

# Lecture 7

Today: define the process theory of linear maps  
 String diagrams + 3 extra ingredients

## BASES

NOTE: in most process theories,

$$\text{if } \forall \downarrow \Psi^A. \begin{array}{c} |B \\ \boxed{f} \\ \downarrow A \\ \downarrow \Psi \end{array} = \begin{array}{c} |B \\ \boxed{g} \\ \downarrow A \\ \downarrow \Psi \end{array} \text{ then } \begin{array}{c} |B \\ \boxed{f} \\ |A \end{array} = \begin{array}{c} |B \\ \boxed{g} \\ |A \end{array}.$$

(extensionality, well-pointed, ...)

DEF A basis for a type  $A$  is a minimal set of states:

$$\mathcal{B} := \left\{ \downarrow \frac{1}{1}^A, \dots, \downarrow \frac{1}{n}^A \right\}$$

such that:

$$\text{if for all } \downarrow \frac{1}{L}^A \in \mathcal{B}. \begin{array}{c} |B \\ \boxed{f} \\ \downarrow A \\ \downarrow \frac{1}{L} \end{array} = \begin{array}{c} |B \\ \boxed{g} \\ \downarrow A \\ \downarrow \frac{1}{L} \end{array} \text{ then } f = g.$$

INTUITION basis states are "reference points" for a process

DEF The dimension of a system is the size of (the smallest) basis.

DEF Two states are orthogonal if  $\langle \phi | \psi \rangle = 0$ .

DEF An orthonormal basis <sup>(ONB)</sup>  $\mathcal{B} = \{ |i\rangle \}_{i=1 \dots n}$  is a basis

where  $\langle j | i \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$   
 $\delta_i^j \leftarrow$  Kronecker  $\delta$ .

DEF self-conjugate ONB:  $|i\rangle^A = \langle i|^A =: |i\rangle^A$

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## MATRICES

Thm For bases:

$$\mathcal{B} = \{ |i\rangle^A \}_{i=1 \dots m} \quad \mathcal{B}' = \{ |i\rangle^B \}_{i=1 \dots n}$$

$$\left( \forall ij. \begin{array}{c} |j\rangle^B \\ \langle i|^B \\ \langle f|^A \\ |i\rangle^A \end{array} = \begin{array}{c} |j\rangle^B \\ \langle i|^B \\ \langle g|^A \\ |i\rangle^A \end{array} \right) \Rightarrow \begin{array}{c} |B \\ \langle f|^A \\ |i\rangle^A \end{array} = \begin{array}{c} |B \\ \langle g|^A \\ |i\rangle^A \end{array}$$

Pf

$$\forall j \left( \forall i. \begin{array}{c} |j\rangle^B \\ \langle i|^B \\ \langle f|^A \\ |i\rangle^A \end{array} = \begin{array}{c} |j\rangle^B \\ \langle i|^B \\ \langle g|^A \\ |i\rangle^A \end{array} \right) \Rightarrow \forall j \left( \begin{array}{c} |j\rangle^B \\ \langle f|^A \\ |i\rangle^A \end{array} = \begin{array}{c} |j\rangle^B \\ \langle g|^A \\ |i\rangle^A \end{array} \right)$$

$$\forall j \left( \begin{array}{c} \boxed{j} \\ \downarrow \\ \boxed{B} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{A} \end{array} = \begin{array}{c} \boxed{j} \\ \downarrow \\ \boxed{B} \\ \downarrow \\ \boxed{g} \\ \downarrow \\ \boxed{A} \end{array} \right) \Rightarrow \forall j \left( \begin{array}{c} \boxed{A} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{B} \\ \downarrow \\ \boxed{j} \end{array} = \begin{array}{c} \boxed{A} \\ \downarrow \\ \boxed{g} \\ \downarrow \\ \boxed{j} \end{array} \right)$$

$$\Rightarrow \begin{array}{c} \boxed{A} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{B} \end{array} = \begin{array}{c} \boxed{A} \\ \downarrow \\ \boxed{g} \\ \downarrow \\ \boxed{B} \end{array} \Rightarrow \begin{array}{c} \boxed{B} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{A} \end{array} = \begin{array}{c} \boxed{B} \\ \downarrow \\ \boxed{g} \\ \downarrow \\ \boxed{A} \end{array} \quad \square$$

NOTATION Let  $f_i^j := \begin{array}{c} \boxed{j} \\ \downarrow \\ \boxed{B} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{A} \\ \downarrow \\ \boxed{i} \end{array} \leftarrow \text{number, depends on bases}$

DEF The set of numbers  $\{f_i^j\}_{i=1..m, j=1..n}$  is called the matrix of  $f$ .

Think of  $f_i^j$   $\leftarrow$  row index  
 $f_i^j$   $\leftarrow$  column index (e.g.  $f_{ji}$  in linear alg.)

e.g. if  $m=3$  and  $n=2$ , matrix of  $f$

$$\begin{pmatrix} f_1^1 & f_1^2 & f_3^1 \\ f_1^2 & f_2^2 & f_3^2 \end{pmatrix}$$

MATRIX TRANSPOSE

$$\begin{pmatrix} f_1^1 & f_1^2 & f_3^1 \\ f_1^2 & f_2^2 & f_3^2 \end{pmatrix}^T = \begin{pmatrix} f_1^1 & f_1^2 \\ f_2^1 & f_2^2 \\ f_3^1 & f_3^2 \end{pmatrix}$$

(with resp. to a self-conj ONB)

Thm The matrix of  $f$   $\begin{matrix} |A \\ \boxed{f} \\ |B \end{matrix} := {}^A \left( \begin{matrix} \boxed{f} \\ |B \end{matrix} \right)_B$  is  $(f^T)_i^j := f_j^i$ .

Pf  $(f^T)_i^j := \begin{matrix} \triangle j \\ | \\ \boxed{f} \\ | \\ \triangle i \end{matrix} \stackrel{\downarrow}{=} \begin{matrix} \triangle j \\ | \\ \boxed{f} \\ | \\ \triangle i \end{matrix} \stackrel{\text{self-conj}}{=} \begin{matrix} \triangle i \\ | \\ \boxed{f} \\ | \\ \triangle j \end{matrix} = f_j^i \quad \square$

or "

Similarly, the conjugate and conjugate-transpose:

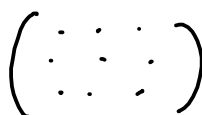
$$(\overline{f})_i^j = \overline{f_j^i} \quad \begin{matrix} \boxed{f} \\ | \\ \triangle i \\ | \\ \triangle j \end{matrix} \mapsto \begin{matrix} \triangle i \\ | \\ \boxed{f} \\ | \\ \triangle j \end{matrix} \quad \left( \begin{matrix} \overline{\cdot} \\ \overline{\cdot} \\ \overline{\cdot} \end{matrix} \right)$$

$$(f^{\dagger})_i^j = \overline{f_j^i} \quad \begin{matrix} \triangle i \\ | \\ \boxed{f} \\ | \\ \triangle j \end{matrix} \mapsto \begin{matrix} \triangle j \\ | \\ \boxed{f} \\ | \\ \triangle i \end{matrix} \quad \left( \begin{matrix} \overline{\cdot} \\ \overline{\cdot} \\ \overline{\cdot} \end{matrix} \right)$$

PROCESS



MATRIX



MATRIX



PROCESS

?

