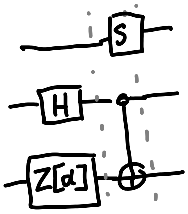
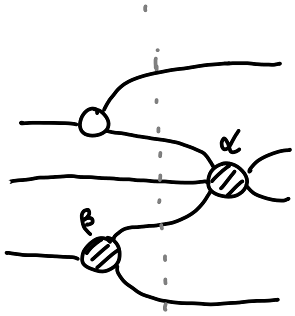
# Lecture 4 — The ZX-calculus

ZX-diagrams are "circuits" made of spiders:

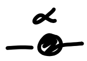




$$(S \otimes CNOT)(I \otimes H \otimes Z[\alpha])$$

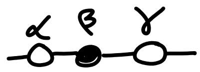


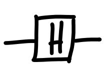
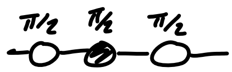

$$(I \otimes X[\alpha]_3^2 \otimes I)(Z[0]_1^2 \otimes I \otimes X[\beta]_1^2)$$

 $= Z[\alpha] := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$ ⟵ Z phase gate.

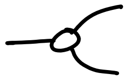
 $= X[\alpha]$ ⟵ X phase gate.

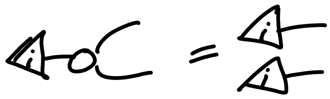
<u>Thm</u> (Euler decomposition) For any single-qubit unitary $U$, $\exists$ angles $\alpha, \beta, \gamma, \theta$ s.t:

$U = e^{i\theta} \cdot$ 

<u>Ex</u>  $= e^{-i\pi/4}$  $=:$ 

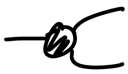
abbreviation

 $= |00\rangle\langle 0| + |11\rangle\langle 1|$ .

"copies Z-basis" 

 $= \langle 0| + \langle 1|$

"deletes Z-basis" 

 $= |{+}{+}\rangle\langle +| + |{-}{-}\rangle\langle -|$

 $= |+\rangle + |-\rangle$

(nb. $\{|x_0\rangle, |x_1\rangle\} = \{|+\rangle, |-\rangle\} =$  )

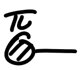
X-basis

## Basis States in ZX

 $= |+\rangle + |-\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle + |0\rangle - |1\rangle\right]$

$= \frac{2}{\sqrt{2}}|0\rangle = \sqrt{2}\cdot|0\rangle$

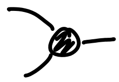
Z basis states

 $= |+\rangle + e^{i\pi}|-\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle - |0\rangle + |1\rangle\right]$

$= \sqrt{2}\cdot|1\rangle$

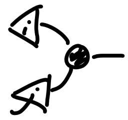
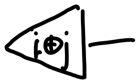
Similarly:  $= \sqrt{2}\cdot|+\rangle$,  $= \sqrt{2}\cdot|-\rangle$

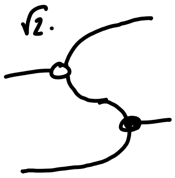
X-basis states

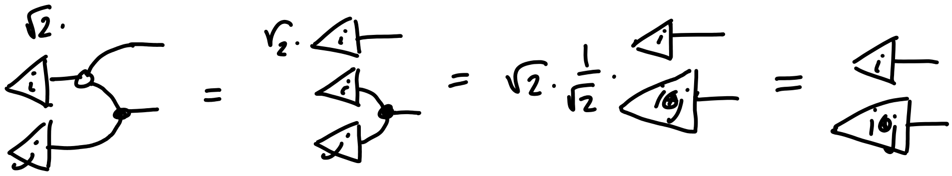
 $= |+\rangle\langle ++| + |-\rangle\langle --| = \ldots$

