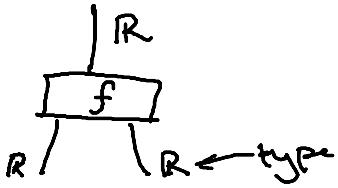
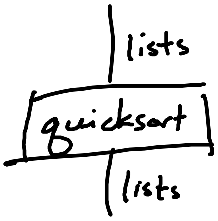
LECTURE 1

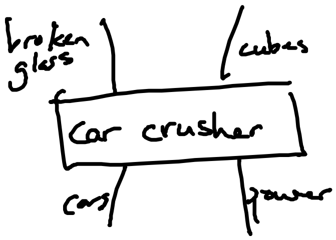
3.1 Processes

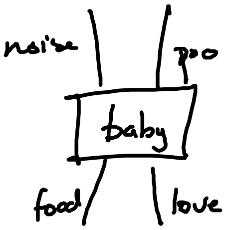
Def A **process** is anything with 0 or more inputs and outputs.

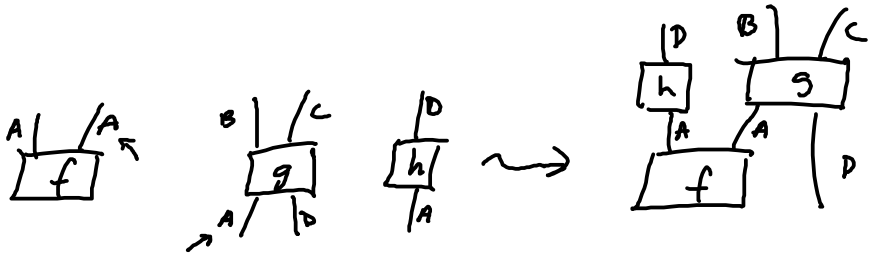
Ex $f(x,y) = x^2+y$









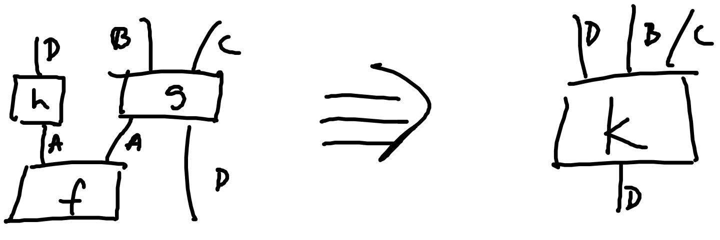


Def A **process theory** consists of:

(i) a collection T of **system-types**

(ii) a collection P of processes

→ (iii) a means of **composing** diagrams of processes.



EXAMPLES

- functions ← types are sets A, B, C
  functions $f : A \to B$

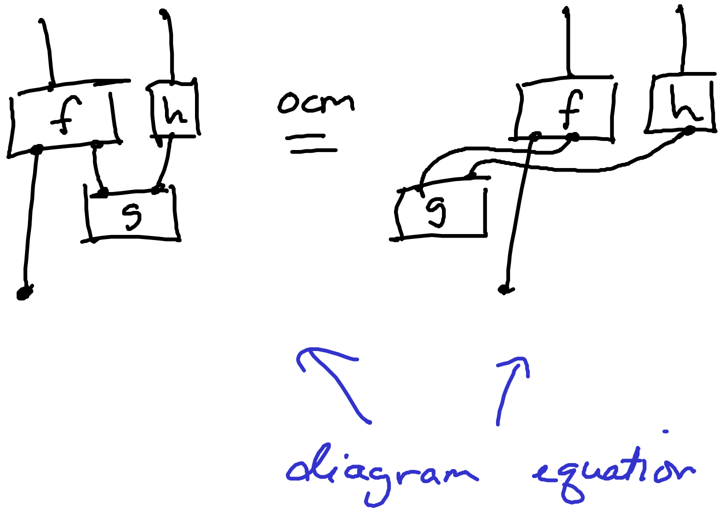
- relations ← types $A, B, C, \ldots$ sets
  procs: $R \subseteq \underline{A} \times \underline{B}$
- linear maps ← types: vector sp.
  ← procs: ''

  dim
  types: numbers
  procs: matrices →
- classical (probabilistic) process
- → quantum processes

LECTURE 2

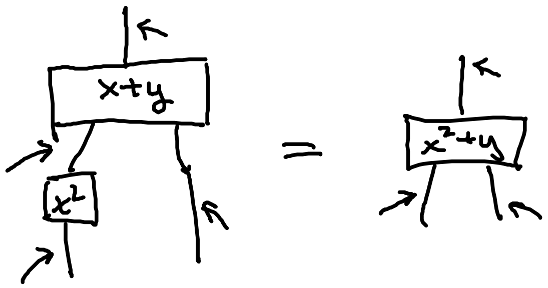
The Golden Rule of Process Theories:

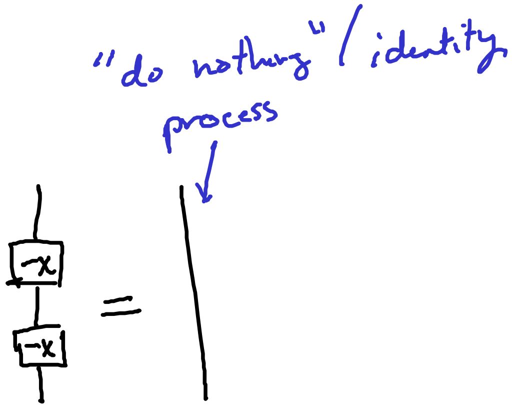
ONLY CONNECTIVITY MATTERS (ocm)

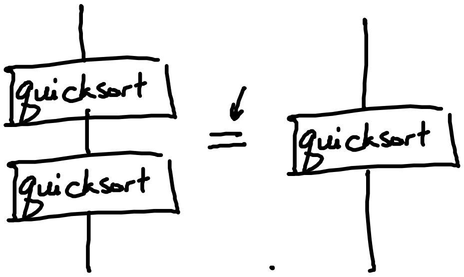


Most p.t.'s have many more (process) equations

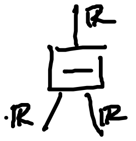
$f(x,y) = x+y$





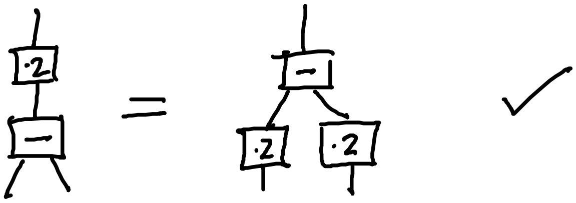


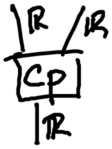
DIAGRAMMATIC REASONING := "using process equations to prove stuff"

Ex 

$:: (m,n) \mapsto m-n$

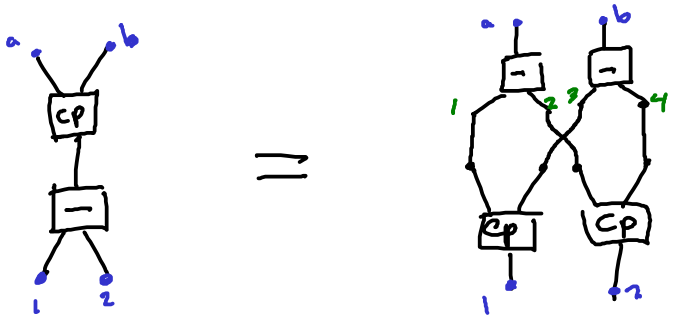
$:: m \mapsto 2m$



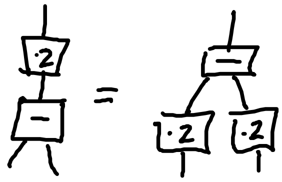


$:: n \mapsto (n,n)$

(1)



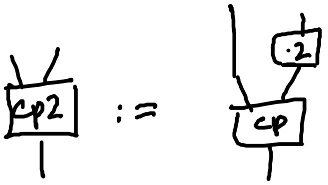
(2)



