Causality and the arrow of time in process theories

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A thesis submitted for the degree of

Doctor of Philosophy

Michaelmas 2021

For my grandmother.

Acknowledgements

First of all, I would like to thank my supervisor Bob Coecke for his help, support and valuable advice. Near him, I changed my mindset regarding the way I view physics and consequently even the every-day world. I am looking forward to more pub-evenings in his house, listening to the ‘eternal noise’ of his guitar.

Next, I would like to thank my supervisor Stefano Gogioso for exposing me to the abstract mathematical thinking and bearing with me at the time when I had no clue what a ‘special dagger symmetric Frobenious algebra’ is!

I could not thank enough John Selby with whom a great amount of this work was developed. I feel extremely lucky to work with him, cause apart from a great scientist he is a fantastic person. I really hope our collaboration continuous beyond this thesis. I would also like to thank Ana Belen Sainz, who, together with John made it possible for me to be a visiting researcher in the University of Gdansk.

I would like to give a special thanks to my colleagues in the Quantum Group as they added moments of joy in the whole PhD experience. In particular, Robin Lorenz for the philosophical discussions, Subhayan Roy Moulik for introducing me to excellent nespresso coffee flavours, Augustin Vantrietvelde and Hler Kristjansson for their kindness, humor and their company at college meals in St Annes, David Reutter for setting the scene for Thursday dinners, Konstantinos Meichanetzidis , Nicola Pinzani, Fatimah Ahmadi, Carlo Maria Scandolo, JS Leemay, Giovanni De Felice, Alexis Toumi, Vojtech Havlicek and many others.

Finally, I would like to thank my professors at UCL, namely Dan Brown, Sougato Bose and Andrew Fisher, who made it possible for me to study in Britain as well as my colleagues, namely Diego Armando Aparicio, Gioele Consani, Conor McKeever, James Seddon and many others who greatly supported me during my time at UCL.

Moving a bit afar from the academic side I would like to thank Vasilis Copetinas–with whom I had the chance to talk some Greek–for the 'short' coffee breaks in St' Annes and Bogdan Cristea who has been a wonderful roommate while I was in Oxford.

A special mention deserves my family and Thomas, that continuously supported me as well as long term friends, namely Zoi-Joyia Spetsieris, Roula Tsiligianni, Sia Andreopoulou and Chrysoula Goupiou.

Finally, I would like to thank Jon Barrett for taking the role of my primary supervisor after Bob's leave.

Abstract

The objective of this thesis is to study the concept of causality and its interplay with the arrow of time in quantum process theories and beyond. The framework of process theories is particularly intuitive since it is accompanied by a diagrammatic formalism. Its formal underpinnings lie in Categorical Quantum Mechanics, a field which has recently seen enormous success both in the foundations of physics and in quantum information theory.

Taking a process-theoretic viewpoint we examine possible ways to time-symmetrize quantum theory. We identify the causality condition as the root of the asymmetry and we proceed by eliminating it in a variety of ways. We furthermore view that process theories can be faithfully captured by operad algebras, which is an alternative to their identification with symmetric monoidal categories. The landscape of operad algebras allows us to introduce novel notions of process theories, namely time neutral process theories. To provide more intuition about them, we establish a link with compact closed categories. Finally, we present an algebraic framework to describe the evolution of fields, perceived as causal process theories, in discretised spacetimes. We form connections with algebraic quantum field theory (AQFT) as well as quantum cellular automata (QCA).

Contents

Introduction

0.1 Background literature

The vast majority of this work relies heavily on the framework of categorical quantum mechanics[AC09, CK15]. The motivation for combining categories with quantum theory is the reassessment of fundamental physics. Indeed, categorical quantum mechanics gives primary focus on the way quantum processes compose and interact with their environment, contrary to the traditional Hilbert space formalism which relies on complex linear structure with the emphasis being given on the states.

In particular, the framework of symmetric monoidal categories is considered suitable to describe physical theories and is accompanied by a diagrammatic formalism [JR91, CK15, BS09]. Diagrams provide a more intuitive understanding of abstract mathematical notions and consequently constitute a more flexible way to reason about things [CK16, SSC21, Sel17]. Due to its emphasis on processes rather than states, the framework of symmetric monoidal categories bears the name *process theories*.

The highlight of process theories is their extreme versatility since the same diagrams can be interpreted in different kinds of monoidal categories. When it comes to the foundations of physics, this feature offers a significant advantage since it provides a wide landscape to explore alternative theories to quantum theory. In the same spirit, it yields intuition of the precise fundamental characteristics that make quantum and classical physics radically different, an intuition that we would not acquire with the traditional Hilbert space formalism[Bae06].

It is noteworthy that the diagrammatic representation of quantum processes has seen enormous success in the field of quantum information, with the development of a graphical calculus, namely ZX calculus, which reasons about interactions between qubits[Bac14, JPV17, NW17]. Of particular relevance to this thesis though, is the success of the process theoretic formalism related to the axiomatization of quantum field theories [Hal11, HM06, HK64, Wit88, Ati88]. There, a quantum field theory is defined as a functor from a suitable category related to spacetime structure to a category related to quantum processes.

0.2 Synopsis of this thesis

In the first chapter, we introduce the category-theoretic concepts and the framework of process theories, that will be used throughout the thesis. We do so by invoking the category **Hilb** of Hilbert spaces as the central example, to demonstrate how the standard formalism of quantum theory can be presented in a category-theoretic language. We present the process theory of quantum physics, **QPhys**, which treats classical systems as internal to quantum theory.

Processes in **QPhys** are completely positive trace preserving (CPTP) maps. We identify trace preservation with the notion of causality in general process theories and we indicate the particular conditions that a process theory has to obey to be considered causal.

In the second chapter, we provide three ways to incorporate time-symmetry in quantum theory from a process-theoretic perspective. The first one has close connections with Refs.[Har21, DBDR20] and restricts the process theory **QPhys** to one that satisfies an additional retrocausality constraint. The second one is, according to our knowledge, a completely new approach, which extends the notions of causality and retrocausality to apply to systems (along with processes). Based on this approach we create a toy model for particle physics, where the causal and retrocausal systems correspond to particles and antiparticles respectively and processes describe interactions between these. Finally, the third approach extends **QPhys** to a super theory that satisfies neither causality nor a retrocausality constraint. To avoid unphysical predictions we can either modify its composition rule, which is a similar approach taken in Refs. [AV08, APTV09, O$^+$08, OC15, SGS$^+$17, SGB$^+$14, OC16, Oec16], or its processes. The material presented in Chapter 2 is a joint forthcoming work with John Selby and Bob Coecke[SSCng].

In the third chapter, we claim that the approach of viewing process theories as symmetric monoidal categories can be proved limited. Inspired by the work of Ref. [PSV21] we argue that process theories are better understood as a particular type of operad algebra. We demonstrate the utility of the operadic perspective by showing that it subsumes causal process theories as well as time neutral process theories. The latter possess no distinction between inputs and outputs of processes, which could be relevant to quantum gravity approaches. To provide a deeper intuition of time neutral process theories we establish a link with compact closed categories. The material presented in Chapter 3 is a joint forthcoming work with John Selby[SSng].

In the fourth chapter, we present an algebraic framework to describe the evolution of quantum fields, perceived as causal process theories, in discretized spacetimes. While in the first two chapters time and space were implicit, in this chapter they become explicit. More specifically, we formulate theory-independent notions of fields over causal orders in a functorial manner. We draw strong connections to quantum cellular automata [Arr19] and algebraic quantum field theory [HK64, HM06]. Our constructions are shown to greatly generalize those in the existing literature. This is a joint published work with Stefano Gogioso and Bob Coecke [GSC21].

Chapter 1

Categorical Quantum Mechanics

1.1 Categories

Categorical quantum mechanics starts with the idea that the behaviour of physical systems should be studied concerning other systems and never in isolation. A prominent example is the measurement process where we acquire knowledge of the physical system of interest through its interaction with a measurement apparatus. Categories provide a natural mathematical language to model such interactions and come with a diagrammatic formalism [JR91, JRD96].

Definition 1.1. A category consists of a set of objects, that correspond to physical systems and a set of morphisms that correspond to physical processes. Every morphism has a source and a target object. If the morphism f has A as a source object and B as a target object, then this is expressed as $f : A \to B$. The morphism f can be interpreted as a process from a physical system A to a physical system B.

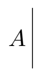
A category obeys the following axioms:

- For two processes $f : A \to B$ and $g : B \to C$, there is a process $g \circ f : A \to C$, namely the composite of f and g. We will say that f and g *compose* *sequentially.*

- Composition of processes is associative, that is $(g \circ f) \circ h = g \circ (f \circ h)$.
- For every system A, there is an identity process $id_A : A \to A$.
- Given $f : A \to B$, there is an identity law, namely $id_B \circ f = f = f \circ id_A$.

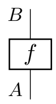
From the definition of a category, we can already realize that processes have a more prominent role than systems. This is in contrast with the standard perspective in physics, where the primer role is on systems (e.g. Hilbert spaces) rather than processes (e.g. linear maps).

To visualise systems (objects) and processes (morphisms) we can use what category theorists call 'string diagrams'. The spirit of string diagrams is that systems are labels of 'strings' or 'wires':

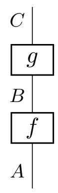


Wires can also be thought of as identity processes $id_A : A \to A$, since by definition systems A are in one to one correspondence with them.

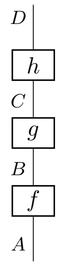
General processes are represented as boxes with an input wire A and an output wire B:



Sequential composition is achieved by composing the output of one box to the input of another. For instance, the composition of $f : A \to B$ with $g : B \to C$ is represented as:



The associativity of composition is latent in the string diagram notation. That is the diagram



represents both $h \circ (g \circ f)$ and $(h \circ g) \circ f$. That is, the associativity law is 'built in' the diagrammatic notation. The same holds for the identity law. The fact that the associativity and identity laws are inherent in the diagrammatic language indicates an advantage over the algebraic language.

Prominent examples of categories are

- **Set**: systems are sets $A, B, ..$ and processes are functions $f : A \to B$ between these. Sequential composition of $f : A \to B$ and $g : B \to C$ is the function $g \circ f : \alpha \mapsto g(f(\alpha))$, with $\alpha \in A$. The identity process is the function $id_A : \alpha \to \alpha$.

- **Rel**: systems are sets $A, B, ..$ and processes are relations $R \subseteq A \times B$. Composition of relations $R : A \to B$ and $T : B \to C$ is the relation

$$\{(\alpha, c) \in A \times C \mid \exists b \in B : (\alpha, b) \in R, (b, c) \in T\}.$$

  The identity process for A is the relation $\{(\alpha, \alpha) \in A \times A | \alpha \in A\}$.

- **Hilb**: systems are Hilbert spaces and processes are linear maps between these. The composition of processes is the composition of linear maps as ordinary functions. The identity process is the identity linear map.

The category **Set** is relevant to classical physics, whereas the category **Hilb** is relevant to quantum physics. The category **Rel** is somewhere in between. Although one would expect that its properties are closer to **Set**, it looks more like **Hilb**. For this reason, it is considered to provide useful insight when studying quantum mechanics from a categorical point of view.