SUPPOSE I HAVE  $\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & g_j^i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ , then I can make the proc:

$\begin{matrix} \triangle j \\ | \\ \boxed{f} \\ | \\ \triangle i \end{matrix} := \begin{matrix} \triangle j \\ | \\ \triangle i \\ | \\ \triangle j \end{matrix} \begin{matrix} \triangle i \\ | \\ \triangle j \end{matrix}$ , then  $\begin{matrix} \triangle i \\ | \\ \boxed{f} \\ | \\ \triangle j \end{matrix}$  has matrix:  $\begin{pmatrix} 0 & 0 & 0 & \dots \\ \dots & 0 & g_j^i & 0 \\ \dots & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 \end{pmatrix}$   $\leftarrow j^{\text{th}} \text{ row}$   
 $\uparrow i^{\text{th}} \text{ column}$

DEF A process theory has sums if, for any two processes of the same type

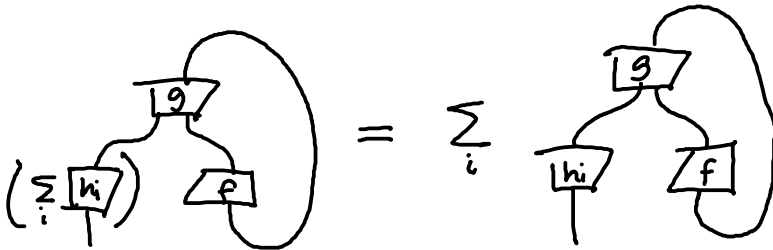


1. associative, commutative, and has unit  $\emptyset$ .

$$\begin{array}{ccc} \uparrow & \nwarrow & \uparrow \\ f+(g+h) = f+(g+h) & f+g = g+f & \begin{array}{c} \uparrow \\ \boxed{f} \\ \downarrow \\ A \end{array}, \begin{array}{c} \uparrow \\ \boxed{\emptyset} \\ \downarrow \\ A \end{array} \rightsquigarrow f+\emptyset = f. \end{array}$$

$$\text{Let } \sum_i \boxed{f_i} := f_1 + \dots + f_n$$

2. Sums distribute over diagrams



3. Sums preserve adjoints

$$\left( \sum_i \boxed{f_i} \right)^{\dagger} = \sum_i \boxed{f_i}^{\dagger}$$

DEF The process theory of linear maps is the theory where:

1. for each  $D \in \mathbb{N}$ , there is a system  $\omega$  ONB of size  $D$ .
2. processes admit sums
3. the numbers are complex numbers.

(ie. every process  $\boxed{\lambda}$  corresponds to a complex number)

→ (we call a system in linear maps a Hilbert space)

# Lecture 8

EXAMPLES OF SUMS DIST. OVER DIAGS.

From the past  
↓

\* linearity.

$$\begin{matrix} B \\ \downarrow \\ \boxed{f} \\ \uparrow \\ A \end{matrix} \circ \left( \lambda \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{\psi} \\ \uparrow \\ \end{matrix} \right) = \lambda \downarrow \begin{matrix} B \\ \downarrow \\ \boxed{f} \\ \uparrow \\ A \end{matrix} \circ \begin{matrix} A \\ \downarrow \\ \boxed{\psi} \\ \uparrow \\ \end{matrix} = \lambda \cdot \left( \begin{matrix} B \\ \downarrow \\ \boxed{f} \\ \uparrow \\ A \end{matrix} \circ \begin{matrix} A \\ \downarrow \\ \boxed{\psi} \\ \uparrow \\ \end{matrix} \right)$$

$$f(\lambda \cdot v) = \lambda \cdot f(v)$$

$$f(v_1 + v_2) = f(v_1) + f(v_2)$$

$$f\left(\sum_i v_i\right) = \sum_i f(v_i)$$

$$\begin{matrix} B \\ \downarrow \\ \boxed{f} \\ \uparrow \\ A \end{matrix} \circ \left( \sum_i \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{v_i} \\ \uparrow \\ \end{matrix} \right) = \begin{matrix} B \\ \downarrow \\ \boxed{f} \\ \uparrow \\ A \end{matrix} \circ \left( \sum_i \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{v_i} \\ \uparrow \\ \end{matrix} \right)$$

$$\begin{matrix} B \\ \downarrow \\ \boxed{f} \\ \uparrow \\ W \end{matrix} \text{ where } \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{w} \\ \uparrow \\ \end{matrix} = \sum_i \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{v_i} \\ \uparrow \\ \end{matrix}$$

$$= \sum_i \begin{matrix} B \\ \downarrow \\ \boxed{f} \\ \uparrow \\ A \end{matrix} \circ \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{v_i} \\ \uparrow \\ \end{matrix} = \sum_i \left[ \begin{matrix} B \\ \downarrow \\ \boxed{f} \\ \uparrow \\ A \end{matrix} \circ \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{v_i} \\ \uparrow \\ \end{matrix} \right]$$

\* linearity of  $\langle \leftarrow | \rightarrow \rangle$

$$\langle w | v_1 + v_2 \rangle = \langle w | v_1 \rangle + \langle w | v_2 \rangle$$

$$\langle w | \sum_i v_i \rangle = \sum_i \langle w | v_i \rangle$$

\* conjugate-linearity of  $\langle \leftarrow | \rightarrow \rangle$

$$\langle \sum_i v_i | w \rangle = \sum_i \langle v_i | w \rangle$$

$$\langle \sum_i \lambda_i v_i | w \rangle = \sum_i \bar{\lambda}_i \langle v_i | w \rangle$$

\* bilinearity of  $\otimes$ .

$$\left( \sum_i \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{f_i} \\ \uparrow \\ \end{matrix} \right) \otimes \downarrow \begin{matrix} B \\ \downarrow \\ \boxed{g} \\ \uparrow \\ \end{matrix} = \sum_i \left[ \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{f_i} \\ \uparrow \\ \end{matrix} \otimes \downarrow \begin{matrix} B \\ \downarrow \\ \boxed{g} \\ \uparrow \\ \end{matrix} \right]$$

$$\downarrow \begin{matrix} A \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \end{matrix} \otimes \left( \sum_i \downarrow \begin{matrix} B \\ \downarrow \\ \boxed{g_i} \\ \uparrow \\ \end{matrix} \right) = \sum_i \left[ \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \end{matrix} \otimes \downarrow \begin{matrix} B \\ \downarrow \\ \boxed{g_i} \\ \uparrow \\ \end{matrix} \right]$$

$$\diamond \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \end{matrix} = \left( \begin{matrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{matrix} \right) \otimes \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \end{matrix} = \sum_{i=0}^1 \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \end{matrix} = \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \end{matrix} + \downarrow \begin{matrix} A \\ \downarrow \\ \boxed{f} \\ \uparrow \\ \end{matrix}$$

Matrix  $M \rightsquigarrow$  process  $f$

Thm For any matrix  $M$  with entries  $m_i^j$ , the process:

$$\left[ \begin{array}{c} B \\ f \\ A \end{array} \right] := \sum_{ij} \diamond m_{ij} \begin{array}{c} \downarrow j \\ \uparrow i \end{array}$$

has matrix  $M$ .