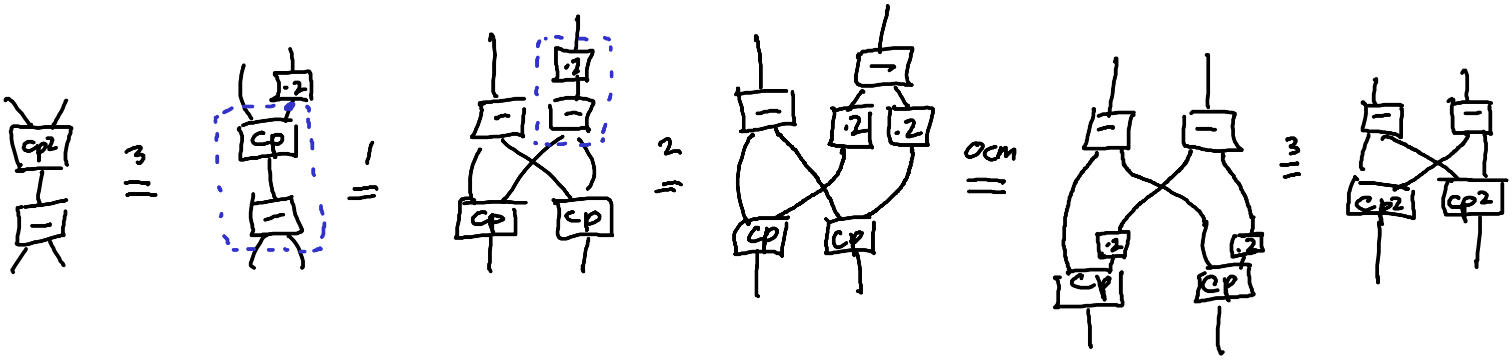
$(m,n) \mapsto m-n \mapsto (m-n, m-n)$

$(m,n) \mapsto (\overset{\downarrow}{m}, \overset{\downarrow}{m}, \overset{\downarrow}{n}, \overset{\downarrow}{n}) \mapsto (m-n, m-n)$

(3)



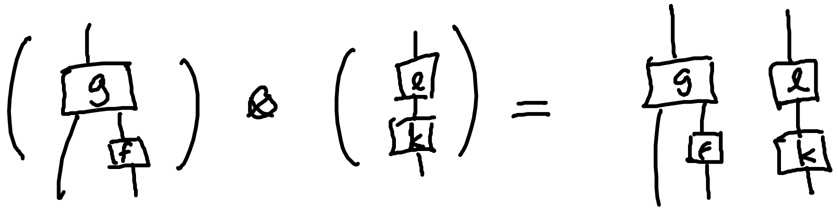
$n \mapsto (n, 2n)$



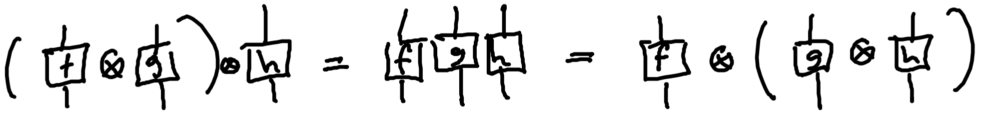
(4)

3.2 Circuit Diagrams

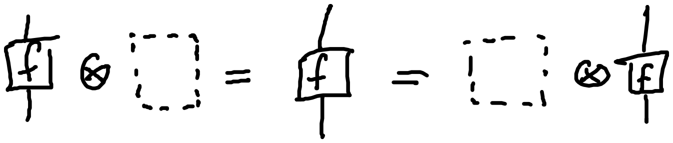
Parallel composition : $f \underset{\uparrow}{\otimes} g$ "f <u>while</u> g"



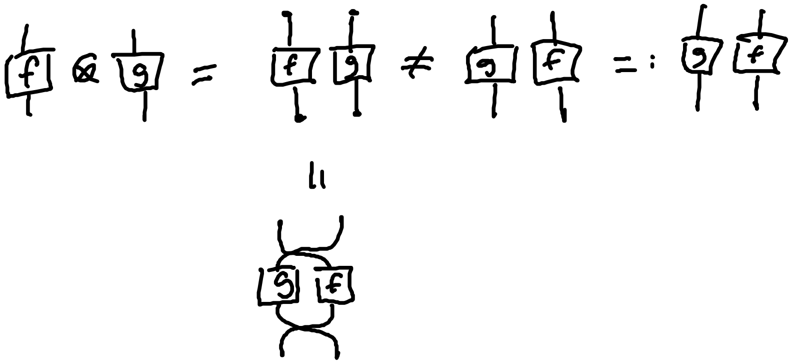
- <u>associative</u>



- <u>unit</u>

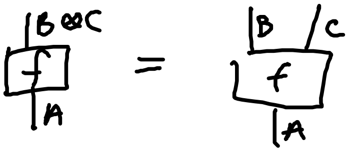


- The <u>order</u> matters.

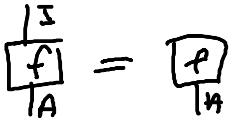


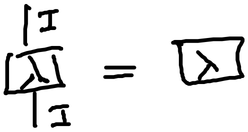
* We can also form <u>joint</u> systems.

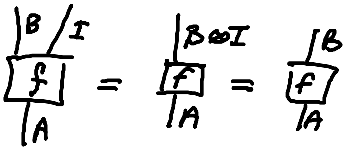
B,C system-types ⇒ B ⊗ C is a system-type



* We write the <u>trivial</u> system as I. A⊗I = A = I⊗A



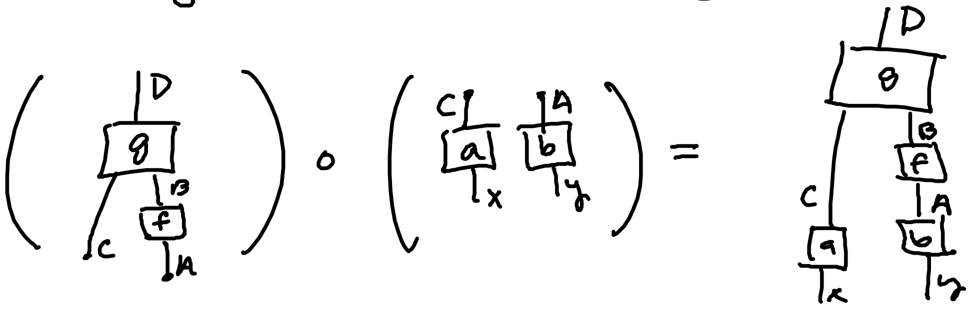




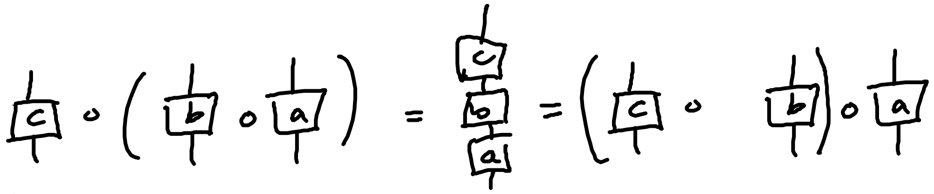
LECTURE 3

Sequential composition

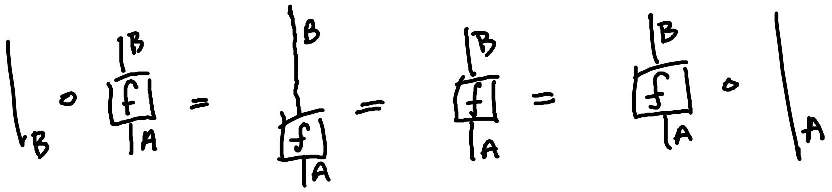
$f \circ g :=$ "f after g"