We have mentioned what a category is and we walked through some examples. The next step is to define mappings between categories.

Definition 1.2. A functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a map that

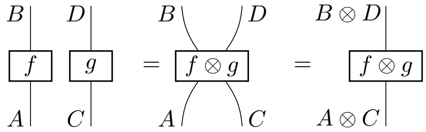
- sends an object A in $\mathcal{C}$ to an object $F(A)$ in $\mathcal{D}$.
- sends a morphism $f : A \to B$ in $\mathcal{C}$ to a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$, such that the following conditions are satisfied:
  - F preserves identities, i.e. $F(id_A) = id_{F(A)}$ for an object A in $\mathcal{C}$.
  - F preserves composition, i.e. $F(g \circ f) = F(g) \circ F(f)$ for morphisms $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$.

In physics, usually, a functor describes a theory: In particular, quantum field theories are axiomatized as functors [Ati88, HK64]. For instance, a topological quantum field theory is a functor $F : \mathrm{Bord_n} \to \mathrm{Vect}$ which maps $n-1$-dimensional manifolds (i.e. a slice of space) to vector spaces of quantum states (i.e. Hilbert spaces) and n-dimensional manifolds (i.e. spacetime) to linear maps.

1.2 Monoidal categories

In physics, we usually want to describe how a joint system constitutes of smaller parts. To manipulate joint systems, we use some sort of tensor product $H \otimes K$ for any pair of systems H and K. As we will see below, categories with a tensor product are called *monoidal* and the corresponding composition of systems, $H \otimes K$ is called *parallel composition.* For example parallel composition in Hilb is the usual tensor product of Hilbert spaces.

Quite generally, given processes $f : A \to B$ and $g : C \to D$ their parallel composition is $f \otimes g : A \otimes C \to B \otimes D$. That is, $f \otimes g$ is a process from the joint system $A \otimes C$ to the joint system $B \otimes D$. Graphically, we can depict parallel composition in multiple ways as follows:



More formally, parallel composition is captured by monoidal categories. These are categories as in definition 1.1 equipped with extra structure.

Definition 1.3. A monoidal category consists of the following ingredients:

- a category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, that performs the following assignment to pairs of systems and processes:

$$\begin{aligned}(A, B) &\mapsto (A \otimes B)\\ (f : A \to B, g : C \to D) &\mapsto (f \otimes g : A \otimes C \to B \otimes D)\end{aligned}$$

- a trivial (or unit) system $I \in \mathcal{C}$
- a natural isomorphism $\alpha_{A,B,C}$ for every triple of objects A, B, C, which is called *the associator*:

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$$

- natural isomorphisms

$$\begin{aligned}\rho_A &: A \otimes I \xrightarrow{\sim} A\\ \lambda_A &: I \otimes A \xrightarrow{\sim} A\end{aligned}$$

  called the right and left *unit laws* respectively.

The associator $\alpha_{A,B,C}$ and the unit laws ρ and λ satisfy certain conditions known as coherency conditions, which in plain words guarantee that we can treat isomorphisms as if they are equalities[CP11, ML98].

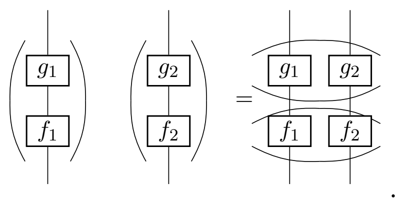
Definition 1.3 obviously requires explanation. First of all, the functor ‘$\otimes'$ assigns a tensor product both to systems and processes automatically placing them on equal footing. For instance, in the case of Hilbert spaces the functor ‘$\otimes$’ assigns to a pair of Hilbert spaces H, H' their tensor product $H \otimes H'$ and to a pair of linear operators $f : H \to K, g : H' \to K'$ their tensor product

$$f \otimes g : H \otimes H' \to K \otimes K'.$$

Physically, the tensor product of processes means that they *run in parallel*: $f \otimes g$ translates as ‘f happens *while* g happens’. This is in contrast with sequential composition, $f \circ g$, which translates as ‘f happens *after* g happens’. However different, sequential and parallel composition distribute over each other, meaning that the following condition holds:

$$(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)$$

That said, f_1 being *in the past* of g_1 *while* f_2 being *in the past* of g_2 , is the same as the pair (f_1, f_2) being in the past of the pair (g_1, g_2). Graphically, the distributivity condition is depicted as



We have now used parentheses to indicate the way diagrams are formed on each side. By dropping them we realize that the distributivity condition comes out naturally in the diagrammatic language which is a manifestation of its advantage over algebraic equations.

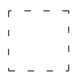
Turning now to the bullet (3) of the definition 1.3, the trivial system acts as a multiplicative identity up to an isomorphism. That is, $A \otimes I \simeq I \simeq I \otimes A$. For instance, in the category Hilb, the unit system is the complex numbers $\mathbb{C}$. In that case, the left and right unit laws are the isomorphisms

$$\rho_H : H \otimes \mathbb{C} \to H$$
$$\lambda_H : \mathbb{C} \otimes H \to H.$$

In general, it is beneficial to think of processes of the form $I \to I$ as the scalars of the theory with id_I being their identity. In our example category Hilb, processes $I \to I$ are linear maps $f : \mathbb{C} \to \mathbb{C}$. They are defined by $f(1)$ since by linearity we have that $f(z) = zf(1)$ for $z \in \mathbb{C}$. Therefore, complex numbers are in correspondence with processes $I \to I$. Graphically, scalars are represented as processes with trivial input or output, i.e.

$$s$$

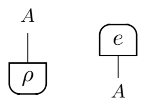
while their identity id_I is either represented as an empty diagram



or it can be omitted for simplicity.

The existence of I allows us to define some special kinds of processes, namely states and effects. States are processes of the form $I \to A$ and they can be perceived as bringing the system A into being. In the case of Hilb for example, every element $\psi \in H$ is in correspondence with states $S_\psi : \mathbb{C} \to H$ via $S_\psi(1) = \psi$. Effects, on the other hand, are processes of the form $A \to I$ and they can be perceived as destroying system A. In the context of quantum theory, we can view them as describing postselection: Many rounds of the experiment can be performed and only the ones with a certain outcome dictated by the particular effect are accepted.

Diagrammatically, states and effects are drawn respectively as



Proceeding to the bullet (4) of the definition naturality of the associator means that there are processes $f : A \to A', g : B \to B', h : C \to C'$ such that the following diagram commutes:

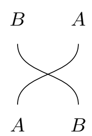
$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "(f \otimes g) \otimes h"'] & A \otimes (B \otimes C) \arrow[d, "f \otimes (g \otimes h)"] \\
(A' \otimes B') \otimes C' \arrow[r, "\alpha'_{A,B,C}"'] & A' \otimes (B' \otimes C')
\end{tikzcd}$$

Similar conditions hold for the right and left unit laws. The usefulness of the naturality condition will be demonstrated in the following when considering a special kind of a monoidal category, namely a symmetric monoidal category.

Definition 1.4. A symmetric monoidal category is a monoidal category with an additional natural isomorphism

$$\sigma_{A,B} := A \otimes B \to B \otimes A$$

The isomorphism $\sigma_{A,B}$ swaps the order of the systems A and B and is represented as a wire crossing:



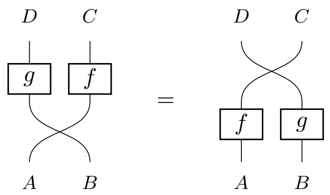
It should satisfy some coherence conditions such as $\sigma_{I,A} \circ \lambda_A = \rho_A$ which ensure that it interplays nicely with the rest of the morphisms in the monoidal category.

The naturality condition for $\sigma_{A,B}$ is the commutative square below:

$$\begin{tikzcd} A \otimes B \arrow[r, "\sigma_{A,B}"] \arrow[d, "f \otimes g"'] & B \otimes A \arrow[d, "g \otimes f"] \\ A' \otimes B' \arrow[r, "\sigma_{A',B'}"'] & B' \otimes A' \end{tikzcd}$$

.

Graphically, the naturality condition is more intuitive being depicted as



This is translated as 'first applying the processes on two systems and then swapping them is the same as first swapping the systems and then applying the processes'.

The example categories Set, Rel and Hilb are symmetric monoidal:

- In **Set**, $\sigma_{A,B} : A \times B \to B \times A$ is defined as $(\alpha, b) \mapsto (b, \alpha)$ for $\alpha \in A$ and $b \in B$.

- In **Rel**, $\sigma_{A,B} : A \times B \to B \times A$ is defined as $(\alpha, b) \sim (b, \alpha)$ for $\alpha \in A$ and $b \in B$.

- In **Hilb**, $\sigma_{H,H'} : H \otimes H' \to H'$ defined as $\psi_1 \otimes \psi_2 \mapsto \psi_2 \otimes \psi_1$ for $\psi_1 \in H$ and $\psi_2 \in H'$.

1.3 Dagger categories

Prominent features of quantum theory, such as unitarity and the braket notation, are tightly connected to the inner product of states in a Hilbert space. However, when we view quantum theory from a categorical perspective, such as when working within the category Hilb of Hilbert spaces and bounded linear operators, the inner product is a feature that does not play a crucial role and is usually ignored. This seems to leave the categorical thinking 'incomplete': on the one hand, the inner product is unimportant when defining the category Hilb but on the other hand it plays an essential role in quantum theory. The answer to this puzzle lies in the categorical formulation of adjoints.

Any bounded linear operator $f : H_1 \to H_2$ between Hilbert spaces H_1 and H_2 admits an adjoint , i.e. a bounded linear map $f^\dagger : H_2 \to H_1$. Taking the adjoint of f can be viewed as a functor $\dagger$: Hilb $\to$ Hilb that acts as the identity on objects and sends every morphism f to its ajoint $f^\dagger$.

Knowing the adjoints amounts to recovering the inner product. Indeed, suppose that ψ is a state in a Hilbert space H. Then there exists an operator $S_\psi : \mathbb{C} \to H$ defined as $S_\psi(1) = \psi$ and such that

$$\langle k, \psi \rangle = S_k^\dagger S_\psi.$$

The right-hand side is a linear map from $\mathbb{C}$ to $\mathbb{C}$, which as we have mentioned in the previous section is in correspondence with complex numbers. The left-hand side is the familiar inner product of states in Hilbert space. Therefore, the dagger structure in Hilb encodes the inner product. Furthermore, this way of viewing the inner product formalizes the bra-ket notation in quantum mechanics since the operators S_ψ and $S_k^\dagger$ correspond to a Dirac ket and bra respectively.

