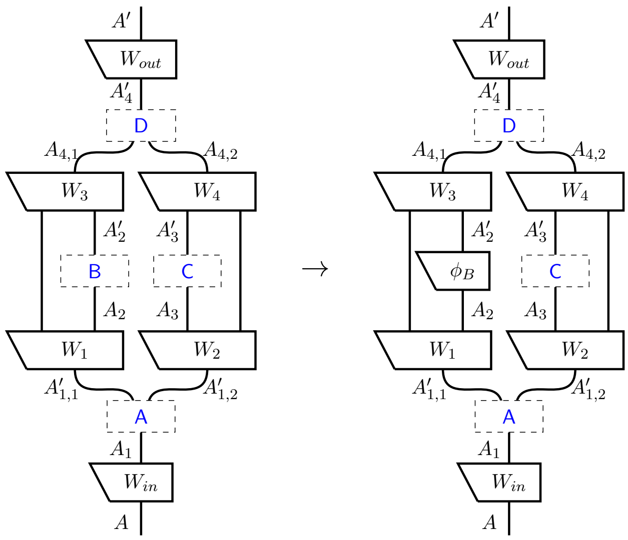
5.2.1 Internalisation and partial internalisation

A rather obvious thing to consider is to have a higher order direct analogue of pre-selection, i.e. imposing an initial state. In first order theory, this will translate as:



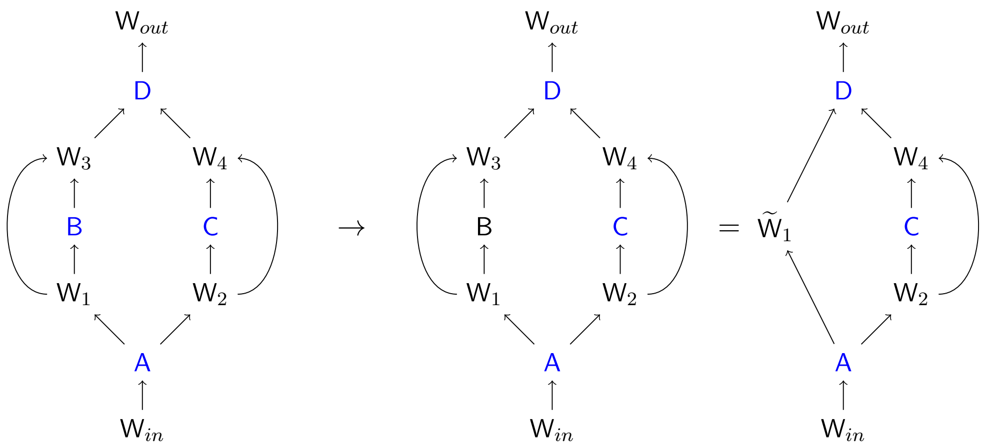
(5.28)

Note that it is in the general case possible to go from ϕ to $\widetilde{\phi}$, but not in the other direction. We can then do the same for supermaps, by imposing one of the input map, e.g.:



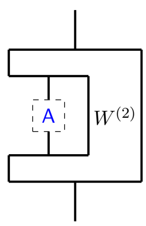
(5.29)

The causal graph in this example changes as follows:



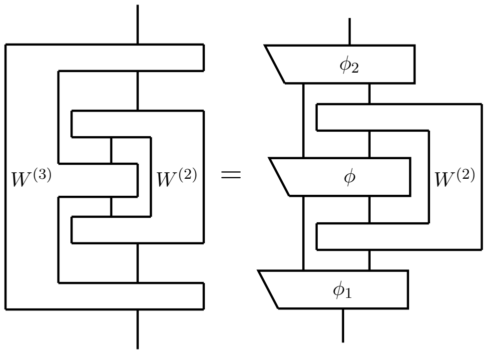
(5.30)

i.e. an external event is “converted” to an internal one; we will then refer to this process as *internalisation* of a node. Another more familiar example of internalisation is the 3-comb. To see this, consider a 2-comb:



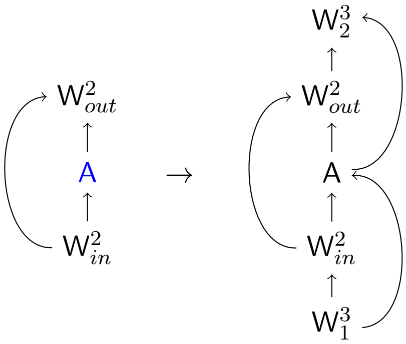
(5.31)

Then recall that a 3-comb takes in a 2-comb and outputs a first order process. Besides, it is also isomorphic to a tripartite channel with memory, so:



(5.32)

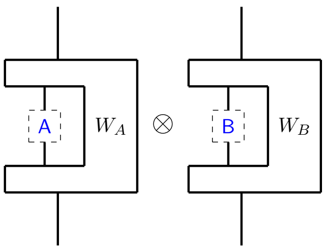
Then, applying the 3-comb $W^{(3)}$ to $W^{(2)}$ does correspond to an internalisation (with extra transformations ϕ_1, ϕ_2):



(5.33)

5.2.2 Composition rules

We now move on to proper 3rd order processes. We would like to look at simple examples first, and in particular at what are the different ways of composing second order processes to form new second order processes. We will in this section focus on the bipartite case with definite causal structure; note that bipartite here means that we are working with two (no-signalling) second order processes, and definite causal structure only states that we can always have a circuit representation involving supermaps, but theses supermaps themselves do not have to have a definite causal structure. In diagrams, we will take inputs of type:



(5.34)

with corresponding causal graph:

$$\begin{array}{ccc} \mathsf{W^{A}}_{out} & \quad & \mathsf{W^{B}}_{out} \\ \uparrow & & \uparrow \\ \mathsf{A} & & \mathsf{B} \\ \uparrow & & \uparrow \\ \mathsf{W^{A}}_{in} & & \mathsf{W^{B}}_{in} \end{array} \tag{5.35}$$

where that A and B can have any kind of structure, and even include some internal processes too. We will denote by $\mathcal{A}$ (respectively $\mathcal{B}$) the event corresponding to the whole process $\mathcal{A} = \mathsf{W}^{\mathsf{A}}_{in} \sqcup \mathsf{W}^{\mathsf{A}}_{out}$ (respectively $\mathcal{B} = \mathsf{W}^{\mathsf{B}}_{in} \sqcup \mathsf{W}^{\mathsf{B}}_{out}$).

Also note that for simplicity, we will drop type labels on diagrams and assume that the depicted morphisms are well-formed.

In first order processes, all events are internal, and hence, the only way to combine quantum maps is to either compose them in parallel via the tensor product ($\otimes \simeq \parr$ in first order theory), or to connect their inputs or outputs, which can then be seen as their *boundaries*.

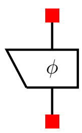


Figure 5.4: Boundaries of a first order process.

For second order processes, there are more ways we can connect supermaps since supermaps have two types of “boundaries”: wires connecting the input maps to the superoperator, and the input and output wires of the resulting process. We will refer to those respectively as the *primary boundaries* and the *secondary boundary*.

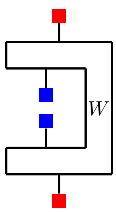
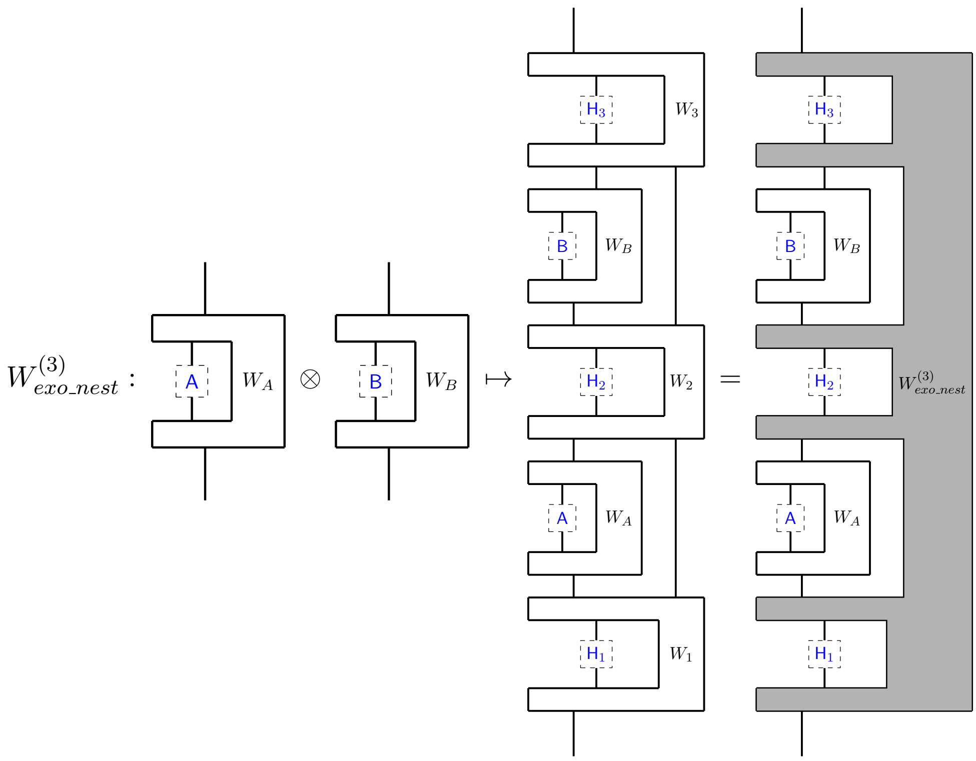


Figure 5.5: Boundaries of a second order process. The primary boundaries are depicted in blue; the secondary boundaries are depicted in red.

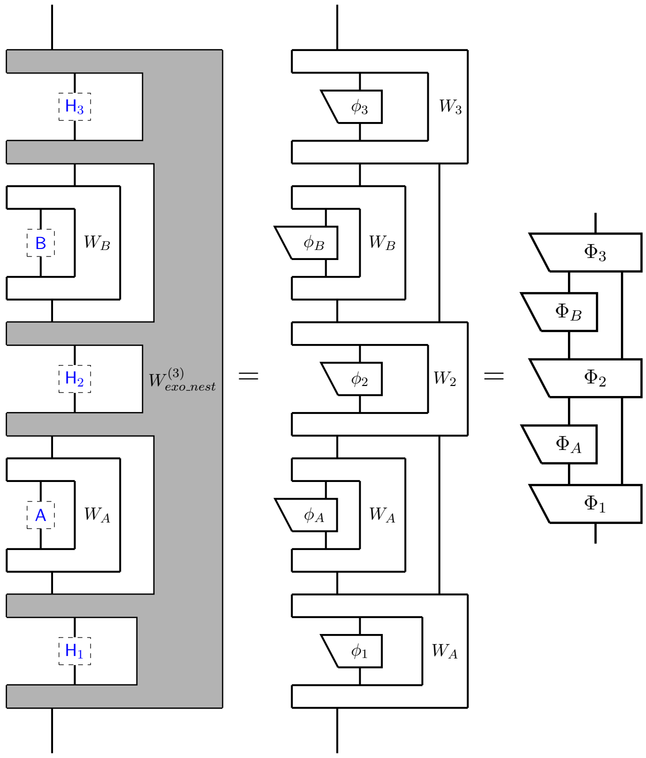
Adding a global order Since second order processes send admissible maps to admissible maps, we can start by “ignoring” the input maps and treat superoperators as simple quantum processes, i.e. only consider the secondary boundaries. Then, since this type of composition only depends on the nature of the output, namely an admissible map. We can hence use what we already know about second order processes, and extend it to third order processes by replacing every first order quantum map by a second order process. The case of bipartite input with definite causal structure is once again easy, since in second order theory, the only type of process satisfying those conditions are 3-combs. So we can extend the combs to a new, purely third order process:



(5.36)

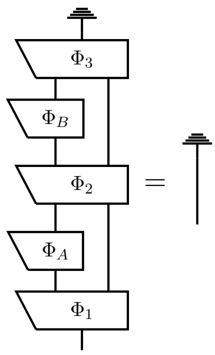
For some second order admissible supermaps W_1, W_2, W_3. For the rest of this chapter, we will highlight the external interventions purely internal to third order processes in grey.