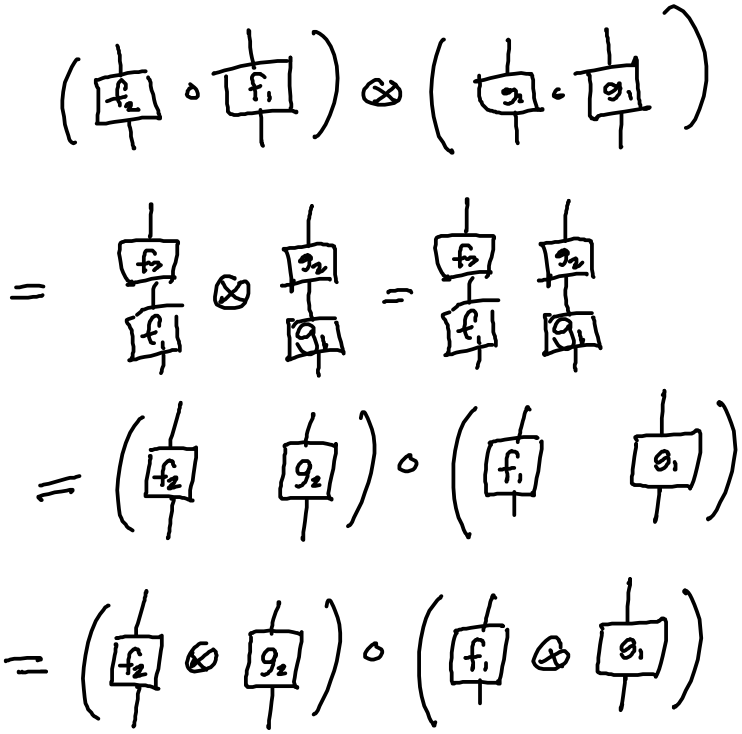
- associative

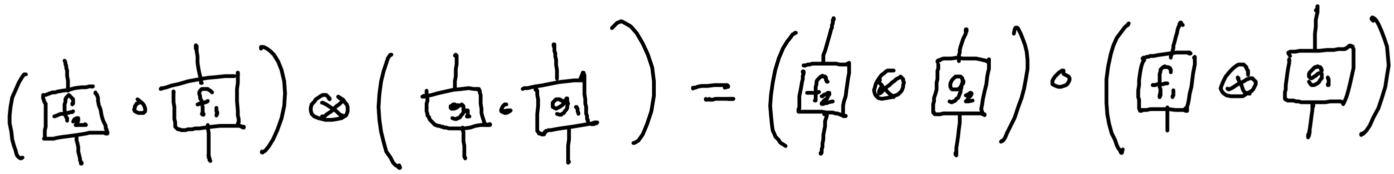


- has a unit :  ← "do nothing / identity" process



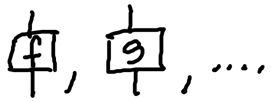
TOGETHER :

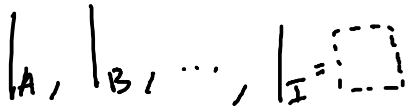


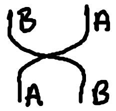


"Interchange law"

DEF A circuit diagram is any diagram built from

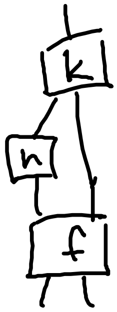
- boxes 

- (identity) wires 

- Swap processes 

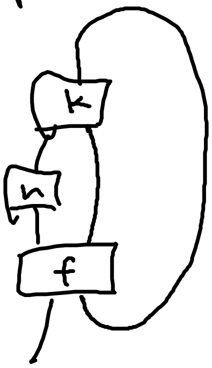
using only ⊗ and ∘.

Equivalently circuit diagrams are any diagram that doesn't have feedback loops.



✓

D.A.G.



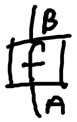
✗

3.3 Functions & relations as process theories.

functions

types: sets

procs: functions



$f: A \to B$

1-element set

$I = \{*\}$

$\circ :=$ fn composition: $(f \circ g)(x) = f(g(x))$

$\otimes :=$ Cartesian product.

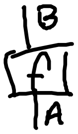
$$A \otimes I = A$$

$$A \times \{*\} = \{(a,*) \mid a \in A\} \cong \{a \mid a \in A\} = A$$

relations

types: sets

procs: relations

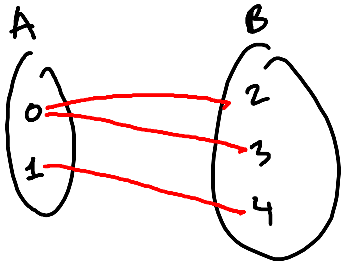


$f \subseteq A \times B$

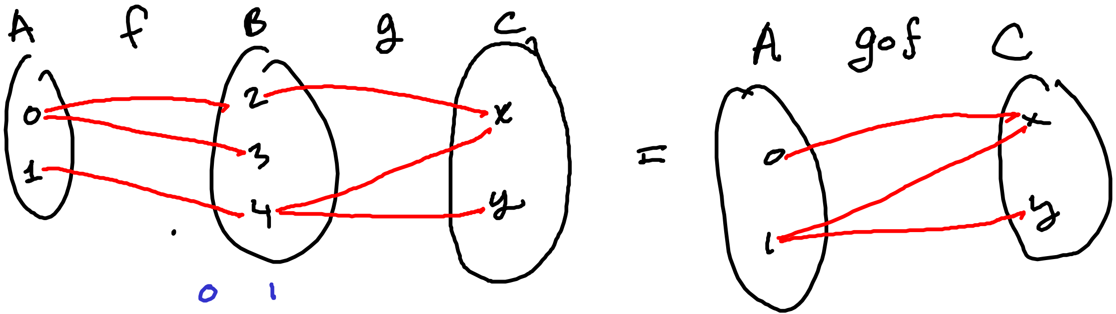
e.g. $A = \{0,1\}$ $B = \{2,3,4\}$

$$f = \{(0,2), (0,3), (1,4)\} \subseteq A \times B$$

write↘ $\{0 \mapsto 2, 0 \mapsto 3, 1 \mapsto 4\}$



$\underline{\circ}$ relation composition.



$$f = \begin{array}{c|cc|c} & 0 & 1 & \\ \hline & 1 & 0 & 2 \\ & 1 & 0 & 3 \\ & 0 & 1 & 4 \end{array} \qquad g = \begin{array}{c|ccc|c} & 2 & 3 & 4 & \\ \hline & 1 & 0 & 1 & x \\ & 0 & 0 & 1 & y \end{array}$$

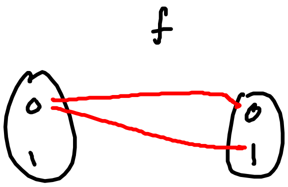
OR
$1+1=1$

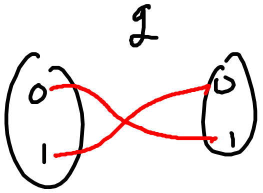
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{array}{c|cc|c} & 0 & 1 & \\ \hline & 1 & 1 & x \\ & 0 & 1 & y \end{array}$$

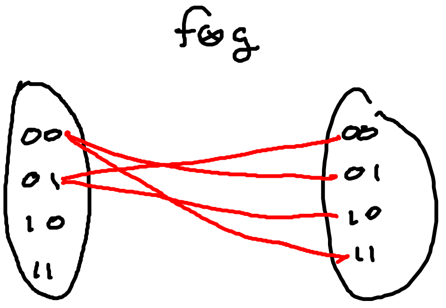
$\underline{\otimes}$ also cartesian product!

$$f\otimes g \subseteq A\times B$$

$f\otimes g :: (a,b) \mapsto (x,y)$ iff $f :: a \mapsto x$ and $g :: b \mapsto y$



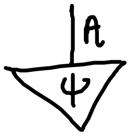


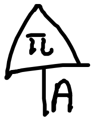


$\mathbb{B}=\{0,1\}$ $\quad \mathbb{B}\times\mathbb{B} = \{\overset{00}{(0,0)}, \overset{01}{(0,1)}, \overset{10}{(1,0)}, \overset{11}{(1,1)}\}$

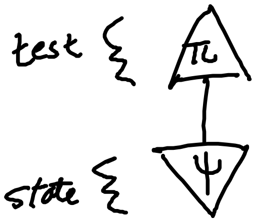
* relations are simple, but still have "quantum-like" features.

