# Quantum Processes and Computation

**Assignment 1, Friday 14 Oct 2022**

**Exercises with answers and grading**

**Exercise 1 (3.4):** We saw in the lecture that **functions** and **relations** are examples of process theories. Give two other examples of a process theory. For each one answer the following questions:

1. What are the system-types?
2. What are the processes?
3. What does it mean to compose them sequentially or in parallel?
4. When should two processes be considered equal?

**Hint**: Note that a single process is not a process theory. In particular, almost any process theory will have an infinite amount of system types (e.g. $A$, $A \otimes A$, $A \otimes A \otimes A, \ldots$). Also: Be creative! You don't have to restrict yourself to mathematics.

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*Many correct answers, for instance*

1. *Cooking: types are food items and their combinations. Processes are cooking actions, like chopping, baking, stirring, etc. Composition is just applying one process after another, or at the same time (sequential and parallel). Two processes are equal if and only if they produce the same output for every choice of ingredients.*
2. *Chemistry: types are chemicals, and processes are chemical reactions. Sequential composition is applying two chemical reactions one after another; parallel composition is applying chemical reactions independently at the same time. Two chemical reactions from chemical A to chemical B are the same if they produce the same chemicals from the same input through the same stages, as their may be several different reactions transforming A into B.*
3. *Math: types are groups, processes are group homomorphisms. Sequential composition is usual composition and parallel composition is cartesian product. Two processes are equal if their composition yields the same group homomorphism.*

*Note that there is a difference between types and states. For instance in the example of cooking, 'vegetable' is a type, while for instance a 'raw potato' could be a state of vegatable. As an example 'cooking' is a process transforming vegetables into vegetables, and transforming the state 'raw potato' into 'cooked potato'.*

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**Exercise 2 (3.10):** Read Section 3.1.3 about diagrams as diagram formulas. Draw the diagrams corresponding to the following diagram formulas:

1. $f_{B_1 C_2}^{C_4} g_{C_4}^{D_3}$
2. $f_{A_1}^{A_1}$
3. $g_{B_1}^{A_1} f_{A_1}^{B_1}$
4. $1_{A_1}^{A_6} 1_{A_2}^{A_5} 1_{A_3}^{A_4}$.

Use the convention that inputs and outputs are numbered from left-to-right.

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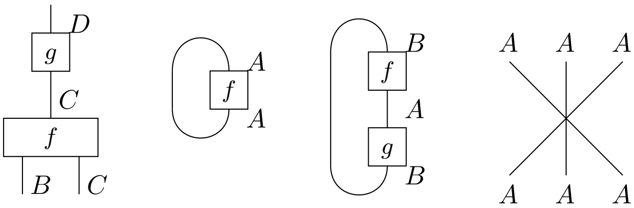


(1)

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**Exercise 3 (3.12):** Give the diagrammatic equations of a process $*$ taking two inputs and one output that express the algebraic properties of being

1. associative: $x * (y * z) = (x * y) * z$
2. commutative: $x * y = y * x$
3. having a unit: there exists a process $e$ (with no inputs) such that $x * e = e * x = x$

**Note:** $x$, $y$ and $z$ should not appear in your final diagrams. They are however useful in trying to figure out what the diagrammatic equation should be.

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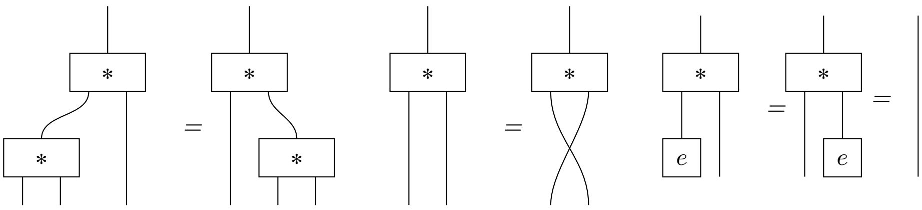
*The diagrams corresponding to associativity, commutativity and having a unit e are respectively:*