It is easy to see that $W^{(3)}_{exo\_nest}$ is 3rd order admissible since: for every W_A, W_B (second order) admissible, and for all ϕ_1, ϕ_2, ϕ_3, ϕ_A, ϕ_B admissible maps, we have:



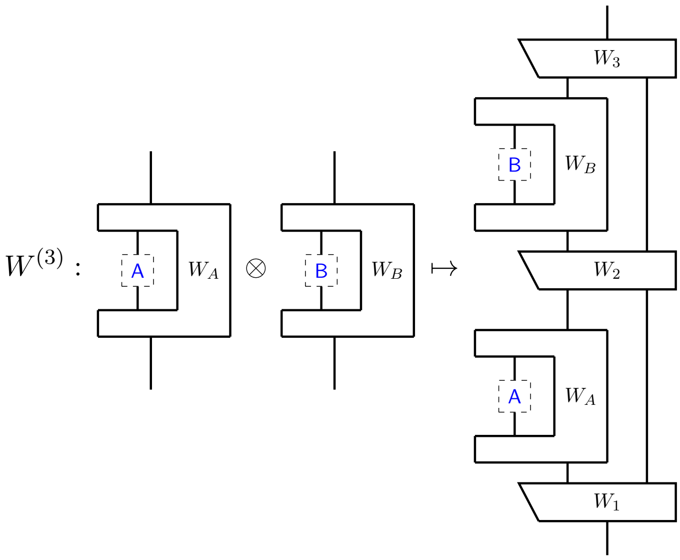
(5.37)

where all the processes $\Phi_{1,2,3,A,B}$ are admissible, hence:



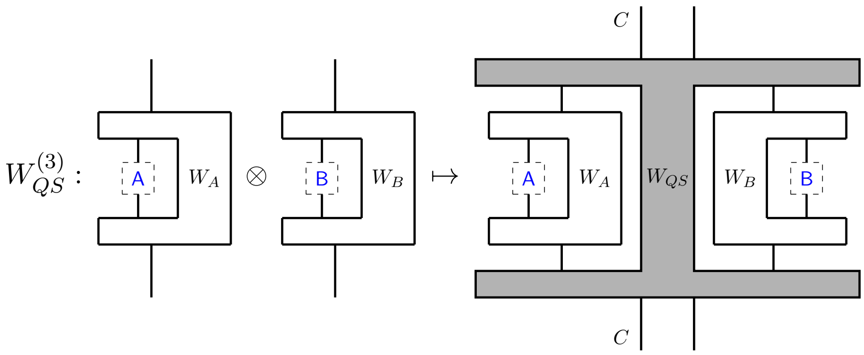
(5.38)

We can from this example make a few preliminary remarks. First of all, since quantum maps (i.e. first order processes) are special cases of supermaps, then we can obtain a special case (5.36) where $W_{1,2,3}$ are simply processes, which gives:



(5.39)

i.e. $W^{(3)}$ is a comb. Secondly, the extension from 2nd order to third order only consists of nesting supermap inside another supermaps: we will call this process *exo-nesting*. From this principle, we see that any supermap (even the switch) can be "converted" into a third order process by nesting supermaps inside the "slots" of the input maps, e.g.:



(5.40)

Now, we note that exo-nesting as defined in (5.36) is always defined, irrespective of the causal relations induced by W_A or W_B. In fact, $W^{(3)}_{exo\_nest}$ acts on the events $\mathcal{A}$ and $\mathcal{B}$ as if all events in them were internal. Hence, denoting the external (w.r.t. $W^{(3)}_{exo\_nest}$) global events in red, and the global events corresponding to the supermaps internal to $W^{(3)}_{exo\_nest}$

as $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, the (minimal) causal graph relative to (5.36) is:

$$\begin{array}{c} \mathcal{H}_3 \\ \uparrow \\ \mathcal{B} \\ \uparrow \\ \mathcal{H}_2 \\ \uparrow \\ \mathcal{A} \\ \uparrow \\ \mathcal{H}_1 \end{array} \tag{5.41}$$

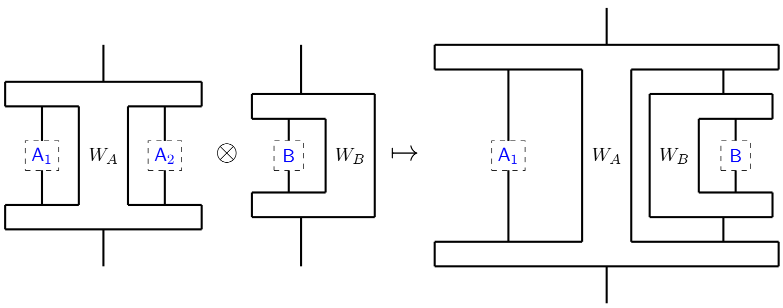
Now, we can see the parallel with standard 3-combs which induce, for A and B no-signalling, the (minimal) causal graph:

$$\begin{array}{c} \mathsf{W}_3 \\ \uparrow \\ \mathsf{B} \\ \uparrow \\ \mathsf{W}_2 \\ \uparrow \\ \mathsf{A} \\ \uparrow \\ \mathsf{W}_1 \end{array}$$

Hence, we say that $W^{(3)}_{exo\_nest}$ as defined in (5.36) induces a *global causal order*.

Adding internal structure We have seen that we can nest supermaps inside others to obtain new ones. Now, given two supermap inputs, one can also nest them inside one another. For example, consider a bipartite superoperator and another arbitrary supermap, we can then have a third order process $W^{(3)}_{intra\_nest}$ acting on those as follows:

$W^{(3)}_{intra\_nest}$:



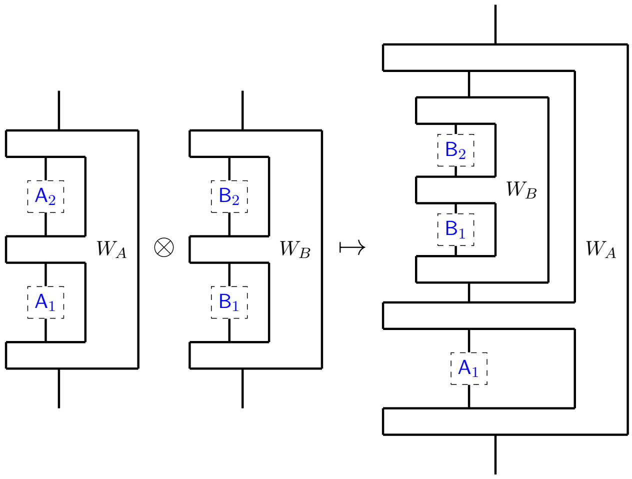
(5.42)

The effect of such a process will, in the causal graph, replace an external event in the causal graph of the first input by an event with a richer structure. For example if both W_A and

W_B in (5.42) have the structure of a comb, i.e.:

$$W_A \models \begin{array}{c} \mathsf{W^A}_{out} \\ \uparrow \\ \color{blue}{\mathsf{A}_2} \\ \uparrow \\ \mathsf{W^A}_1 \\ \uparrow \\ \color{blue}{\mathsf{A}_1} \\ \uparrow \\ \mathsf{W^A}_{in} \end{array} \qquad W_B \models \begin{array}{c} \mathsf{W^B}_{out} \\ \uparrow \\ \color{blue}{\mathsf{B}_2} \\ \uparrow \\ \mathsf{W^B}_1 \\ \uparrow \\ \color{blue}{\mathsf{B}_1} \\ \uparrow \\ \mathsf{W^B}_{in} \end{array}$$

Then (5.42) becomes:



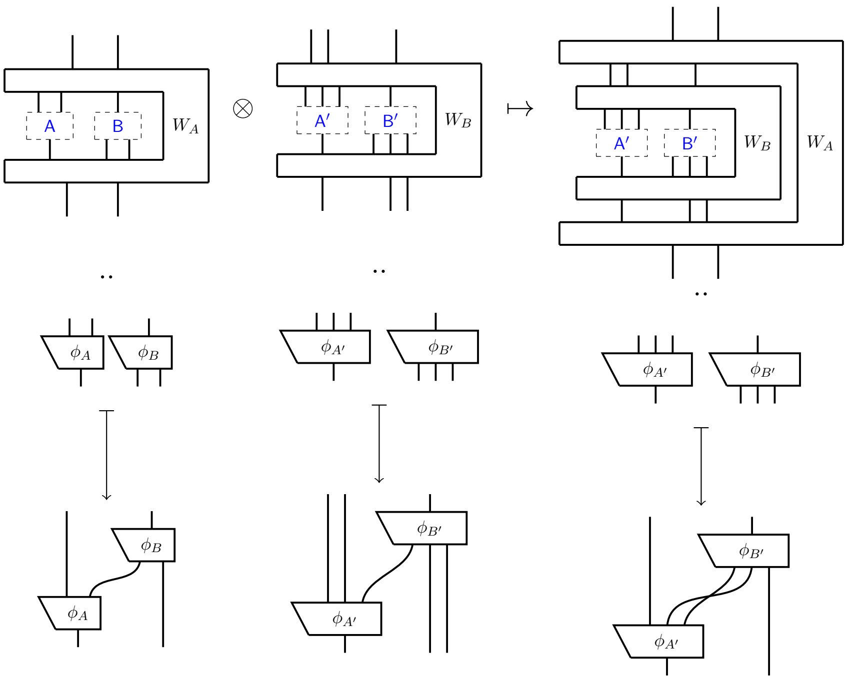
(5.43)

and the causal graph becomes:

$$\begin{array}{c} \mathsf{W^A}_{out} \\ \uparrow \\ \color{blue}{\mathsf{A}_2} \\ \uparrow \\ \mathsf{W^A}_1 \\ \uparrow \\ \color{blue}{\mathsf{A}_1} \\ \uparrow \\ \mathsf{W^A}_{in} \end{array} \qquad \begin{array}{c} \mathsf{W^B}_{out} \\ \uparrow \\ \color{blue}{\mathsf{B}_2} \\ \uparrow \\ \mathsf{W^B}_1 \\ \uparrow \\ \color{blue}{\mathsf{B}_1} \\ \uparrow \\ \mathsf{W^B}_{in} \end{array} \quad \mapsto \quad \begin{array}{c} \mathsf{W^A}_{out} \\ \uparrow \\ \mathsf{W^B}_{out} \\ \uparrow \\ \color{blue}{\mathsf{B}_2} \\ \uparrow \\ \mathsf{W^B}_1 \\ \uparrow \\ \color{blue}{\mathsf{B}_1} \\ \uparrow \\ \mathsf{W^B}_{in} \\ \uparrow \\ \mathsf{W^A}_1 \\ \uparrow \\ \color{blue}{\mathsf{A}_1} \\ \uparrow \\ \mathsf{W^A}_{in} \end{array} \tag{5.44}$$

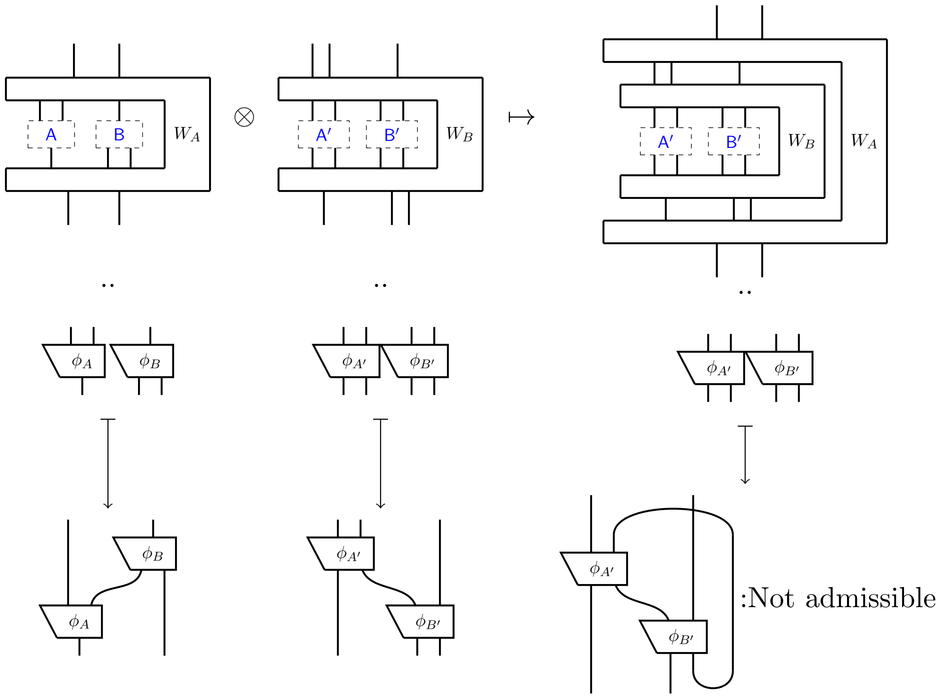
In other words $\mathsf{A}_2 \mapsto \mathcal{B}$, and every element X of $\mathcal{B}$ has, after application of $W^{(3)}_{intra\_nest}$, $\mathsf{X} \sqsubseteq \mathsf{A}_2$. Hence, we can differentiate the processes defined in (5.36) and the one in consideration here as follows: given two processes with global events $\mathcal{A}, \mathcal{B}$, the higher order comb gives a superoperator with a global (external) causal order of the form $\mathcal{A} \to \mathcal{B}$, whereas the process described in (5.42) gives rise to a causal order of the form $\mathcal{B} \sqsubseteq \mathcal{A}$.