3.4 Special processes

States  $\psi : I \to A$ "preparation"

Effects  $\pi : A \to I$ "testing for a property"

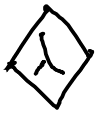
Generalised Born rule



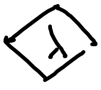
$\Longrightarrow$ answer

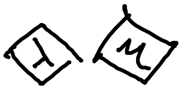
Depending on the thy

- → yes or no
- → "possible" or "impossible"
- → probability.

Numbers  $\lambda : I \to I$

(most important numbers come from the Born rule)

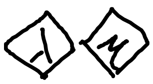
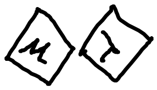
Q how is  like a number?

(partial) A 2 numbers can be "multiplied": $\lambda \cdot \mu :=$ .

Q Why is "·" like multiplication?

Assoc. $\lambda\cdot(\mu\cdot\xi) =$  $= (\lambda\cdot\mu)\cdot\xi$

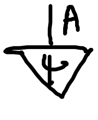
Unit. $\lambda\cdot 1 := \lambda\cdot$  $=$  $= \lambda$

Comm. $\lambda\cdot\mu =$  $=$  $= \mu\cdot\lambda$

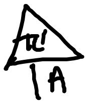
⟹ "numbers" in a process they always form a commutative monoid.

Ex $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{B}$, $[0,1]$, etc, etc.

In functions states are the same a elements of a set.

 $\psi : \{*\} \to A \qquad \psi(*) = a \in A.$

... but effects + numbers are boring!

 $=$  $\pi : A \to \{*\}$

... but in relations, they are interesting again!

LECTURE 4

In relations, states $\psi : I \to A$ are relations $\psi \subseteq \{*\} \times A$
$\cong A$

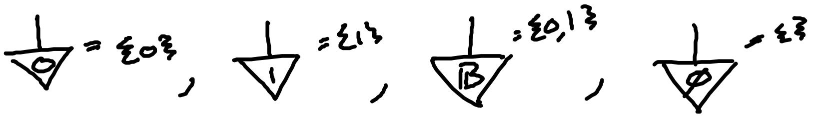
⇒ states correspond to subsets.

$$\underbrace{\{a,c,d\}}_{\psi} \subseteq A := \{a,b,c,d\}$$

$$\psi :: \begin{cases} * \mapsto a \\ * \mapsto c \\ * \mapsto d \end{cases}$$ ← non-deterministic state.

e.g: $A := \mathbb{B} = \{0,1\}$

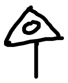
4 possible states:



Effects $\pi : A \to \{*\}$ $\pi \subseteq A \times \{*\}$

⇒ effects are also $\underbrace{\text{subsets of } A}_{\text{"questions about our state"}}$.

e.g. $A := \mathbb{B}$

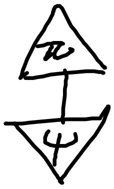


"are you 0?"

"are you 1?"

"true"

"false"

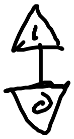


← number $\lambda : \{*\} \to \{*\}$ $\lambda \subseteq \{*\} \times \{*\}$

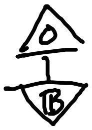
$\lambda = \{\ \}$ $\lambda : \{* \mapsto *\}$

"0" impossible "1" possible } possibilities.

$= 1$ "possible"



$= 0$ "impossible"



$= 1$ "sometimes"/"possible"

$= 1$ "possible"

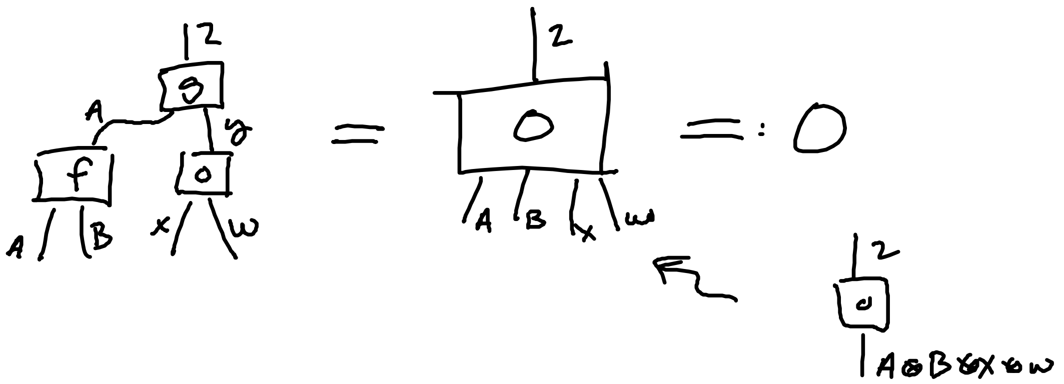
Possibilities are a coarse-graining of probabilities.

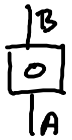
"impossible" ⟷ prob $= 0$

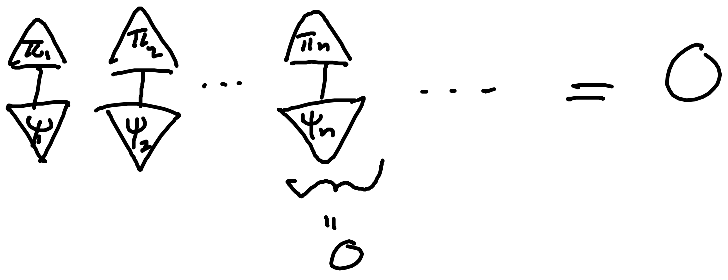
"possible" ⟷ prob $\neq 0$

Zero processes

DEF A process thy has zero processes if it has a special process  for every A, B that "eats everything."



Ex For relations,  $= \{\}$.

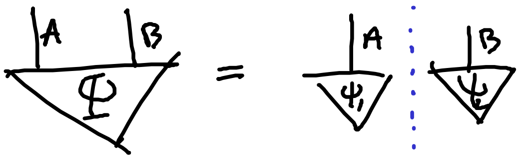


Chapter 4: String diagrams

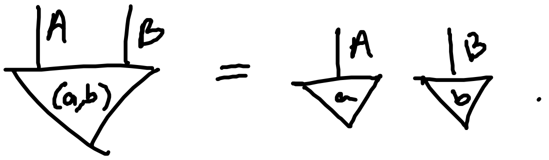
"separable vs. $\overset{}{\underset{\text{non}}{\hat{}}}$separable"

4.1

DEF a $\otimes$-separable state $\Psi : I \to A \otimes B$ is a state s.t. there exist $\psi_1 : I \to A$, $\psi_2 : I \to B$ where:

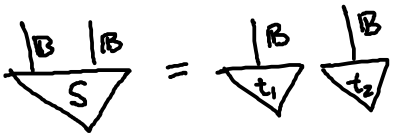


In functions, states are elements. $\Rightarrow$ all states are $\overset{\otimes}{\text{sep'l}}$.



in relations, separable states correspond to cartesian products of subsets.

e.g. if

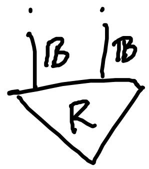


for $t_1 = \{0\} \subseteq \mathbb{B}$, $t_2 = \{0,1\} \subseteq \mathbb{B}$

then $S = t_1 \times t_2 = \{(0,0),(0,1)\} \subseteq \mathbb{B} \times \mathbb{B}$.

BUT! NO ALL SUBSETS $S \subseteq \mathbb{B} \times \mathbb{B}$ are cart. pr's of subsets $t_1 \subseteq \mathbb{B}$, $t_2 \subseteq \mathbb{B}$.