(2)

*Distributivity can't be represented easily, because the lefthandside $(x + y) * z$ has 3 inputs, while $(x * z) + (y * z)$ has four inputs, 2 of them being equal (the $z$). This can be fixed by introducing a 'copy' operation.*

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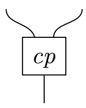
**Exercise 4 (3.15):** Using the copy operation:




$$:: \quad n \mapsto (n, n)$$

write down the diagram representing distributivity: $(x + y) * z = (x * z) + (y * z))$? Here, $+$ and $*$ are processes that take two inputs and and one output.

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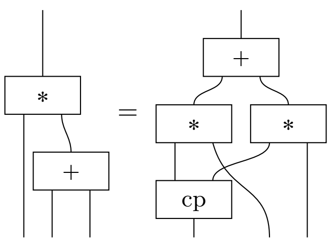

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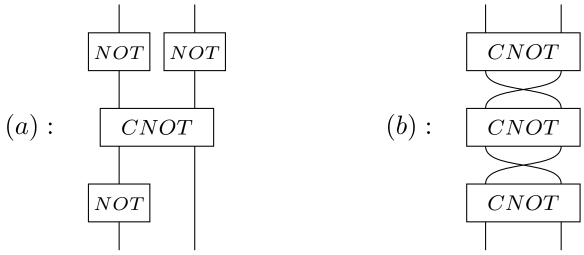
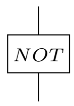
**Exercise 5 (3.30):** First compute the values of the following functions, then give the commonly used name of these functions:

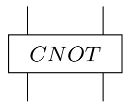



where:




$$:: \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$$

and




$$:: \begin{cases} (0,0) \mapsto (0,0) \\ (0,1) \mapsto (0,1) \\ (1,0) \mapsto (1,1) \\ (1,1) \mapsto (1,0) \end{cases}$$

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*These functions are defined in terms of how they map the elements of the set where they're defined, so it's enough to calculate it by "brute force". For the first one, calculating $(NOT \otimes NOT) \circ (CNOT) \circ (NOT \otimes Id)$ step by step:*

$$(0,0) \mapsto (1,0) \mapsto (1,1) \mapsto (0,0) \tag{3}$$
$$(0,1) \mapsto (1,1) \mapsto (1,0) \mapsto (0,1) \tag{4}$$
$$(1,0) \mapsto (0,0) \mapsto (0,0) \mapsto (1,1) \tag{5}$$
$$(1,1) \mapsto (0,1) \mapsto (0,1) \mapsto (1,0) \tag{6}$$

*Which is equal to CNOT. For the second one, calculating CNOT,SWAP,CNOT,SWAP,CNOT:*

$$(0,0) \mapsto (0,0) \mapsto (0,0) \mapsto (0,0) \mapsto (0,0) \mapsto (0,0) \tag{7}$$
$$(0,1) \mapsto (0,1) \mapsto (1,0) \mapsto (1,1) \mapsto (1,1) \mapsto (1,0) \tag{8}$$
$$(1,0) \mapsto (1,1) \mapsto (1,1) \mapsto (1,0) \mapsto (0,1) \mapsto (0,1) \tag{9}$$
$$(1,1) \mapsto (1,0) \mapsto (0,1) \mapsto (0,1) \mapsto (1,0) \mapsto (1,1) \tag{10}$$

*which is the SWAP map.*
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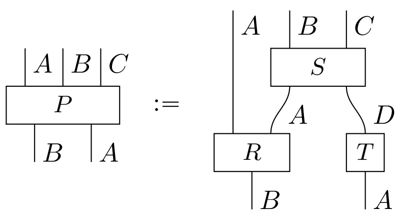
**Exercise 6 (3.31):** Suppose $A$, $B$, $C$, and $D$ are sets and $P$ is a relation given by:

$$\begin{aligned} A &= \{a_1, a_2, a_3\} \\ B &= \mathbb{B} \\ C &= \{\textbf{red}, \textbf{green}\} \\ D &= \mathbb{N} \end{aligned}$$