The notion of adjoints does not concern only Hilb, but can be generalized to arbitrary categories:

Definition 1.5. A dagger category is a category $\mathcal{D}$ equipped with a functor $^\dagger : \mathcal{D} \to \mathcal{D}$, which acts as the identity on objects and associates every morphism $f : A \to B$ to its adjoint $f^\dagger : B \to A$. Furthermore, the following conditions should be satisfied for every $f : A \to B$ and $g : B \to C$:

- $id_A = id_A^\dagger$
- $(g \circ f)^\dagger = f^\dagger \circ g^\dagger : C \to A$
- $(f^\dagger)^\dagger = f$

If, furthermore, $f \circ f^\dagger = id_B$ and $f^\dagger \circ f = id_A$, then f is a *unitary* map.

Turning to our example categories, Rel is a dagger category whereas Set is not.

- In **Rel**, given the relation $R : A \to B$, the dagger is defined as the relational converse $R^\dagger : B \to A$. That said,

$$R^\dagger = \{(b, \alpha) \in B \times A | (\alpha, b) \in R\}$$

- We will demonstrate that **Set** is not a dagger category through a counterexample: Suppose that Set has a dagger structure. Consider a non-empty set A and let the function $f : \emptyset \to A$. Then, $f^\dagger$ is the function $f^\dagger : A \to \emptyset$. This is a contradiction, since there are no functions from a non-empty set to an empty set.

1.4 Dagger-Compact categories

A dagger compact category is a symmetric monoidal dagger category that is also compact closed. In what follows, we will analyse what this means [Sel07, AC04, AC09].

Definition 1.6. A dagger symmetric monoidal dagger category is simply a symmetric monoidal category with a dagger structure. More precisely,

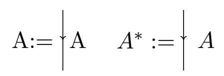
- For every pair of morphisms f, g we have that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- The associator, the left and right unit laws and symmetry are unitary maps.

So far, we have discussed string diagrams for symmetric monoidal categories. Compact closed categories, on the other hand, provide an additional feature to the diagrammatic formalism: They provide us with the ability to bend the wires around, interconverting inputs with outputs.

Definition 1.7. A compact closed category comprises the following ingredients:

- a symmetric monoidal category $\mathcal{C}$

- For every object $A \in C$, there is a dual object A^*. They are represented as wires pointing upwards and downwards respectively:



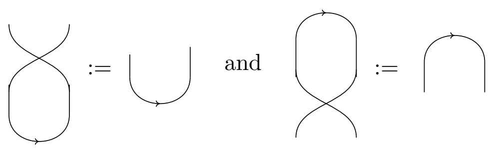
- There are morphisms $\eta_A : I \to A^* \otimes A$, called the cup or unit and $\epsilon_A : A \otimes A^* \to I$ called the cap or the counit, drawn as

  In a sense, the compact structure adds orientation to the diagrams. The cup $\eta_A : I \to A^* \otimes A$ is a state of $A^* \otimes A$ and in particular it represents an entangled state.

- Cups and caps must satisfy the following conditions known as snake equations:

  They convey that bending wires with a cup and a cap is the same with a single wire.

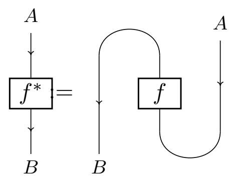
Furthermore, the composition of cups and caps with the symmetry isomorphism leads to the following equations:



Definition 1.8. A dagger compact category is a dagger symmetric monoidal category that is also compact closed and for which the equation $\epsilon_A^\dagger = \eta_{A^*}$ holds.

In a dagger compact category, there are two kinds of involutive symmetries.

The first one comes from the dagger and sends a process $f : A \to B$ to its adjoint $f^\dagger : B \to A$. The second one comes from the compact structure and sends a process $f : A \to B$, to its transpose $f^* : B^* \to A^*$ defined as



The transpose of a map $g : H \to H'$, $g^* : H'^* \to H^*$, is realized as acting on effects rather than states. Indeed, if $e : H' \to \mathbb{C}$ is an element of H'^*, then g^* acts as $g^*(e) := e \circ g$ taking the effect e to the effect $g \circ e : H \to \mathbb{C}$.

Finite quantum mechanics is naturally formed in the language of dagger compact categories [AC04, AC09]. In particular, the category FHilb of finite dimensional Hilbert spaces and bounded operators is dagger compact closed: The dual of a Hilbert space H is H^*, i.e. the space of linear functionals $H \to \mathbb{C}$. The cap $\epsilon : H \otimes H^* \to \mathbb{C}$ is defined as

$$\epsilon : |\psi\rangle \otimes \langle k| \mapsto \langle k|\psi\rangle$$

and the cup $\eta : \mathbb{C} \to H^* \otimes H$ is defined as

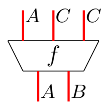
$$\eta : 1 \to \sum_i \langle i| \otimes |i\rangle$$

where $|i\rangle$ is an orthonormal basis.

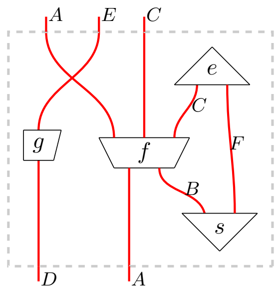
1.5 Process theories

The main objective of the categorical quantum mechanics programme was to reassess the foundations of quantum theory by introducing new mathematics. In the previous sections, we saw that this has been achieved successfully by describing quantum mechanics in the language of dagger-compact categories [AC04, AC09], which is accompanied by a diagrammatic calculus. This pictorial representation of systems and processes falls into a wide class of theories, known as *process theories* and it is the highlight of the categorical quantum mechanics programme [CK16, Coe05, Coe09].

A process theory [CK16, SSC21, Sel17] is defined as a collection of *systems* and *processes* obeying certain constraints that we mention below. For example,



is a process with *input systems* A and B and *output systems* A, C, and C. Processes must be closed under *wirings*. For instance, the diagram

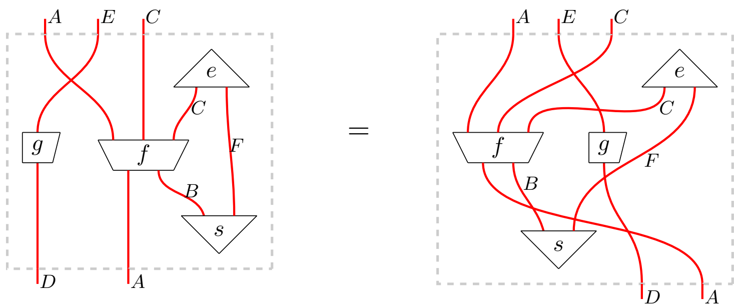


corresponds to another process in the theory.

Wirings are subject to the following constraints:

i. outputs cannot be wired to outputs, and inputs cannot be wired to inputs;
ii. acyclicity;
iii. system labels match.

Furthermore, two diagrams are equal if they have the same *connectivity*. For example,



Remark 1.9. Every symmetric monoidal category (SMC) is a process theory. However, the converse is not necessarily true since we have not made any requirement on the existence of identity processes within a process theory. We argue in Ref. [SSng] that process theories are more faithfully captured by a particular kind of operad algebra.

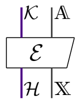
The most advertised advantage that process theories have to offer apart from being intuitive, is their extreme generality. For instance, we can interpret the diagrams intended for quantum theory (such as those in Hilb) in the context of different categories and thus learn what is particular about quantum.

There are many frameworks apart from process theories in the literature that have attempted to explore alternative theories to quantum. These include work in quantum logic[BvN36, BC84, CHK13] and in operational theories[Har01b, Bar07]. However, those approaches describe physics from the perspective of isolated systems, since compound systems constitute a technical challenge. Process theories on the other hand take the compositionality of systems and their interaction with their environment as fundamental notions shedding new light on quantum foundations. For instance, in the context of quantum theory, process theories embrace classical information, treating it as internal to the theory. This is better illustrated in the process theory **QPhys**, in the following subsection.

1.5.1 The process theory of Quantum Physics

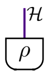
We introduce the process theory which describes (finite-dimensional) quantum theory, **QPhys**. There are two different kinds of systems in **QPhys**: Quantum systems, denoted by purple wires and labelled by a (finite-dimensional) Hilbert space $\mathcal{H}$, and classical systems, denoted by grey wires and labelled by a (finite) set, $\mathbb{A}$. Quantum systems are the fundamental systems of interest within quantum theory, whilst classical systems represent how we interact with the quantum world. That is, they represent control variables on devices and pointers which encode measurement outcomes.

A general quantum instrument is denoted by:



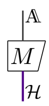
It has a quantum system $\mathcal{H}$ together with a classical control variable $\mathbb{X}$ as an input and a quantum system $\mathcal{K}$ and a classical outcome variable $\mathbb{A}$ as an output. Formally this can be understood as a completely positive trace preserving (CPTP) map between complex matrix algebras, $\mathcal{E} : \mathcal{B}[\mathcal{H}] \otimes \left(\bigoplus_{x\in\mathbb{X}} \mathcal{B}[\mathbb{C}]\right) \to \mathcal{B}[\mathcal{K}] \otimes \left(\bigoplus_{y\in\mathbb{Y}} \mathcal{B}[\mathbb{C}]\right)$. If there is no classical input or output then this is expressed as the singleton set $\star := \{*\}$ and if there is no quantum input or output then this is expressed as the one-dimensional Hilbert space $\mathbb{C}$.

A process with no inputs and a quantum output



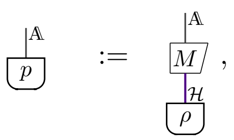
corresponds to a CPTP map $\rho : \mathcal{B}[\mathbb{C}] \to \mathcal{B}[\mathcal{H}]$. Such processes are in one-to-one correspondence with trace-one elements of $\mathcal{B}[\mathcal{H}]$, i.e. quantum states.

A process with a quantum input and a classical output,



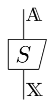
corresponds to a CPTP map $M : \mathcal{B}[\mathcal{H}] \to \bigoplus_{a\in\mathbb{A}} \mathcal{B}[\mathbb{C}]$. These are in one-to-one correspondence with sets of positive operators in $\mathcal{B}[\mathcal{H}]$ indexed by $a \in \mathbb{A}$, $\{M_a\}_{a\in\mathbb{A}}$ and such that $\sum_{a\in\mathbb{A}} M_a = \mathbb{1}_{\mathcal{H}}$. In other words, they are in one-to-one correspondence with destructive POVM measurements.

When we compose a quantum state with a quantum measurement we end up with a process that has only classical outputs,



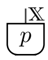
corresponding to a CPTP map $p : \mathcal{B}[\mathbb{C}] \to \bigoplus_{a\in\mathbb{A}} \mathcal{B}[\mathbb{C}]$. This is simply the sequential composition of the CPTP maps ρ and M, i.e., $p = M \circ \rho$. Processes of this form are in one-to-one correspondence with probability distributions over the set $\mathbb{A}$ and in particular, to the probability distribution defined by $p(a) = \mathsf{tr}(M_a \rho)$ for all $a \in \mathbb{A}$. Hence, we see that the Born rule is encoded as a special case of sequential composition of CPTP maps.