If we restrict to the substitution of one (atomic) external event in W_A by W_B, then $W^{(3)}_{intra\_nest}$ is always well-defined. However, one can consider replacing more than one external node in the causal graph of W_A by W_B; this is possible as long as the causal order of W_B is compatible with the sub-order which is getting replaced in W_A, i.e. if it does not induce a causal loop. For example, the following does give rise to an admissible second order process:



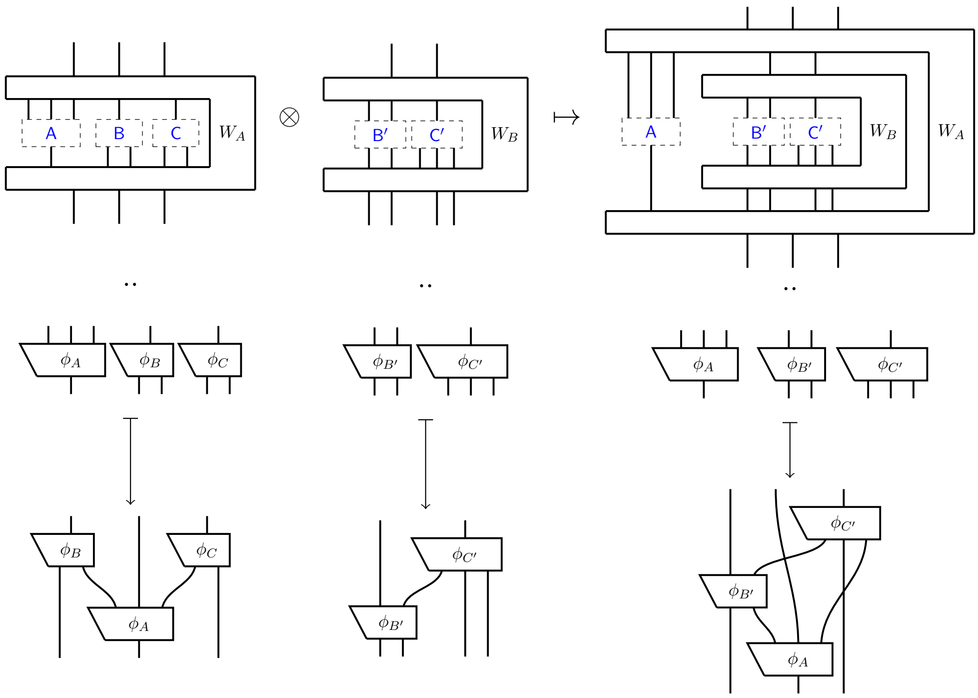
(5.45)

while reversing the causal ordering of A' and B' in W_B does not:



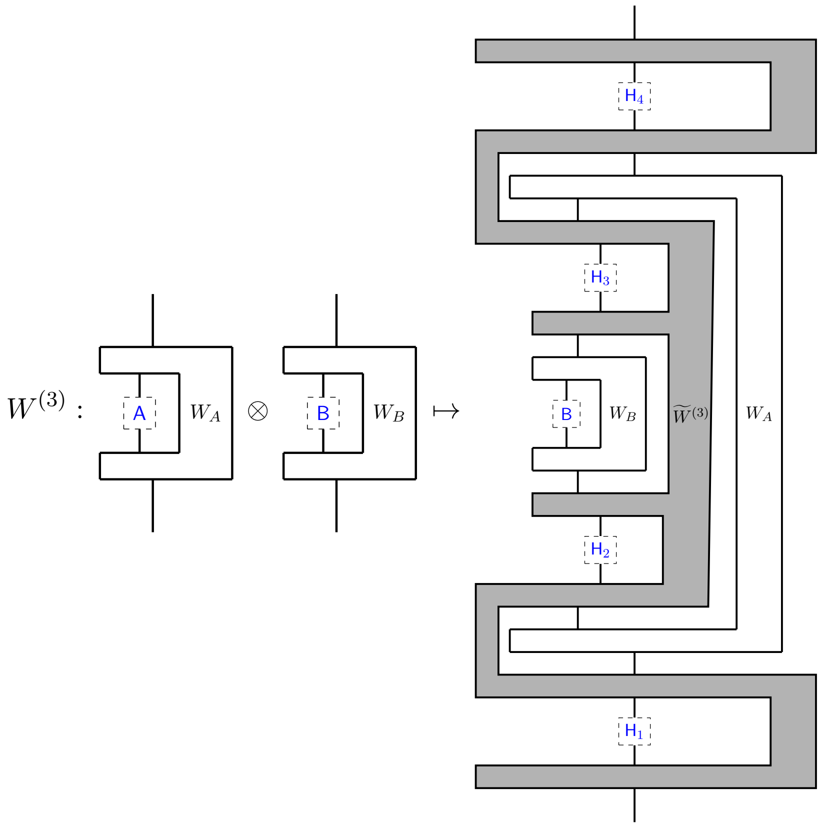
(5.46)

Furthermore, since no-signalling events are compatible with every causal order, then the following is also valid:



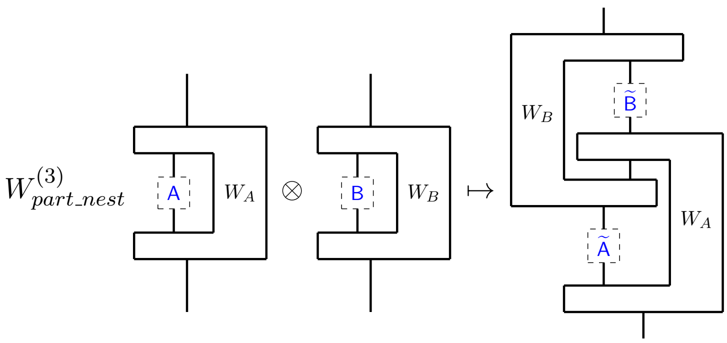
(5.47)

One can also use the two previously discussed way of combining, namely exo-nesting and "intra"-nesting as follows.:



(5.48)

One last thing on this subject is that we can not only nest supermap within each other, but also *partially* nest them, for example:



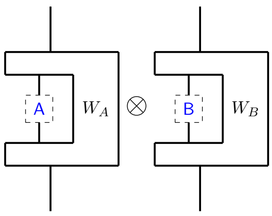
(5.49)

In this case, we see that the resulting causal graph is of the form:

$$\begin{array}{c} \mathsf{W^B}_{out} \\ \uparrow \\ \widetilde{\mathsf{B}} \\ \uparrow \\ \mathsf{W^A}_{out} \\ \uparrow \\ \mathsf{W^B}_{in} \\ \uparrow \\ \widetilde{\mathsf{A}} \\ \uparrow \\ \mathsf{W^A}_{in} \end{array}$$

So in this case the events A and B are sent, under $W^{(3)}_{part\_nest}$ to respectively $\widetilde{\mathsf{A}} \sqcup \mathsf{W^B}_{in}$ and $\mathsf{W^A}_{out} \sqcup \widetilde{\mathsf{B}}$. This is a first example of a third order process making two supermap non-separable. We will now see more example of such third order maps.

Altering the input structure So far, we have seen how to embed supermaps into a global order, or how to insert extra structure on a superoperator; we will now see how, in some cases, it is possible to completely change the structure of the input maps of two (or several) given supermaps. We will in particular focus on the case when two supermaps:

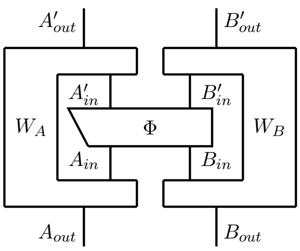


$: \big[(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}) \multimap (\mathbf{A}_{out} \multimap \mathbf{A}'_{out})\big] \otimes \big[(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}) \multimap (\mathbf{B}_{out} \multimap \mathbf{B}'_{out})\big]$

can be extended to a supermap of the type:

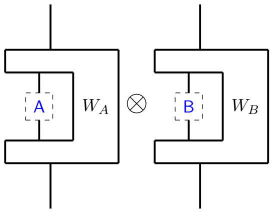
$$\big[(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}) \parr (\mathbf{B}_{in} \multimap \mathbf{B}'_{in})\big] \multimap \big[\mathbf{A}_{out} \otimes \mathbf{B}_{out} \multimap \mathbf{A}'_{out} \otimes \mathbf{B}'_{out}\big]$$

i.e. supermaps for which:



is admissible for *any* admissible map $\Phi : A_{in} \otimes B_{in} \to A'_{in} \otimes B'_{in}$.

Remark 25. From (4.25), we known that for any type $\mathbf{A}, \mathbf{B}$ there exists a canonical embedding $\mathbf{A} \otimes \mathbf{B} \hookrightarrow \mathbf{A} \parr \mathbf{B}$. Hence:



$$: \left[(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}) \multimap (\mathbf{A}_{out} \multimap \mathbf{A}'_{out})\right] \parr \left[(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}) \multimap (\mathbf{B}_{out} \multimap \mathbf{B}'_{out})\right]$$

Now, we have:

$$\begin{aligned}
&\left[(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}) \multimap (\mathbf{A}_{out} \multimap \mathbf{A}'_{out})\right] \parr \left[(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}) \multimap (\mathbf{B}_{out} \multimap \mathbf{B}'_{out})\right] \\
&= \left(\left[(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}) \multimap (\mathbf{A}_{out} \multimap \mathbf{A}'_{out})\right]^* \otimes \left[(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}) \multimap (\mathbf{B}_{out} \multimap \mathbf{B}'_{out})\right]^*\right)^* \\
&= \left(\left(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}\right) \otimes \left(\mathbf{A}_{out} \multimap \mathbf{A}'_{out}\right)^* \otimes \left(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}\right) \otimes \left(\mathbf{B}_{out} \multimap \mathbf{B}'_{out}\right)^*\right)^* \\
&\simeq \left(\left(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}\right) \otimes \left(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}\right) \otimes \left(\mathbf{A}_{out} \multimap \mathbf{A}'_{out}\right)^* \otimes \left(\mathbf{B}_{out} \multimap \mathbf{B}'_{out}\right)^*\right)^* \\
&\simeq \left(\left(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}\right) \otimes \left(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}\right) \otimes \left[\left(\mathbf{A}_{out} \multimap \mathbf{A}'_{out}\right) \parr \left(\mathbf{B}_{out} \multimap \mathbf{B}'_{out}\right)\right]^*\right)^* \\
&= \left(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}\right) \otimes \left(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}\right) \multimap \left(\mathbf{A}_{out} \multimap \mathbf{A}'_{out}\right) \parr \left(\mathbf{B}_{out} \multimap \mathbf{B}'_{out}\right)
\end{aligned}$$

and since all the types $\mathbf{A}_{out}$, $\mathbf{A}'_{out}$, $\mathbf{B}_{out}$, $\mathbf{B}'_{out}$ are first order, then we have:

$$\begin{aligned}
\left(\mathbf{A}_{out} \multimap \mathbf{A}'_{out}\right) \parr \left(\mathbf{B}_{out} \multimap \mathbf{B}'_{out}\right) &= \mathbf{A}^*_{out} \parr \mathbf{A}'_{out} \parr \mathbf{B}^*_{out} \parr \mathbf{B}'_{out} \\
&\simeq \left(\mathbf{A}_{out} \otimes \mathbf{B}_{out}\right)^* \parr \left(\mathbf{A}'_{out} \otimes \mathbf{B}'_{out}\right) \\
&= \left(\mathbf{A}_{out} \otimes \mathbf{B}_{out}\right) \multimap \left(\mathbf{A}'_{out} \otimes \mathbf{B}'_{out}\right) \qquad (5.50)
\end{aligned}$$

Therefore $[(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}) \multimap (\mathbf{A}_{out} \multimap \mathbf{A}'_{out})] \parr [(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}) \multimap (\mathbf{B}_{out} \multimap \mathbf{B}'_{out})]$ is isomorphic to :

$$\left(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}\right) \otimes \left(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}\right) \multimap \left(\mathbf{A}_{out} \otimes \mathbf{B}_{out} \multimap \mathbf{A}'_{out} \otimes \mathbf{B}'_{out}\right) \qquad (5.51)$$

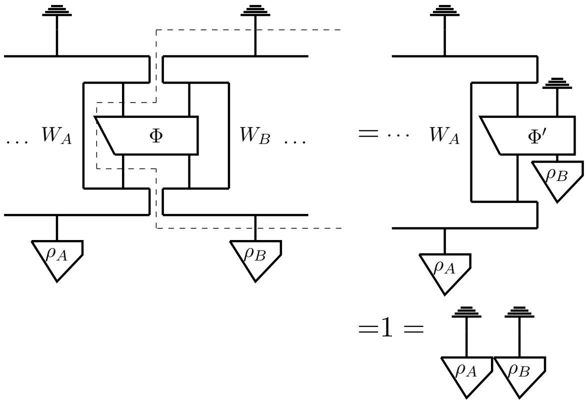
So the required extendability condition corresponds to the situation when pair of supermaps[3] gives rise to an admissible operator in (5.51) via the embedding:

$$\left(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}\right) \otimes \left(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}\right) \hookrightarrow \left(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}\right) \parr \left(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}\right)$$

Also note that we have seen in the example (5.46) that this is not always the case, but it is always the case when considering only one external event each. To see this, one notes

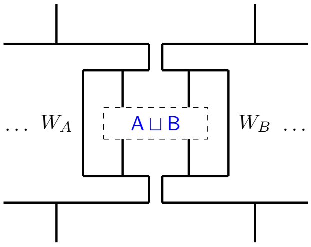
[3] As seen as a superoperator in $[(\mathbf{A}_{in} \multimap \mathbf{A}'_{in}) \multimap (\mathbf{A}_{out} \multimap \mathbf{A}'_{out})] \parr [(\mathbf{B}_{in} \multimap \mathbf{B}'_{in}) \multimap (\mathbf{B}_{out} \multimap \mathbf{B}'_{out})]$.

that for every W_A, W_B admissible, we have the following:



(5.52)

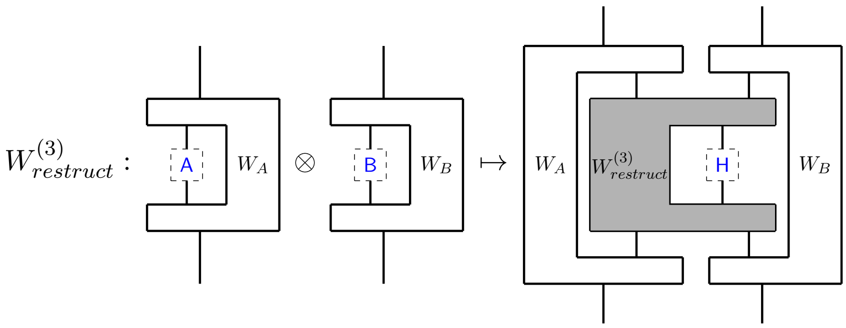
Since, assuming we have enough admissible states, all second order admissible processes are "completely" admissible (see Corollary 1). Hence, if we have enough admissible states, we indeed obtain that:



(5.53)

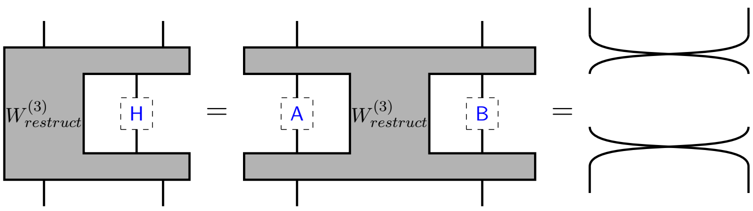
is second order admissible.

