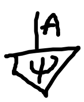
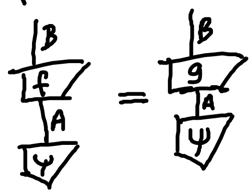
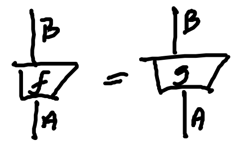
Lecture 7

Today: define the process theory of linear maps
= string diagrams + 3 extra ingredients

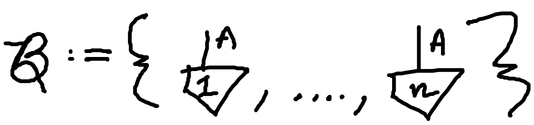
Bases

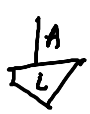
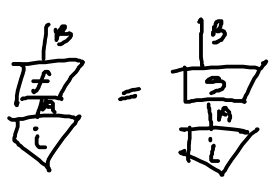
NOTE: in most process theories,

if $\forall$  .  then  .

(extensionality, well-pointed, ...)

DEF A basis for a type A is a minimal set of states:

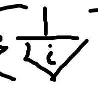


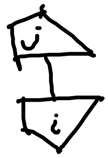
such that:
if for all  $\in \mathcal{B}$.  then $f = g$.

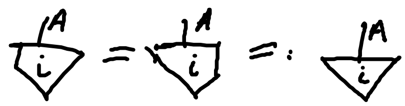
INTUITION basis states are "reference points" for a process

DEF The dimension of a system is the size of (the smallest) basis.

DEF Two states are <u>orthogonal</u> if  $= 0$.

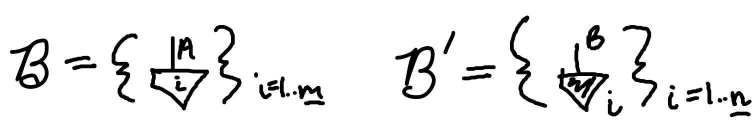
DEF An orthonormal basis (ONB) $\mathcal{B} = \{$  $\}_{i=1\ldots n}$ is a basis

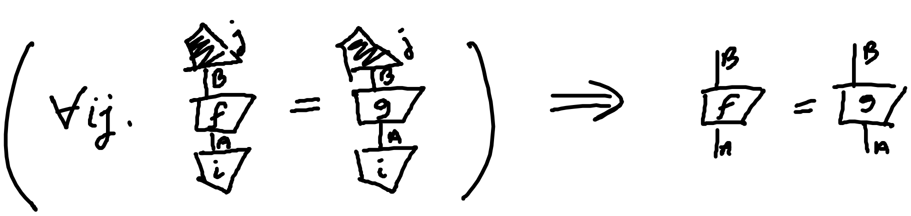
where  $= \underbrace{\begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}}_{\delta_i^j}$ $\Leftarrow$ Kronecker δ.

DEF self-conjugate ONB: 

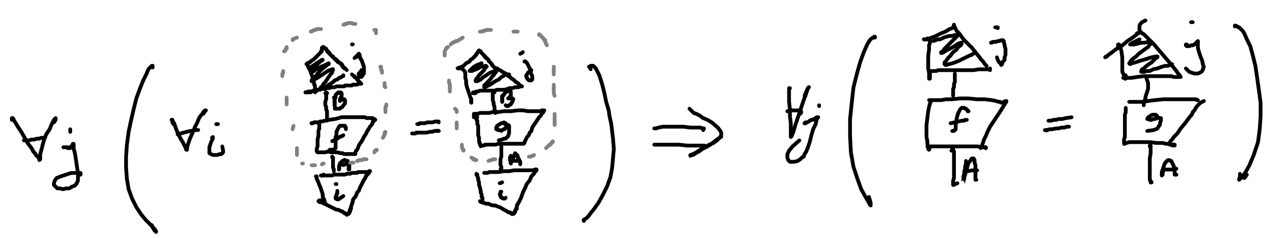
MATRICES

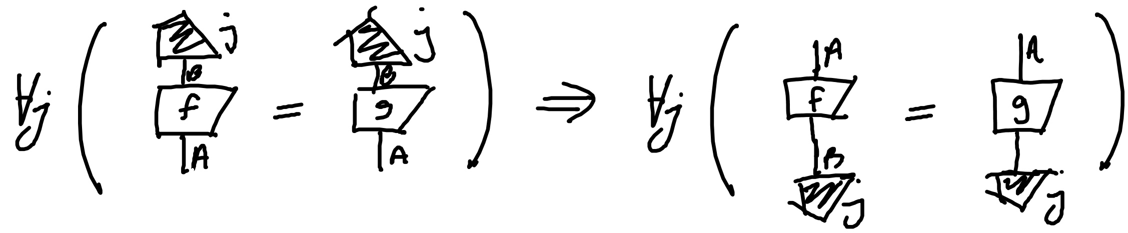
THM For bases:

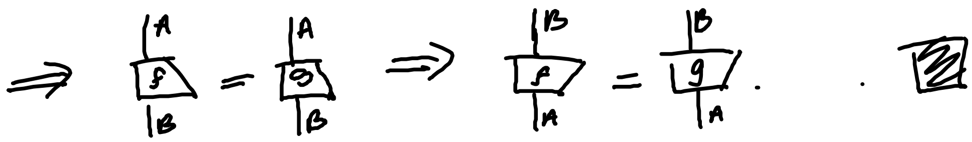


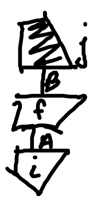


Pf







NOTATION Let $f_i^j :=$  $\leftarrow$ number, depends on bases

DEF The set of numbers $\{f_i^j\}_{i=1..m,\, j=1..n}$ is called the *matrix* of f.

Think of f_i^j (j ← row index, i ← column index) (e.g. f_{ji} in linear alg.)

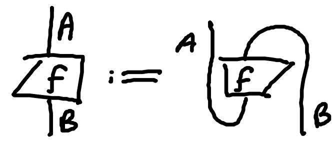
e.g. if $m=3$ and $n=2$, matrix of f

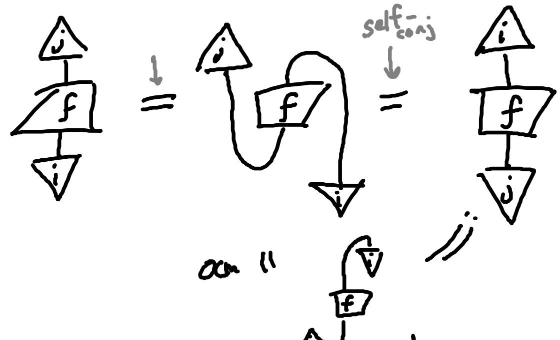
$$\begin{pmatrix} f_1^1 & f_2^1 & f_3^1 \\ f_1^2 & f_2^2 & f_3^2 \end{pmatrix}$$

MATRIX TRANSPOSE

$$\begin{pmatrix} f_1^1 & f_2^1 & f_3^1 \\ f_1^2 & f_2^2 & f_3^2 \end{pmatrix}^T := \begin{pmatrix} f_1^1 & f_1^2 \\ f_2^1 & f_2^2 \\ f_3^1 & f_3^2 \end{pmatrix}$$

(with resp. to a self-conj ONB)

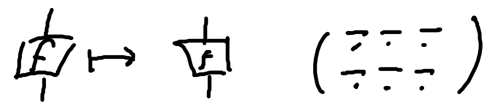
Thm The matrix of  is $(f^T)^j_i = f^i_j$.