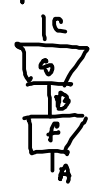
Pf matrix of  $f$  has entries  $f_i^j := \begin{array}{c} \downarrow j \\ \left[ \begin{array}{c} f \\ \uparrow i \end{array} \right] \end{array}$ .

$$f_i^j = \begin{array}{c} \downarrow j \\ \left[ \begin{array}{c} f \\ \uparrow i \end{array} \right] \end{array} = \left( \sum_{kl} m_k^l \begin{array}{c} \downarrow l \\ \left[ \begin{array}{c} \uparrow j \\ \uparrow i \end{array} \right] \end{array} \right) = \sum_{kl} m_k^l \begin{array}{c} \downarrow l \\ \left[ \begin{array}{c} \uparrow j \\ \uparrow i \end{array} \right] \end{array} = \sum_{k,l} m_k^l \delta_i^k \delta_l^j$$

$$= \sum_{k \neq i, l} m_k^l \delta_i^k \delta_l^j + \sum_l m_i^l \underbrace{\delta_i^i}_{=1} \delta_l^j = \sum_l m_i^l \delta_l^j = m_i^j \quad \square$$

NOTE  $\rightsquigarrow \left[ \begin{array}{c} \\ f \\ A \end{array} \right]$  has matrix  $\begin{pmatrix} 1 & 0 \\ 0 & \dots \end{pmatrix}$ , so  $\left[ \begin{array}{c} \\ f \\ A \end{array} \right] = \sum_j \delta_i^j \begin{array}{c} \downarrow j \\ \left[ \begin{array}{c} \uparrow i \end{array} \right] \end{array} = \sum_i \begin{array}{c} \downarrow i \\ \left[ \begin{array}{c} \uparrow i \end{array} \right] \end{array}$


COROLLARY  $\left[ \begin{array}{c} \\ f \\ A \end{array} \right] = \sum_i \begin{array}{c} \downarrow i \\ \left[ \begin{array}{c} \uparrow i \end{array} \right] \end{array}$

Thm If  $f$  has matrix  $M_f$  and  $g$  has matrix  $M_g$ , then the matrix of  $g \circ f$  is  $M_g M_f$ .  

matrix multiplication

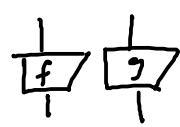
Pf

$$M_g M_f = \begin{pmatrix} g^1 & \dots & g^n \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \sum_k g_k^j f_k^i & \dots \\ \dots & \dots & \dots \end{pmatrix} \leftarrow j$$

↑  
i

need to show the matrix of  has entries  $\sum_k g_k^j f_k^i$ .  
 $\{ \downarrow_k \}_{k=1..n}$  basis for B

$$(M_{g \circ f})^j_i = \begin{matrix} \uparrow j \\ \downarrow i \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \downarrow i \end{matrix} = \sum_{k=1}^n \begin{matrix} \uparrow j \\ \text{---} \\ \downarrow k \\ \text{---} \\ \downarrow i \end{matrix} = \sum_k g_k^j f_k^i \quad \square$$

Thm The matrix of  is the Kronecker product.  
tensor product

$$\begin{pmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{pmatrix}_{2 \times 2} \otimes \begin{matrix} G \\ \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix}_{2 \times 2} \end{matrix} = \begin{pmatrix} f_1^1 G & f_2^1 G \\ f_1^2 G & f_2^2 G \end{pmatrix}$$

$$= \begin{pmatrix} f_1^1 g_1^1 & f_1^1 g_2^1 & \dots & f_2^1 g_1^1 & f_2^1 g_2^1 \\ f_1^2 g_1^1 & f_1^2 g_2^1 & \dots & f_2^2 g_1^1 & f_2^2 g_2^1 \\ f_1^1 g_1^2 & f_1^1 g_2^2 & \dots & f_2^1 g_1^2 & f_2^1 g_2^2 \\ f_1^2 g_1^2 & f_1^2 g_2^2 & \dots & f_2^2 g_1^2 & f_2^2 g_2^2 \end{pmatrix}_{4 \times 4}$$



$$\begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}_{m \times n} \otimes \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}_{m' \times n'} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}_{mm' \times nn'}$$

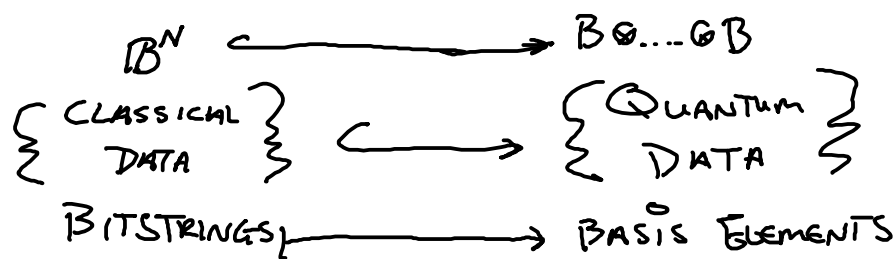
The reason: if  $\left\{ \begin{array}{|c|} \hline A \\ \hline \end{array} \right\}_{i=1..m}$  is a basis for A and if  $\left\{ \begin{array}{|c|} \hline B \\ \hline \end{array} \right\}_{j=1..m'}$  is a basis for B, then  $\left\{ \begin{array}{|c|c|} \hline A & B \\ \hline \end{array} \right\}_{i=1..m, j=1..m'}$  is a basis for  $A \otimes B$ .

$$\left( \forall ij \begin{array}{|c|} \hline F \\ \hline \end{array} = \begin{array}{|c|} \hline G \\ \hline \end{array} \right) \Rightarrow \begin{array}{|c|} \hline c \\ \hline \end{array} = \begin{array}{|c|} \hline c \\ \hline \end{array}$$

Ex assume B has basis  $\left\{ \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \end{array} \right\}$  ← 2-dim

Then  $B \otimes B$  has basis  $\left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \right\}$  ← 4-dim

and  $\underbrace{B \otimes \dots \otimes B}_N$  has basis  $\left\{ \begin{array}{|c| \dots |} \hline \vdots \\ \hline \end{array}, \begin{array}{|c| \dots |} \hline \vdots \\ \hline \end{array}, \dots, \begin{array}{|c| \dots |} \hline \vdots \\ \hline \end{array} \right\}$



↖  $2^N$ -dim

# Lecture 9

linear maps  $\rightsquigarrow$  quantum maps

1. systems  $\mathbb{C}^n$  for every  $n \in \mathbb{N}$  with an ONB  $\{|\downarrow\rangle, \dots, |\uparrow\rangle\}$
2. sums
3. numbers are complex numbers  $\mathbb{C}$

\* in particular, the trivial system  $\mathbb{I} = \mathbb{C}$  ( $= \mathbb{C}^1$ )  
has an ONB  $\{|\downarrow\rangle\}$

$$\mathbb{C}^m \otimes \mathbb{C}^n = \mathbb{C}^{m \cdot n}$$

$$\left( \begin{array}{c} |\downarrow\rangle \\ \vdots \end{array} = \begin{array}{c} |\uparrow\rangle \\ \vdots \end{array} \right) \Rightarrow |\downarrow\rangle = |\uparrow\rangle$$

Therefore:

$$\begin{array}{c} \mathbb{B} \\ \boxed{F} \\ \mathbb{A} \end{array} \leftrightarrow \underbrace{\left( \begin{array}{c} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{array} \right)}_{\dim A} \} \dim B$$

so  $\begin{array}{c} \mathbb{A} \\ \boxed{\Psi} \\ \mathbb{I} \end{array}$  are column vectors,  $\begin{array}{c} \mathbb{A} \\ \boxed{\pi} \end{array}$  row vectors,  $\begin{array}{c} \mathbb{C} \\ \boxed{\tau} \end{array} \leftrightarrow (\lambda)$

What's special about  $\mathbb{C}$ ?