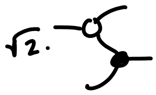
$$= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 00| + |0\rangle\langle 11| + |1\rangle\langle 01| + |1\rangle\langle 10|\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \text{XOR}$$

i.e.:  $= \frac{1}{\sqrt{2}}$ 

<u>Ex</u> CNOT = 



So:  $:: |i,j\rangle \mapsto |i, i\oplus j\rangle$

<u>Thm</u> (universality) any n-qubit unitary can be constructed using only:

- single qubit gates
- CNOT

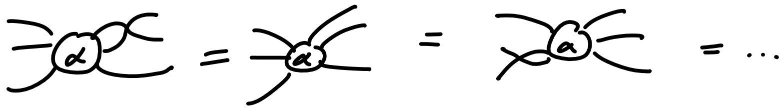
<u>Cor</u> Any n-qubit unitary can be constructed as a ZX-diagram.

# ZX Rewriting

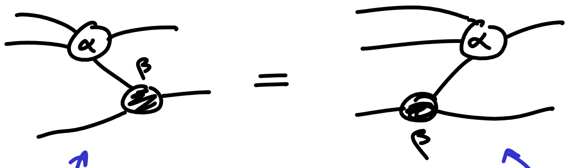
ZX diagrams have "extreme" OCM.

They are invariant under:

- SWAPPING SPIDER-LEGS:



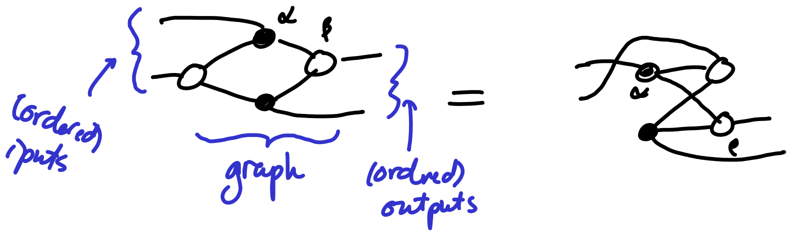

- CHANGING DIRECTION



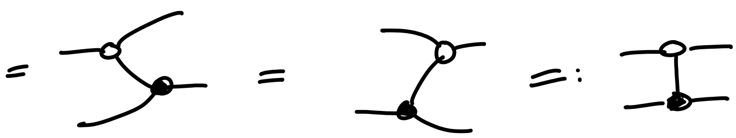

$$(I\otimes X[\beta]^{1}_{2})(Z[\alpha]^{2}_{2}\otimes I) = (Z[\alpha]^{1}_{3}\otimes I)(I\otimes I\otimes X[\beta]^{2}_{1})$$

$\Rightarrow$ they can be treated as <u>undirected</u> <u>graphs</u> (w lists of inputs & outputs)

e.g.

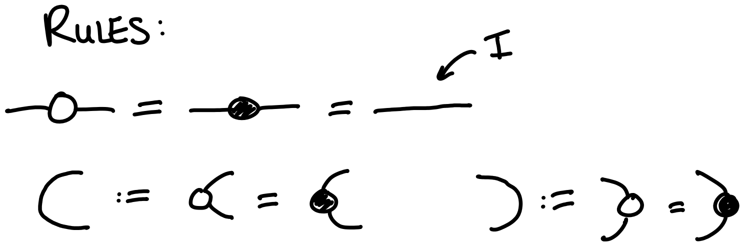



e.g. CNOT =

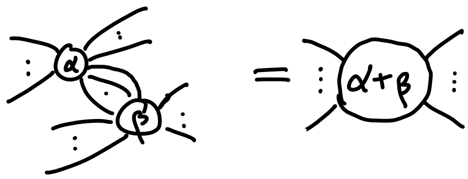


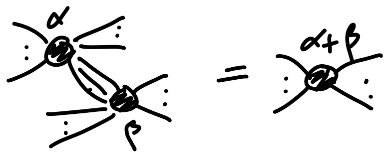
ZX-diagrams have rewrite rules called the ZX-calculus

(0) "WIRE" RULES:

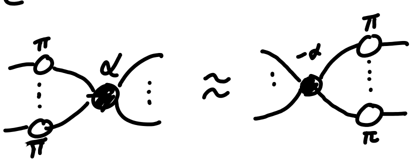



(1) SPIDER-FUSION

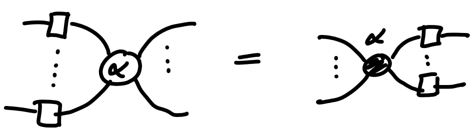


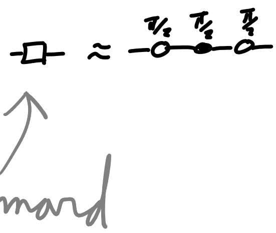

(2) π-rule*:



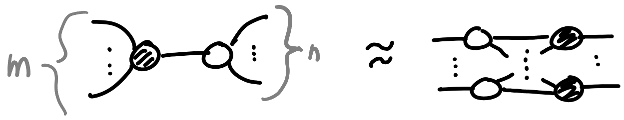

(3) COLOUR CHANGE:




where

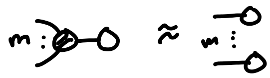


## (4) Strong complementarity




Special cases: $m=0 \Rightarrow$


$n=0 \Rightarrow$




COPY RULES

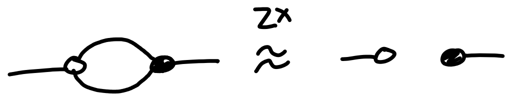
$m=2, n=2 \Longrightarrow$


(EU) rule :