When an extension is possible, then we have more freedom in defining new supermaps by correlating the input channels; in particular, we can impose *any* type of signalling conditions on the events. One example is to define the third order map, acting as before on $W_A \otimes W_B$ as:



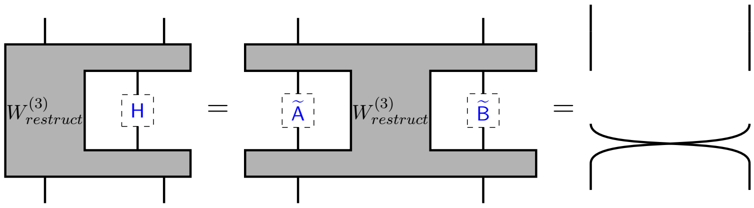
(5.54)

In terms of events, we see that in this case $\mathsf{A} \sqcup \mathsf{B}$ is mapped to a new global $\mathcal{H}$, which is in general not separable. So as in (5.49), the two supermaps are now themselves non-separable. Some interesting examples of such processes are swapping the events A and B when:



(5.55)

Or swapping the input (equivalently output) Hilbert spaces of A and B, while keeping the output (respectively input) Hilbert spaces constants:



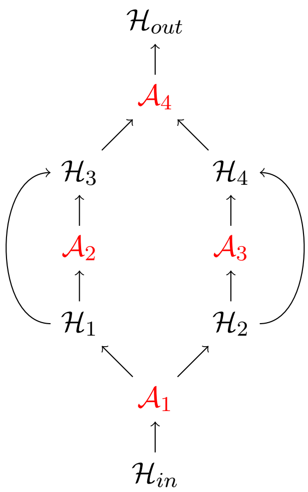
(5.56)

i.e. for $\mathsf{A} = (A, A')$, $\mathsf{B} = (B, B')$, $\mathsf{A} \sqcup \mathsf{B} \mapsto \widetilde{\mathsf{A}} \sqcup \widetilde{\mathsf{B}}$ where $\widetilde{\mathsf{A}} = (A, B')$ and $\widetilde{\mathsf{B}} = (B, A')$.

We have now seen a few examples of third order processes acting on two no-signalling supermaps. We have also seen that some of these compositions have a natural extension to more than two supermap-inputs (e.g. exo-nesting). We will more generally see in the next section how all these composition rules extend to n-inputs third order processes.

5.3 Multipartite processes

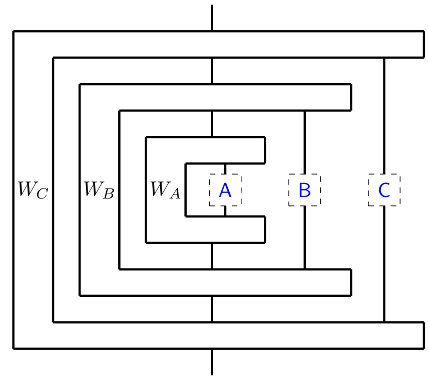
First recall that exo-nesting is the direct generalisation of supermaps into third order processes; hence, we can fairly easily extend this type of composition to an arbitrary number of supermap-inputs. The resulting supermap can, if we keep the definite causal structure assumption, be represented by a causal graph, e.g.:



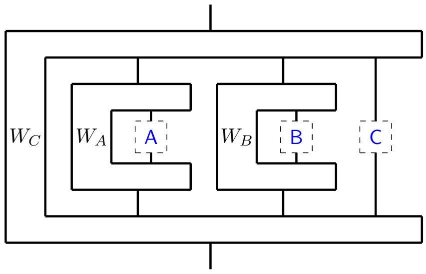
(5.57)

in which every node is allowed to have both external and internal events.

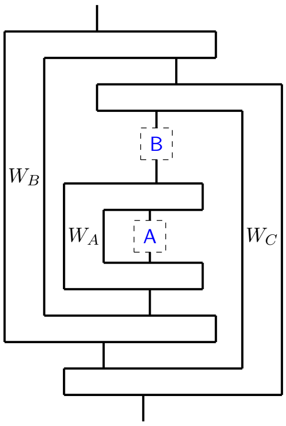
In addition, the inclusion relation $\sqsubseteq$ also defines a partial order; one can also obtain a DAG representation if considering nesting only. For example, for three input supermaps with associated global events $\mathcal{A}, \mathcal{B}, \mathcal{C}$, one can have:



$$: \mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{C} \mapsto \mathcal{A} \dashrightarrow \mathcal{B} \dashrightarrow \mathcal{C} \tag{5.58}$$



$$: \mathcal{A}, \mathcal{B} \sqsubseteq \mathcal{C} \mapsto \begin{matrix} \mathcal{A} \dashrightarrow \mathcal{C} \\ \mathcal{B} \dashrightarrow \mathcal{C} \end{matrix} \tag{5.59}$$



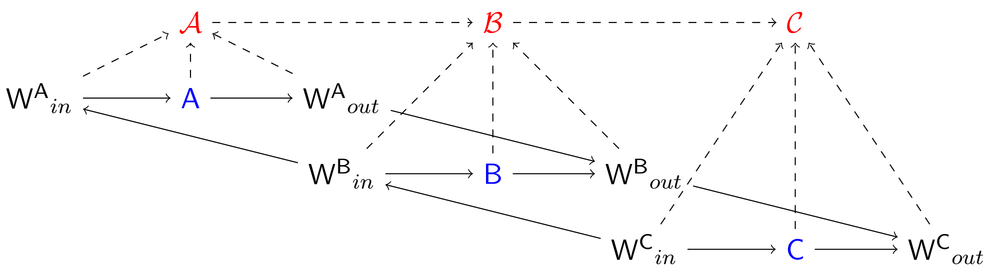
$$: \mathcal{A} \sqsubseteq \mathcal{B}, \mathcal{C} \mapsto \begin{matrix} \mathcal{A} \dashrightarrow \mathcal{B} \\ \mathcal{A} \dashrightarrow \mathcal{C} \end{matrix} \tag{5.60}$$

We note that the acyclicity of the obtained graph comes from the fact that for any events $\mathcal{A}, \mathcal{B}$, $\mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{A}$ iff $\mathcal{A} = \mathcal{B}$. To see this we know:

$$\mathcal{A} \sqsubseteq \mathcal{B} \implies \text{for some supermap } W_A : \mathcal{A} = \mathsf{W^A}_{in} \sqcup \mathsf{W^A}_{out} \sqsubseteq \mathcal{B}$$
$$\mathcal{B} \sqsubseteq \mathcal{A} \implies \text{for some supermap } W_B : \mathcal{B} = \mathsf{W^B}_{in} \sqcup \mathsf{W^B}_{out} \sqsubseteq \mathcal{A} = \mathsf{W^A}_{in} \sqcup \mathsf{W^A}_{out}$$

which implies that $\mathcal{B} = \mathsf{W^B}_{in} \sqcup \mathsf{W^B}_{out} = \mathsf{W^A}_{in} \sqcup \mathsf{W^A}_{out} = \mathcal{A}$.

Remark 26. We can then combine the two "types" of DAG to represent both the signalling and inclusion relations. For example (5.58) will have:



Finally, if one restricts itself to solely restructure the input structure, the only restriction is to preserve the existing signalling conditions.

Besides, since every third order maps send supermaps to supermaps, then we can use all these different ways of combining supermaps recursively to obtain more general third order processes. (See examples in the Appendix B.1).

Once again, I must emphasise that this is not an exhaustive list of all possible third order processes, or I do not make any claim about the generality of those composition rules. We have however seen that these examples already give rise to entirely new types of processes. Moreover, we obtain, notable via the inclusion $\sqsubseteq$ another dimension to act on, which gives a much larger variety of processes than what is present in first and second order theory.

We will now move on two third order processes with quantum uncertainty.

5.4 Adding uncertainty

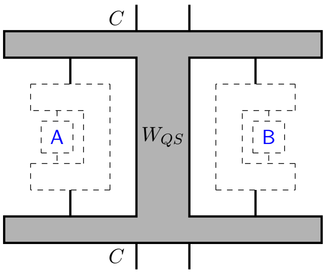
Let us now see how we can add uncertainty and "quantumness" to those third order processes. Recall that second order processes with indefinite causal structure, such as the switch, arises from the incompatibility of the two causal orders with $\mathsf{A} \prec \mathsf{B}$ and $\mathsf{B} \prec \mathsf{A}$. Now, as expected, since we have more ways of combining supermaps than ordinary quantum maps, there are more ways of obtaining "incompatible" connections.

The aim of this section is to describe some examples of third order processes for which its "circuit" representation is not deterministic, using controlled operations.

5.4.1 Examples arising from the quantum switch

A "cheap" way of obtaining third order processes with indefiniteness is to use known quantum supermaps with indefinite causal structure, such as the switch, and the previously described composition rules to obtain third order processes. We have seen in Section 5.2.2

that the quantum switch itself can be extended to a third order process:

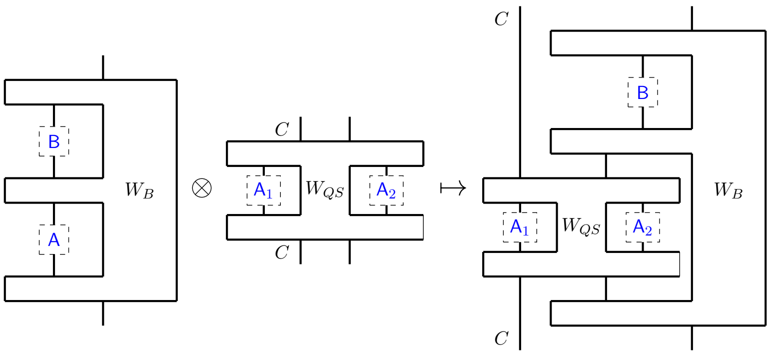


(5.61)

which gives a global indefinite causal structure.

Similarly, one can "add" indefiniteness to a given process by setting the switch as a subroutine. For example, given any map inducing a definite causal structure, e.g. a 3-comb, substituting one of the inputs with the switch introduces some indefiniteness in the resulting supermap:

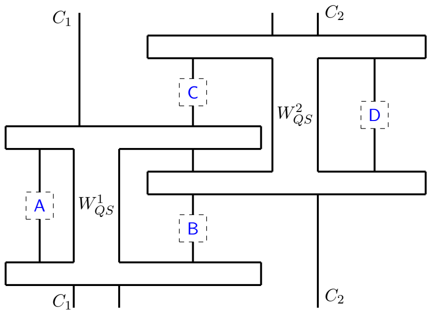
$W^{(3)}_{inta\_nest}$:



(5.62)

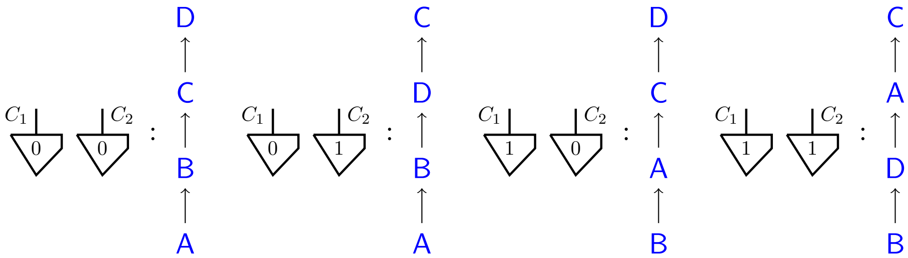
In this example, if W_B is indeed a comb, we will have $\mathsf{A}_1 \sqcup \mathsf{A}_2 \prec \mathsf{B}$, but the causal order between A_1 and A_2 will be uncertain.

We can similarly partially nest a quantum switch within another supermap; the effect of this is particularly visible when partially nesting a switch within another:



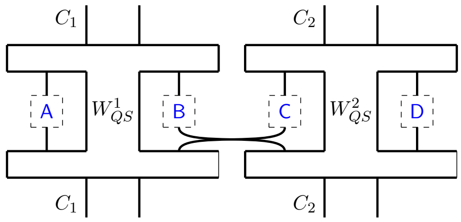
(5.63)

What we obtain is a supermap with indefinite causal structure induced by *two* control systems. Now, for every basis state in $C_1 \otimes C_2$, we obtain the corresponding (external) causal graphs:



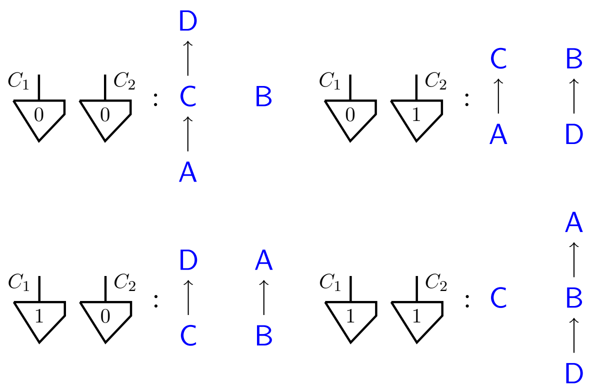
(see Appendix B.2 for details).