- In general, we can write a process in matrix form:

$$\begin{array}{c} \mathbb{B} \\ \boxed{F} \\ \mathbb{A} \end{array} = \sum_{ij} m_{ij} \begin{array}{c} |\downarrow\rangle \\ \uparrow \\ |\uparrow\rangle \end{array}$$

M-N complex numbers

Thm (diagonalisation) if  $\frac{|A}{F} = \frac{|A}{A}$ , then there exists some ONB  $\{ \downarrow_i^A \}$  such that:

eigenbasis

$$\frac{|A}{F} = \sum_j r_j \downarrow_j^A \iff \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}$$

dim(A) parameters

$$\begin{matrix} \downarrow_f \\ \downarrow_i \end{matrix} = \left( \sum_j r_j \downarrow_j^A \right) = r_i \begin{matrix} \downarrow_i \\ \downarrow_i \end{matrix} = r_i \begin{matrix} \downarrow_i \\ \downarrow_i \end{matrix}$$

eigenvector of  $f$

↑  
eigenvalue.

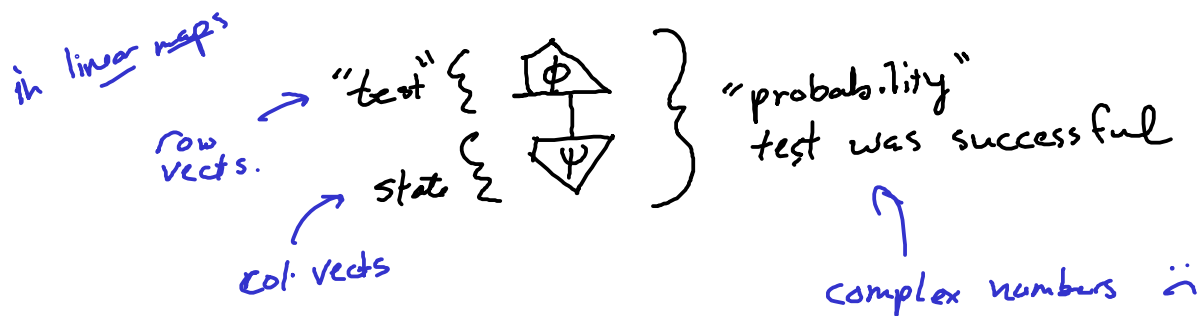
Furthermore:  $f$  self-adjoint  $\implies r_i = \bar{r}_i$  (real numbers)

if  $f$  is positive  $\implies r_i \geq 0$  (+ive real numbers)

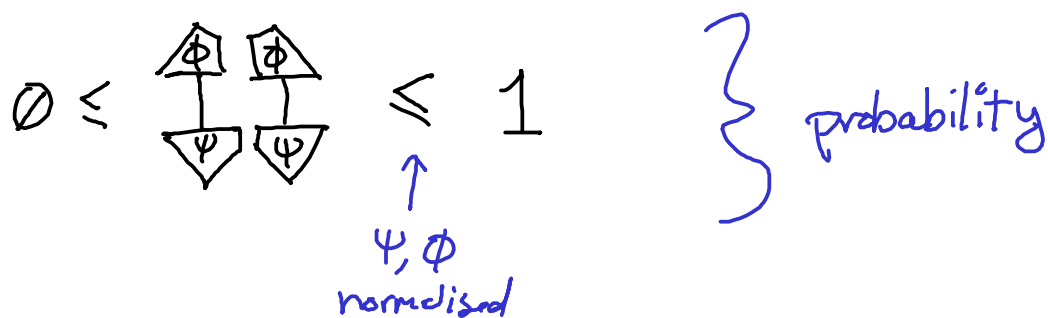
Thm In linear maps,  $\exists \downarrow \cdot \lambda = \begin{matrix} \downarrow \\ \downarrow \end{matrix}$  iff  $\lambda$  is real,  $\geq 0$ .

Pf (book)

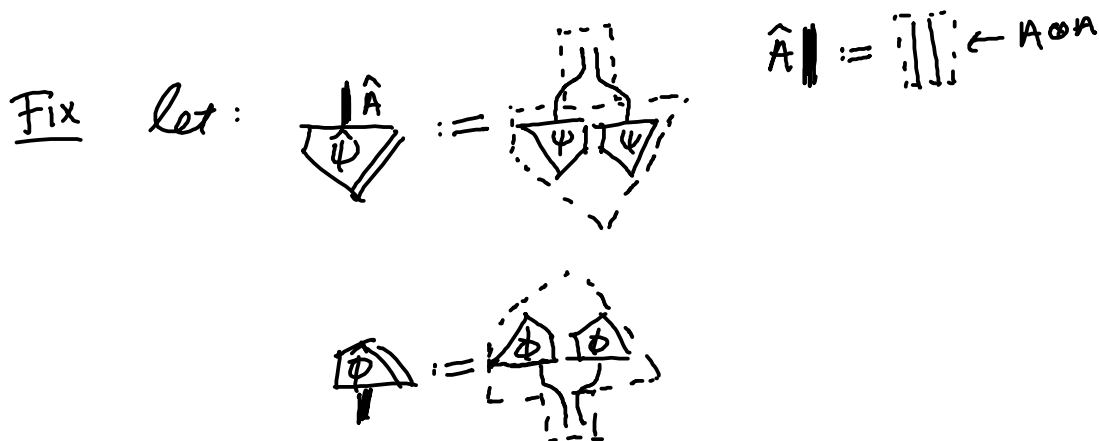
Recall the Generalised Born rule:

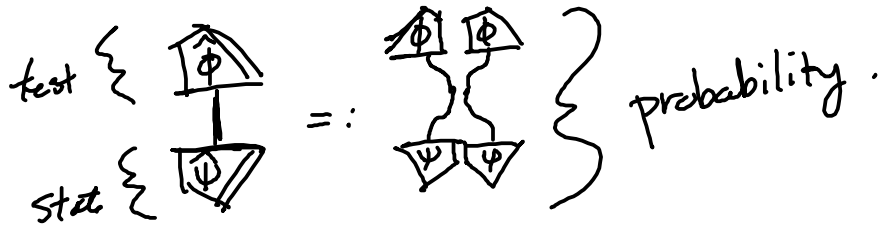


Fix: for any complex number  $y$ ,  $\bar{y}y = |y|^2$  is a +ive number

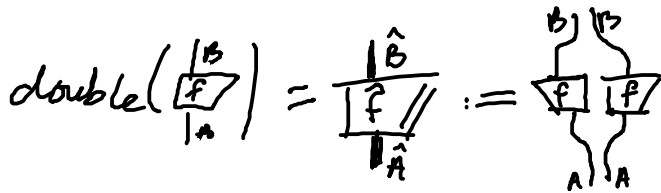


...but we've lost the elegance of the Born rule.