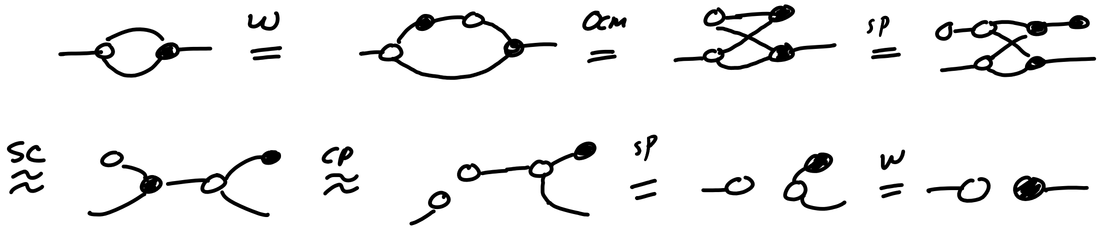
# Lecture 5

## Rewriting examples

**Thm** (Complementarity)

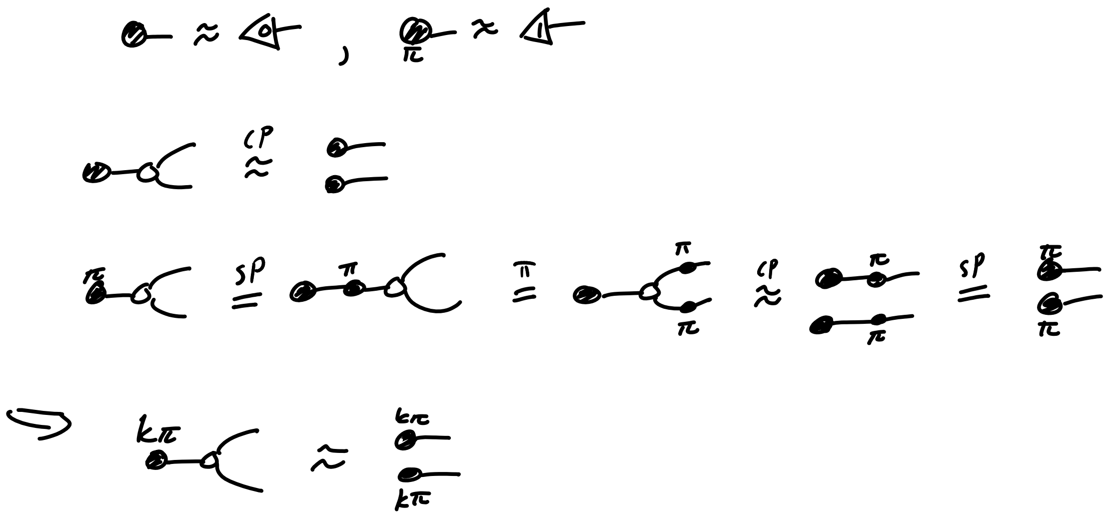


**Pf**

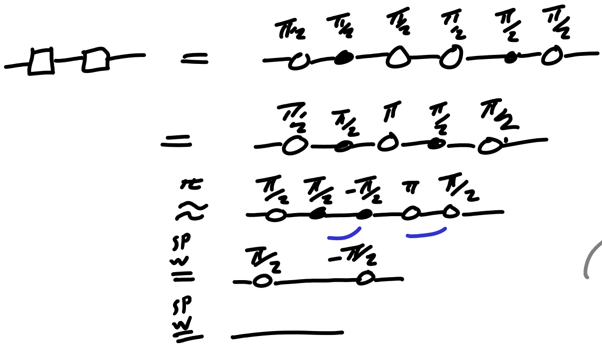


□

**Ex** Basis state copy:

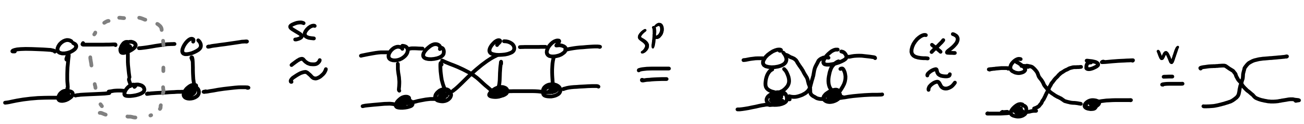


**Ex** HH



(because $\pi + \frac{\pi}{2} \equiv -\frac{\pi}{2} \pmod{2\pi}$)

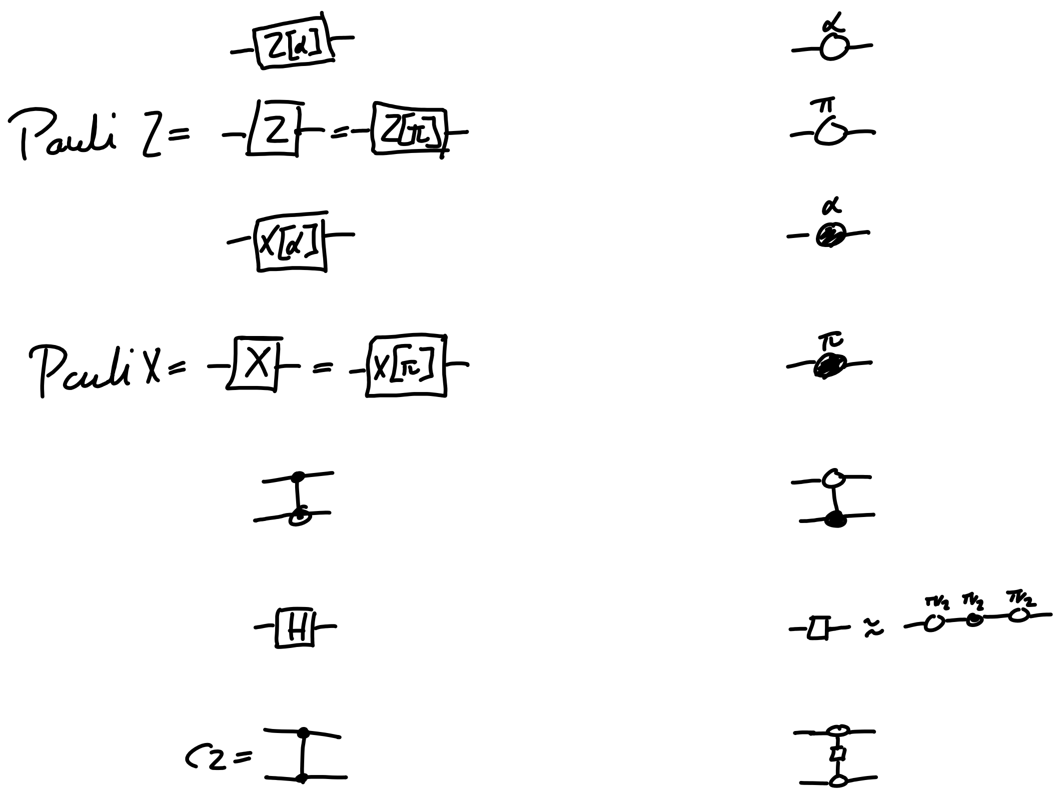
**Ex** 3CNOT:



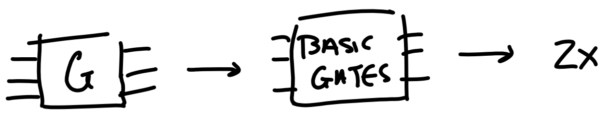
# ZX Dictionary

CIRCUITS ⟶ ZX-diagrams

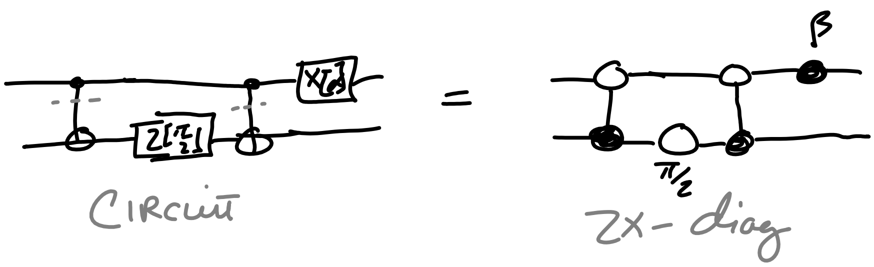
gate        diagr




other stuff
(e.g. CCZ, Tof, ...)




Ex

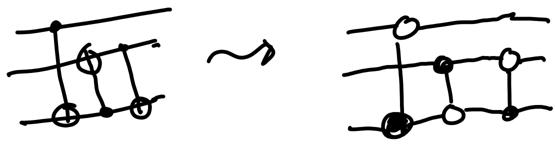


# CNOT Circuits & Phase Free ZX Diagrams

Circuits made just out of 

ZX-Diags made out of  and 

Prop Any CNOT circuit is equal to a phase free ZX-diagram.



Q: What about the converse?

Today: (Unitary) phase-free ZX-diags $\rightsquigarrow$ CNOT circuits.

$$\text{CNOT}\,|x,y\rangle \mapsto |x, x\oplus y\rangle$$

$$\text{CNOT}\,|x,y\rangle \mapsto |f_1(x,y), f_2(x,y)\rangle \quad \text{where} \quad \begin{cases} f_1(x,y) = x \\ f_2(x,y) = x\oplus y. \end{cases}$$

Def A function of the form $f(x_1,\dots,x_n) = x_{i_1} \oplus \dots \oplus x_{i_k}$ is called a parity map.

# Parities.

<u>Def</u> The field $\mathbb{F}_2$ has elements $\{0,1\}$ where:

$$x \cdot y := x \wedge y \qquad x + y = x \oplus y \qquad (\text{ie. } x+y \bmod 2)$$

Sometimes we call some $x \in \mathbb{F}_2$ a parity.

$$\mathrm{par}(\vec{b}) = \overbrace{\sum_i}^{\text{in } \mathbb{F}_2} b_i$$

$\mathrm{par}(\vec{b}) = 0$ means $\vec{b}$ has an <u>even</u> # of <u>1's</u>

$\mathrm{par}(\vec{b}) = 1$ means <u>odd</u> #.