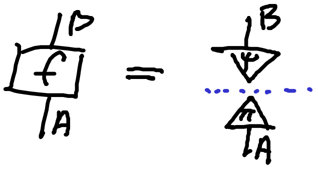
⇒ <u>relations</u> has non-separable states.

<u>Ex</u> 

$$:: \begin{cases} * \mapsto (0,0) \\ * \mapsto (1,1) \end{cases}$$ is <u>not</u> separable.

(if R was sep'l, $(0,0)\in R$, $(1,1)\in R \Rightarrow (0,1)\in R$ and $(1,0)\in R$)

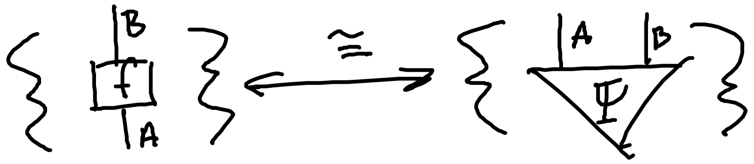
<u>DEF</u> a ⊗-separable process $f: A\to B$ is a proc. such that there exists a state $\psi: I\to B$ and an effect $\pi: A\to I$ such that



(<u>DEF</u> <u>Trivial</u> <u>process</u> <u>thy</u> is a process thy where all processes ⊗-separate. ← "nothing ever happens")

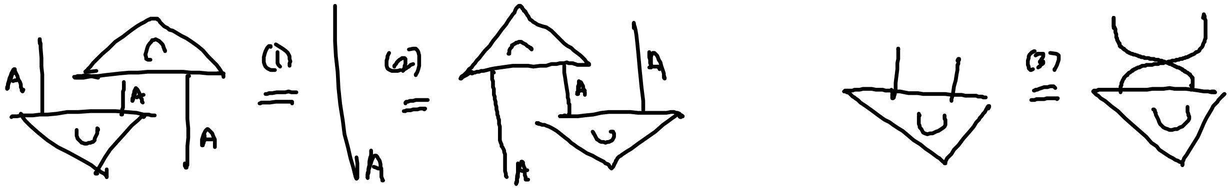
In <u>relations</u> (and in Q.T.) there are the same amount of processes $\underbrace{f: A\to B}_{f\subseteq A\times B}$ as there are states $\underbrace{\Psi: I\to A\otimes B}_{\Psi\subseteq \{*\}\times A\times B}$

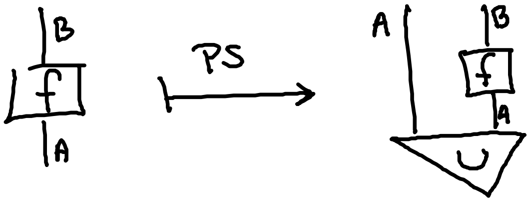
PROCESS-STATE DUALITY

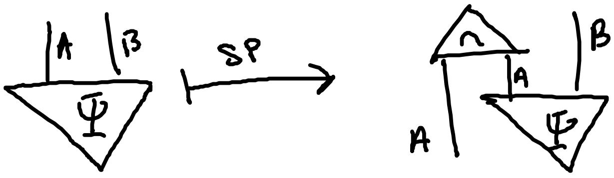


(in Q.T. Choi-Jamoitkowski isomorphism)

DEF A process theory admits string diagrams if it has a special state & effect for every A, satisfying:

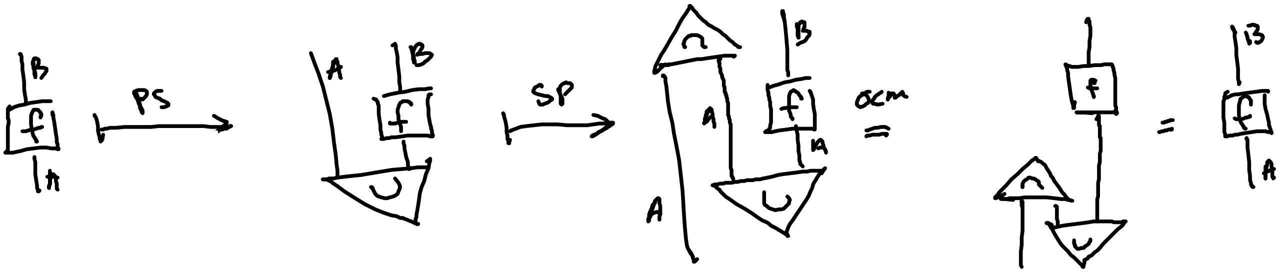




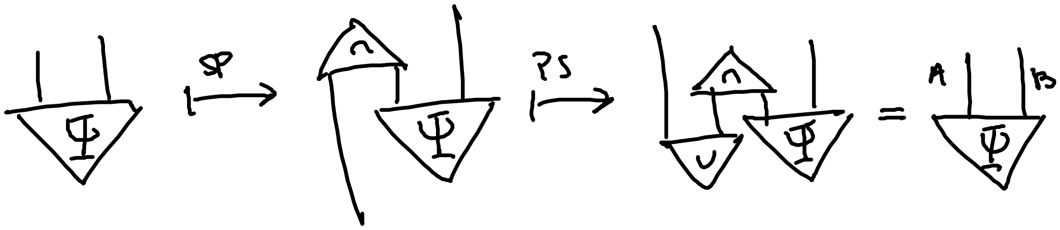


THM PS & SP are inverses.

Pf



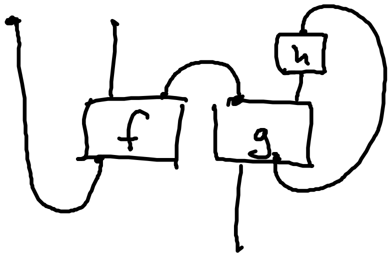
Sim'ly



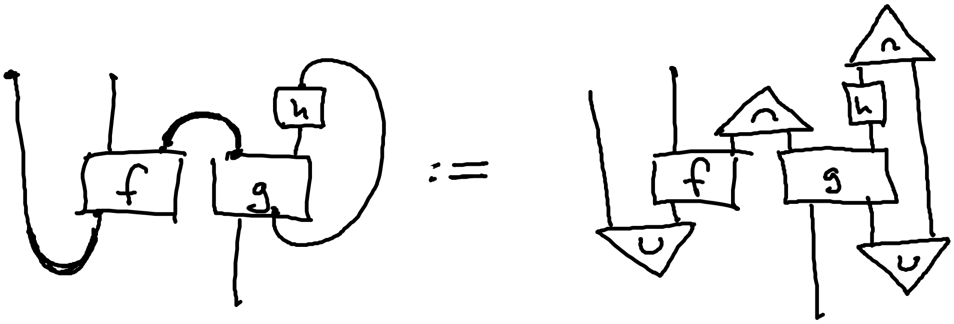
∎

DEF A string diagram is a circuit diagram + caps + cups.

OR equivalently it is a diagram which allows us to connect any input or output to any input or output.

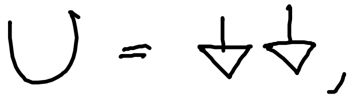
e.g. 

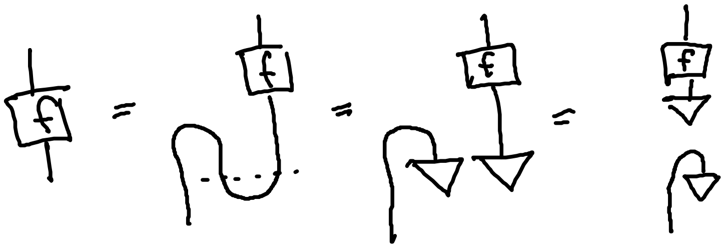
is a string diagram.



THM NO-GO FOR UNIVERSAL SEPARABILITY.

IF A P.T. ADMITS STRING DIAGRAMS
& ALL STATES ⊗-SEPARATE
THEN THE P.T. IS TRIVIAL.