A process with only classical inputs and outputs,

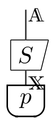


,

corresponds to a CPTP map $S : \bigoplus_{x\in\mathbb{X}} \mathcal{B}[\mathbb{C}] \to \bigoplus_{a\in\mathbb{A}} \mathcal{B}[\mathbb{C}]$. These are in one-to-one correspondence with stochastic maps from $\mathbb{X}$ to $\mathbb{A}$. They therefore map probability distributions over $\mathbb{X}$ to probability distributions over $\mathbb{A}$ via:

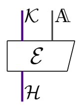


$\mapsto$



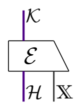
.

Other special cases that are common in the literature are:



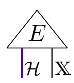
,

, and

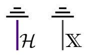


They describe non-destructive measurements, classically controlled state preparations, and classically controlled CPTP maps respectively.

Finally, we consider the set of processes that have no outputs:



They correspond to CPTP maps $E : \mathcal{B}[\mathcal{H}] \otimes \left(\bigoplus_{x\in\mathbb{X}} \mathcal{B}[\mathbb{C}]\right) \to \mathcal{B}[\mathbb{C}]$. Note, however, that, due to the trace-preservation condition, these processes are unique. On the quantum side, they correspond to the (partial) trace while on the classical side to marginalisation. Since they are unique, we will introduce a special symbol to denote them:



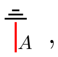
A special case of the above processes is when there is no input system. Then, there is a unique process with neither input nor output which corresponds to the unique CPTP map from $\mathcal{B}[\mathbb{C}]$ to itself. This can be thought of as the scalar 1 which maps, for example, $\rho \mapsto 1 \cdot \rho = \rho$. We diagrammatically denote the scalar 1 by the empty diagram:

1.6 Causal process theories

The trace-preservation condition for quantum processes in **QPhys** can be elegantly expressed in process-theoretic terms. Furthermore, trace-preservation is an equivalent way to express that **QPhys** is a *causal* theory.

Definition 1.10 (Causality [CDP10, CK16]). A process theory is *causal* if for every system A, there is a unique process from A to the trivial system I, known as *discarding* map.

We denote discarding maps in general process theories by

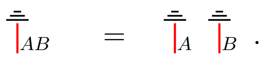


(1.1)

a diagrammatic representation we have already met in the special case of **QPhys**.

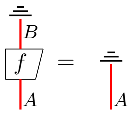
Below we provide the conditions that any causal process theory has to obey:

i. Discarding a composite system is the same as discarding its components:



(1.2)

ii. Discarding the output of a process is the same as discarding its input:



(1.3)

This is the abstract characterisation of trace preservation of quantum theory. We will refer to it as the *causality condition*.

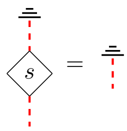
iii. There is a unique process with no inputs nor outputs, denoted by the empty diagram:

(1.4)

It is a special case of the (unique) discarding effect with trivial input. The uniqueness of the empty diagram implies that the theory is *deterministic*, i.e.the only scalar in the theory is the empty diagram.

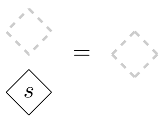
In what follows, we show that causality implies determinism.

Proof. Consider any scalar s in the process theory. The causality postulate takes the form



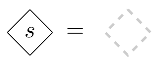
(1.5)

where the dashed edges indicate trivial inputs and outputs. The discarding effect with trivial input is the empty diagram. Thus, the causality postulate becomes



(1.6)

In order for (1.6) to hold, we must have:

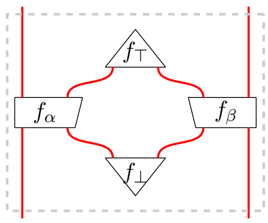


(1.7)

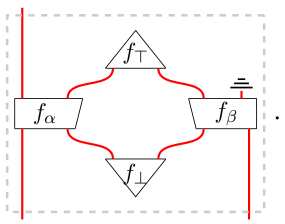
Thus, every scalar in a causal theory is the empty diagram. □

Remark 1.11. While causality implies determinism, the converse is not necessarily true. In Sec. 2.7 we will define the process theory **QNeut** that is deterministic but not causal.

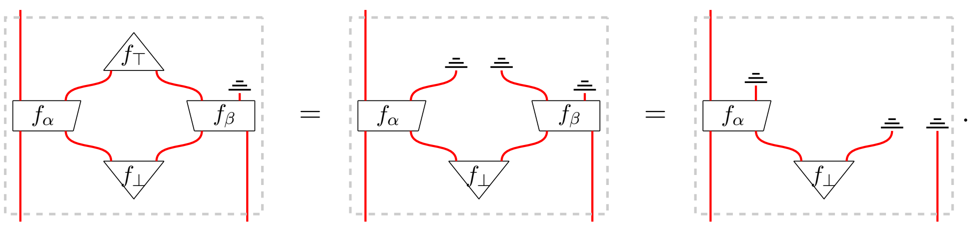
The causality condition ensures that the theory is no-signalling between causally separated regions [Coe14] and, more generally, is compatible with relativistic causal structure [KHC17]. More specifically, suppose that we have two space-like separated parties which may not be able to directly signal to one another but might have a common past and future, where their light cones intersect. This would be represented as a diamond-shaped diagram:



Then, to describe the local physics from the perspective of the left-hand party, we discard the right-hand output which is inaccessible to them:



Then, using the fact that the only effect is the discarding map, and that f_β satisfies eq. (1.3), we find that:



That is, the right-hand input is simply discarded and is disconnected from the left-hand side. Thus, the left-hand party can not infer the input of the right-hand party from his input-output pair.

Remark 1.12. In a category-theoretic language, a causal process theory can be thought of as a symmetric monoidal category in which the monoidal unit is terminal [Coe14].

Chapter 2

Time symmetry in process theories

It is a commonly held belief that quantum theory, or indeed nature itself, is time-symmetric. This claim, however, requires qualification – whilst the unitary dynamics of the theory may be time-symmetric, the theory as a whole certainly is not. This is exemplified by the fact that while there are many pure states in the theory – corresponding to rays in a Hilbert space – there are no pure effects that can be realised without invoking post-selection, i.e., conditioning on the outcomes when performing measurements. Beyond the pure theory, we find that every density matrix corresponds to a preparable state while there is but a single effect – discarding – that can be implemented without post-selection. Indeed as demonstrated in Ref. [CGS17] the time reverse of quantum theory is a remarkably different theory: It has but a single state and describes a theory of eternal noise.

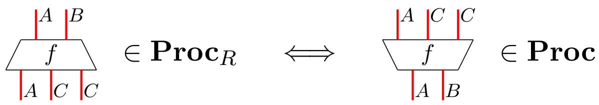
Time-asymmetry within quantum theory is not a new observation. It has been attempted several times in the literature to formulate quantum theory in an explicitly time-symmetric or even time-neutral way. There are many different motivations for doing so. Firstly, for philosophical reasons, we may believe that nature should not have a preferred direction of time at a fundamental level, and hence, quantum theory as a fundamental description of nature should be time-symmetric. If this is the case, the asymmetry that we observe in our experiments must be an emergent phenomenon, perhaps arising due to particular choices of boundary conditions for the universe[OC15]. Secondly, in our attempts to reconcile the theories of quantum theory and gravity there are hints that modifying the role of time in quantum theory will be essential. In particular, it may be the case that we cannot have a predetermined causal structure and so the 'past' and 'future' could be inextricably mixed[Har07]. If that is the case then formulating quantum theory in a time-neutral way may be essential to making progress on the unification of quantum theory and general relativity. The final reason however is much more pragmatic. To perform calculations for quantum theory we often allow ourselves to go beyond the physically realisable processes (for example, embedding density matrices within the space of all Hermitian matrices). In doing so, one can end up with a time-symmetric 'theory of calculation' [Har13] which contains the time-asymmetric 'theory of physics' as a subtheory.

In this thesis, we consider the possible ways in which we can describe quantum theory in a time-symmetric/neutral way. We see that the existing works in this direction [$O^{+}08$, Oec16, OC16, OC15, APTV09, AV08] have a particularly concise process-theoretic description. Moreover, they all correspond to essentially the same process theory, differing only in choices of convention and philosophical perspective rather than in their fundamental mathematics.

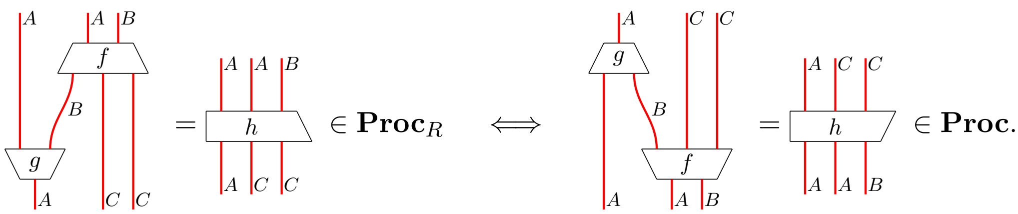
In Sections from 2.1 to 2.5 we provide the relevant tools needed to time-symmetrize quantum theory from a process theoretic perspective. In Section 2.6 we explore three different approaches to obtain a time-symmetric version of quantum theory. The first has close connections with existing literature, whilst the second is a novel approach. Finally, the third approach is analyzed further in Section 2.7, where we recast various other approaches to time symmetry and time neutrality in two different (but equivalent) ways within a process-theoretic formalism.

2.1 Time reversed process theories

From a process theoretic viewpoint **Proc** there is a very simple way to reverse the arrow of time. We simply read the diagrams in the theory from top to bottom rather than bottom to top. The inputs of a process are then considered outputs and the outputs are considered inputs. It is often, however, more convenient to take an active view of time reversal – that is, we keep the bottom to the top reading of the diagrams but we flip all of the diagrams upside down. This defines a new process theory $\mathbf{Proc}_R$ which has the same systems as **Proc** and processes that go in the opposite direction. That is:



Composing the time-reversed processes is the time reverse of composing the original processes. For example:



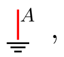
It is therefore straightforward to see that time-reversing a theory twice leaves it invariant. That is, $\mathbf{Proc}_{R_R} = \mathbf{Proc}$.

Remark 2.1. Categorically time reversal is the contravariant functor $R : \mathbf{Proc} \to \mathbf{Proc_R}$.

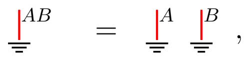
If the process theory, **Proc** is a causal process theory, then, as noted in Ref.[CGS17], the time-reversed theory will be remarkably different. It describes a theory in which there is a single state for every system and is left invariant by every transformation – it is a theory of eternal noise.

Definition 2.2 (Retrocausality)**.** We say that a process theory is *retrocausal*, if for each system A, there is a unique state.