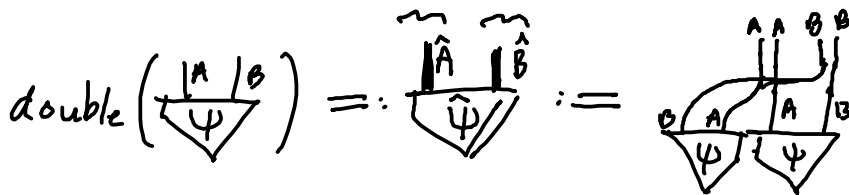
more generally, we define:



DEF The theory of pure quantum maps has:

- \* types  $\hat{A}$  for every Hilbert space  $A$  (i.e.  $\mathbb{C}^n$  for all  $n$ )
- \* processes  $\hat{f}: \hat{A} \rightarrow \hat{B}$  for every linear map  $f: A \rightarrow B$ .

NOTE, our CONVENTION FOR  $\begin{array}{c} A \quad B \\ \triangle \downarrow \\ \Psi \end{array}$  WILL BE:



$$\text{double}(\cup) = \text{diagram of two circles joined at a point} = \text{diagram of two circles with arrows} =: \cup \hat{A}$$

$$\text{double}(\cap) = \text{diagram of two circles overlapping} =: \cap \hat{A}$$

$$\text{diagram of a loop} = \text{diagram of a loop with arrows} = \parallel = \hat{A}$$

Thm Doubling preserves string diagrams.

$$\text{double}(\text{diagram with nodes } \psi, g, h \text{ and wires } A, B, C) = \text{diagram with nodes } \hat{\psi}, \hat{g}, \hat{h} \text{ and wires } \hat{A}, \hat{B}, \hat{C}$$

Pf (sketch)

\*  $\circ$  and  $\otimes$  (e.g.  $\text{double}(g \circ f) = \text{double}(g) \circ \text{double}(f)$ )

\* swaps, cups, caps

\* adjoints

(book)



\* on states,  $\text{diagram of } \psi \text{ with wire } A = \text{diagram of } \phi \text{ with wire } A \Rightarrow \text{diagram of } \hat{\psi} \text{ with wire } \hat{A} = \text{diagram of } \hat{\phi} \text{ with wire } \hat{A}$ , but

$$\text{diagram of } \psi \text{ with wire } A = \text{diagram of } \phi \text{ with wire } A$$

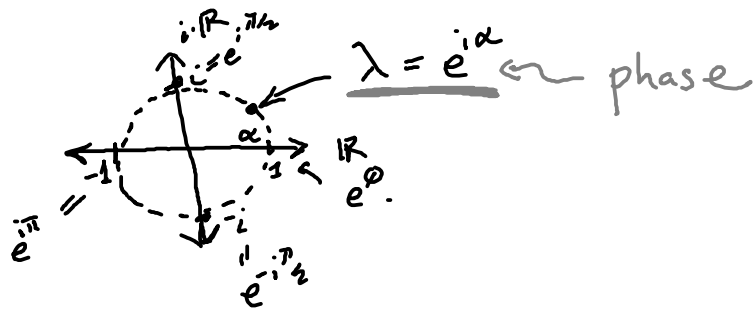
what about the converse?

\*let  $\lambda$  be a complex  $\neq$  s.t.  $\bar{\lambda} \cdot \lambda = 1$  (eg.  $\lambda = i$ )

$$\text{double}(\lambda \cdot \downarrow \Psi) = \downarrow \Psi \underbrace{\bar{\lambda} \lambda}_{=1} \downarrow \Psi = \downarrow \Psi \downarrow \Psi = \text{double}(\downarrow \Psi)$$

"doubling eats numbers where  $\bar{\lambda} \cdot \lambda = 1$ ."

\* What numbers  $\lambda \in \mathbb{C}$  satisfy  $\bar{\lambda} \lambda = |\lambda|^2 = 1$ ?



global phase is redundant data.

$$\hat{\Phi} = \text{double}(\Psi) = \text{double}(e^{i\alpha} \cdot \Psi)$$

$$\underline{T_{HM}} \quad \frac{1}{\hat{F}} = \frac{1}{\mathcal{G}} \iff \frac{1}{F} = e^{i\alpha} \frac{1}{\mathcal{G}}$$

↑ global phase

PF already showed  $\Leftarrow$ , so n.t.s.  $\Rightarrow$ . Let

$$\begin{matrix} \beta & \beta \\ | & | \\ \text{F} & \text{F} \\ | & | \\ \lambda & \lambda \end{matrix} = \begin{matrix} \beta & \beta \\ | & | \\ \mathcal{G} & \mathcal{G} \\ | & | \\ \lambda & \lambda \end{matrix}$$

and define:

$$\lambda := \text{loop}(F, F) \quad \text{and} \quad \mu := \text{loop}(\mathcal{G}, F)$$

$$\bar{\lambda} \lambda = \text{loop}(F, F, F, F) = \underbrace{\text{loop}(F, \mathcal{G})}_{\bar{\mu}} \underbrace{\text{loop}(\mathcal{G}, F)}_{\mu} = \bar{\mu} \mu$$

$$\lambda \cdot \frac{1}{F} = \text{loop}(F, F) \frac{1}{F} = \text{loop}(F, \mathcal{G}) = \bar{\mu} \cdot \frac{1}{\mathcal{G}}$$

$$\Rightarrow \frac{1}{F} = \frac{\bar{\mu}}{\lambda} \cdot \frac{1}{\mathcal{G}} = e^{i\alpha} \cdot \frac{1}{\mathcal{G}} \quad (\text{for some } \alpha), \quad \square$$

$$\left| \frac{\bar{\mu}}{\lambda} \right|^2 = \frac{\bar{\mu}}{\lambda} \cdot \frac{\mu}{\bar{\lambda}} = \frac{\bar{\mu} \mu}{\lambda \bar{\lambda}} = 1$$



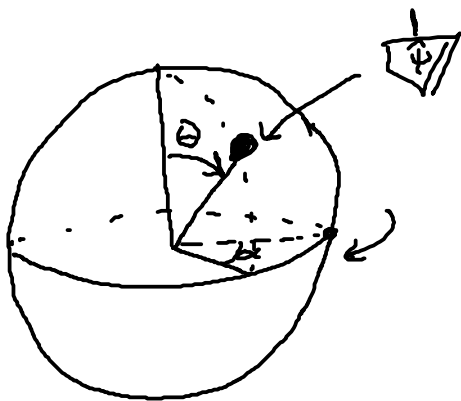
# Lecture 10

Pure quantum states are  $\downarrow\psi$  for qubits

$$\downarrow\psi = \text{double} \left( \overset{x+iy}{\downarrow} a \downarrow_0 + \overset{x'+iy'}{\downarrow} b \downarrow_1 \right)$$

$$\psi \text{ normalised} \Leftrightarrow \underbrace{|a|^2 + |b|^2 = 1}_{\text{Cartesian}} \Leftrightarrow \begin{aligned} a &= \cos \frac{\theta}{2} \cdot e^{i\phi} \\ b &= \sin \frac{\theta}{2} \cdot e^{i\alpha} \end{aligned} \underbrace{\hspace{10em}}_{\text{polar}}$$

$$\text{double}(\downarrow\psi) = \text{double} \left( e^{-i\beta} \cdot \downarrow\psi \right) = \text{double} \left( \overset{\downarrow}{\cos \frac{\theta}{2}} \downarrow_0 + e^{i\alpha} \overset{\downarrow}{\sin \frac{\theta}{2}} \downarrow_1 \right)$$



Bloch sphere.