Parities for subsets of bits:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = b_1 \oplus b_3 \oplus b_4$$

Multiple parities at once:

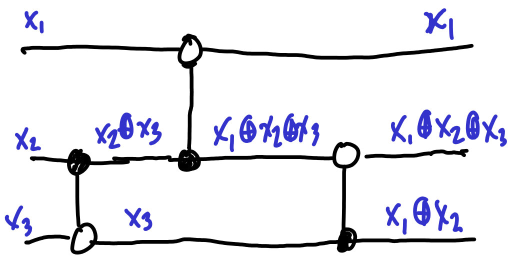
$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{parity matrix.}} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} b_1 \oplus b_3 \oplus b_4 \\ b_2 \oplus b_3 \\ b_1 \oplus b_4 \\ b_4 \end{pmatrix}$$

<u>Thm</u> The action of a CNOT circuit on basis elements is defined by an invertible parity matrix:

$$C|b_1,\dots,b_n\rangle = |c_1,\dots,c_n\rangle$$

where
$$P\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

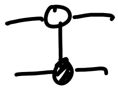
e.g.

$C =$ 


$$C|x_1, x_2, x_3\rangle = |x_1, x_1 \oplus x_2 \oplus x_3, x_1 \oplus x_2\rangle$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{P} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \oplus x_2 \oplus x_3 \\ x_1 \oplus x_2 \end{pmatrix}$$

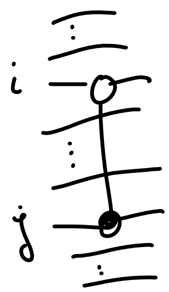
Special case: Single CNOT.



$|x,y\rangle \mapsto |x, x\oplus y\rangle$

$$\underbrace{\begin{pmatrix}1&0\\1&1\end{pmatrix}}_{P}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\x\oplus y\end{pmatrix}$$

More generally:




$\longleftrightarrow$

$$\begin{array}{c c}
 & \\
j \rightarrow & \begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & \square & & 1 & \\ & & & & 1\end{pmatrix} \\
 & \uparrow\, i
\end{array} = \underbrace{E^{ij}}_{\text{elementary matrix}}$$

$$E^{ij}A = A'$$

row $j$ = row $j$ + row $i$

$$A E^{ji} = A'$$

col $j$ := col $i$ + col $j$

Suppose $P E^{i_1 j_1} \cdots E^{i_k j_k} = I$,

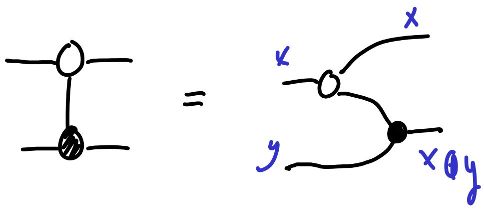
then $P = E^{i_k j_k} \cdots E^{i_1 j_1}$

parity matrix

CNOT gates!

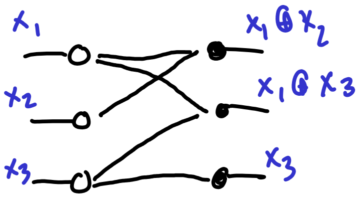
Algorithm: CNOT-SYNTH:

* Start w/ Parity matrix $P$, empty circ. $C$.
* Do Gauss-Jordan reduction of columns of $P$.
  - whenever an elem. col operation $E^{ji}$ is applied, append $\mathrm{CNOT}^{ji}$ to $C$.
* $C$ now implements $P$.