Compute $P$ first for $R, S, T$ given by:

$$R :: \begin{cases} 1 \mapsto (a_1, a_1) \\ 1 \mapsto (a_1, a_2) \end{cases} \qquad S :: \begin{cases} (a_1, 5) \mapsto (0, \textbf{red}) \\ (a_1, 5) \mapsto (1, \textbf{red}) \\ (a_2, 6) \mapsto (1, \textbf{green}) \end{cases} \qquad T :: \begin{cases} a_1 \mapsto 200 \\ a_3 \mapsto 5 \end{cases}$$

and then for $R, S, T$ given by:

$$R :: \begin{cases} 0 \mapsto A \times \{a_2, a_3\} \\ 1 \mapsto A \times \{a_2, a_3\} \end{cases} \qquad S :: \begin{cases} (a_1, 0) \mapsto \mathbb{B} \times \{\textbf{red}, \textbf{green}\} \\ (a_1, 1) \mapsto \mathbb{B} \times \{\textbf{red}, \textbf{green}\} \\ (a_1, 2) \mapsto \mathbb{B} \times \{\textbf{red}, \textbf{green}\} \\ \quad \vdots \end{cases} \qquad T :: \begin{cases} a_1 \mapsto \mathbb{N} \\ a_2 \mapsto \mathbb{N} \\ a_3 \mapsto \mathbb{N} \end{cases}$$

**Hint:** This exercise is in fact well-defined, and does not contain typos. Please read Section 3.3.3 if you are confused.

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*The parallel composition of $R$ and $T$ is*

$$\begin{cases} (1, a_1) \mapsto (a_1, a_1, 200) \\ (1, a_1) \mapsto (a_1, a_2, 200) \\ (1, a_3) \mapsto (a_1, a_1, 5) \\ (1, a_3) \mapsto (a_1, a_2, 5) \end{cases} \tag{11}$$

*The only output pair that matches $S$ is $(a_1, 5)$, so that the output is indeed*

$$P :: \begin{cases} (1, a_3) \mapsto (a_1, 0, \textit{\textbf{red}}) \\ (1, a_3) \mapsto (a_1, 1, \textit{\textbf{red}}) \end{cases} \tag{12}$$

*For the second case we note that $S$ only relates elements where the first input is $a_1$ while the output of $R$ connecting to $S$ will always be $a_3$, so $P$ must be the empty relation: $P = \varnothing$.*
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**Exercise 7 (3.38 & 3.40):** Suppose that there is a zero process $0 : A \to B$ for all possible types $A$ and $B$ (see Section 3.4.2).

(a) Show that the family of zero processes is unique. That is, show that if there exists *another* family of zero processes $0' : A \to B$ for all types $A, B$ such that $0' \circ f = 0' = f \circ 0'$ for all processes $f$, then for all $A, B$, $0 : A \to B$, and $0' : A \to B$ we have $0 = 0'$.

(b) We call two processes $f$ and $g$ with the same inputs and outputs *equal up to a number* (written $f \approx g$) if there exist non-zero numbers $\lambda, \mu$ such that $\lambda f = \mu g$. Suppose a process theory has *no zero divisors.* That is, it satisfies the following property: $\lambda f = 0$ if and only if $\lambda = 0$ or $f = 0$. Show that $f \approx 0$ if and only if $f = 0$.

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*Suppose $0$ and $0'$ are zero processes. We have $0 \circ 0' = 0'$, but also $0 \circ 0' = 0$. For the second one, if $f \approx 0$, then there exists a $\lambda \neq 0$ such that $\lambda f = 0$. Now, by the assumption that there are no zero-divisors we have $\lambda = 0$ or $f = 0$. Since we know that $\lambda \neq 0$, we must have $f = 0$.*

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