Things not preserved by doubling.

▷ numbers in the theory of <sup>prob</sup> quantum maps are positive real numbers, not  $\mathbb{C}$ ! (probabilities)

$$\boxed{\lambda} = \text{double}(\triangleleft \lambda) = \bar{\lambda} \cdot \lambda \geq 0.$$

▷ doubled ONB's are not bases!

e.g.  $\triangleleft \psi = \sum_j \triangleleft \psi_j$  vs.  $\triangleleft \phi = \sum_j e^{i\alpha_j} \triangleleft \psi_j$

in linear maps,  $\triangleleft \psi = 1 \neq e^{i\alpha_i} = \triangleleft \phi \Rightarrow \triangleleft \psi \neq \triangleleft \phi$

in quantum maps,  $\forall i, \triangleleft \psi = 1 = 1 = \triangleleft \phi$ , but  $\triangleleft \psi \neq \triangleleft \phi$

The set  $\{\triangleleft i\}_i$  describes an ONB measurement.

▷ Sums are not preserved.

$$\begin{aligned}
 & \text{double} \left( \sum_{i=1}^k \boxed{f_i} \right) \\
 & \parallel \\
 & \left( \sum_{j=1}^k \boxed{f_j} \right) \left( \sum_{i=1}^k \boxed{f_i} \right) \\
 & \parallel \\
 & \sum_{i,j=1}^k \boxed{f_j} \boxed{f_i} \\
 & \leftarrow k^2 \text{ terms} \\
 & \underbrace{\hspace{10em}} \\
 & \text{SUPERPOSITION}
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{i=1}^k \boxed{f_i} \\
 & \parallel \\
 & \sum_{i=1}^k \boxed{f_i} \boxed{f_i} \leftarrow k \text{ terms} \\
 & \underbrace{\hspace{10em}} \\
 & \text{MIXTURE}
 \end{aligned}$$

6.2 Doubling "makes space" for an interesting new process.

DEF The discarding process is the effect that "discards" any normalised state  $\hat{\psi}$ :

$$\text{disc.} \rightarrow \boxed{\hat{\psi}} = \boxed{\phantom{\hat{\psi}}}$$

n.b. for linear maps there is no such thing as discarding!

Pf Suppose  $\exists \dot{T}$  s.t.

$$\forall \psi. \begin{array}{c} \psi \\ \downarrow \\ \psi \end{array} = 1 \Rightarrow \begin{array}{c} \dot{T} \\ \downarrow \\ \psi \end{array} = 1$$

$$\begin{aligned} \text{Then: } 1 &= \dot{T} \circ \left[ \frac{1}{\sqrt{2}} (\downarrow + \downarrow) \right] \\ &= \frac{1}{\sqrt{2}} \left[ \begin{array}{c} \dot{T} \\ \downarrow \\ \psi \end{array} + \begin{array}{c} \dot{T} \\ \downarrow \\ \psi \end{array} \right] = \frac{1}{\sqrt{2}} [1 + 1] = \sqrt{2} \quad \square \end{aligned}$$

Doubling to the rescue!

$$\begin{array}{c} \dot{T} \\ \downarrow \\ \psi \end{array} = \begin{array}{c} \downarrow \\ \psi \end{array} \downarrow \begin{array}{c} \downarrow \\ \psi \end{array} = \begin{array}{c} \psi \\ \downarrow \\ \psi \end{array} = 1 \quad \text{normalized} \quad \text{☺}$$

(n.b.  $\dot{T} := \cap$  is uniquely fixed by  $\begin{array}{c} \psi \\ \downarrow \\ \psi \end{array} = 1$ )

Thm 6.31

$$\dot{T}_{\hat{A}} := \cap_{\hat{A}} \neq \cap_{\hat{A}} = \cap_{\hat{A}} \cap_{\hat{A}}$$

In particular:  $\hat{\mathbb{T}}_{\hat{A} \otimes \hat{B}} = \hat{\mathbb{T}}_{\hat{A}} \hat{\mathbb{T}}_{\hat{B}} \rightarrow \hat{\mathbb{T}}_{\hat{I}} = \boxed{\phantom{\hat{\mathbb{T}}_{\hat{I}}}}$

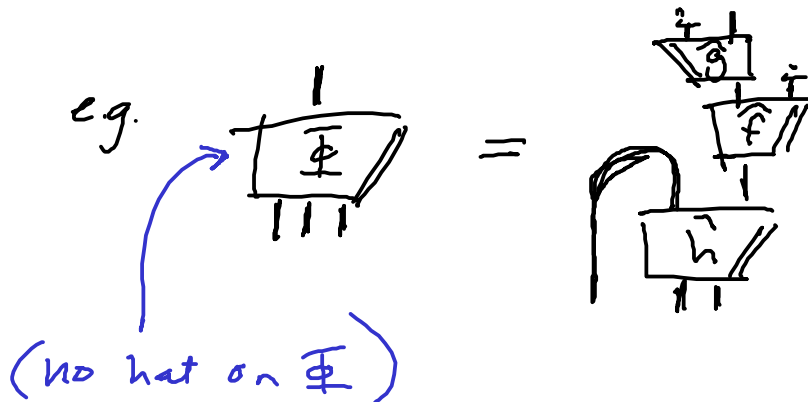
Discarding is not a pure q. map!  $\leftarrow$  Prop 6.27.

IDEA:  $\cap \neq \text{double}(\hat{\mathbb{T}}) = \hat{\mathbb{T}} \hat{\mathbb{T}}$

DEF The process theory of quantum maps has:

types:  $\hat{A}$  doubled Hilbert spaces

procs: <sup>string</sup> diagrams of  $\hat{\mathbb{T}}_{\hat{A}} + \hat{\mathbb{T}}$ .



# CAUSALITY

DEF A process is called causal if:

$$\begin{array}{c} \dot{\hat{B}} \\ \downarrow \\ \boxed{\Phi} \\ \uparrow \\ \dot{\hat{A}} \end{array} = \dot{\hat{A}}$$

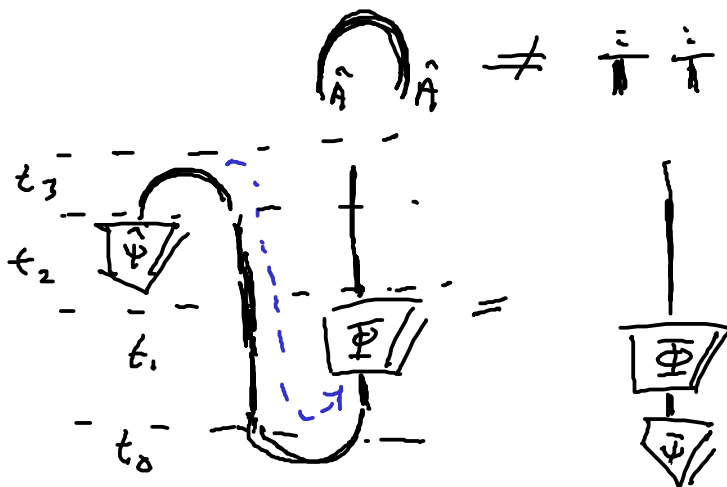
"If the output of a process is discarded, it doesn't matter which process occurred."