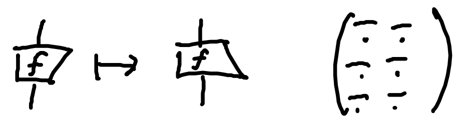
Pf $(f^T)^j_i :=$  $= f^i_j$. ∎

self-conj

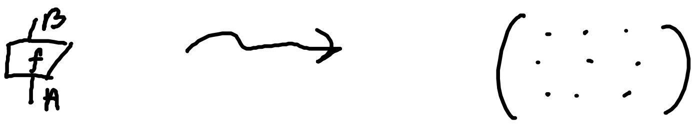
also

Similarly, the conjugate and conjugate-transpose:

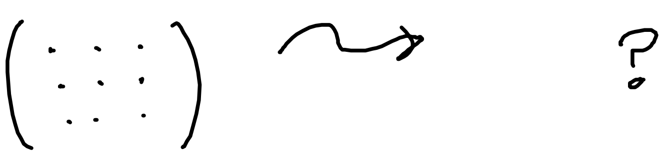
$(\overline{f})^j_i = \overline{f^j_i}$ 

$(f^\dagger)^j_i = \overline{f^i_j}$ 

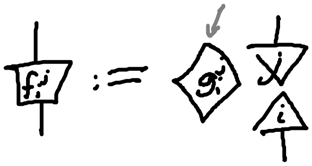
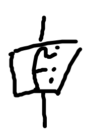
PROCESS $\qquad$ MATRIX



MATRIX $\qquad$ PROCESS



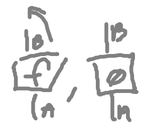
SUPPOSE I HAVE , then I can make the proc:

, then  has matrix:

$$\begin{array}{cc} \begin{pmatrix} \emptyset & \emptyset & \emptyset & \cdots \\ \cdots & \emptyset & g^j_i & \emptyset \\ \cdots & \emptyset & \emptyset & \emptyset \end{pmatrix} & \leftarrow j^{th} \text{ row} \\ \uparrow i^{th} \text{ column} & \end{array}$$

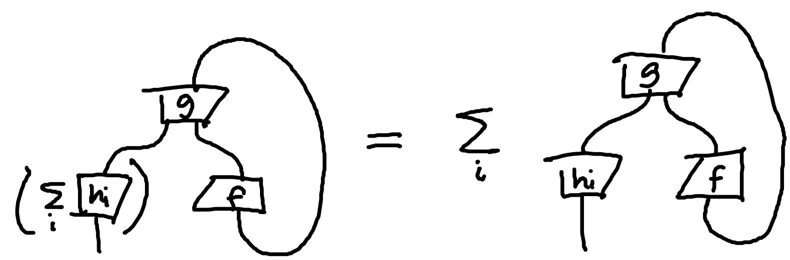
DEF A process theory <u>has sums</u> if, for any two processes of the same type

1. associative, commutative, and has unit 0.

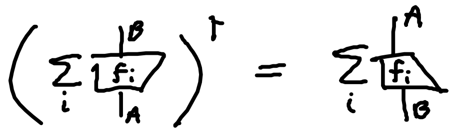
$f+(g+h) = f+(g+h)$ $\qquad$ $f+g = g+f$ $\qquad$  $\rightsquigarrow f + 0 = f$.

Let  $:= f_1 + \dots + f_N$

2. Sums <u>distribute</u> over diagrams



3. Sums <u>preserve</u> adjoints



DEF The process theory of <u>linear</u> <u>maps</u> is the theory where:

1. for each $D \in \mathbb{N}$, there is a system w/ ONB of size D.
2. processes admit sums
3. the numbers are complex numbers.
   (ie. every process $\langle\lambda\rangle$ corresponds to a complex number)

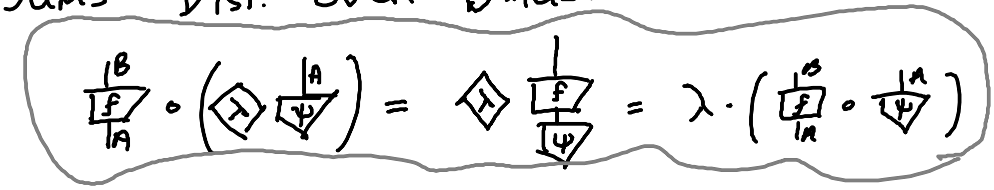
→ (we call a system in <u>linear</u> <u>maps</u> a Hilbert space)

Lecture 8

from the past

EXAMPLES OF SUMS DIST. OVER DIAGS.

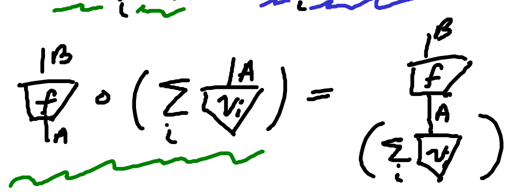
* linearity.

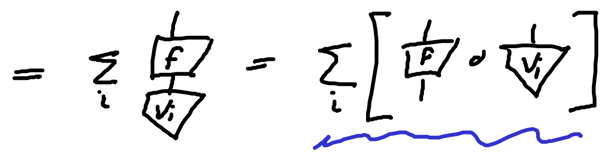


$$f(\lambda \cdot v) = \lambda \cdot f(v)$$

$$f(v_1 + v_2) = f(v_1) + f(v_2)$$

$$f\Big(\sum_i v_i\Big) = \sum_i f(v_i)$$





* linearity of $\langle - | - \rangle$

$$\langle w | v_1 + v_2 \rangle = \langle w | v_1 \rangle + \langle w | v_2 \rangle$$

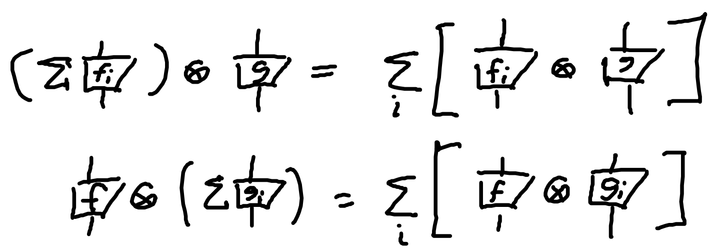
$$\langle w | \textstyle\sum_i v_i \rangle = \sum_i \langle w | v_i \rangle$$

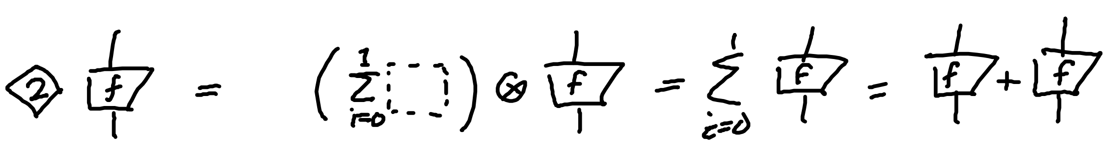
* conjugate-linearity of $\langle - | - \rangle$

$$\langle \textstyle\sum_i v_i | w \rangle = \sum_i \langle v_i | w \rangle$$

$$\langle \textstyle\sum_i \lambda_i v_i | w \rangle = \sum_i \bar{\lambda}_i \langle v_i | w \rangle$$

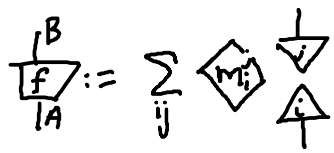
* bilinearity of $\otimes$.



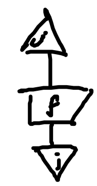


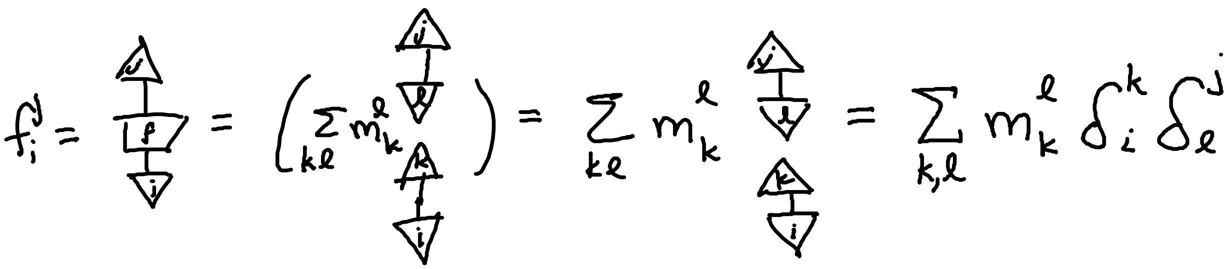
Matrix M $\rightsquigarrow$ process f

Thm For any matrix M with entries m_i^j, the process:

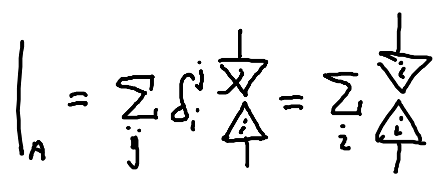


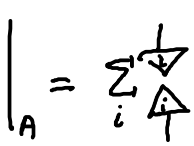
has matrix M.