Finally, we recall that it is possible to associate one event from two supermaps. Hence, one can define the following:



(5.64)

Once again, the basis states of $C_1 \otimes C_2$ give rise to the following (external) causal graphs:



(see Appendix B.3 for details.)

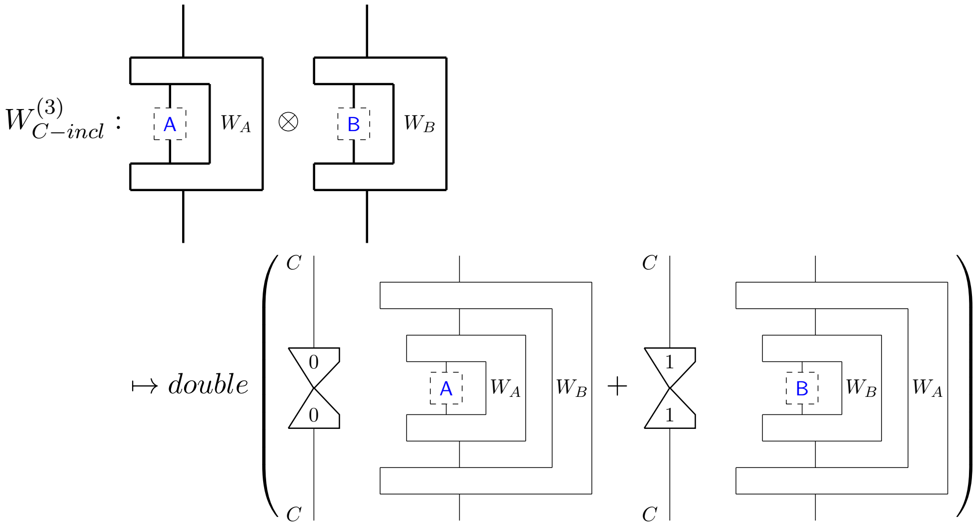
All these examples are not exactly new since they are all derived from the quantum switch. We have however seen in the previous section that, when considering 3rd order processes, we have more potential to obtain uncertainty. We will see some examples of such 3rd order maps now.

5.4.2 New examples

The main thing to notice from the previous "indefinite" processes, is that the uncertainty is restricted to the signalling relations only. Besides, recall that we can also induce an ordering w.r.t. $\sqcup$, or even completely change the input structure of two (or more) supermaps.

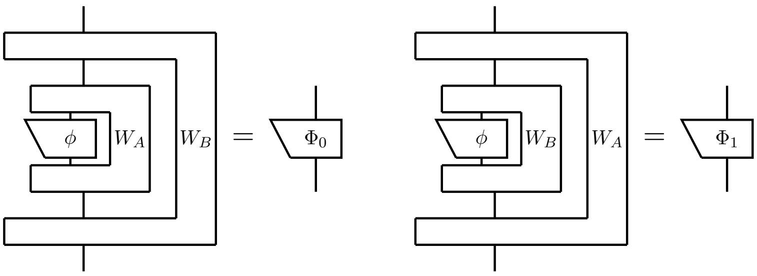
We will now discuss a few examples involves those two composition rules. Also, for simplicity and to make the described processes fairly intuitive, we will once again restrict to controlled operations.

Controlled inclusion From the inclusion relation, we can define a process which will proceed in a similar way as the quantum switch does, but by replacing signalling relation $\rightarrow$ by $\sqsubseteq$. We then define the *controlled inclusion* third order operator:



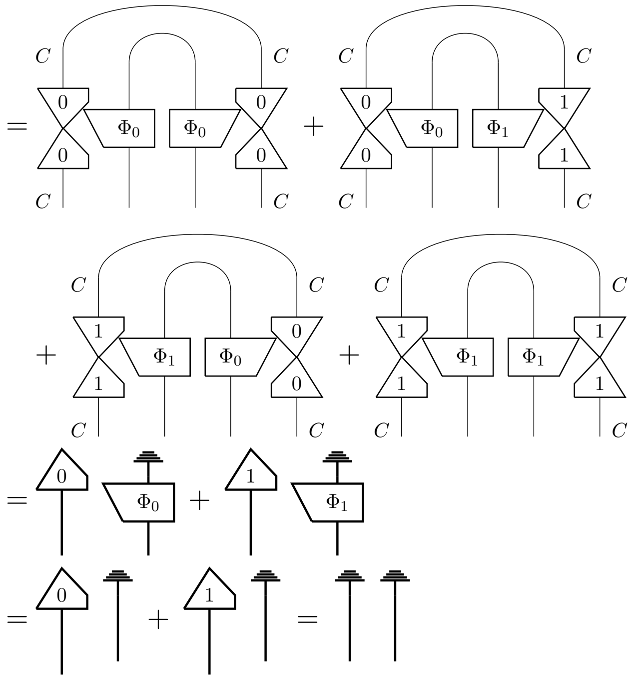
(5.65)

Now, we can see that assuming that both W_A and W_B are (second order) admissible, for every ϕ admissible, we have:



with Φ_0 and Φ_1 admissible maps. Hence, we have:

$$Tr\left[W^{(3)}_{C-incl}\left(W_A \otimes W_B\right)(\phi)\right] =$$

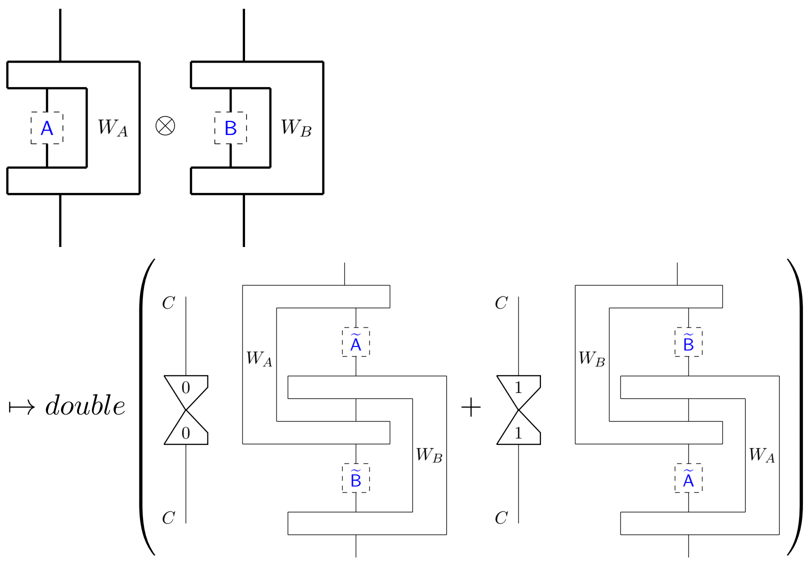


(5.66)

So $W^{(3)}_{C-incl}$ is indeed (third order) admissible. Moreover, quantum superpositions (respectively mixture) $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqsubseteq \mathcal{A}$ is induced by superpositions (respectively mixture) of the basis states $|0\rangle$ and $|1\rangle$ in the control system.

Note that we can also define a partial version of (5.67):

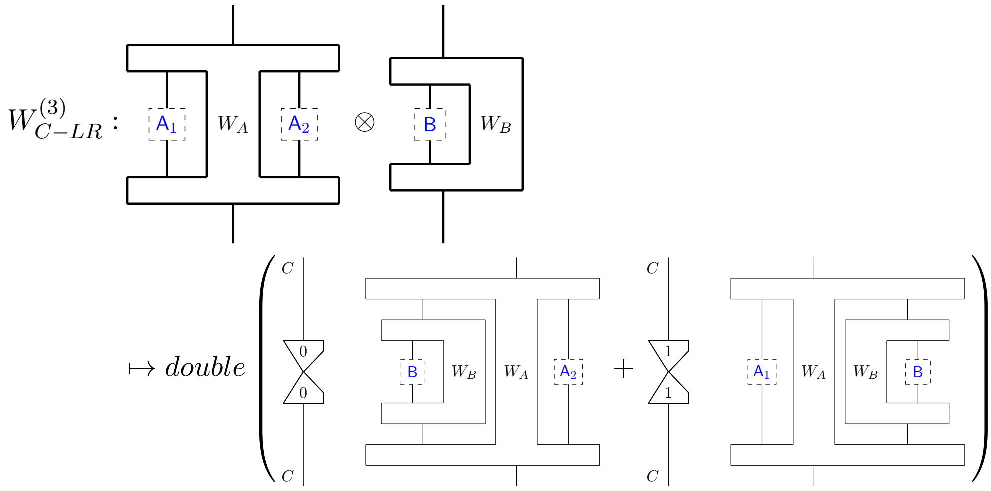
$$W^{(3)}_{C-part\_nest} :$$



(5.67)

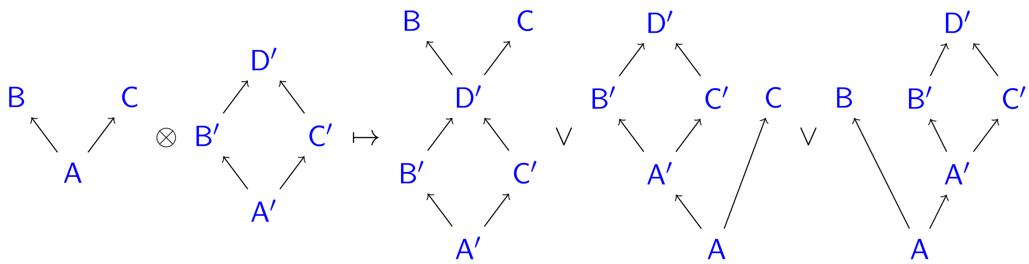
which as before is third order admissible.

Left-Right mixing Another way of using nesting as a source of indefiniteness is to consider a bipartite (or n-partite for any $n > 1$) supermap W_A and define the following controlled operation:



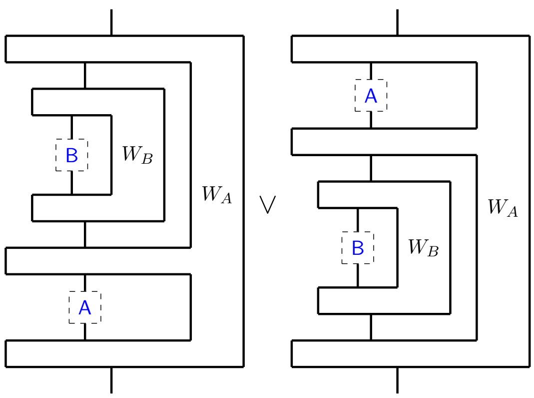
(5.68)

Once again, it is easy to see that $W^{(3)}_{C-LR}$ is third order admissible. For an n-partite superoperator W_A, this operation will induce uncertainty in which of the "slots" W_B will be nested on; i.e. which external event of W_A will be replaced by $\mathcal{B}$. This will indeed generate a process for the causal graph will be uncertain. E.g. (representing the external causal graph, for simplicity):



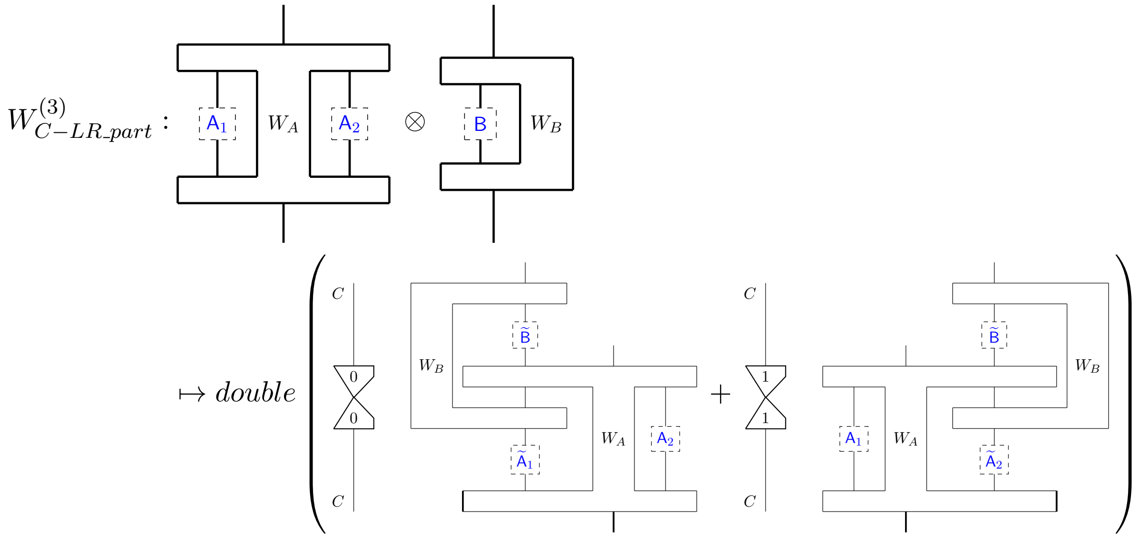
(5.69)

Note that when W_A is a comb, this essentially reduces to a special case of (5.61):



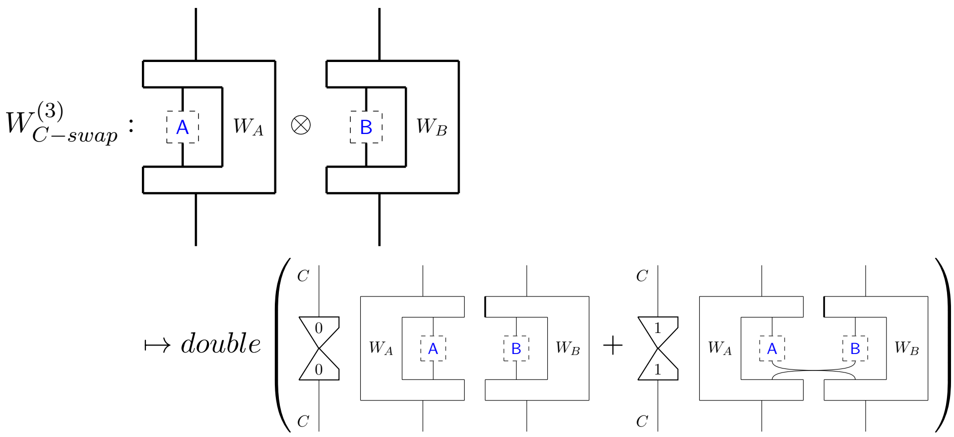
(5.70)

Once again, we can also have the "partialisation" of (5.68):



(5.71)

Controlled swap Finally, we can use the last "type" of composition, by making, when allowed, the input structure indefinite. Since can with this composition rule, literally change the input structure to anything we want, there are infinitely many ways of defining third order process with some indefiniteness in the input structure. We will however only consider one example, that I personally found interesting, namely the *controlled swap*:



(5.72)

Here, what is uncertain is the localisable parties themselves. Note that all the operations are still localised, but, as in the quantum switch, the connections are not deterministic.