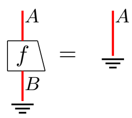
The relevant result of Ref. [CGS17] states that the time reverse of a causal theory is a retrocausal theory. In particular, in any retrocausal theory, the unique state for every system A is denoted as



and can be thought of as a state of uniform noise. Uniqueness implies that states compose as



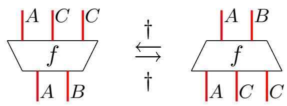
and moreover that every process satisfies the *retrocausality* constraint:



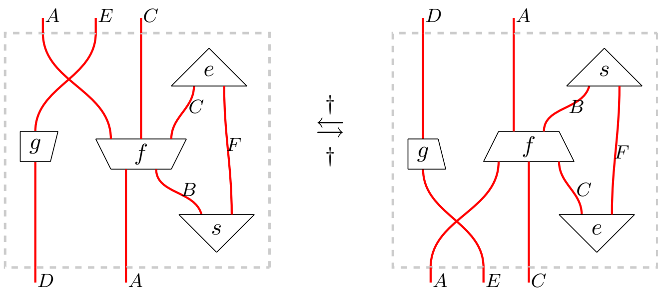
2.2 Time-symmetric process theories

In the works [O$^+$08, Oec16, OC16, OC15, APTV09, AV08] quantum theory is formulated in a time-symmetric way. From a process-theoretic perspective, what this means is that time-reversal is *internal* to the process theory. That said, the time-reversed theory is the same as the original theory. Formally, we say that a process theory is time-symmetric if and only if it permits a *dagger*.

Definition 2.3 (Dagger)**.** A dagger is a map, †, from the process theory to itself that *reflects diagrams*. Specifically, it acts on processes as



and on diagrams as

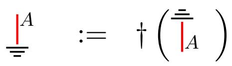


If a process theory has a dagger, then for every process $f : A \to B$ there exists a process $f^\dagger : B \to A$ (that is, the symbolic notation for the upside-down f). The process $f^\dagger$ can be interpreted as the time-reversed of process f. Hence, we say that process theories with daggers are time-symmetric.

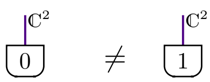
Remark 2.4. If the process theory is representing a SMC then any involutive contravariant endofunctor $\dagger$: **Proc** $\to$ **Proc** that acts as the identity on objects provides a dagger for the process theory. The existence of a dagger implies that **Proc** and **Proc**$_R$ are covariantly isomorphic. We define the covariant isomorphism simply by $R \circ \dagger$: **Proc** $\to$ **Proc**$_R$. It is covariant as both $\dagger$ and R are contravariant. Thus, covariance is the key to having a time-symmetric theory.

Theorem 2.5. *[CGS17, Thm. 3] A time-symmetric theory is causal if and only if it is retrocausal.*

We have already mentioned that there is a unique state for each system in the theory given by the dagger of the discarding map:

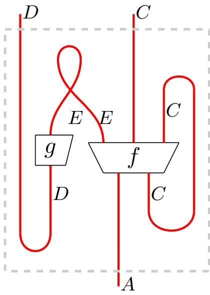


It is clear that **QPhys** is not a time-symmetric theory – it is causal but not retrocausal. Indeed, there are multiple states for any (non-trivial) system, such as the computational basis states of a qubit:



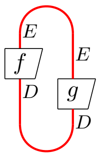
2.3 Time-neutral process theories

In the work of [OC16], the authors aim to go a step beyond time symmetry and create a *time neutral* version of quantum theory. Process-theoretically this means that we want to forget about the distinction between inputs and outputs. This is possible if the theory has *cups & caps*. Consequently, time neutral theories allow for a freer notion of wiring which neglects the input-output structure. For example, the following diagram is permissible:



.

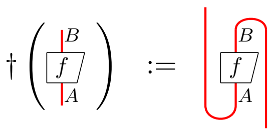
Moreover, there is no meaningful causal order that can be assigned to the processes within a diagram as this freer notion of wiring allows for cycles. For example, in the diagram below



,

f is both in the 'causal future' and the 'causal past' of g.

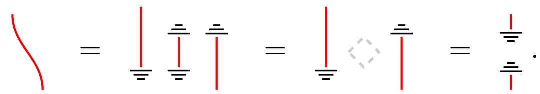
Process-theoretically time-neutrality is a stronger notion than time-symmetry. That is, any time-neutral theory is also necessarily time-symmetric since we can define a dagger using cups and caps:



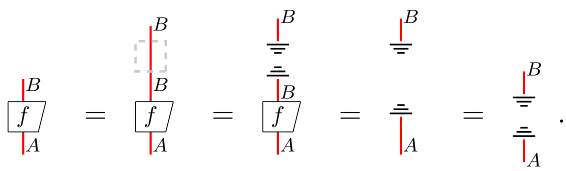
Theorem 2.6. *If a time-neutral theory is causal then there is a unique process between any two systems.*

Proof. As time neutrality implies time symmetry, we immediately have that causality implies retrocausality. Hence, for every system, we have a unique state and effect. In particular, this means that:

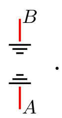
The snake equation implies that:



Therefore, for any f we have:

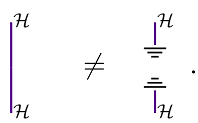


Thus, there is a unique process per pair of systems in the theory, namely:



□

It is clear that the process theory **QPhys** is not a time-neutral theory as there are multiple distinct processes with the same inputs and outputs. For example:



Remark 2.7. Categorically the cups and caps, in this case, correspond to the unit and counit in a compact closed category in which the objects are equal to their dual.

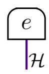
One could argue that process theories with cups and caps are not truly time-neutral since the individual processes still have a distinction between input and output systems, even if we can now freely interchange them. In Chapter 3 we will see how to go to fully time-neutral process theories, where there is no distinction between inputs and outputs.

2.4 Example : Quantum Calculations

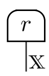
We have seen that our first example **QPhys** is a causal process theory that fails to be either time-neutral or time-symmetric. There is, however, a closely related theory **QCalc** which is a supertheory of **QPhys** that is both time-symmetric and time-neutral. This is the theory in which we often perform calculations about quantum physics – for example, when computing the probabilities of measurement outcomes.

The systems in **QCalc** are the same as those of **QPhys** and thus we will use the same diagrammatic notation for them as we did in **QPhys**. The entire difference between the two theories is then in the definition of processes. In **QPhys** processes were defined as CPTP maps. However, in **QCalc** we drop the trace preservation condition and thus allow for arbitrary completely positive maps. In particular, states of quantum systems $\mathcal{H}$ are given by arbitrary positive operators and so are not necessarily trace-1, and states of classical systems $\mathbb{X}$ are arbitrary functions over $\mathbb{X}$ valued in $\mathbb{R}^+$ rather than probability distributions over $\mathbb{X}$.

Notably, this theory is not a causal process theory. Indeed, many processes have a system $\mathcal{H}$ as an input and no output:



They correspond to arbitrary CP maps $e : \mathcal{B}[\mathcal{H}] \to \mathcal{B}[\mathbb{C}]$. Similarly, many processes have a system $\mathbb{X}$ as an input and no output:



They correspond to arbitrary CP maps $r : \bigoplus_{x\in\mathbb{X}} \mathcal{B}[\mathbb{C}] \to \mathcal{B}[\mathbb{C}]$.

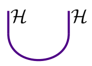
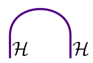
The fact that it is not a causal process theory removes the obstacles that we had with **QPhys** regarding time symmetry and time neutrality. Indeed, **QCalc** has a dagger as well as cups and caps. More specifically, a suitable dagger for **QCalc** is given by the Hermitian adjoint, $\dagger_H$ as argued in Refs. [SC17, SSC21] because it has the property that when applied to a state it defines an effect which tests for that state:

$$\dagger_H :: \rho \mapsto \mathsf{tr}(\rho\ \cdot\ ),$$

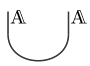
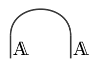
Moreover, it inverts reversible dynamics for any unitary supermap $\mathcal{U}$.:

$$\dagger_H :: \mathcal{U} \mapsto \mathcal{U}^{-1},$$

As far as the cups and caps in **QCalc** are concerned, we express them as

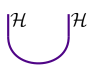
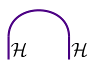
 $\sim \sum_{ij} |ii\rangle\langle jj| \sim$  ,

for some basis $|i\rangle \in \mathcal{H}$. That is, the cup is a supernormalised version of the Bell state and the cap is a Bell effect. For classical systems $\mathbb{A}$ we express the cup and cap as

 $\sim \sum_{a\in\mathbb{A}} |a\rangle\langle a| \sim$  ,

i.e., as the supernormalised perfectly correlated state and the perfectly correlated effect respectively.

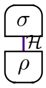
Note that the dagger given by the Hermitian adjoint interacts with the cups and caps in the way we would expect, namely:

 $\overset{\dagger_H}{\underset{\dagger_H}{\leftrightarrows}}$  .

Since we have dropped the trace preservation condition, CP maps in **QCalc** can in addition be trace-decreasing or trace-increasing. The trace-decreasing maps can be seen as processes that occur in some branch of a causal process. However, the presence of the trace-increasing maps leads to a theory that gives nonsensical predictions since it permits ‘probabilities’ that are greater than 1. One example of this comes from the cups and caps themselves:

(2.1)

As the scalars that are greater than one do not have any physical interpretation, the process theory **QCalc** cannot be a good description of nature. It is, however, extremely useful as a theory in which we perform calculations relevant to quantum theory. For example, if we want to compute the probability of some measurement outcome given a state, then we can simply compose the associated effect with the state:



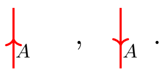
$= \mathsf{tr}(\sigma\rho)$

This gives a sensible probability provided that the state and effect are both trace non-increasing.

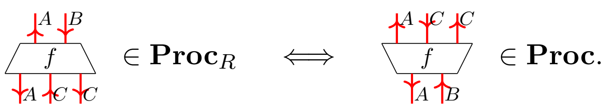
2.5 Process theories with dual systems

So far, we have assumed that systems are invariant under time reversal. In this section, we present a process theory in which this assumption does not hold and provide the category of representation as an illustrative example.

To indicate that systems possess a time orientation we add arrows to the wires representing them:

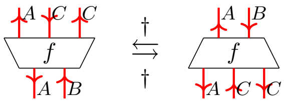


We say that these systems are *dual* to one another, and can be symbolically denoted by $A^{\uparrow}$ and $A^{\downarrow}$ respectively. It is then clear that when we consider time reversal, systems will get mapped to their duals. For example,

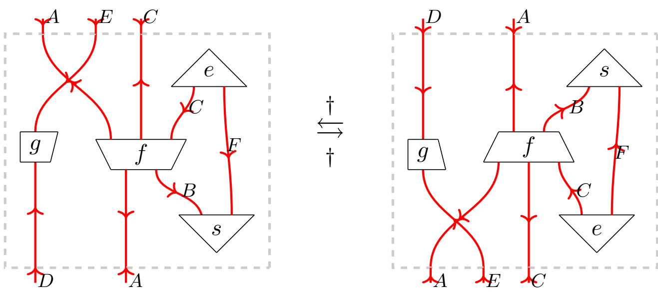


This means that the notion of the dagger and hence the condition for time symmetry for process theories with duals should be refined.

Definition 2.8 (Dagger for process theories with duals)**.** A dagger is a map, †, from the process theory to itself that *reflects diagrams*. In particular, it acts on systems and processes as

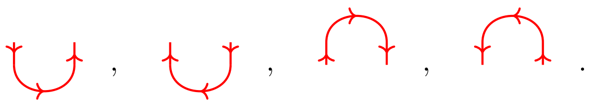


and moreover, on diagrams as



We can then say that a process theory with duals is time-symmetric iff it has a dagger.