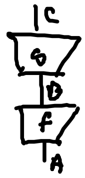
Pf matrix of f has entries $f_i^j :=$  .



$$= \underbrace{\sum_{k\neq i,\ell} m_\ell^k \delta_i^k \delta_\ell^j}_{=0} + \sum_\ell m_i^\ell \underbrace{\delta_i^i}_{=1} \delta_\ell^j = \sum_\ell m_i^\ell \delta_\ell^j = m_i^j \qquad \blacksquare$$

NOTE $\rightsquigarrow$  has matrix $\begin{pmatrix}1 & & 0\\ & 1 & \\ 0 & & \ddots\end{pmatrix}$, so 

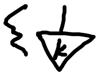
COROLLARY 

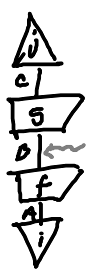
Thm If f has matrix M_f and g has matrix M_g, then the matrix of  is $\underbrace{M_g \cdot M_f}$ ← matrix multiplication.

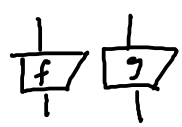
Pf

$$M_g M_f = \begin{pmatrix} & & \\ g^j_1 & \cdots\cdots & g^j_N \\ & & \end{pmatrix} \begin{pmatrix} & f^1_i & \\ & \vdots & \\ & f^N_i & \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdots & \sum_k g^j_k f^k_i & \cdots \\ \cdot & \cdot & \cdot \end{pmatrix} \leftarrow j \\ \uparrow \\ i \end{array}$$

need to show the matrix of  has entries $\sum_k g^j_k f^k_i$.

$\{$  $\}_{k=1..n}$ basis for B

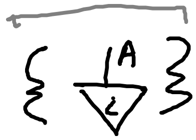
$$(M_{g\circ f})^j_i = $$  $$= \sum_{k=1}^{n}$$  $$= \sum_k g^j_k f^k_i \, . \quad \blacksquare$$

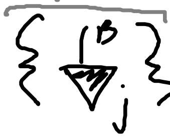
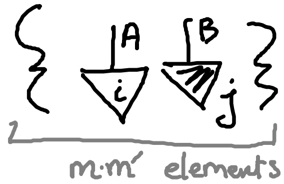
Thm The matrix of  is the $\underbrace{\underline{\text{Kronecker}}\ \text{product}}_{}$. ← tensor product

$$\underset{2\times 2}{\begin{pmatrix} f^1_1 & f^1_2 \\ f^2_1 & f^2_2 \end{pmatrix}} \otimes \underset{2\times 2}{\overbrace{\begin{pmatrix} g^1_1 & g^1_2 \\ g^2_1 & g^2_2 \end{pmatrix}}^{G}} = \begin{pmatrix} f^1_1 G & f^1_2 G \\ f^2_1 G & f^2_2 G \end{pmatrix}$$

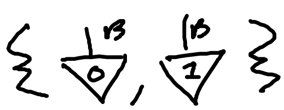
$$= \underset{4\times 4}{\left(\begin{array}{cc|cc} f^1_1 g^1_1 & f^1_1 g^1_2 & f^1_2 g^1_1 & f^1_2 g^1_2 \\ f^1_1 g^2_1 & f^1_1 g^2_2 & f^1_2 g^2_1 & f^1_2 g^2_2 \\ \hline f^2_1 g^1_1 & f^2_1 g^1_2 & f^2_2 g^1_1 & f^2_2 g^1_2 \\ f^2_1 g^2_1 & f^2_1 g^2_2 & f^2_2 g^2_1 & f^2_2 g^2_2 \end{array}\right)}$$

$$\underbrace{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}_{m \times n} \otimes \underbrace{\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}}_{m' \times n'} = \underbrace{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}_{mm' \times nn'} .$$

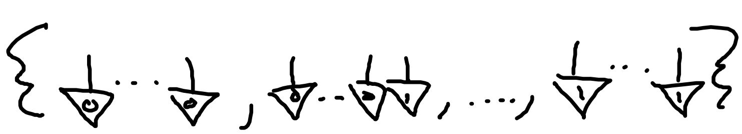
<u>The</u> <u>reason</u>: if  $_{i=1..m}$ is a basis for A and

if  $_{j=1...m'}$ is a basis for B, then  $_{i=1..m,\, j=1..m'}$

is a <u>basis</u> for $A \otimes B$.

<u>Ex</u> assume B has basis  ← 2-dim'l

then $B \otimes B$ has basis  ← 4-dim'l

and $\underbrace{B \otimes \ldots \otimes B}_{N}$ has basis 

2^N-dim'l

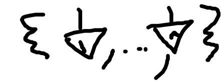
$\mathbb{B}^N \longleftrightarrow B \otimes \ldots \otimes B$

$\{$ CLASSICAL DATA $\} \longleftrightarrow \{$ QUANTUM DATA $\}$

BITSTRINGS $\longmapsto$ BASIS ELEMENTS

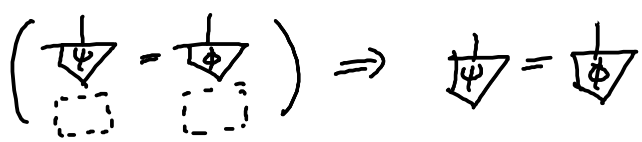
Lecture 9

linear maps $\rightsquigarrow$ quantum maps

1. systems $\mathbb{C}^n$ for every $n \in \mathbb{N}$ with an ONB 
2. sums
3. numbers are complex numbers $\mathbb{C}$

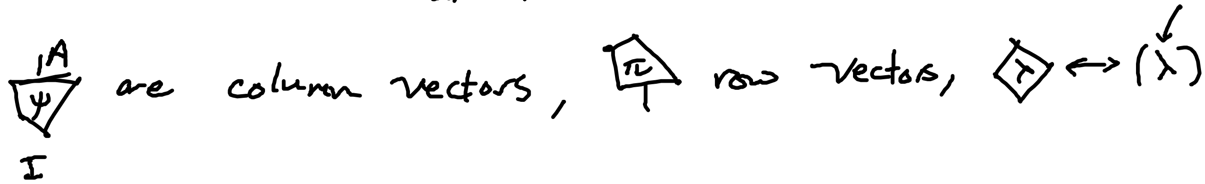
* in particular, the $\underbrace{\text{trivial system}}$ $I = \mathbb{C}$ $(= \mathbb{C}^1)$ has an ONB $\{ \boxed{} \}$ $=1$

$\mathbb{C}^m \otimes \mathbb{C}^n = \mathbb{C}^{m \cdot n}$



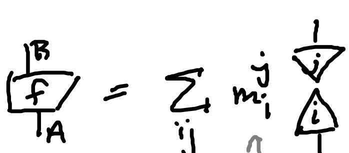
Therefore:

so

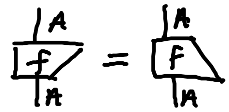


What's special about $\mathbb{C}$?

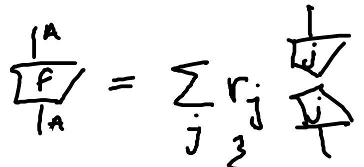
— In general, we can write a process in matrix form:



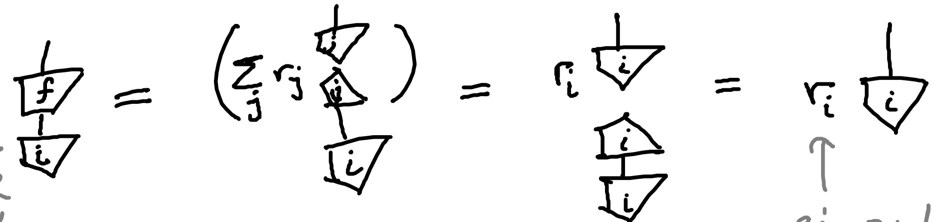
M·N complex numbers

Thm (diagonalisation) if , then there exists some ONB $\{$$\}_i$ such that:

eigenbasis

 $\longleftrightarrow \begin{pmatrix} r_1 & & \emptyset \\ & \ddots & \\ \emptyset & & r_N \end{pmatrix}$

dim(A) parameters

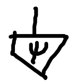


eigenvector of f

eigenvalue.