More general parity maps:

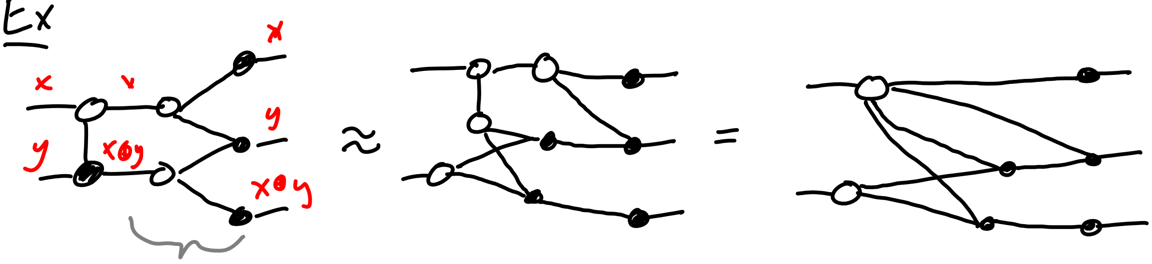
$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \oplus x_2 \\ x_1 \oplus x_3 \\ x_3 \end{pmatrix}$$




← implements P!

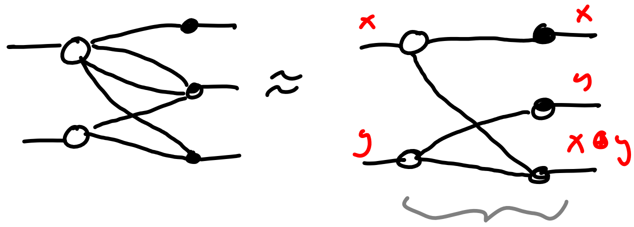
# Lecture 6

## Ex



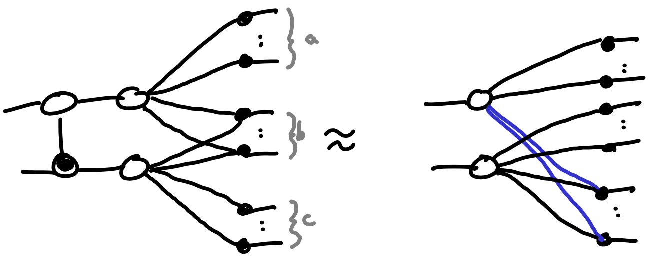

$$\underbrace{\phantom{xx}}_{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

$=$




$$\underbrace{\phantom{xx}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}}$$

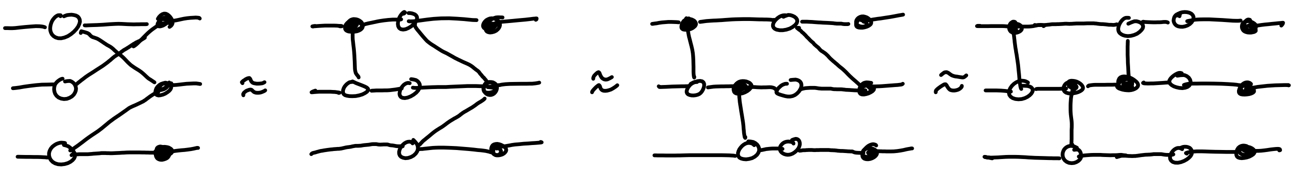
## Lem 4.2.3




$$\begin{array}{c} a\{ \\ b\{ \\ c\{ \end{array} \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \underset{c_1 = c_1 + c_2}{\overset{c_1 = c_1 + c_2}{\rightleftarrows}} \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix}$$

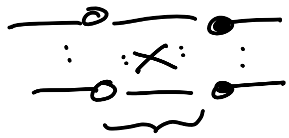
## Ex

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{c_2 = c_2 + c_1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{c_3 = c_3 + c_2} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{c_1 = c_1 + c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



**Def** A spider is called
* an <u>input</u> spider if it is conn. to an input
* an <u>output</u> spider - - - output
* an <u>interior</u> spider otherwise.