Along these lines, cups and caps themselves possess a time orientation



while satisfying the obvious diagrammatic equations. A process theory with duals is then said to be time-neutral iff it possesses the above cups and caps.

2.5.1 Example: Quantum Representations of Groups

An example of a category with duals is the category of representations of a group G within quantum theory, $\mathbf{QRep}_G$. It has representations of a group G on systems in **QCalc** as objects and some special kind of processes within **QCalc**, called intertwiners, as morphisms.

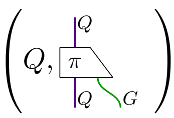
In Sec. 2.6.2 we motivate $\mathbf{QRep}_G$ physically by creating a toy model for particle physics, where a particle is identified with a representation of G on a quantum system $\mathcal{B}(\mathcal{H})$. Generally speaking, a particle is defined in the literature as a particular irreducible representation of the Poincare group, which has infinite dimensionality. In our case, however, we construct a toy model by assuming that we have only finite representations of the group G. We conjecture though that the generalization to groups with infinite representations is also possible.

This toy model has the potential to put particle physics under a new light, since it provides a neat description for the interaction of particles with classical systems introducing, for example, the very concept of measurements within particle physics. In addition, it creates a passage from the standard pure state to mixed state particle physics.

We now define the category $\mathbf{QRep}_G$ using a diagrammatic notation for groups and their representations that we introduce in the Appendix.

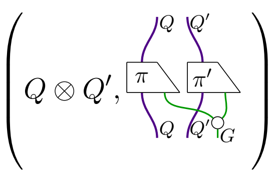
Definition 2.9. Consider a group G. The category $\mathbf{QRep}_G$ consists of the following data:

- Objects are pairs

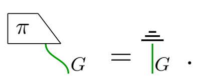


  where Q is an object in **QCalc** and π_Q is a causal representation (see Eq. (4.65)) of G on Q.

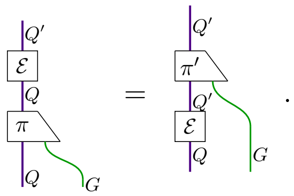
- The tensor product of (Q, π_Q) and $(Q', \pi'_{Q'})$ is given by



- The tensor unit is the pair $(\mathbb{C}, \pi_{\mathbb{C}})$, where $\pi_{\mathbb{C}}$ is the trivial representation:



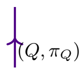
- Morphisms from (Q, π_Q) to $(Q', \pi'_{Q'})$ are intertwiners in **QCalc**. That is, they are CP maps, $\mathcal{E}$, satisfying the covariance condition:



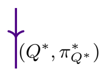
- Composition is the familiar composition of processes $\mathcal{E}$ as in **QCalc**. It can be easily checked that indeed the composition of intertwiners is an intertwiner.

- The identity is the identity process which can also be seen to be an intertwiner.

- Finally, if we denote the system (Q, π_Q) as



then we can represent the dual system as



where the π^* is the conjugate representation defined in Eq. (4.72).

In (Eq. (4.76)) we prove that cups and caps are intertwiners and hence, $\mathbf{QRep}_G$ is a compact closed category with duals. Like **QCalc**, however, $\mathbf{QRep}_G$ cannot be directly interpreted as a theory of physics – that is, it does not necessarily make sensible probabilistic predictions. We, therefore, need to find a condition on the processes in $\mathbf{QRep}_G$, akin to the restriction of **QCalc** to **QPhys** via the causality condition. In Sec. 2.6.2, we propose a way to implement this in a way that preserves time symmetry of $\mathbf{QRep}_G$.

2.6 Three approaches to time symmetry

We have seen in the previous sections that the causality condition in **QPhys** serves as the main obstacle towards time symmetry and time neutrality. In particular, time symmetry together with causality implies that there should be a single state per system, and time neutrality together with causality imply that there should be a single transformation between any pair of systems, neither of which is true within **QPhys**.

Having identified the root of time asymmetry within quantum theory, we can then ask how we could obtain a time-symmetric theory. We have identified several ways to approach the problem:

1. We can restrict **QPhys** to a subtheory that additionally satisfies the retrocausality constraint – that is, every system is both causal and retrocausal. This approach is related to the works of Refs. [Har21] and [DBDR20]

2. We extend the systems in **QPhys** to have time-symmetric counterparts for every system – that is, every system is either causal or retrocausal. This approach is relevant to our toy model of particle physics.

3. We can extend the processes in **QPhys** to a supertheory in which the causality constraint no longer holds – that is, every system is neither causal nor retrocausal. To avoid unphysical predictions we can:

    1. Modify the composition rule. This is closely related to the works of Refs. [AV08, APTV09, O+08, OC15, SGS+17, SGB+14, OC16, Oec16] .

    2. Modify the processes. This is equivalent to (1) but is a more elegant and adaptable presentation of the theory.

The first two approaches lead to a time-symmetric theory (but not a time-neutral theory) and thus we discuss them in this section, while the third leads to a time-neutral theory and thus we discuss it in the following section.

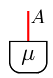
2.6.1 Causal and retrocausal

Given that causality is the main obstacle towards time symmetry within quantum theory, perhaps the most obvious way to time symmetrise the theory is by restricting the processes to those that additionally satisfy a retrocausality condition.

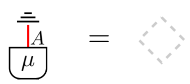
Definition 2.10 (Bicausality)**.** A process theory that is both causal (Def. 1.10) and retrocausal (Def. 2.2) is said to be *bicausal*[1]. Bicausality implies that there exists a unique effect and a unique state for each system.

In such theories, the simple argument against time symmetry in causal theories (namely, that there are more states than effects) breaks down. Therefore, it is plausible that bicausal theories can be considered time-symmetric. In the following, we explore how to construct bicausal theories out of causal theories.

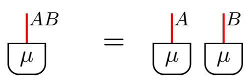
Given any causal process theory, we construct a subtheory that additionally satisfies a retrocausality constraint. To do so, we pick a particular state for each system that we demand to be unique. The candidate states should satisfy certain consistency conditions for the resulting theory to be well-defined. We denote them as:



Since they belong to a causal theory, they automatically satisfy the causality constraint:

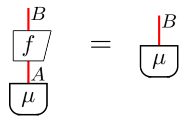


Furthermore, the resulting theory should be close under composition. Therefore the following condition should hold:



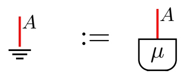
[1]Known as double causality in [Har21].

We further restrict the allowed processes within the theory to those that satisfy:



(2.2)

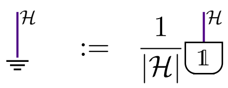
Having so restricted the theory, the remaining subtheory satisfies the retrocausality constraint where we take



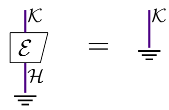
for all systems A.

Generally speaking, there is no reason to believe that this approach will surely result in a time-symmetric theory. There are process theories (e.g. those that are not self-dual on objects as in Sec. 2.6.2) in which this is not possible, at least not without imposing further constraints on the sets of processes.

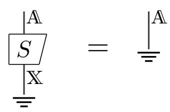
Returning to the case of **QPhys** the natural choice to make concerning the unique state is $\mu_{\mathcal{H}} := \frac{1}{|\mathcal{H}|}\mathbb{1}_{\mathcal{H}}$, i.e. the maximally mixed state for the system:



It is simple to verify that these indeed satisfy the compositionality condition of eq. (2.2). The constraint that is then imposed on the processes of **QPhys** to define the subtheory is:



This means that $\mathcal{E}$ maps the maximally mixed state to the maximally mixed state, i.e. that $\mathcal{E}$ is a *unital* CPTP map. In the special case that the inputs and outputs are classical, we find that



This means that $\mathcal{S}$ is a bistochastic map[2]. The subtheory of unital CPTP maps

[2]Bistochastic maps are typically taken to be square matrices but this constitutes the natural notion which applies also to the non-square case [Har21].

is denoted as **QPhys**$|_{unital}$.

One may then be tempted to define the dagger using the Hermitian adjoint, $\dagger_H$, as we did for **QCalc**. However, the Hermitian adjoint maps the discarding map to the supernormalised state $\mathbb{1}_{\mathcal{H}}$ rather than the unique state $\frac{1}{|\mathcal{H}|}\mathbb{1}_{\mathcal{H}}$. To take care of the normalisation issues we define the dagger via:

$$\dagger\left(\begin{array}{c} \mathcal{K}\quad \mathbb{A} \\ \boxed{\ \mathcal{E}\ } \\ \mathcal{H}\quad \mathbb{X} \end{array}\right) := \dagger_H\left(\begin{array}{c} \mathcal{K}\quad \mathbb{A} \\ \boxed{\ \mathcal{E}\ } \\ \mathcal{H}\quad \mathbb{X} \end{array}\right)\frac{|\mathcal{K}||\mathbb{A}|}{|\mathcal{H}||\mathbb{X}|} \tag{2.3}$$

Proposition 2.11. **QPhys**$|_{unital}$ *is a time-symmetric process theory with the dagger being defined as in eq.* (2.3)

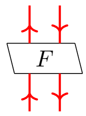
On the face of it, however, **QPhys**$|_{unital}$ does not seem to be a good candidate to describe our world: there is but a single state and effect for every system. Nevertheless, the theory is not entirely trivial as it still has interesting transformations. In particular, it still contains unitary evolution. Moreover, **QPhys**$|_{unital}$ makes classical 'predictions' in the form of bistochastic matrices. However, it is not straightforward to conjecture how these bistochastic matrices suffice to explain our everyday experiences. For example, they do not allow for copying classical information, which is an operation that we would expect to be able to implement.

2.6.2 Causal or retrocausal

In contrast to the previous section (in which we tried to impose both the causality and retrocausality condition for every system), we will formulate a theory in which every system satisfies either the causality or the retrocausality condition.

To do so, we work with process theories with duals, where we will view a system $A^{\uparrow}$ as a causal system and $A^{\downarrow}$ as its retrocausal counterpart. To enforce this interpretation we demand that systems $A^{\uparrow}$ have a unique effect and systems $A^{\downarrow}$ have a unique state. Hence, processes from $A^{\uparrow} \to B^{\uparrow}$ necessarily satisfy the causality condition (Def. 1.10), whilst processes from $A^{\downarrow} \to B^{\downarrow}$ satisfy the retrocausality condition (Def. 2.2).

A general process, however, has both causal as well as retrocausal inputs and outputs. Diagrammatically it is denoted as:

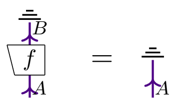


We, therefore, ask what 'causality' type of condition should this process satisfy so that the (retro)causality conditions are satisfied for the systems $A^{\uparrow}$ and $A^{\downarrow}$ individually.