Furthermore: f self-adjoint $\Rightarrow$ $r_i = \overline{r_i}$ (real numbers)

if f is positive $\Rightarrow$ $r_i \geq 0$ (+-ive real numbers)

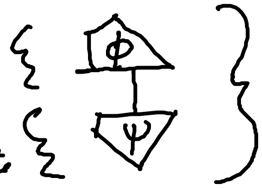
Thm In linear maps, $\exists$ . $\lambda =$  iff λ is real, ≥ 0.

Pf (book)

Recall the Generalised Born rule:

in linear maps

row vects. → "test"

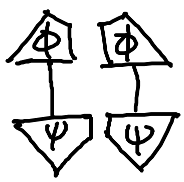


col. vects → state

"probability" test was successful

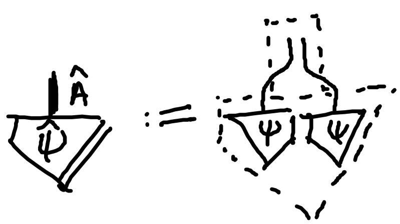
complex numbers ∵

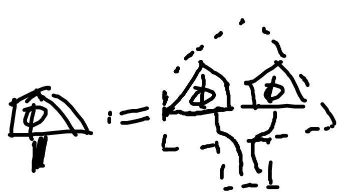
<u>Fix</u>: for any complex number ψ, $\bar{\psi}\psi =: |\psi|^2$ is a +-ive number

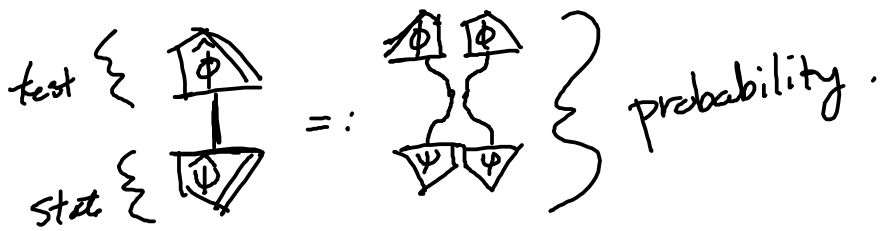
$0 \leq$  ≤ 1 } probability

ψ, ϕ normalised

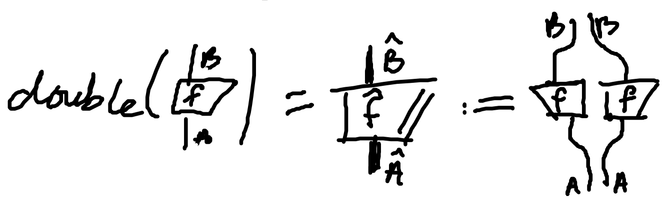
... but we've lost the elegance of the Born rule.

<u>Fix</u> let: 



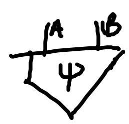


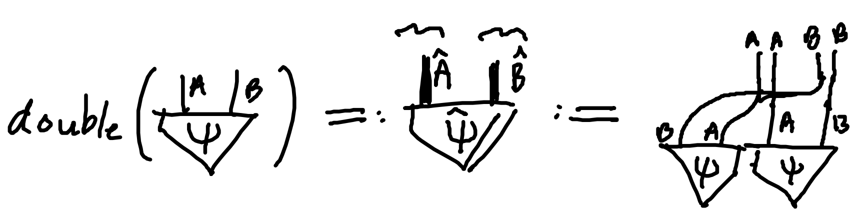
More generally, we define:

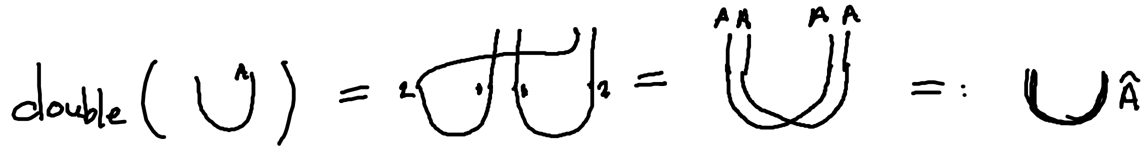


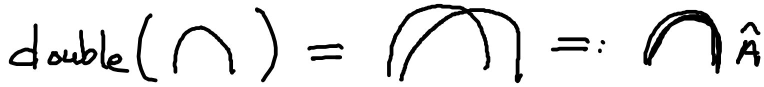
DEF The theory of <u>pure</u> <u>quantum</u> <u>maps</u> has:

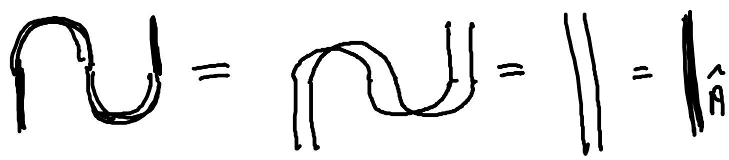
* types $\hat{A}$ for every Hilbert space A (i.e. $\widehat{\mathbb{C}^n}$ for all n)
* processes $\hat{f} : \hat{A} \to \hat{B}$ for every linear map $f : A \to B$.

NOTE, OUR CONVENTION FOR  WILL BE:

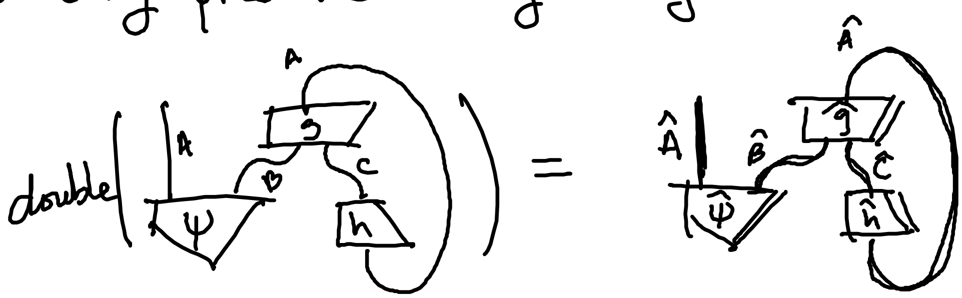








<u>THM</u> Doubling preserves string diagrams.



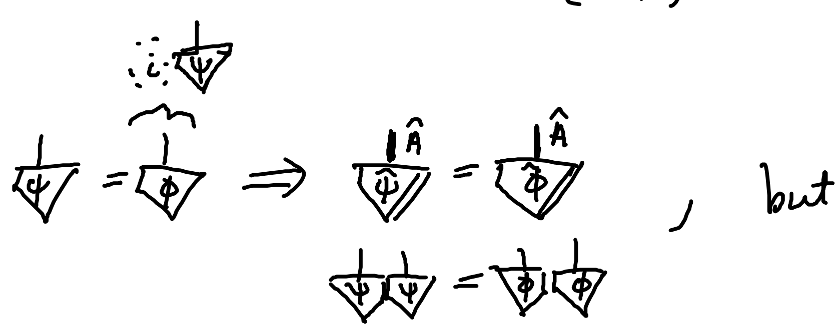
<u>Pf</u> (sketch)

* $\circ$ and $\otimes$ (e.g. double$(g \circ f)$ = double$(g) \circ$ double(f))

* swaps, cups ✓, caps ✓

* adjoints (book) ∎

* on states,



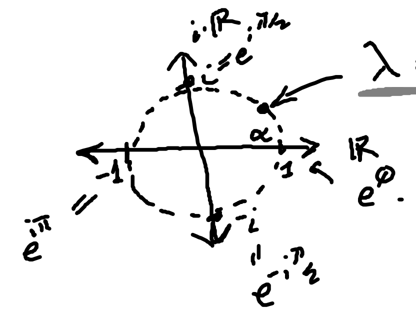
what about the converse?