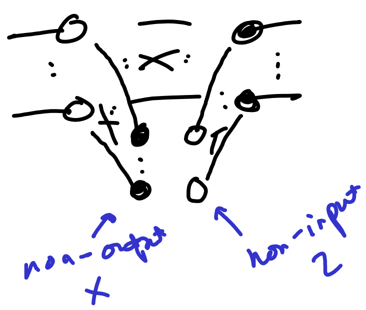
**Def** A phase-free ZX-diagram is in <u>parity normal form</u>
- every Z spider is conn. to exactly 1 input
- every X - - - - - - output
- no wires between spiders of the same type
- no parallel wires



P parity matrix.

**Def** A phase-free ZX diag. is in generalised parity form if:

1. every input is conn. to a Z spider
2. every output ... X spider
3. no wires Z-Z or X-X
4. no parallel wires
5. no wires b/w interior Z-spiders and interior X-spiders.



**Algorithm 2:** Reduction to generalised PNF.

1. apply (sp), (comp), and ○ = ● = 2 as much as possible.
2. try to apply (sc) to a pair ○—● where:
   - ○ is not an input
   - ● is not an output
3. if step 2 applied (sc), goto step 1. otherwise:
4. use (id) to make sure every input is conn. to Z & output conn. to X.

**Thm** (sketch) Alg 2 terminates efficiently in generalised PNF.

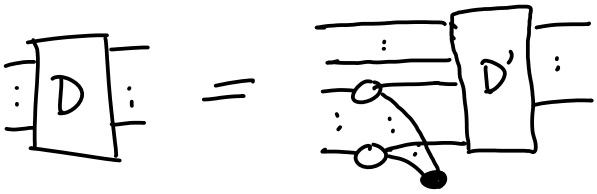
**Pf** · Each iteration of steps 1-3 removes non-input Z spiders (and non-output X-spiders) without making new ones.

$\Rightarrow$ # iterations bounded by # of spiders

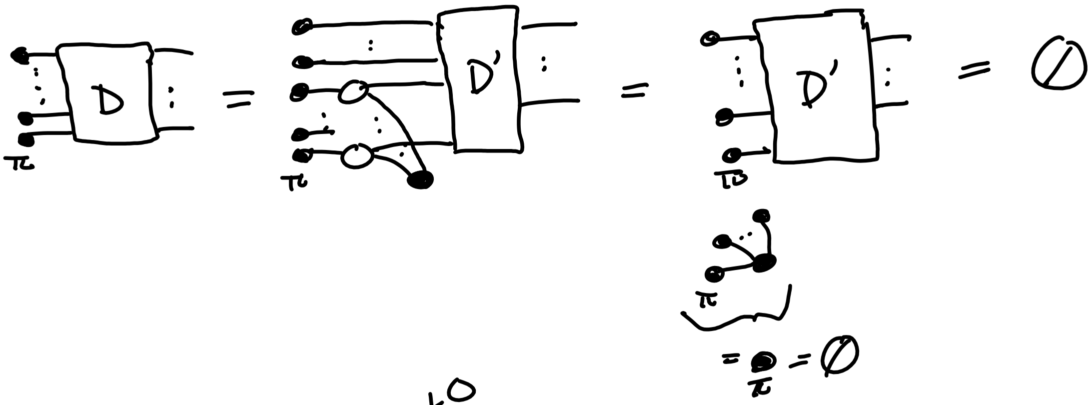
- after the loop, conds 3-5 are satisfied.
- after step 4, conds 1-2 are satisfied. $\boxed{\text{☑}}$

**Prop** If D is unitary and in generalised PNF, then it is in PNF.

Pf If D has an interior X spider, then:



So:



So, there exists $|\psi\rangle \neq 0$ s.t. $D|\psi\rangle = 0$. But:

$$D^\dagger D|\psi\rangle = |\psi\rangle \neq 0. \quad \lightning$$

So D has no interior X-spiders. Similarly,

D has no
- Z-sp's connected to >1 input
- interior X sp's
- X-sp's connected to >1 output. $\square$

$$\underset{\text{ZX}}{\overset{\text{Unitary}}{\text{Phase-free}}} \overset{*}{\Longrightarrow} \text{PNF} \Longrightarrow \text{CNOT circuit}.$$