"A process can only influence its future."

In particular, causal effects:

$$\begin{array}{c} \boxed{\pi} \\ \downarrow \\ \dot{\hat{A}} \end{array} = \dot{\hat{A}}$$

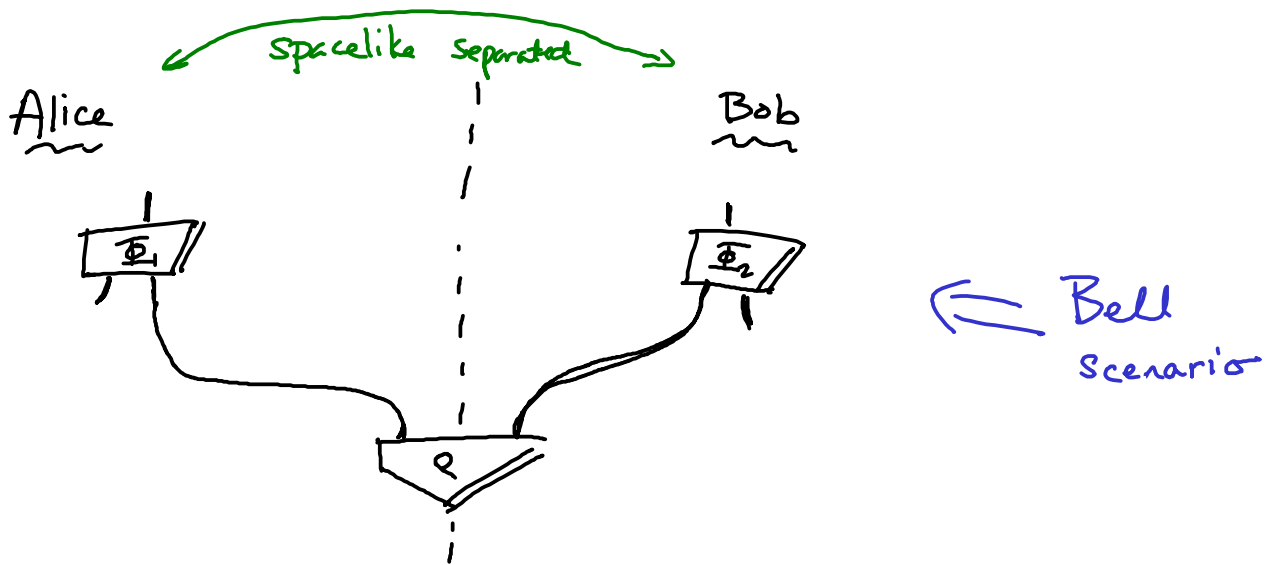
EXAMPLE OF A PROCESS WHICH IS NOT CAUSAL:



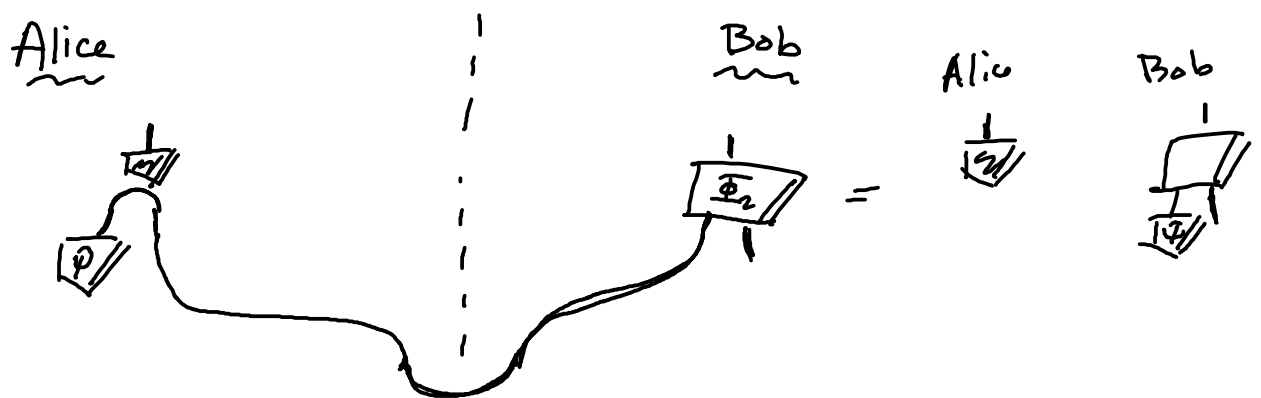
# Lecture 11

## 6.3 Causality + non-signalling.

No signalling (special relativity) "Nothing can travel faster than the speed of light." can



Spse: processes didn't need to be causal.

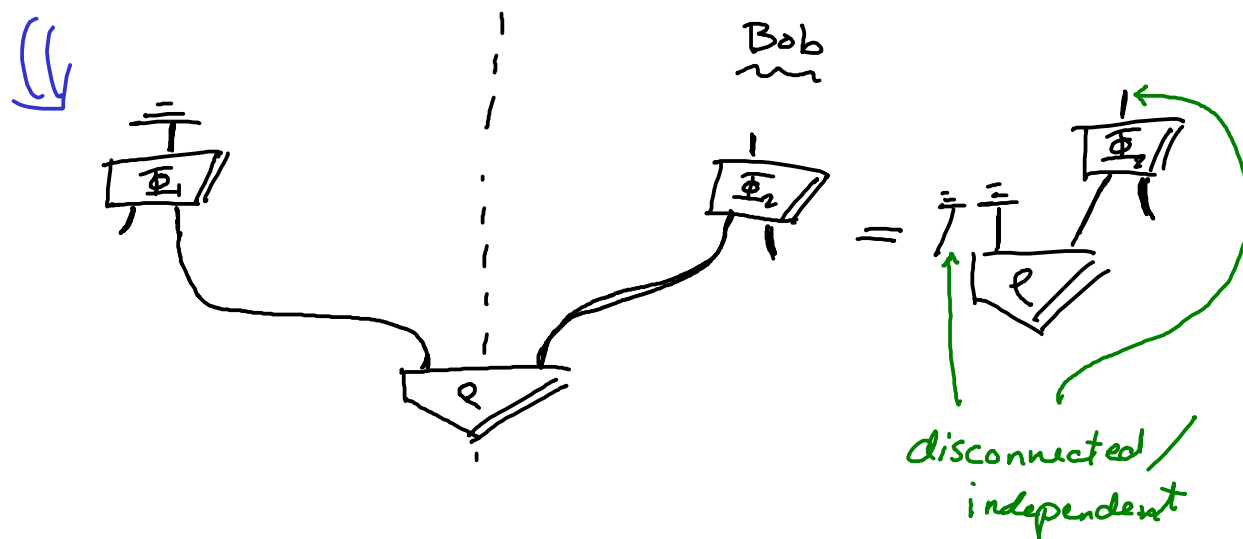


Violates special relativity!

X

Back in "our universe" (as predicted by QT) processes do need to be causal.

From Bob's perspective, Bob can't access Alice's output.



$\Rightarrow$  Bob cannot access Alice's input.

CAUSALITY  $\Leftrightarrow$  NO-SIGNALLING

\* CAUSALITY RULES OUT  $\cap$ .

\* So why did we learn about  $\cap$ ? ... and  $\cap = 1$ , etc.