* let λ be a complex # s.t. $\bar{\lambda}\cdot\lambda = 1$ (e.g. $\lambda = i$)

$$\text{double}(\lambda \cdot \psi) = \psi\, \underset{\sim}{\bar{\lambda}\lambda}\, \psi = \psi\,\psi = \text{double}(\psi)$$

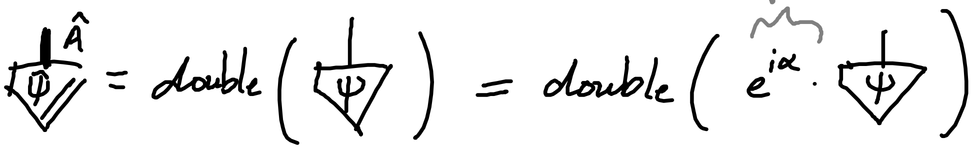
"doubling eats numbers where $\bar{\lambda}\cdot\lambda = 1$."

* What numbers $\lambda \in \mathbb{C}$ satisfy $\bar{\lambda}\lambda = |\lambda|^2 = 1$?

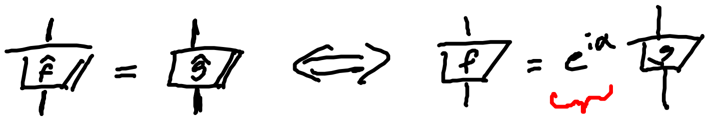


$\lambda = e^{i\alpha}$ ← phase

global phase is redundant data.

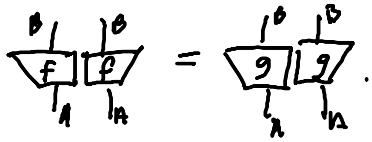


THM

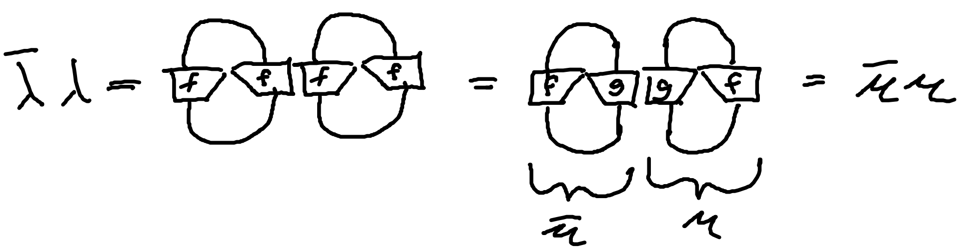


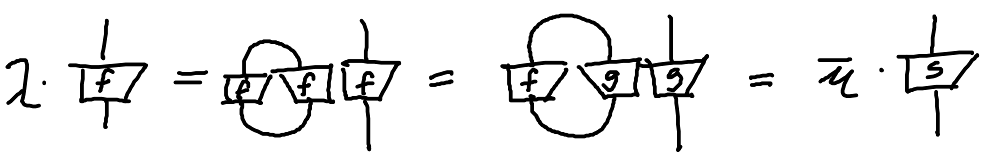
global phase

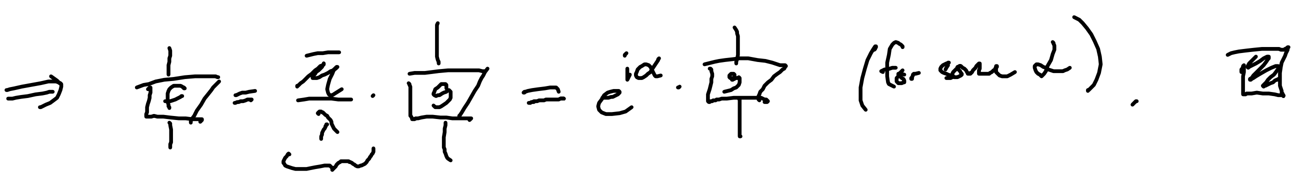
Pf already showed $\Leftarrow$, so n.t.s. $\Rightarrow$. Let



and define:







$$\left|\frac{\bar{\mu}}{\lambda}\right|^2 = \frac{\bar{\mu}}{\lambda}\cdot\frac{\mu}{\bar{\lambda}} = \frac{\bar{\mu}\mu}{\bar{\lambda}\lambda} = 1$$

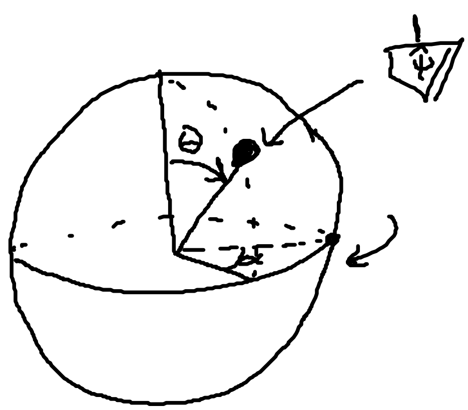
Lecture 10

Pure quantum states are: ψ for qubits

$$\psi = \text{double}\left(\overset{x+iy}{a}\; 0 + \overset{x'+iy'}{b}\; 1 \right)$$

$$\psi \text{ normalised} \iff \underbrace{|a|^2 + |b|^2 = 1}_{\text{cartesian}} \iff \underbrace{\begin{array}{l} a = \cos\frac{\theta}{2} \cdot e^{i\beta} \\ b = \sin\frac{\theta}{2} \cdot e^{i\gamma} \end{array}}_{\text{polar}}$$

$$\text{double}(\psi) = \text{double}\left(e^{-i\beta} \cdot \psi\right) = \text{double}\left(\cos\frac{\theta}{2}\; 0 + e^{i\alpha} \sin\frac{\theta}{2}\; 1\right)$$

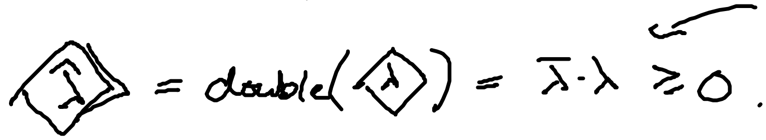


Bloch sphere.

Things not preserved by doubling.

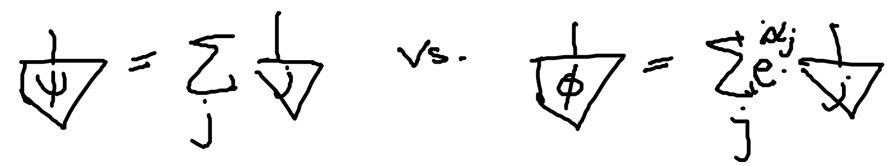
▷ numbers in the thy of pure quantum maps are positive real numbers, not $\mathbb{C}$!

(probabilities)

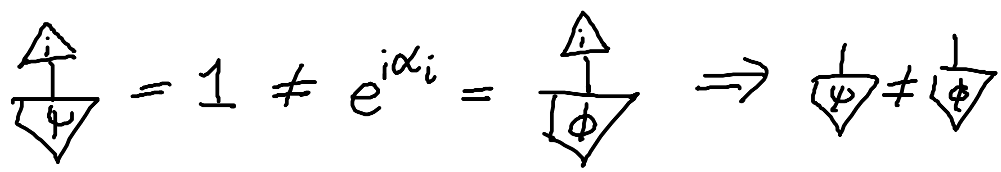


▷ doubled ONB's are not bases!

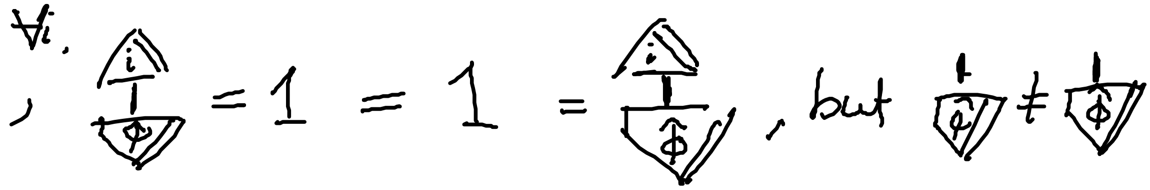
e.g.