Pf ASSUME , then for any $f : A \to B$:



← ∘-separates

⇒ all f ∘-separate, thus the p.t. is trivial. ∎

all states ⊗-separate

⇒ functions does not admit string diagrams.

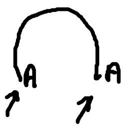
(but relations does!)

LECTURE 5

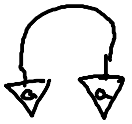
In relations,

$:= \{ * \mapsto (a,a) \mid a \in A \}$

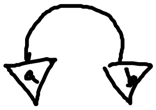
$\{(a,a) \mid a \in A\} \subseteq \underline{A \otimes A} := \{(a,b) \mid a,b \in A\}$



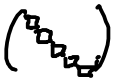
$:= \{ (a,a) \mapsto * \mid a \in A \}$



$= 1$

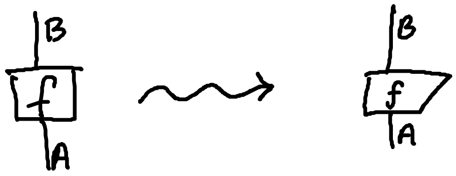


$= 0$ $b \neq a$

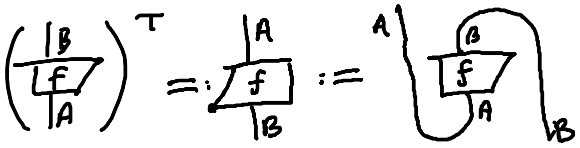


4.2 Transposition & trace of a process

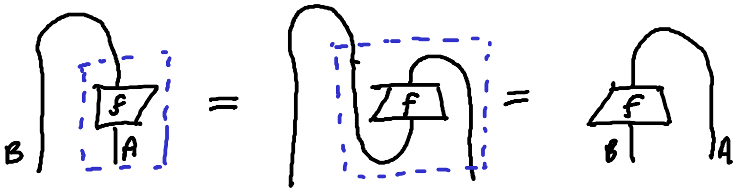
$f : A \to B$



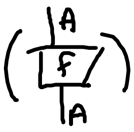
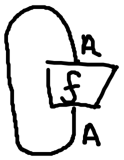
DEF (Transpose)

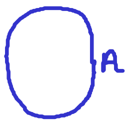


↜ $f^T : B \to A$



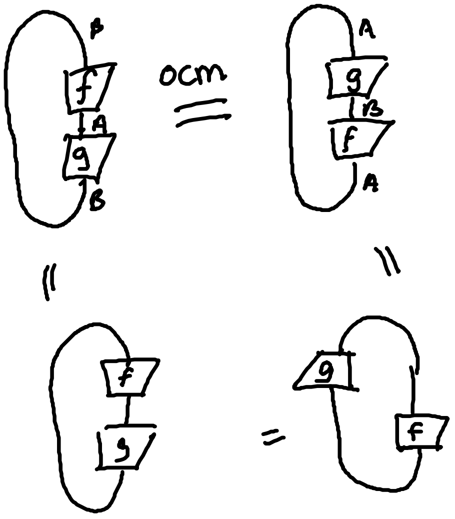
DEF (TRACE)

tr() := 

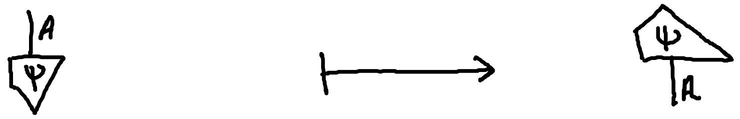
 =: dim(A)

CYCLICITY

$tr(f\circ g) = tr(g\circ f)$

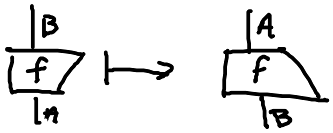


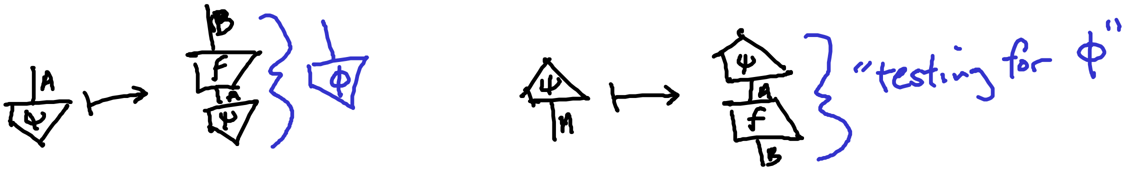
4.3.1 ADJOINTS.

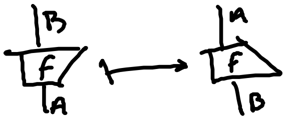


"being in state ψ"
"process that prepares ψ"

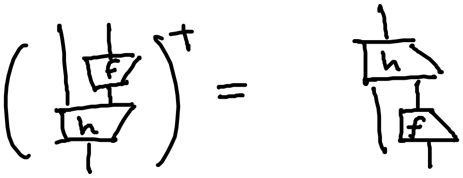
"testing for ψ"

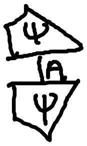
EXTENDS TO PROCESSES:  =: $f^\dagger : B \to A$

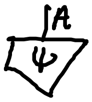


DEF an adjoint is a mapping 

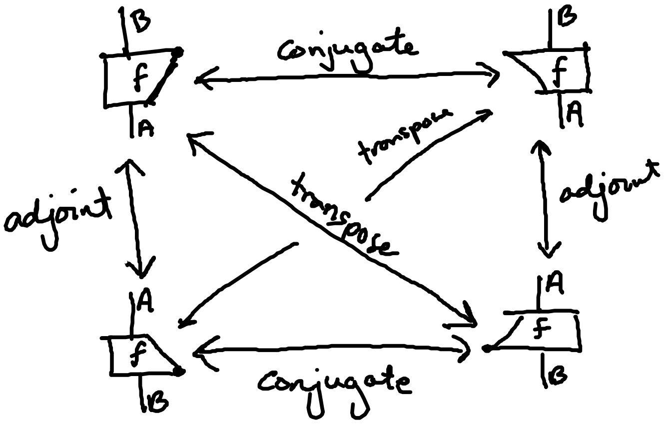
that is:

- involutive $(f^{\dagger})^{\dagger} = f$
- reflects diagrams 

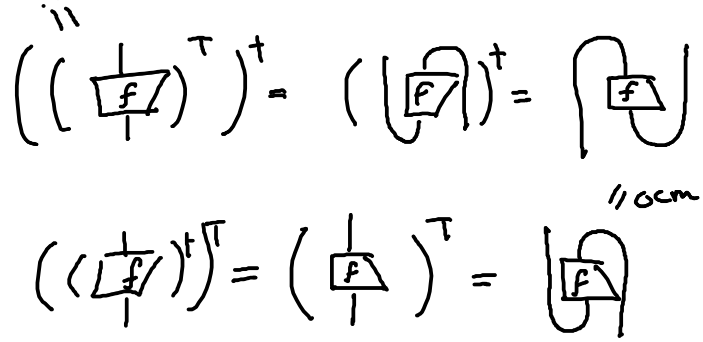
$(\cup)^{\dagger} = \cap$
$(\cap)^{\dagger} = \cup$
- definite 

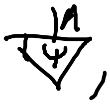
$= 0 \iff$ 

$= 0$



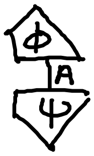
$\bar{f} : A \to B$



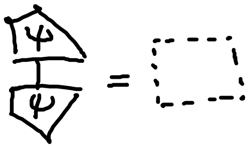
DEF For 2 states 

, 

, the inner product is a number:

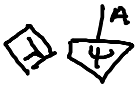
$\langle \phi | \psi \rangle :=$ 

physics convention

and a state is normalised if $\langle \psi | \psi \rangle = 1$. i.e. 

Prop The inner product is:

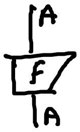
1. conjugate-symmetric $\overline{\langle\phi|\psi\rangle} = \langle\psi|\phi\rangle$
2. preserves numbers (2^{nd} arg.) 