To get to grips with this condition we work within our example category $\mathbf{QRep}_G$. In this case, systems $A^{\uparrow}$ correspond to pairs (Q, π_Q) and systems $A^{\downarrow}$ to dual pairs $(Q^*, \pi^*_{Q^*})$. This correspondence can be thought of as a toy model for particle physics: particles are defined as the (causal) pairs (Q, π_Q) while antiparticles as (retrocausal) pairs $(Q^*, \pi^*_{Q^*})$. In a sense, this takes seriously Feynmann's interpretation of antiparticles as being particles travelling back in time. The flexibility of our construction allows particles to interact with classical systems a feature that does not exist in the standard particle physics literature.

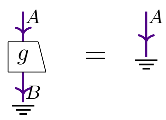
Below we formulate the conditions that process theory with particles and antiparticles needs to satisfy to be both causal and time-symmetric.

As mentioned above, if we only have causal inputs and outputs then the process should be causal:



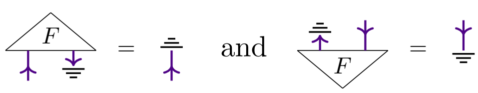
(2.4)

Similarly, if we only have retrocausal inputs and outputs then the process should be retrocausal:



(2.5)

Conditions (2.4) and (2.5) imply that effects or states with both causal and retrocausal systems should satisfy:



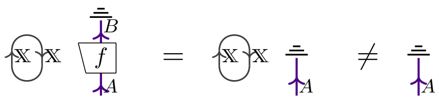
(2.6)

Examples of such processes are given by cups and caps :

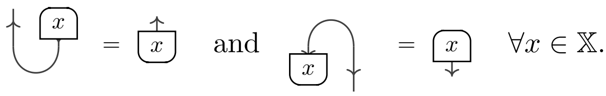

(2.7)

However, this example indicates the need for further constraints on states and effects over eq. (2.6) as we can use these cups and caps to violate Eqs. (2.4) and (2.5). Specifically, as noted in Eq. (2.1), if we compose a cup with a cap then we end up with a scalar other than the empty diagram:

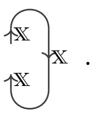
If we furthermore compose either of these with a causal process, f, we will acquire a process that violates the causality condition, i.e.:



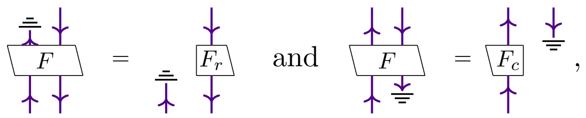
The example (2.7), however, also gives us a hint about the necessary and sufficient condition that should be imposed on general processes to ensure that Eqs. (2.4) and (2.5) always hold. In particular, the classical cup can be viewed as perfect signalling from a retrocausal to a causal system, whilst the classical cap can be viewed as perfect signalling from causal to a retrocausal system:



Therefore, signalling between the causal and retrocausal systems leads us to problems. This is perhaps not surprising as many well-known paradoxes arise from the closed time loops which we could construct via



To prevent such loops from occurring we impose non-signalling conditions on our processes. What this means is that every process F must satisfy both

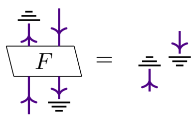


(2.8)

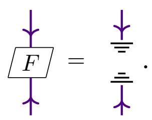
for some retrocausal F_r and some causal F_c.

Note that we could impose no-signalling in one direction, and therefore avoid loops, but to maintain time symmetry, we demand no-signalling in both directions.

The no-signalling conditions (2.8) imply that a general process in $\mathbf{QRep}_G$ must satisfy the condition:



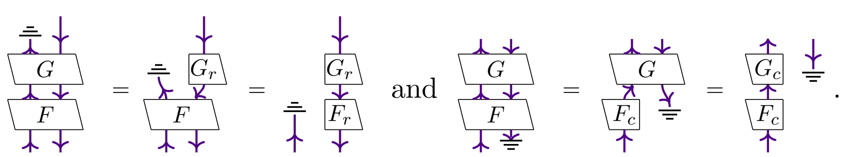
It rules out interactions that send information from a particle to an antiparticle since,



Note also that there is a single scalar satisfying this constraint,

which means that the theory is deterministic.

Finally, this condition is closed under composition. For example, if F and G are non-signalling then so is their sequential composite:



Thus, restricting to such processes does indeed define a legitimate process theory.

We have therefore argued about the constraints that a toy model of particle physics, with particles viewed as finite representations of a group G, needs to satisfy to be both causal and time-symmetric. This leads us to the following definition:

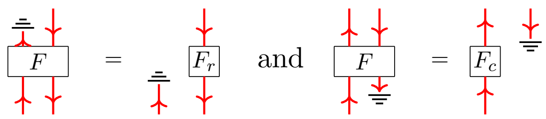
Definition 2.12. We define the process theory $\mathbf{QPart}_G$ as the restriction of

$\mathbf{QRep}_G$ to the subtheory of processes satisfying the non-signalling conditions (2.8) from particles to antiparticles and vice versa.

It is straightforward that $\mathbf{QPart}_G$ inherits the dagger from $\mathbf{QRep}_G$ and hence $\mathbf{QPart}_G$ is a time-symmetric process theory.

Returning now to general process theories, we take the no-signalling conditions (from causal to retrocausal and vice versa) as the definition of a well-behaved theory with causal and retrocausal systems:

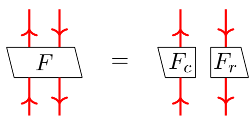
Definition 2.13 (Dual-causal)**.** A process theory with dual systems is said to be dual-causal if it is no-signalling from causal to retrocausal and vice versa, namely if



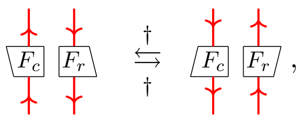
for all processes F.

Given any causal process theory, there is a fairly boring way to construct a time-symmetric theory by forbidding any interactions between causal and retrocausal systems.That said, we demand that

$\forall F\ \exists F_c$ and F_r such that



where F_c belongs to the original causal process theory **Proc** and F_r belongs to the retrocausal theory $\mathbf{Proc}_R$. We then define a dagger by using the time-reversal map R :



Now $F_c \in \mathbf{Proc}_R$ is a retrocausal process and $F_r \in \mathbf{Proc}$ is a causal process.

Remark 2.14. Categorically what we are defining is $\mathbf{Proc} \times \mathbf{Proc}_R$. Note that we can use the R functor to define a dagger as $\mathbf{Proc} \times \mathbf{Proc}_R \cong \mathbf{Proc}_R \times \mathbf{Proc}$.

This however does not seem like a particularly useful or insightful theory. For it to be of interest we need non-trivial interactions between causal and retrocausal systems. We would like to build a generic construction that takes a causal process theory to a dual-causal process theory such that there are non-trivial interactions between causal and retrocausal systems. We leave this construction for future research.

2.7 Two equivalent approaches to time neutrality

The above approaches may lead to interesting time-symmetric theories, even though they do not address time neutrality. The only time-neutral theory that we have encountered so far is the process theory **QCalc**. However, since it does not always make valid probabilistic predictions it can not be considered a suitable physical theory. This section addresses whether it is possible to adapt **QCalc** such that it remains time-neutral and at the same time makes sensible probabilistic predictions.

There are various approaches in the literature aiming (directly or indirectly) at formulating a time-symmetric formulation of quantum theory such as Refs. [AV08, APTV09, O$^+$08, OC15, SGS$^+$17, SGB$^+$14, OC16, Oec16]. Whilst they have many philosophical differences, they are captured by the general formalism we present below.

These approaches are typically presented as a modification of the measurement postulate of quantum theory. They begin, by extending the set of measurements to allow for measurements, M, in which the POVM elements do not necessarily sum to the identity:

$$M = \{\{M_a\}_{a\in\mathbb{A}} | M_a \geq 0\}$$

Then they ensure that the probabilistic predictions are sensible via a suitable modification of the Born rule. Namely, if the "probability distribution" over measurement outcomes predicted by the standard quantum formalism is not normalised, i.e., if

$$N_M(\rho) := \sum_{a\in\mathbb{A}} \mathsf{tr}(M_a\rho) \neq 1,$$

then the new rule for computing probabilities is as follows:

$$\mathrm{Prob}(a|\rho) := \frac{1}{N_M(\rho)}\mathsf{tr}(M_a\rho)$$

The new rule differs from the standard Born rule whenever $N_M(\rho) \neq 1$ and ensures that the theory makes valid probabilistic predictions. (At least, aside from the case of $N_M(\rho) = 0$ which is treated as a special case, to return to later.)

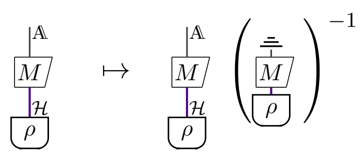
Simply modifying the probability rule is not, however, particularly satisfying from a process-theoretic perspective. The reason is that it is not manifestly compositional. What we show in this section is that we can recover the modified Born rule in a manifestly compositional way. We do so via the construction of two (equivalent) new process theories.

The first process theory that we construct to achieve this can be defined by starting from **QCalc** and modifying the composition rule of its processes such that it reduces to the modified Born rule when a state is composed with a measurement. The second process theory that we construct achieves this in a simpler, more elegant, and more adaptable way by defining an appropriate quotienting of **QCalc**.

2.7.1 Modified composition rule

Within **QCalc** the modified Born rule can be presented as

Standard rule



Modified rule

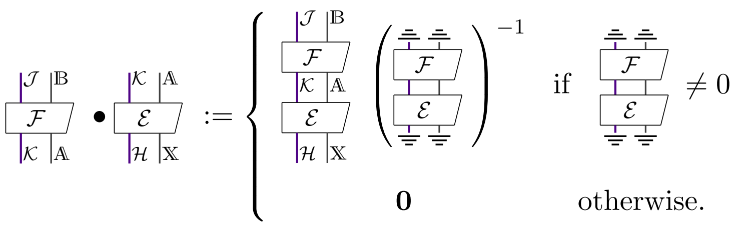
for a CP map M from a quantum to a classical system and a quantum state ρ. Provided that the normalisation factor is non-zero, this defines a valid probability distribution over $\mathbb{A}$.

As mentioned above, we want to define a process theory that has the same processes and systems as **QCalc** but in which composition is redefined. The new composition rule $\bullet$ should be such that

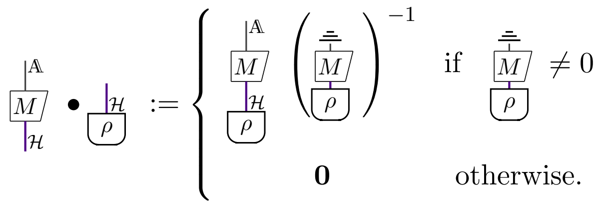
$$M \bullet \rho := M \circ \rho(\mathsf{tr}(M \circ \rho))^{-1},$$

where $\circ$ is the composition rule in **QCalc**. We denote the new process theory by **QCalc**$^\bullet$.