We have in this last chapter investigated several ways of composing supermaps to obtain new supermaps. We have in particular seen that not only act on the signalling relations, but also the structure of supermaps themselves. This can notably be used to generate new families of processes intrinsically quantum, such as the controlled inclusion or the controlled swap.

Conclusion

We have in this project explored consequences of including the causality principle in the quantum mechanics, and how this lead to the study of higher order processes. We have then introduced the current literature on higher order quantum mechanics, in particular the $Caus[\mathbf{CPM}]$ construction and how it can be used to investigate quantum causality.

All of this served as motivation for the last chapter, in which we presented the main result of this project. We have in particular introduced some concrete instances of definite and indefinite third order processes.

Further directions An obvious way of extending this project would be to explore the applications of third order processes, obtain for more general results about them and potentially investigate their physical realisation. More importantly, an urging question is to ask what third order processes have to offer on top of the existing quantum mechanical framework. In particular, the discussion on generalised events, and notably the $\sqsubseteq$ relation suggests that third order processes can be useful to gain an insight into the internal structure of quantum maps. On this note, there is quite a lot of things that still need to be studied regarding the $\sqcup$ construction and its features.

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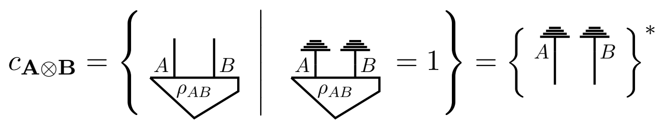
A. Results on $Caus$[CPM].

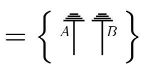
A.1 Admissible states

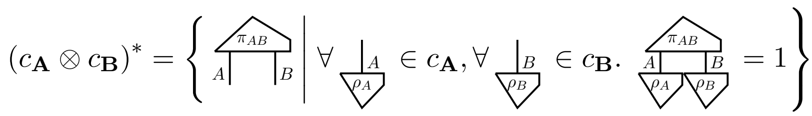
One very sensible question to ask whether the definition of the tensor product of types $\mathbf{A}$ and $\mathbf{B}$ does indeed give the correct set of admissible states. For example, for first order types $\mathbf{A}, \mathbf{B}$, we have by definition:

$$c_{\mathbf{A}\otimes\mathbf{B}} = (c_{\mathbf{A}} \otimes c_{\mathbf{B}})^{**} \tag{A.1}$$

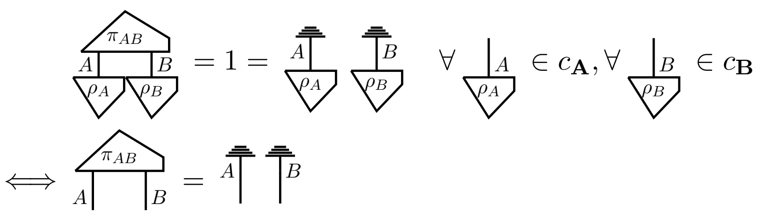
Now, for every two first order system $\mathbf{A}$ and $\mathbf{B}$, we have by definition:



So we only need to prove is that $(c_{\mathbf{A}} \otimes c_{\mathbf{B}})^* =$ . Now:



But then, from Proposition 3, we have:



A.2 Closedness of $c_{\mathbf{X}}$

We start with a preliminary result that is proved in [31]

Lemma 2. For any set of states c (not necessarily associated with any type), we have:

$$c^* = c^{***} \tag{A.2}$$

Proof. We start from the general fact[31]:

$$c \subseteq c^{**} \tag{A.3}$$

for every set of states c. In particular, taking $\widetilde{c} = c^*$, this gives:

$$c^* \subseteq c^{***} \tag{A.4}$$

Now, we also know that taking the dual reverses inclusion, i.e. for any c, d set of states:

$$c \subseteq d \implies d^* \subseteq c^* \tag{A.5}$$

Hence, from (A.2), we get:

$$c^{***} \subseteq c^* \tag{A.6}$$

And with (A.3), this indeed gives $c = c^{***}$. □

Hence, we have for any pair of types $\mathbf{A}, \mathbf{B}$:

$$c_{\mathbf{A}\multimap\mathbf{B}} = (c_{\mathbf{A}\otimes\mathbf{B}^*})^* = (c_{\mathbf{A}} \otimes c^*_{\mathbf{B}})^{***} = (c_{\mathbf{A}} \otimes c^*_{\mathbf{B}})^* \tag{A.7}$$

So using the notation from Section 3.1.1, the type corresponding to admissible processes from the type $\mathbf{A}$ to $\mathbf{B}$ is given by:

$$\mathbf{A} \multimap \mathbf{B} = (A \to B, (c_{\mathbf{A}} \otimes c^*_{\mathbf{B}})^*) = (A^* \otimes B, (c_{\mathbf{A}} \otimes c^*_{\mathbf{B}})^*) \tag{A.8}$$

Using this observation and Lemma 2, it follows that every set $c_{\mathbf{X}}$ has the property:

Corollary 2. For any type $\mathbf{X}$ defined iteratively from first order systems and $\multimap$, we have:

$$c^{**}_{\mathbf{X}} = c_{\mathbf{X}} \tag{A.9}$$

i.e. the set $c_{\mathbf{X}}$ is *closed* for any well-defined type $\mathbf{X}$.

Proof. Note that every type $\mathbf{X} \simeq \mathbf{I} \multimap \mathbf{X}$. Hence:

$$c^{**}_{\mathbf{X}} = c^{**}_{\mathbf{I}\multimap\mathbf{X}} = (c_{\mathbf{I}} \otimes c_{\mathbf{X}})^{***} = c_{\mathbf{I}\multimap\mathbf{X}} = c_{\mathbf{X}} \tag{A.10}$$

□

A.3 Linear orders from combs

Theorem 6.A n-partite channel $A_1 \otimes A_2 \otimes \ldots \otimes A_n \to A'_1 \otimes A'_2 \otimes \ldots \otimes A'_n$ admits the minimal causal graph:

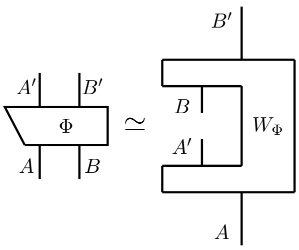
$$\begin{array}{c} \mathsf{A}_n \\ \uparrow \\ \vdots \\ \uparrow \\ \mathsf{A}_2 \\ \uparrow \\ \mathsf{A}_1 \end{array}$$

then it can be deformed into an n-comb of type:

$$\mathbf{A}_1 \multimap (\mathbf{A}'_1 \multimap (\ldots) \multimap \mathbf{A}_n) \multimap \mathbf{A}'_n$$

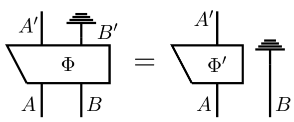
The proof proceeds by induction:

Base case: One way signalling bipartite channels Let us start we a fairly easy example: we want to show that every one-way signalling channel $\mathsf{A} \preceq \mathsf{B}$ can be deformed as a 2-comb:



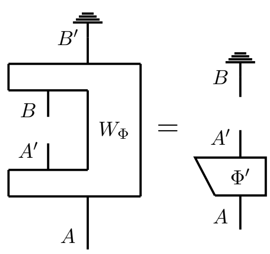
$$: \mathbf{A} \multimap (\mathbf{A}' \multimap \mathbf{B}) \multimap \mathbf{B}' \tag{A.11}$$

If we assume tht Φ indeed satisfies $\mathsf{A} \preceq \mathsf{B}$, i.e.:



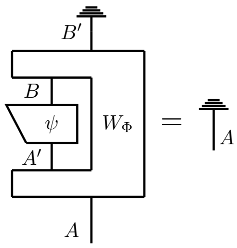
(A.12)

Then, this implies that the induced supermap W_Φ satisfies:



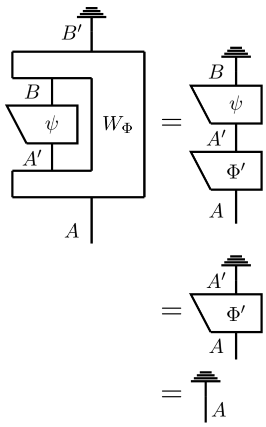
(A.13)

with Φ' admissible. Now, $W_\Phi : \mathbf{A} \multimap (\mathbf{A'} \multimap \mathbf{B}) \multimap \mathbf{B'} \simeq (\mathbf{A'} \multimap \mathbf{B}) \multimap (\mathbf{A} \multimap \mathbf{B'})$ iff:



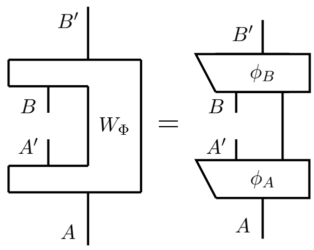
(A.14)

for every $\psi : A' \to B$ admissible map. But then, by applying the causality principle to ψ and Φ', we obtain:



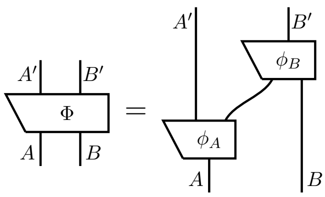
(A.15)

So W_Φ is indeed a 2-comb. On the other hand, if we know that $W_\Phi : \mathbf{A} \multimap (\mathbf{A'} \multimap \mathbf{B}) \multimap \mathbf{B'}$, since **CPM** is precausal, then W_Φ decomposes as follows:



(A.16)

Transforming back to the bipartite channel then gives:

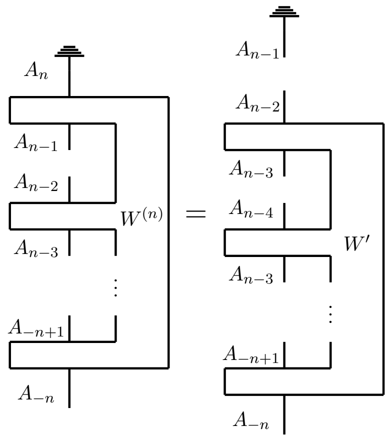


(A.17)

which is exactly the condition $\mathsf{A} \preceq \mathsf{B}$.

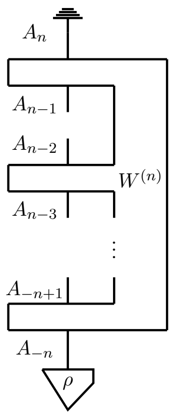
Generalisation to n-partite channel In order to prove the Theorem 6 for an arbitrary n-partite channel, we need to prove some intermediate results:

Lemma 3. Every n-combs satisfies:



(A.18)

Proof. The results is mainly based on the assumption that **CPM** has enough admissible states. Indeed, for every state $\rho : \mathbf{A}_{-n}$, and $W^{(n)} : \mathbf{C}^{(n)} \simeq \mathbf{C}^{(n-1)} \multimap (\mathbf{A}_{-n} \multimap \mathbf{A}_n)$, we have:

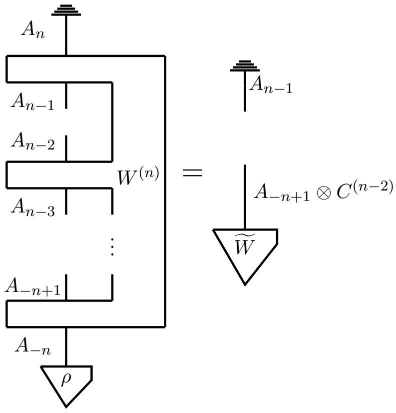


$: \mathbf{C}^{(n-1)} \multimap \mathbf{I}$ (A.19)

Now, we have:

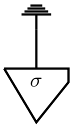
$$\begin{aligned}
\mathbf{C}^{(n-1)} \multimap \mathbf{I} \simeq & \mathbf{C}^{(n-1)^*} \\
= & \Big(\mathbf{A}_{-n+1} \multimap \mathbf{C}^{(n-2)} \multimap \mathbf{A}_{n-1}\Big)^* \\
= & \mathbf{A}_{-n+1} \otimes \Big(\mathbf{C}^{(n-2)} \multimap \mathbf{A}_{n-1}\Big)^* \\
= & \mathbf{A}_{-n+1} \otimes \mathbf{C}^{(n-2)} \otimes \mathbf{A}_{n-1}^*
\end{aligned}$$

Now, since $\mathbf{A}_{n-1}$ is first order, then $c^*_{\mathbf{A}_{n-1}} \equiv$ {  }, which implies that:



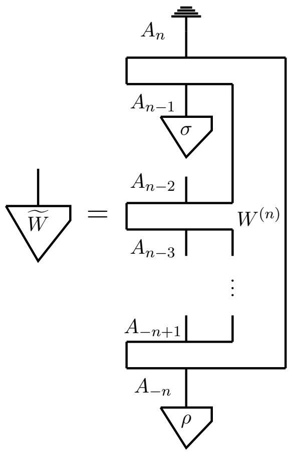
(A.20)

Now, for every admissible state $\sigma : \mathbf{A}_{n-1}$, we have:



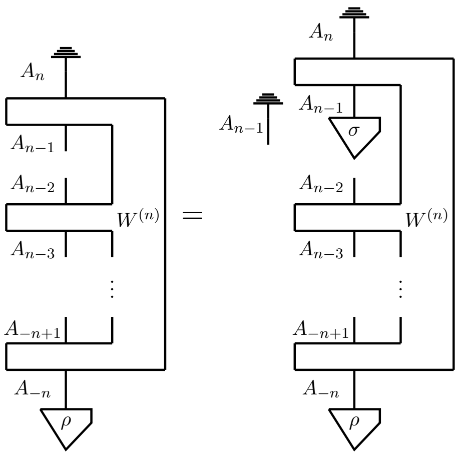
$= 1$ (A.21)

Hence, we can obtain an expression for $\widetilde{W}$ from (A.20) as:

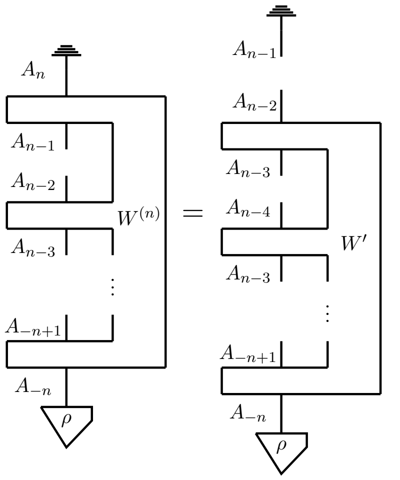


(A.22)

Pluging this back to (A.20), we obtain:



which can be furthermore reduced to:

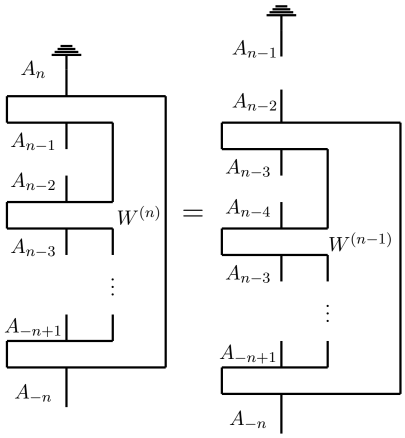


(A.23)

And since **CPM** has enough admissible states, then we get the required relation (A.18). □

The form of (A.18) since it is similar to (2.11) in the definition of n-partite one-way signalling processes, up to deformation. We however do not know whether W' is itself a comb; we will now prove an even stronger statement:

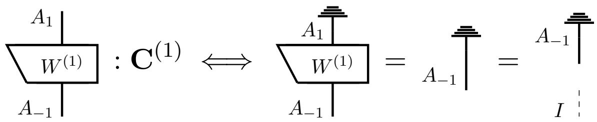
Lemma 4. For all $n \geq 1$, there exists $W^{(n-1)} : \mathbf{C}^{(n-1)}$ s.t.:



(A.24)

iff $W^{(n)} : \mathbf{C}^{(n)}$.

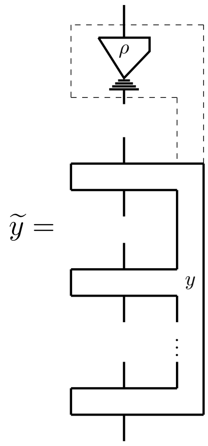
Proof. The proof is trivial for $n = 1$ since $\mathbf{C}^{(1)} = \mathbf{A}_{-1} \multimap \mathbf{A}_1$ is a first order process. By definition this means:



(A.25)

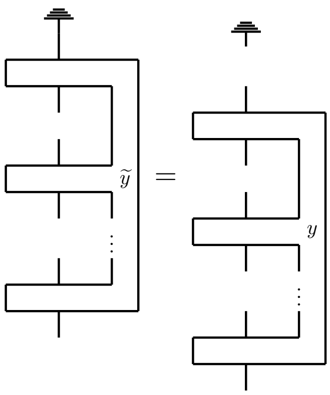
where $\mathbf{I} \equiv \mathbf{C}^{(0)}$.

We will, for simplicity omit the labels of the wires in the rest of this proof to make the diagrams more readable. Suppose that (A.24) holds for all $N \leq n$, for some $n \geq 1$. In addition let's assume that we have $W^{(n+1)} : \mathbf{C}^{(n)}$, and let $y : \mathbf{C}^{(n-1)}$ be any $n-1$-comb. Then we can define the auxiliary process:



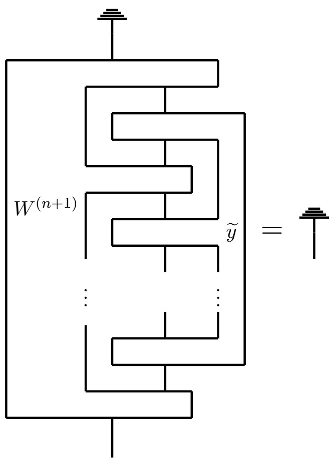
(A.26)

for some admissible state ρ. In addition, causality of ρ implies that:



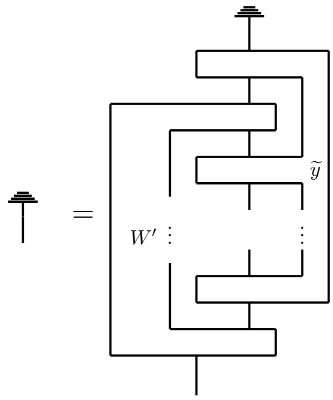
(A.27)

From the induction hypothesis and the assumption $y : \mathbf{C}^{(n-1)}$, we then deduce $\widetilde{y} : \mathbf{C}^{(n)}$. Now, from the definition of combs, we have:



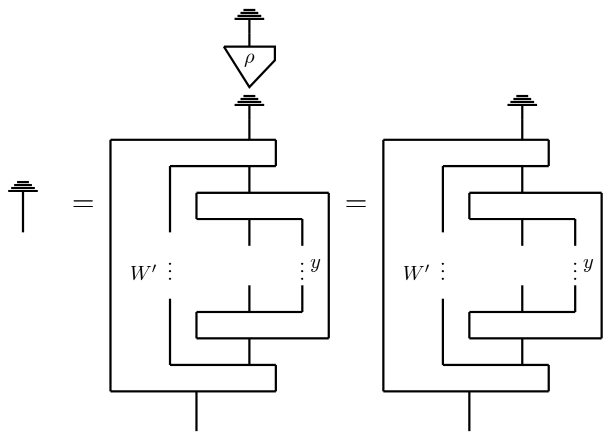
(A.28)

We can now use Lemma 3 since $W^{(n+1)}$ is a comb, which gives:



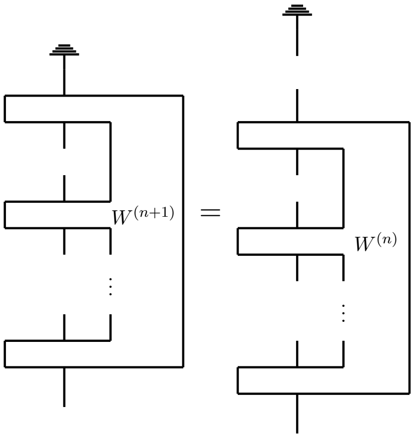
(A.29)

Using this, the definition of $\widetilde{y}$, and causality of ρ we obtain:



(A.30)

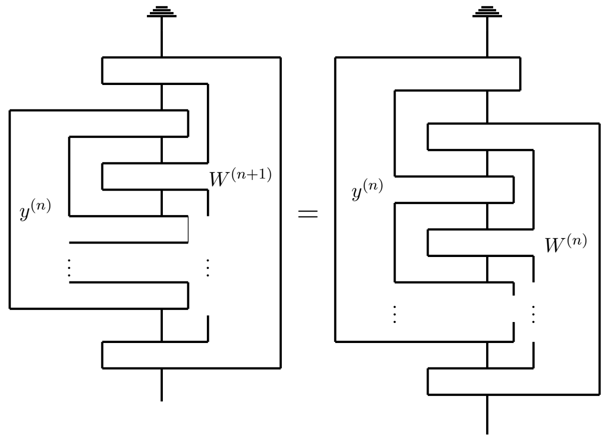
And since this is valid for every $y : \mathbf{C}^{(n-1)}$, we conclude that $W' : \mathbf{C}^{(n)}$. On the other hand, if we assume :



(A.31)

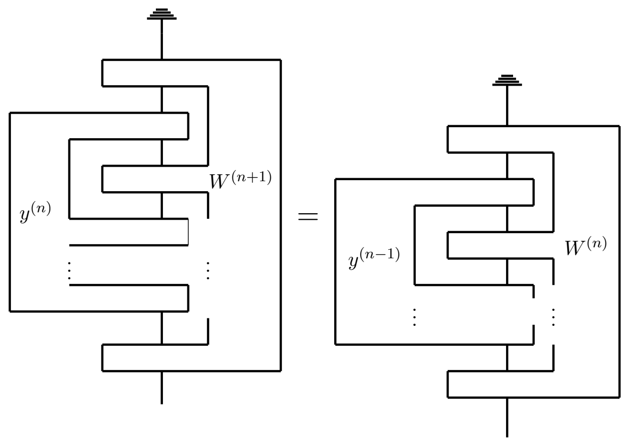
with $W^{(n)} : \mathbf{C}^{(n)}$, and (A.24) holds for all $N \leq n$. We want to show that $W^{(n+1)}$ is an

$n+1$-comb. Let $y^{(n)} : \mathbf{C}^{(n)}$ be any n-comb. Then:



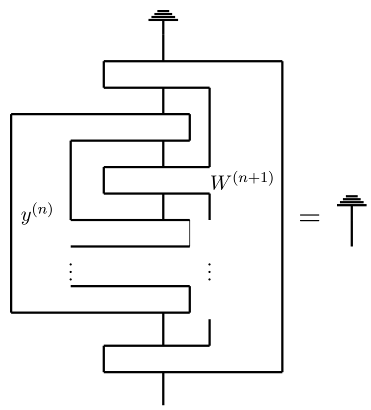
(A.32)

Now, since $y^{(n)}$ is an n-comb, then it also satisfies (A.24), i.e. there exists $y^{(n-1)} : \mathbf{C}^{(n-1)}$ s.t.:



(A.33)

and hence, since we had $W^{(n)} : \mathbf{C}^{(n)}$ by assumption, then we conclude:

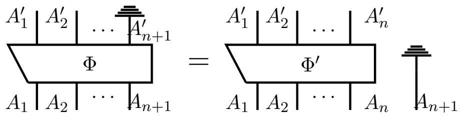


(A.34)

so $W^{(n+1)}$ is indeed an $n+1$-comb. □

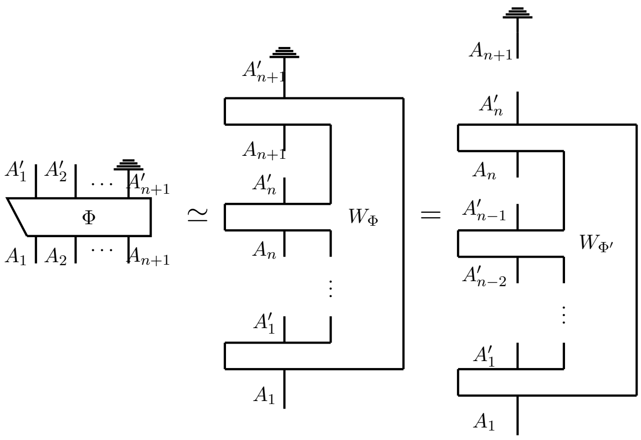
We are now ready to prove Theorem 6. We know that it is valid for $n = 2$; so by induction we suppose that Theorem 6 is valid for all $N \leq n$, for some $n \geq 2$. Hence, a

$n+1$-partite channel Φ is one-way signalling iff:



(A.35)

with Φ' one-way signalling. Hence, from the induction hypothesis, we know that Φ' can be deformed into an n-comb:



(A.36)

where $W_{\Phi'} : \mathbf{C}^{(n)}$. Then, by applying Lemma A.24, it follows that W_Φ is an $n+1$-comb.

A.4 Proof of the embedding $e : \mathbf{X} \hookrightarrow \mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_n \multimap \mathbf{A}'_1 \otimes \ldots \otimes \mathbf{A}'_m$

This proof is mainly taken from [31].

For any type $\mathbf{X}$, we have $\mathbf{X} = (X, c_{\mathbf{X}})$, where X is a Hilbert space. Hence, X does always reduce to the Hilbert space of a first order process:

$$A_1 \otimes \ldots \otimes A_n \to A'_1 \otimes \ldots \otimes A'_m$$

so we only need to check that there exists a map $e : \mathbf{X} \to \mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_n \multimap \mathbf{A}'_1 \otimes \ldots \otimes \mathbf{A}'_m$.