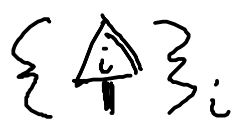
in linear maps,



in quantum maps,

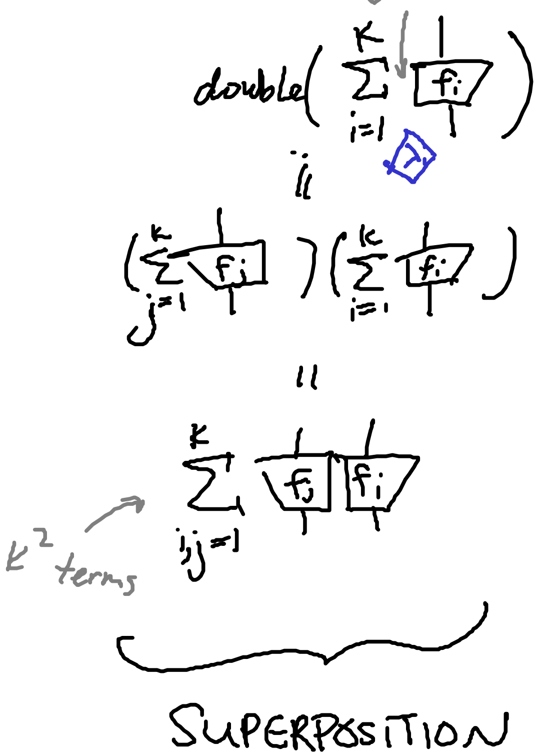


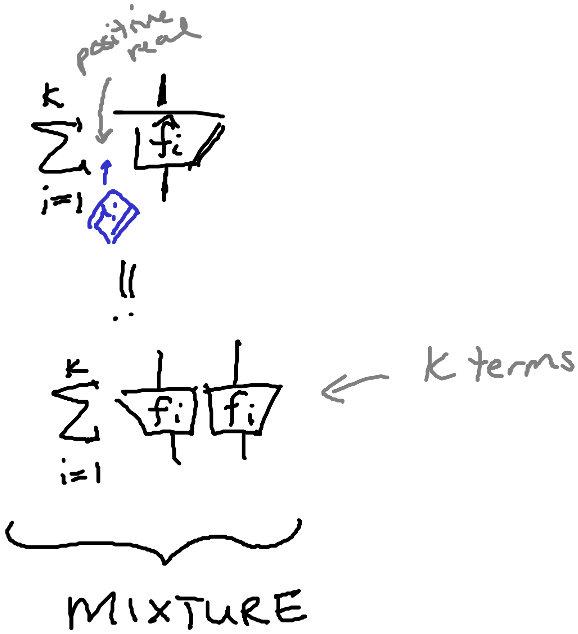
The set



describes an ONB measurement.

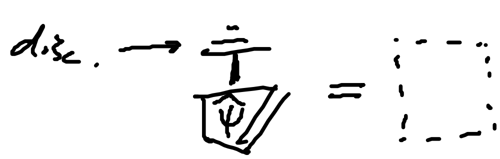
▷ Sums are not preserved.





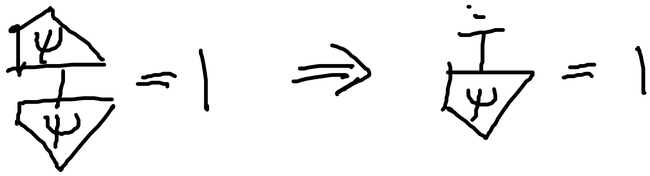
6.2 Doubling "makes space" for an interesting new process.

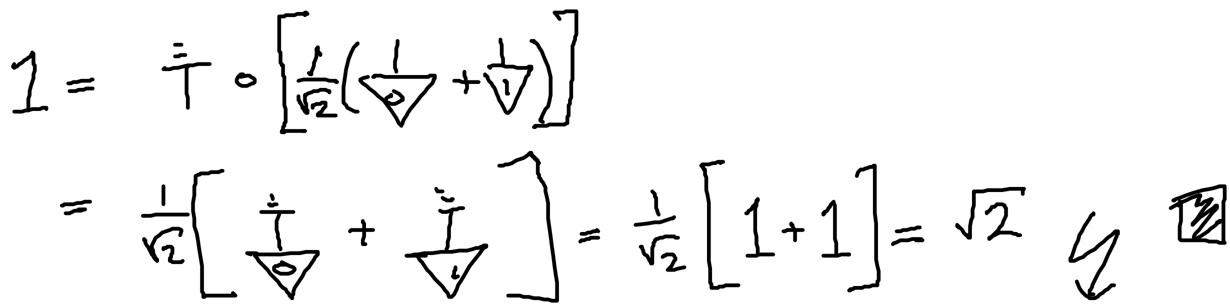
DEF The discarding process is the effect that "discards" any normalised state $\hat{\psi}$:



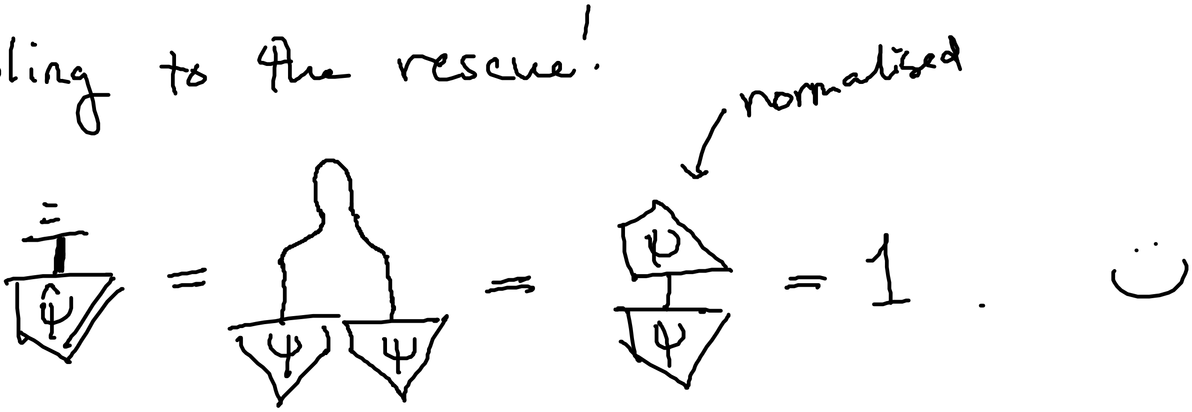
n.b. for <u>linear maps</u> there is no such thing as discarding!

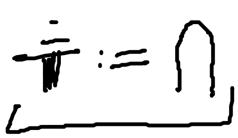
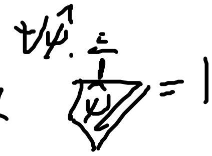
<u>Pf</u> Suppose $\exists\ \dot{\bar{T}}$ s.t.

$\forall \psi$. 

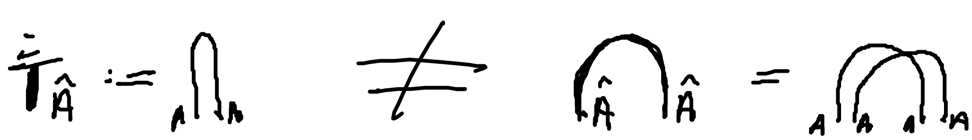
Then: 

Doubling to the rescue!



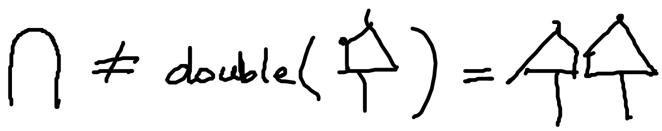
(n.b.  is uniquely fixed by )

Thm 6.31



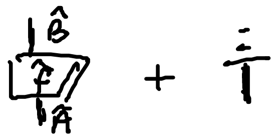
In particular: 

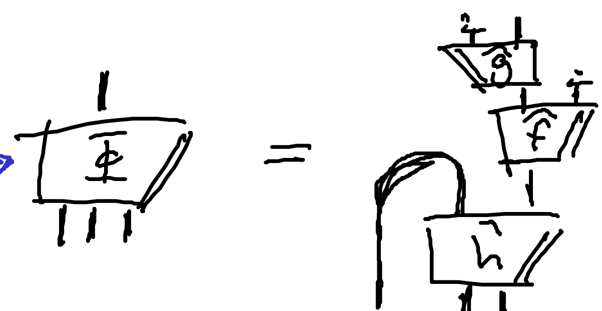
Discarding is <u>not</u> a pure q. map! ← Prop 6.27.