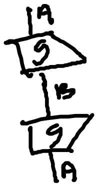
$\lambda\cdot\psi :=$ 

$$\Rightarrow \langle\phi|\lambda\cdot\psi\rangle = \lambda\cdot\langle\phi|\psi\rangle$$

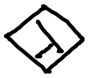
3. Conjugates numbers (1^{st} arg.) $\langle\lambda\cdot\phi|\psi\rangle = \bar{\lambda}\cdot\langle\phi|\psi\rangle$
4. is positive definite.

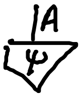
DEF a process $f:A\to A$ is positive if $\exists\ g: A\to B$ s.t.

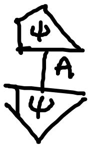
$f =$ 

$=$ 

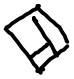
$=: g^\dagger\circ g$

Ex Positive number. 

s.t. $\exists$. 

s.t $\lambda =$ 

.

λ st. $\exists\ \eta: I\to I$. 

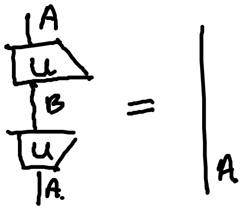
$=$ 

$\underline{\lambda} = \bar{\eta}\eta$

LECTURE 6

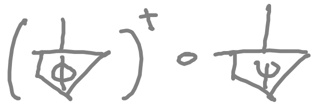
DEF An $\overset{\text{same length}}{\text{isometry}}$ 

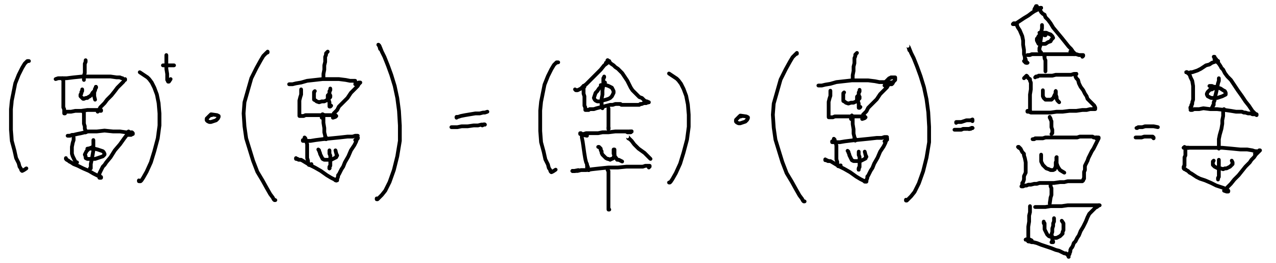
is a proc. satisfying:

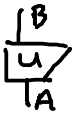


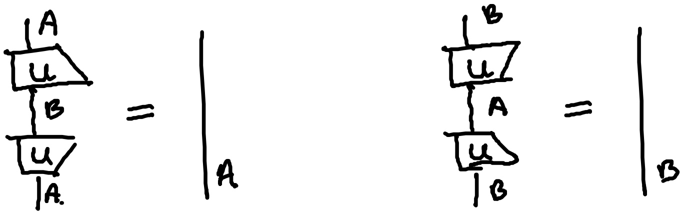
.

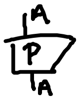
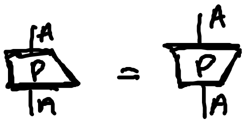
Prop Isometries preserve inner products: $\langle \phi | \psi \rangle = \langle U\phi | U\psi \rangle$

n.b. $\langle \phi | \psi \rangle :=$ 

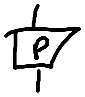
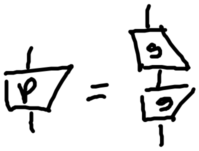
Pf  ▨

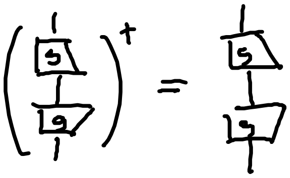
DEF A unitary  is a process that sats:



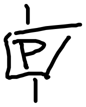
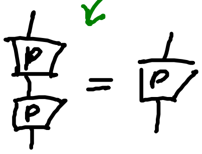
DEF A process  is called self-adjoint if  .

Prop Positive processes are self-adjoint.

Pf  is +-ive, there exists  s.t.  .

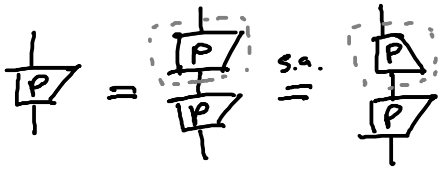
But 

$(g^\dagger \circ g)^\dagger = g^\dagger \circ g^{\dagger\dagger} = g^\dagger \circ g$. ▨

DEF A process  is a projector if it is s.a. + .

idempotent

Prop Projectors are positive.

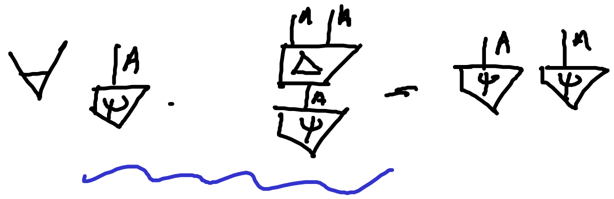
Pf 

. Thus P is +-ive (where g:=P) ▨

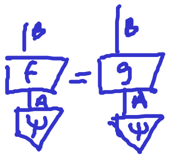
"Quantum-like features from string diagrams."

No-Cloning → quantum $\underset{\text{states}}{\underset{\updownarrow}{\underbrace{\text{data}}}}$ cannot be copied

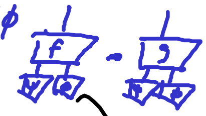
In quantum theory, there exists no process $\Delta : A \to A\otimes A$ where:

(a) ∀ 

"enough states"

(b) ∀ $\psi : I \to A$. 

⇒ f=g

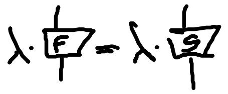
(c) ∀ψ,φ 

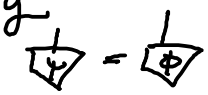
⇒ f=g

2 PROOFS (MAKING DIFFERENT ASSUMPTIONS)

"CLASSIC" PROOF

WOOTERS + ZUREK : 1982

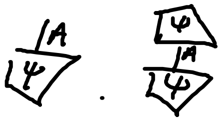
- $\lambda \neq 0$, 

⇒ f=g
- Δ is an isometry
- $\langle \psi | \phi \rangle = 1$ ⇒ 

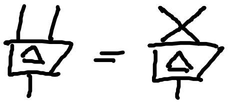
normalised ψ, φ

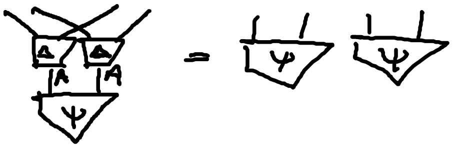
"CAPS + CUPS" PROOF

ABRAMSKY 2010

(1) ∃ 

= 1.