Now, any type $\mathbf{X}$ is constructed iteratively from first order systems and the symbols $\multimap, \otimes, \parr, \_^*$. Moreover, we have for every type $\mathbf{A}, \mathbf{B}$:

$$\mathbf{A} \multimap \mathbf{B} = \mathbf{A}^* \parr \mathbf{B} \tag{A.37}$$

Hence, the decomposition of $\mathbf{X}$ can be reduced to a type containing only first order systems, and the tensor products $\otimes, \parr, \_^*$. Now, the dual operator might apply to more than one first order type, so one can use the reduction rules:

$$(\mathbf{A} \otimes \mathbf{B})^* = \mathbf{A}^* \parr \mathbf{B}^* \qquad (\mathbf{A} \parr \mathbf{B})^* = \mathbf{A}^* \otimes \mathbf{B}^* \tag{A.38}$$

to obtain an expression which only involves $\otimes, \parr$ and the dual $\_\_^*$ applied to first order systems only. Hence, we can now use the previously derived morphism φ from (4.25) to obtain a new type only containing first order systems, their duals and the tensor product $\parr$. Note that every transformation so far only involved isomorphisms and the map φ, hence the underlying Hilbert spaces have not changed. Moreover, since the obtained "new" type only contains the tensor product $\parr$, then one can use associativity of the tensor product and symmetry of $Caus[\mathbf{CPM}]$ to show that there exists a map:

$$\mathbf{X} \to \mathbf{A}_1^* \parr \mathbf{A}_2^* \parr \ldots \parr \mathbf{A}_n^* \parr \mathbf{A}_1' \parr \mathbf{A}_2' \parr \ldots \parr \mathbf{A}_m' \tag{A.39}$$

Now, since we have $\mathbf{A}^* \parr \mathbf{B}^* = (\mathbf{A} \otimes \mathbf{B})^*$, then:

$$\mathbf{A}_1^* \parr \ldots \parr \mathbf{A}_n^* \parr \mathbf{A}_1' \parr \ldots \parr \mathbf{A}_m' = (\mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_n)^* \parr \mathbf{A}_1' \parr \mathbf{A}_2' \parr \ldots \parr \mathbf{A}_m'$$

Now, it has been shown in [31] that for first order systems, we have:

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{A} \parr \mathbf{B} \tag{A.40}$$

Hence, we get:

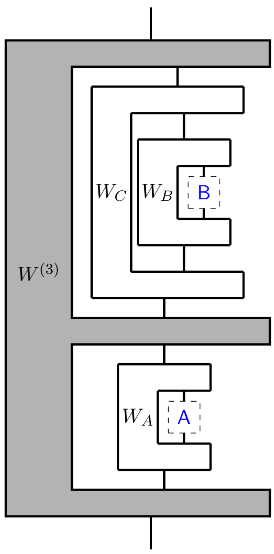
$$(\mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_n)^* \parr \mathbf{A}_1' \parr \ldots \parr \mathbf{A}_m' = (\mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_n)^* \parr \left(\mathbf{A}_1' \otimes \ldots \otimes \mathbf{A}_m'\right)$$

i.e. there exists a map:

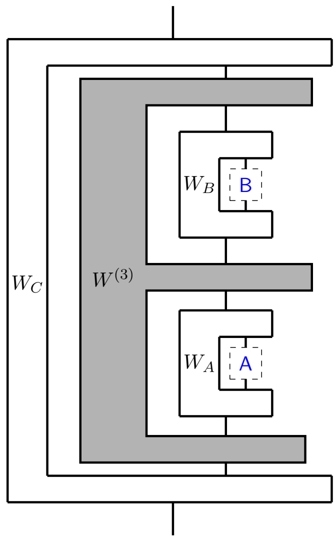
$$e : \mathbf{X} \to (\mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_n) \multimap \left(\mathbf{A}_1' \otimes \ldots \otimes \mathbf{A}_m'\right) \tag{A.41}$$

B. 3rd order processes - Further Examples

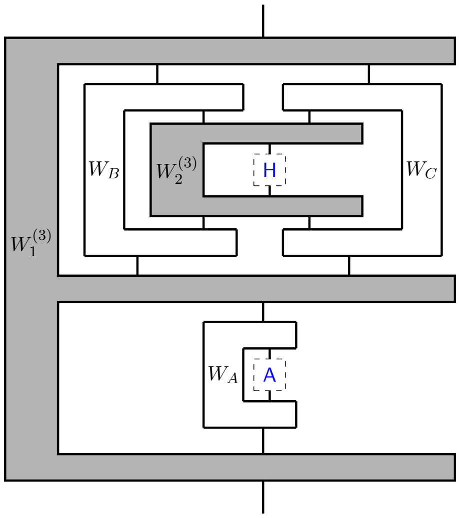
B.1 Mixing composition types



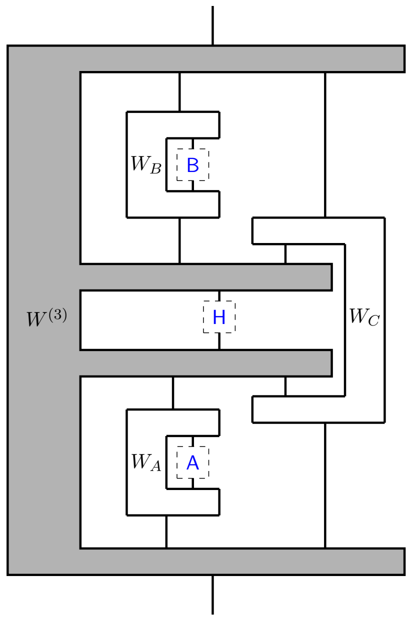
$\mathcal{A} \to (\mathcal{B} \sqsubseteq \mathcal{C})$



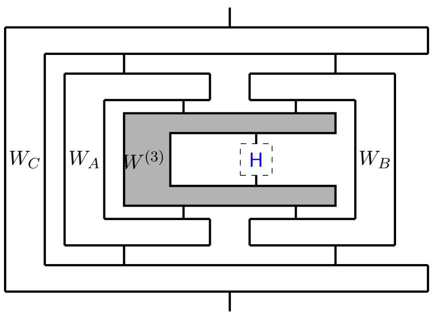
$(\mathcal{A} \to \mathcal{B}) \sqsubseteq \mathcal{C}$



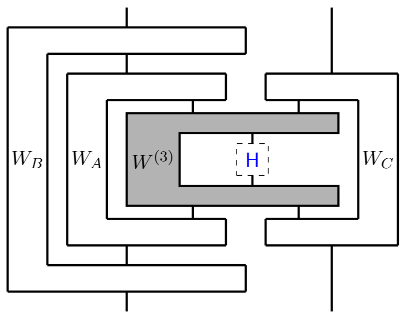
$\mathcal{A} \to (\mathcal{B} \sqcup \mathcal{C})$



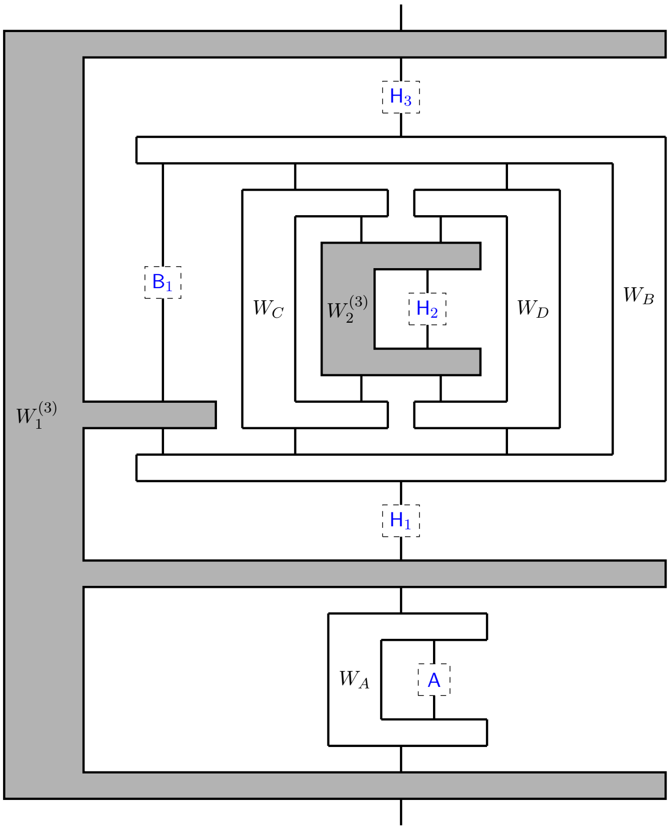
$(\mathcal{A} \to \mathcal{B}) \sqcup \mathcal{C}$



$(\mathcal{A} \sqcup \mathcal{B}) \sqsubseteq \mathcal{C}$

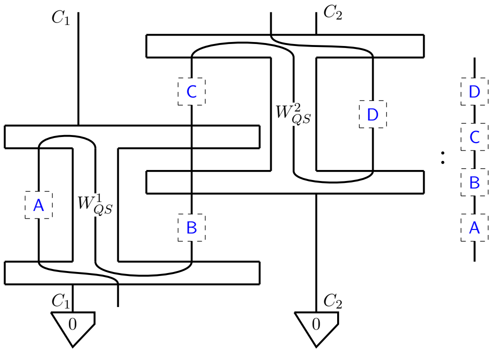


$(\mathcal{A} \sqsubseteq \mathcal{B}) \sqcup \mathcal{C}$

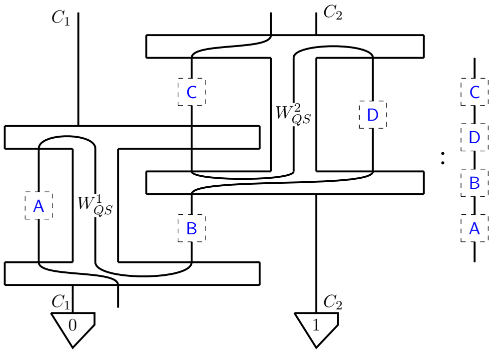


$$\mathcal{A} \to ((\mathcal{C} \sqcup \mathcal{D}) \sqsubseteq \mathcal{B})$$

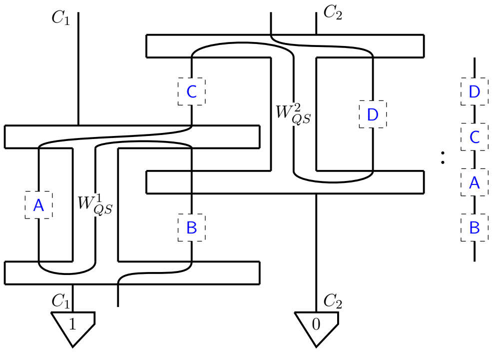
B.2 Controlled partial nesting



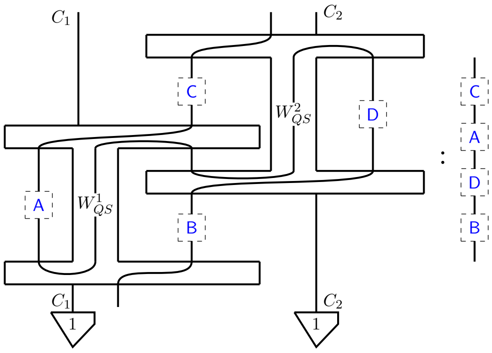
(B.1)



(B.2)

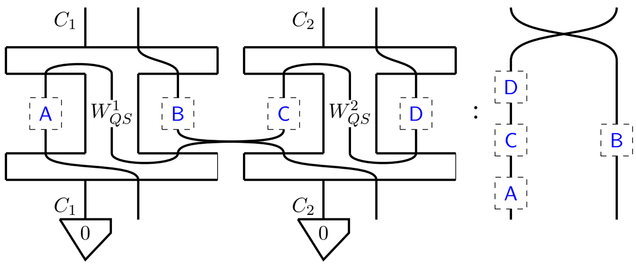


(B.3)

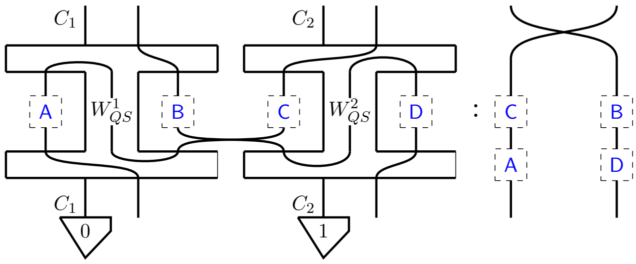


(B.4)

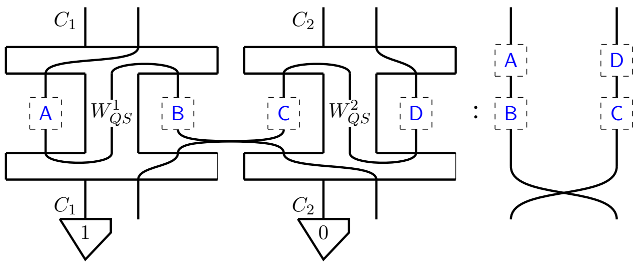
B.3 Controlled swap



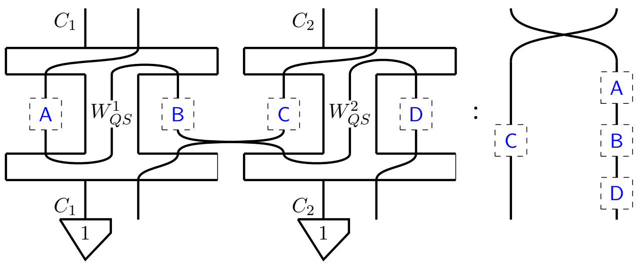
(B.5)



(B.6)



(B.7)



(B.8)