IDEA: 

<u>DEF</u> The process theory of <u>quantum maps</u> has:

types: $\hat{A}$ doubled Hilbert spaces

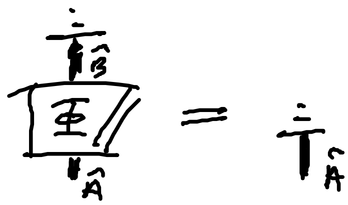
procs: string diagrams of  .

e.g. 

(no hat on Φ)

CAUSALITY

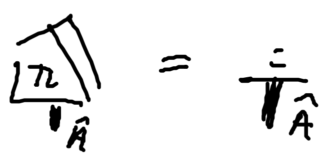
DEF A process is called causal if:



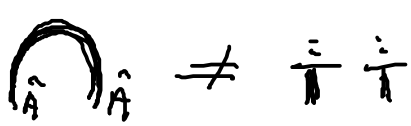
"If the output of a process is discarded, it doesn't matter which process occurred."

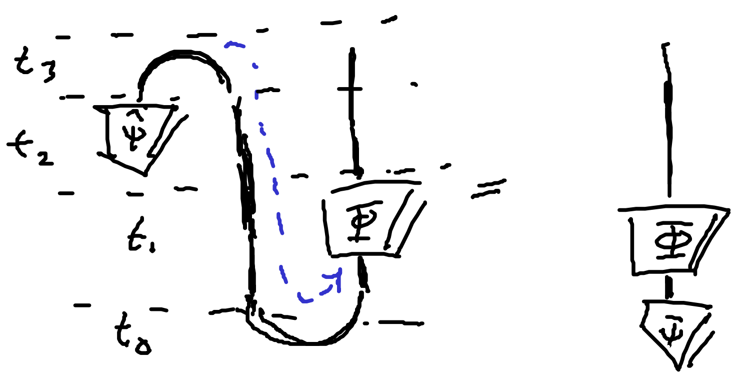
"A process can only influence its future."

In particular, causal effects:



EXAMPLE OF A PROCESS WHICH IS NOT CAUSAL:

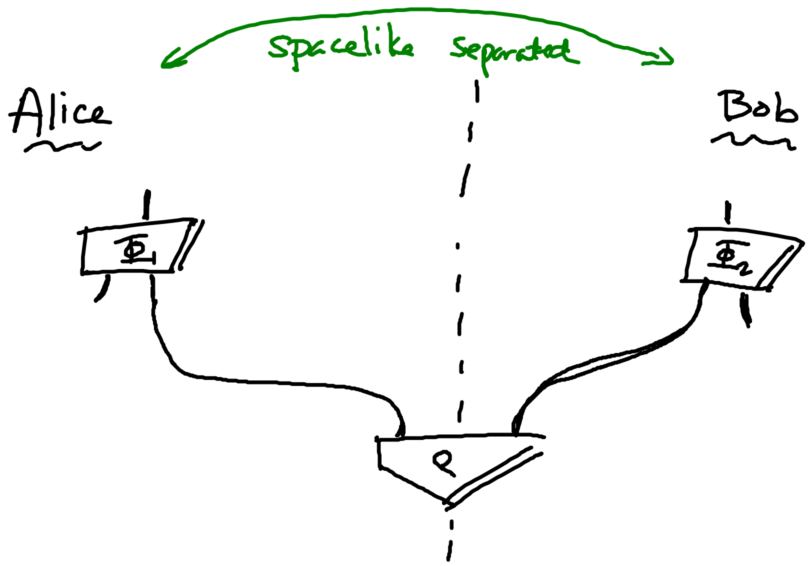




Lecture 11

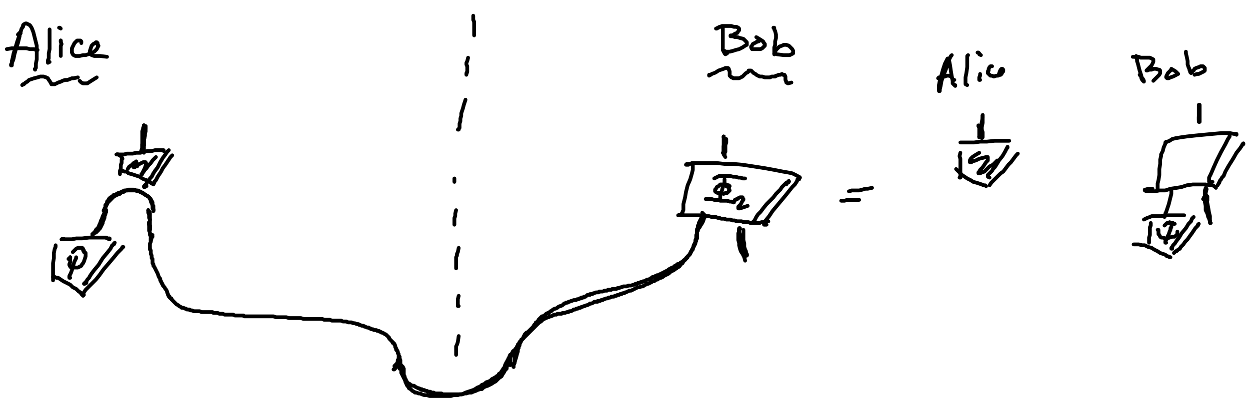
6.3 Causality + non-signalling.

No signalling (special relativity) "Nothing can travel faster than the speed of light."



Bell scenario

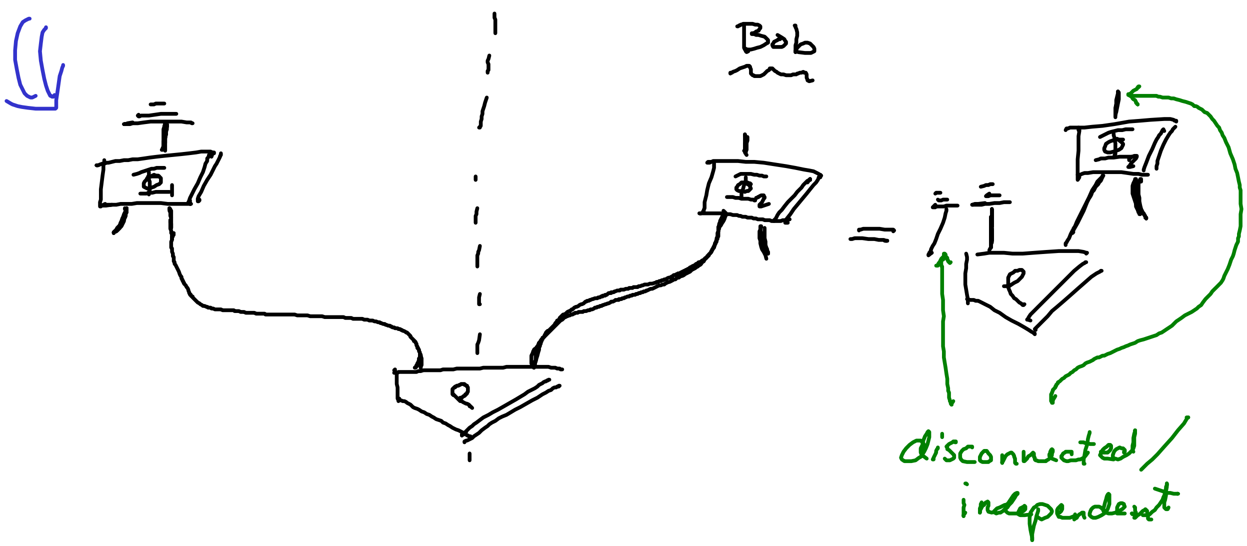
Spse: processes didn't need to be causal.



Violates special relativity!