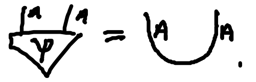
(2) 

[(3) 

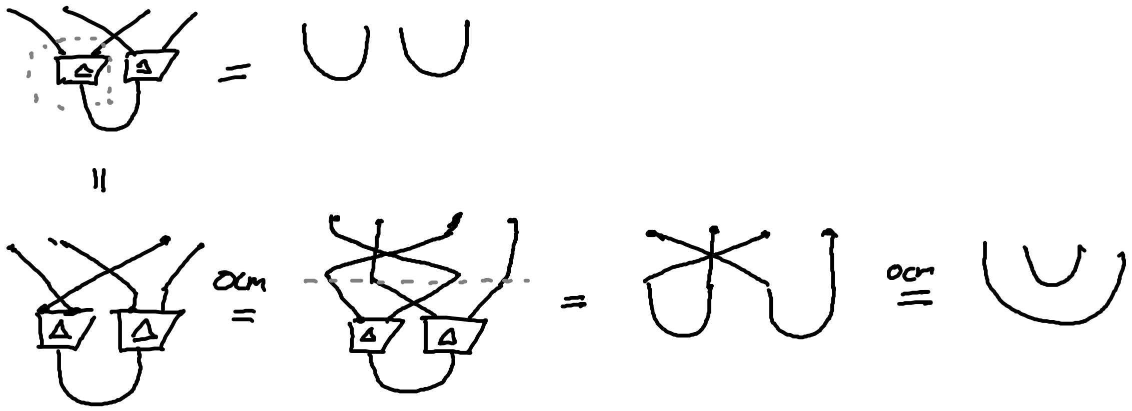
]

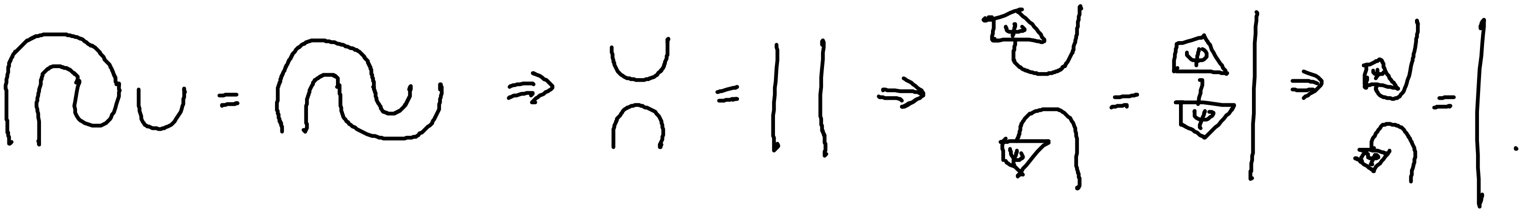
Thm If a process thy admits string diagrams, and all types A have a process sats. (1), (2), + (3), then the proc. thy is trivial.

Pf

Let 

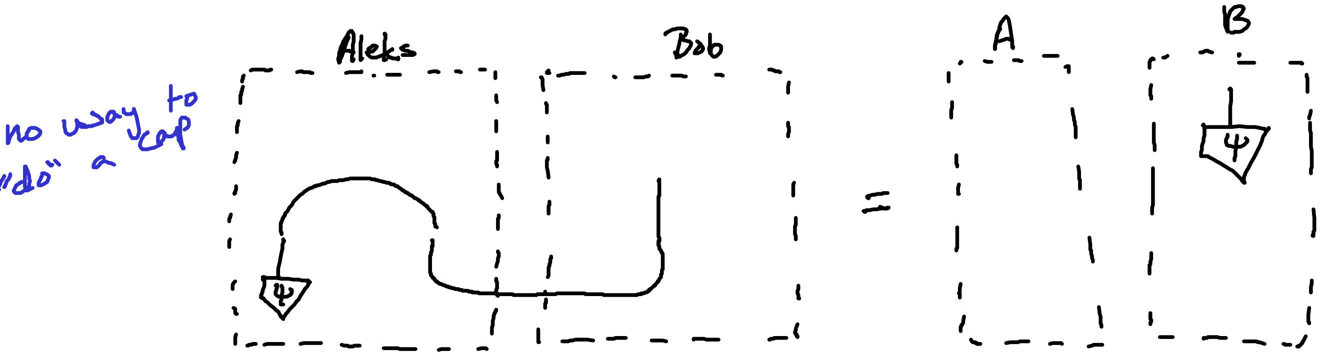
. Then —



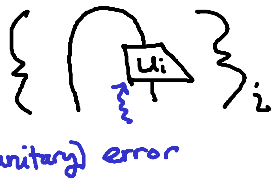


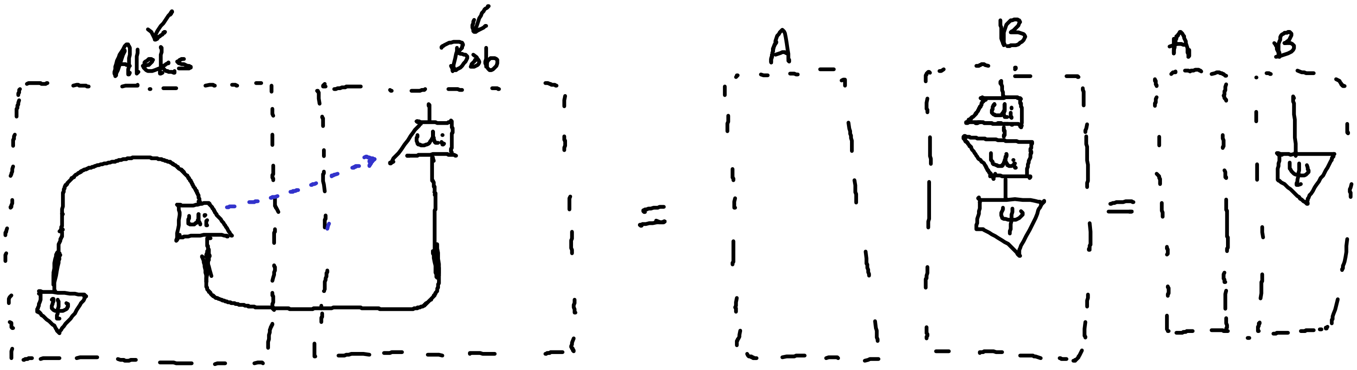
⇒ any process is 0-separable. (ie. P.T. is trivial). ∎

TELEPORTATION

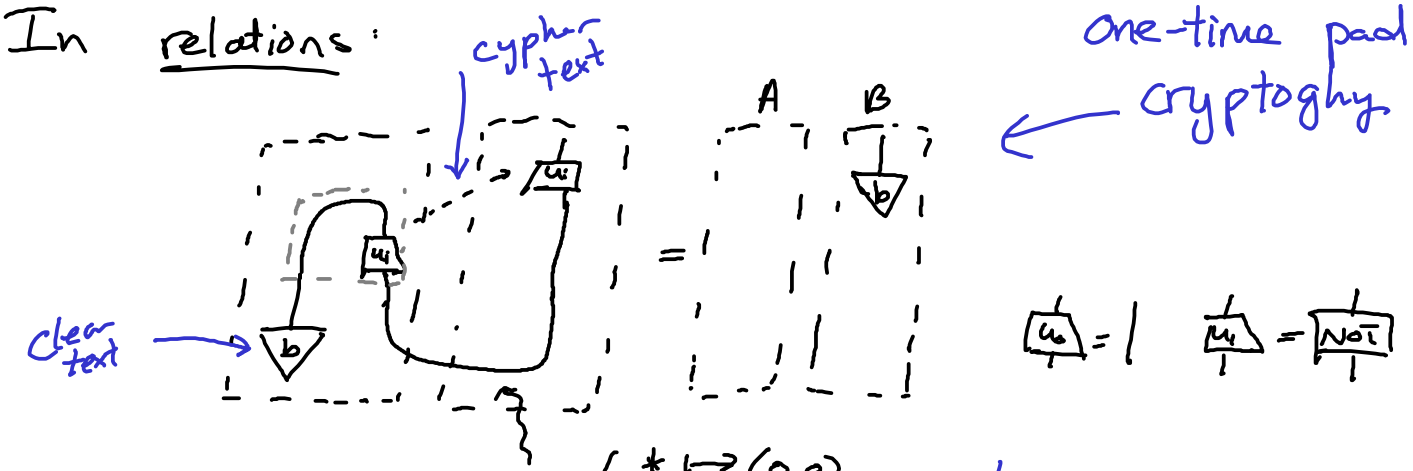


But I am allowed a non-deterministic process:





In relations:



$$\left\{ \begin{array}{l} * \mapsto (0,0) \\ * \mapsto (1,1) \end{array} \right.$$ ← key