" A process is deterministically physically realisable  
 $\Leftrightarrow$   
 it is causal. "



# 6.4 (Non-deterministic) Quantum processes

ASSIDE. MIXING.

NOTE:  $\bar{\mathbb{I}} = \mathbb{I} = \begin{array}{|c|} \hline \text{?} \\ \hline \end{array} = \sum_i \begin{array}{|c|} \hline \text{?} \\ \hline \end{array} = \sum_i \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} = \sum_i \begin{array}{|c|} \hline \uparrow \\ \hline \end{array}$

So:  $\begin{array}{|c|} \hline \text{?} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{?} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{?} \\ \hline \end{array}$

$\begin{array}{|c|} \hline \Phi \\ \hline \end{array} = \begin{array}{|c|} \hline f \\ \hline \end{array} = \sum_i \begin{array}{|c|} \hline \text{?} \\ \hline \end{array} = \sum_i \begin{array}{|c|} \hline \text{?} \\ \hline \end{array}$  ← MIXTURE

⇒ DISC. CAN BE REPLACED BY MIXING.

Conversely:

$\begin{array}{|c|} \hline \Phi \\ \hline \end{array} = \sum_i \begin{array}{|c|} \hline f_i \\ \hline \end{array}$

let  $\begin{array}{|c|} \hline f \\ \hline \end{array} := \sum_j \begin{array}{|c|} \hline f_j \\ \hline \end{array} \downarrow$ , then:  $\begin{array}{|c|} \hline \text{?} \\ \hline \end{array} = \sum_j \delta_j \begin{array}{|c|} \hline f_j \\ \hline \end{array} = \begin{array}{|c|} \hline f \\ \hline \end{array}$

so:  $\begin{array}{|c|} \hline \text{?} \\ \hline \end{array} = \begin{array}{|c|} \hline f \\ \hline \end{array}$

$\begin{array}{|c|} \hline \Phi \\ \hline \end{array} = \sum_i \begin{array}{|c|} \hline f_i \\ \hline \end{array} = \sum_i \begin{array}{|c|} \hline \text{?} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{?} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{?} \\ \hline \end{array}$

⇒ MIXTURE CAN BE REPLACED BY  $\bar{\mathbb{I}}$ .

/ ASSIDE ON MIXTURE .... FOR NOW.

(possibly non-deterministic)

DEF A quantum process is a set  $\{ \boxed{\Psi_i} \}_i$  of quantum maps such that:

$$\sum_i \boxed{\Psi_i} = \mathbb{I}$$

EX1 CAUSAL QUANTUM MAPS.  $\rightarrow \{ \boxed{\Phi} \}$ . a.k.a. deterministic quantum processes.

EX2 Random state preparation,

\* Alice flips a fair coin and prepares:

—  $\boxed{\Psi_0}$  if heads.

—  $\boxed{\Psi_1}$  if tails.

This is repr. by the quantum process:  $\left\{ \frac{1}{2} \cdot \boxed{\Psi_0}, \frac{1}{2} \cdot \boxed{\Psi_1} \right\}$

classical probs

quantum states

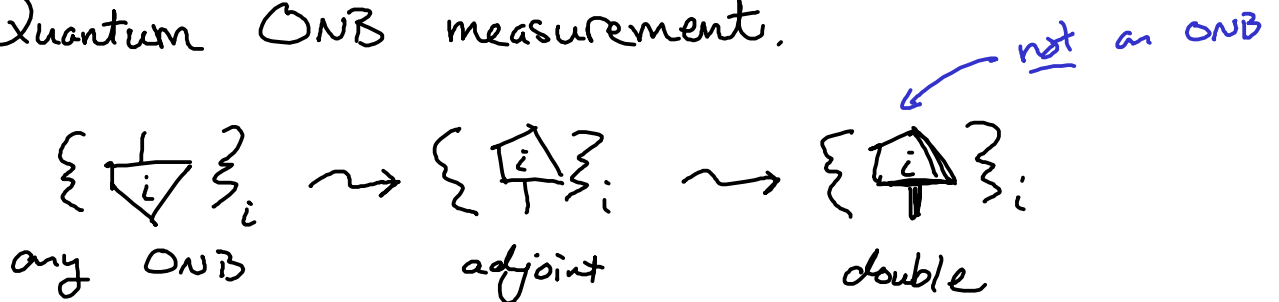
Non-deterministic process  $\rightarrow$  we do know which branch happened once the process happens.

vs. a mixture  $\rightarrow$  we don't know which branch happened.

If Alice does  $\{\frac{1}{2} \cdot \begin{array}{|c|} \hline \downarrow \\ \hline \end{array}, \frac{1}{2} \cdot \begin{array}{|c|} \hline \uparrow \\ \hline \end{array}\}$  and sends her state to Bob, but does not reveal the coin flip, Bob sees a mixed state:

$$\begin{array}{|c|} \hline \downarrow \\ \hline \end{array} = \sum_i \frac{1}{2} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array}$$

Ex3 Quantum ONB measurement.



$$\sum_i \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} = \mathbb{I} \quad \checkmark \text{ quantum process}$$

So if Alice has a state  $\begin{array}{|c|} \hline \uparrow \\ \hline \end{array}$ , then she can measure  $\{\begin{array}{|c|} \hline \uparrow \\ \hline \end{array}\}_i$ .

$\swarrow$  causal

$$\text{Prob}(i | \hat{\psi}) := \frac{\begin{array}{|c|} \hline \uparrow \\ \hline \end{array}}{\begin{array}{|c|} \hline \uparrow \\ \hline \end{array}} \quad \text{Born rule.}$$

$$\sum_i \text{Prob}(i | \hat{\psi}) = \sum_i \frac{\begin{array}{|c|} \hline \uparrow \\ \hline \end{array}}{\begin{array}{|c|} \hline \uparrow \\ \hline \end{array}} = \frac{\mathbb{I}}{\begin{array}{|c|} \hline \uparrow \\ \hline \end{array}} = 1 \quad \Rightarrow \{\text{Prob}(i | \hat{\psi})\}_i \text{ is a prob. distr.}$$

In Ex2, Bob's state was  $\downarrow \rho = \frac{1}{2} \cdot \downarrow \hat{\psi}_0 + \frac{1}{2} \cdot \downarrow \hat{\psi}_1$ .

So, Bob can also use the Born rule:

$$\begin{aligned} \text{Prob}(i|\rho) &= \frac{\uparrow i}{\downarrow \rho} = \frac{1}{2} \cdot \frac{\uparrow i}{\downarrow \hat{\psi}_0} + \frac{1}{2} \cdot \frac{\uparrow i}{\downarrow \hat{\psi}_1} \\ &= \frac{1}{2} \cdot \text{Prob}(i|\hat{\psi}_0) + \frac{1}{2} \cdot \text{Prob}(i|\hat{\psi}_1) \end{aligned}$$

Eg. if  $\frac{\uparrow i}{\downarrow \hat{\psi}_0} = \frac{\uparrow i}{\downarrow i}$  and  $\frac{\uparrow i}{\downarrow \hat{\psi}_1} = 0$ , then:

$$\text{Prob}(i|\rho) = \frac{1}{2} \cdot \frac{\uparrow i}{\downarrow i} + \frac{1}{2} \cdot \frac{\uparrow i}{\downarrow \hat{\psi}_1} = \frac{1}{2}.$$