But "in our universe" (as predicted by QT) processes <u>do</u> need to be causal.

From Bob's perspective, Bob can't access Alice's output.



$\Rightarrow$ Bob cannot access Alice's input.

CAUSALITY $\Longleftrightarrow$ NO-SIGNALLING

* CAUSALITY RULES OUT $\cap$.

* So why did we learn about $\cap$? ...and $N = 1$, etc.

" A process is $\overbrace{\text{deterministically}}$ physically realisable

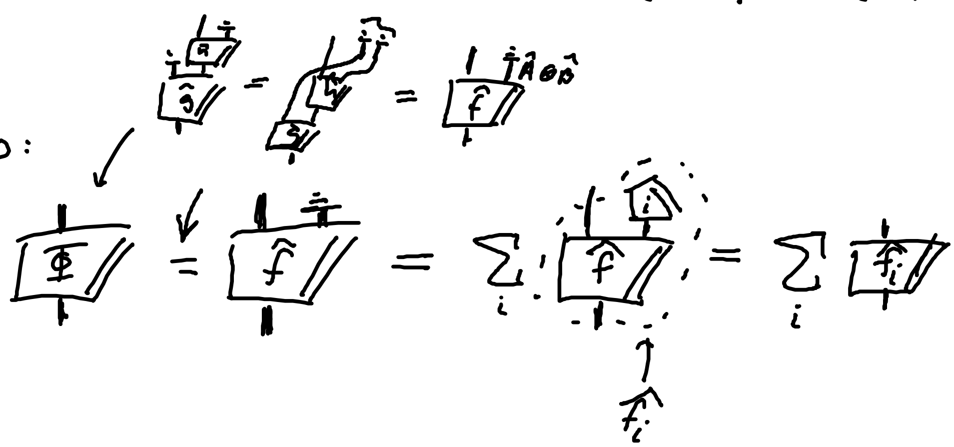
$\Longleftrightarrow$

it is causal. "

6.4 (Non-deterministic) Quantum processes

ASSIDE. MIXING.

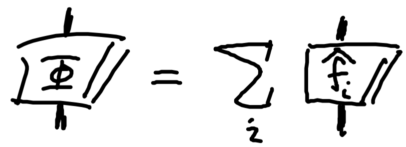
NOTE: 

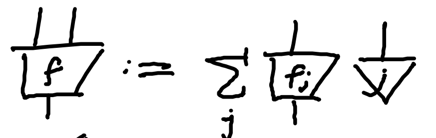
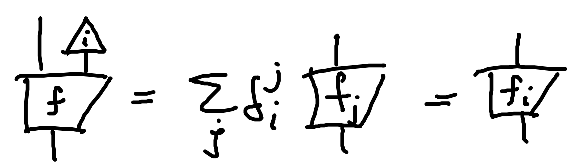
So: 

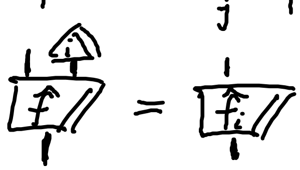
MIXTURE

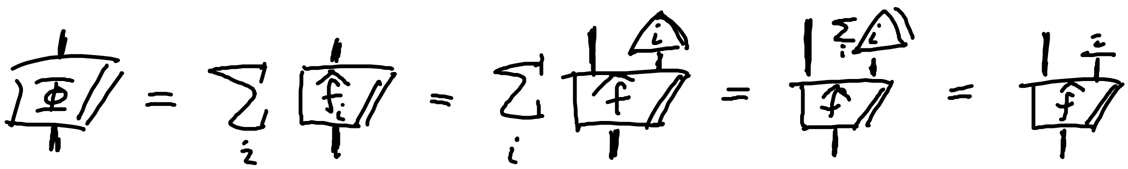
⇒ DISC. CAN BE REPLACED BY MIXING.

Conversely:



let  , then: 

So: 



⇒ MIXTURE CAN BE REPLACED BY $\bar{\top}$.

/ ASSIDE ON MIXTURE .... FOR NOW.

DEF A (possibly non-deterministic) <u>quantum</u> <u>process</u> is a set $\{\boxed{\Phi_i}\}_i$ of quantum maps such that:

$$\sum_i \overset{\bar{\top}}{\boxed{\Phi_i}} = \bar{\top}$$

Ex1 CAUSAL QUANTUM MAPS. $\to \{\boxed{\Phi}\}$. a.k.a. deterministic quantum processes.

Ex2 Random state preparation,

* Alice flips a fair coin and prepares:
  - $\boxed{\psi_0}$ if heads.
  - $\boxed{\psi_1}$ if tails.

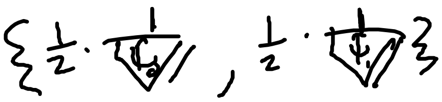
This is repr. by the quantum process: $\left\{\frac{1}{2}\cdot\boxed{\psi_0}, \frac{1}{2}\cdot\boxed{\psi_1}\right\}$

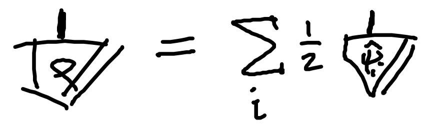
classical probs

quantum states

Non-deterministic process $\to$ we do know which branch happened once the process happens.

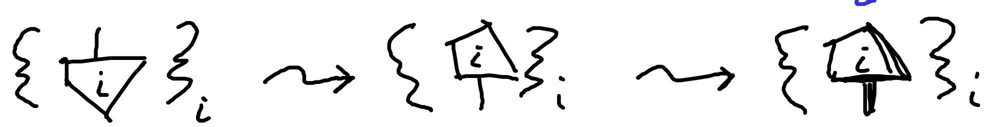
vs. a mixture $\to$ we don't know which branch happened.

If Alice does  and sends her state to Bob, but does not reveal the coin flip, Bob sees a mixed state:



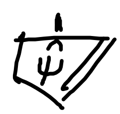
Ex3 Quantum ONB measurement.

not an ONB

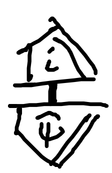


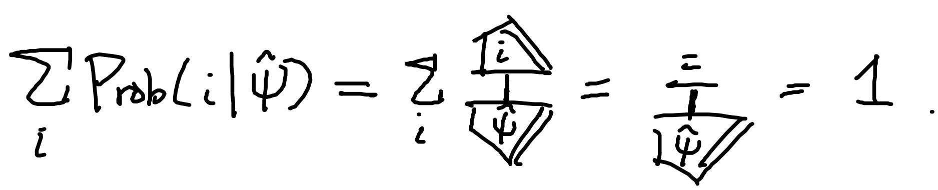
any ONB   adjoint   double

✓ quantum process

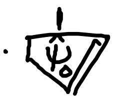
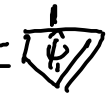
So if Alice has a state , then she can measure .

causal

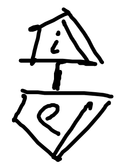
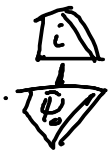
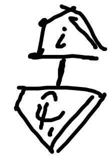
$\mathrm{Prob}(i \mid \hat{\psi}) :=$  Born rule.



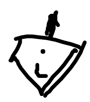
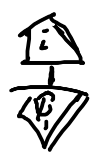
$\Rightarrow \{\mathrm{Prob}(i \mid \hat{\psi})\}_i$ is a prob. distr.

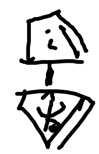
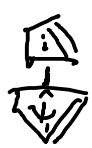
In Ex2, Bob's state was  $= \frac{1}{2} \cdot$  $+ \frac{1}{2}$  .

So, Bob can also use the Born rule:

$\mathrm{Prob}(i \mid \rho) =$  $= \frac{1}{2} \cdot$  $+ \frac{1}{2} \cdot$ 

$$= \frac{1}{2} \cdot \mathrm{Prob}(i \mid \hat{\psi}_0) + \frac{1}{2} \cdot \mathrm{Prob}(i \mid \hat{\psi}_1)$$

Eg. if  $=$  and  $= 0$, then:

$\mathrm{Prob}(i \mid \rho) = \frac{1}{2} \cdot$  $+ \frac{1}{2}$  $= \frac{1}{2}$.