To obtain a consistent process theory, however, we cannot simply redefine composition for the special case where a state is composed with a CP map. Instead, we must redefine composition in general, and obtain the above as a particular instance of the new rule. Explicitly, we define $\bullet$ via:

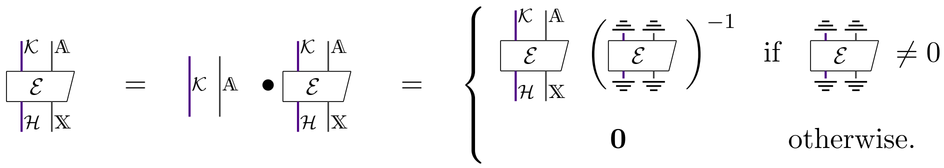


Then, as a special case, we have

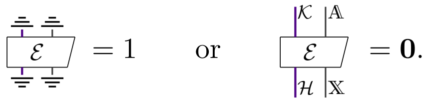


That said, the probabilities that we expect from the time neutral theories are reproduced.

For **QCalc**$^\bullet$ to define a valid process theory, various conditions must be satisfied. In particular one can show by direct computation, that $\bullet$ is associative and that it interacts suitably with parallel composition. The more interesting case, however, comes from considering the identity processes. In any process theory we have that $\mathbb{1}_B \circ f = f = f \circ \mathbb{1}_A$ for every process $f : A \to B$. However, in our case, if we try to impose this condition with • we find that



This holds only in the special cases where



(2.9)

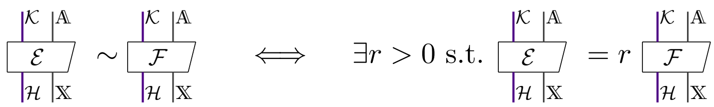
Therefore, to define the process theory **QCalc**$^\bullet$, we must both modify the composition rule *and* restrict the set of allowed processes to the above special cases.

Definition 2.15 (**QCalc**$^\bullet$)**.** The process theory **QCalc**$^\bullet$ has the same objects as **QCalc** and as processes the subset of processes in **QCalc** satisfying Eq. (2.9). Sequential composition is defined as in Eq. (2.7.1) and parallel composition is as in **QCalc**.

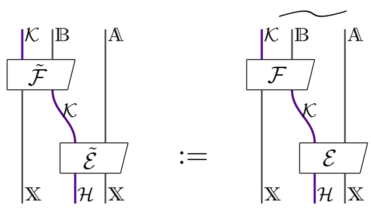
After modifying **QCalc** to obtain **QCalc**$^\bullet$, it is no longer immediately clear that time neutrality or even time symmetry of **QCalc** have been preserved. It turns out that this is the case. We return to this in the following section once we have the more elegant characterisation of this process theory.

2.7.2 Modified processes

The construction of **QCalc**$^\bullet$ of the previous section can be more elegantly captured by defining a new theory in which the processes correspond to equivalence classes of processes in **QCalc**. Specifically, we define two processes to be equivalent if they are equal up to non-zero scalar:



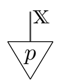
We denote the equivalence class for a process $\mathcal{E}$ as $\tilde{\mathcal{E}}$. The new process theory, **QCalc**/$\sim$, has the same systems as **QCalc** and processes that correspond to equivalence classes of processes in **QCalc** under the above equivalence relation. Composition within **QCalc**/$\sim$ is defined as



(2.10)

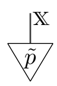
where $\mathcal{E}$ is an arbitrary element of $\tilde{\mathcal{E}}$ and $\mathcal{F}$ is an arbitrary element of $\tilde{\mathcal{F}}$.

The scalars of **QCalc**/$\sim$ are severely restricted: While scalars in **QCalc** are $\mathbb{R}^+$, in **QCalc**/$\sim$ they are equivalent to $\mathbb{Z}_2$ as we have only two equivalence classes $\{\tilde{0}, \tilde{1}\}$, where $\tilde{0} = \{0\}$ and $\tilde{1} = (0, \infty)$. This may at first glance seem problematic since we still want our time-symmetric theory to make probabilistic predictions. The resolution is to consider the probabilistic predictions as being encoded into classical states rather than in individual scalars. This is the case for instance, with the causal theory **QPhys**, which has only a single scalar. In particular, probabilities in **QPhys** are encoded in



.

We can similarly ask what predictions are made by the theory **QCalc**/$\sim$ in terms of processes with a classical output, i.e., what are the processes of the form



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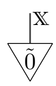
It is not hard to see that these will be in one-to-one correspondence with probability distributions with one extra classical state left over, namely, the zero-state:



This theory, therefore, makes predictions that can be interpreted probabilistically most of the time. We simply view the equivalence class containing a probability distribution p as describing the same prediction with the probability distribution itself. The exception to this interpretation is the zero state for which there is no obvious interpretation as a probability distribution. If we want the process theory to only represent physical processes, then this serves as a challenge as it is difficult to give a physical interpretation of a zero-process. In a nutshell, the existence of the zero state means that the process theory is not deterministic. We return to this shortly. Before doing so, we show that this theory is time-symmetric, and moreover time-neutral:

Proposition 2.16. *The Hermitian adjoint* $\dagger_H$ *is a dagger for* **QCalc**$/\sim$.

Proof. It is easy to see that $\dagger_H$ preserves equivalence classes as $\dagger_H(r\mathcal{F}) = r\,\dagger_H(\mathcal{F})$ for all processes $\mathcal{F}$ and scalars r. □

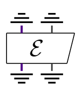
Proposition 2.17. **QCalc**$/\sim$ *is time-neutral.*

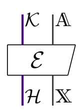
Proof. The equivalence class containing the cup and the equivalence class containing the cap will define cups and caps for **QCalc**$/\sim$. These satisfy the snake equations, which immediately follows from the definition of composition of equivalence classes in Eq. (2.10). □

2.7.3 Comparing QCalc/∼ and QCalc$^\bullet$

In this section, we compare the process theory **QCalc**$/\sim$ with the process theory **QCalc**$^\bullet$ and we show that they are equivalent process theories.

To begin, note that any process in **QCalc** is in the same equivalence class as one satisfying one of the conditions in Eq. (2.9). Moreover, each equivalence class will contain a unique such element. The crux of this is the following property of processes in **QCalc**

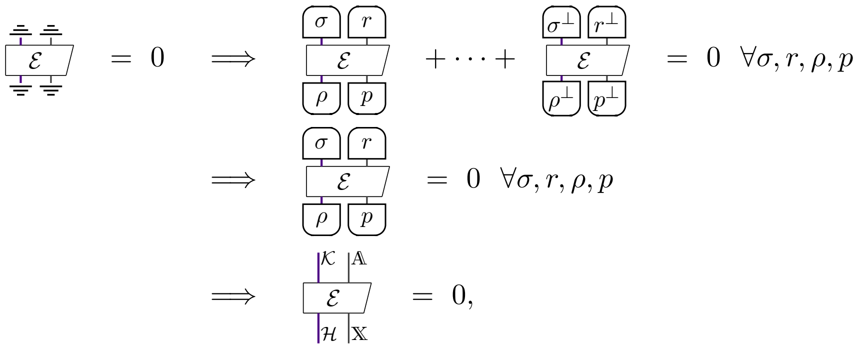


$= 0 \iff$


$= 0.$

which follows from the fact that **QCalc** is locally tomographic. We provide the relevant proof below:

Proof.



(2.11)

(2.12)

(2.13)

In the first step, we use that the maximally mixed state and the discarding effect are internal to the cones of states and effects respectively. In the second step, we use that if a sum of non-negative terms is zero then each term is zero and in the final step that **QCalc** is locally tomographic. □

The processes in **QCalc**$^\bullet$ can therefore be viewed as a particular conventional choice of representative elements for the equivalence classes. The modified composition rule can then be derived by replacing each process with its equivalence class, then composing the equivalence classes and finally picking the representative element for the equivalence class of the composite.

We, therefore, infer that

$$\mathbf{QCalc}/\!\sim \;\;\cong\;\; \mathbf{QCalc}^\bullet,$$

as **QCalc**$^\bullet$ is simply a way to describe **QCalc**/$\sim$ using representative elements of the equivalence classes. Moreover, the inelegant nature of **QCalc**$^\bullet$ can be viewed as a consequence of the somewhat arbitrary nature in which the representative elements are chosen. Picking a different convention for how to pick a representative element would lead to a distinct composition rule (and hence a different probability rule). Nevertheless, it would ultimately be describing the same process theory.

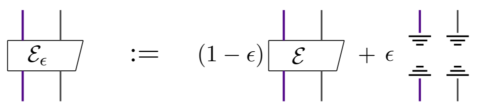
This equivalence, together with Props. 2.16 and 2.17, immediately tells us that **QCalc**$^\bullet$ is also time-symmetric and time neutral as we claimed earlier.

2.7.4 Determinism

As we have already mentioned, in **QCalc**/$\sim$ (or equivalently **QCalc**$^\bullet$) we have a problem with determinism – we have zero-processes that describe things that cannot occur. In other approaches, this issue has been handled in an arbitrary way by simply stating that when $N_M(\rho) = 0$ then $\text{Prob}(a|\rho) = 0$ for all $a \in \mathbb{A}$. However, how are we to operationally understand a measurement in which all of the possible outcomes occur with probability zero?

The problem with determinism manifests in our approach by the fact that we have a pair of scalars in **QCalc**/$\sim$ rather than the single scalar that we would expect in a deterministic theory. Furthermore, our constructions have a somewhat arbitrary nature to them: the modified composition rule in **QCalc**$^\bullet$ includes two cases depending on whether or not a zero appears. In addition, when we quotient in **QCalc**/$\sim$, we do so concerning non-zero scalars rather than arbitrary scalars. In the following, we present a solution that results in a deterministic theory and where these zero-cases naturally do not arise. This is achieved by considering a different starting point from **QCalc**.

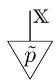
We begin by conjecturing that in any real-world experiment, we will never manage to completely suppress all sources of noise. That is, in the lab we never actually prepare a pure state or perform a projective measurement (at least not on the system of interest). In particular, we take the set of experimentally realisable quantum processes to be those of the form



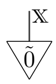
$1 > \epsilon > 0$ (2.14)

i.e, processes that have a non-zero epsilon of noise. Indeed, we can define a restriction of **QCalc**, denoted as **QCalc**$|_{noise}$, which has only processes of the form (2.14). It is straightforward to verify that this restriction defines a valid process theory as its processes are closed under composition.

QCalc$|_{noise}$ and **QCalc**, are equivalent to one another, at least from an operational point of view. There is no real-world experiment that could distinguish between a process $\mathcal{E}$ and the noisy version $\mathcal{E}_\epsilon$ provided that ϵ is small enough. (We can approximate $\mathcal{E}$ using processes $\mathcal{E}_\epsilon$ and taking the limit of $\epsilon \to 0$). In some sense then, choosing between working with **QCalc** rather than **QCalc**$|_{noise}$ is purely a matter of convenience. However, when we move over to the respective time-neutral theories, **QCalc**$/\!\sim$ and **QCalc**$|_{noise}/\!\sim$ we obtain strikingly different theories. This is highlighted by considering the scalars of the theory. In particular, the scalars in **QCalc** are $[0,\infty)$ whilst in **QCalc**$|_{noise}$ they are $(0,\infty)$. This implies that in **QCalc**$/\!\sim$ we have two scalars $\{\tilde{0},\tilde{1}\}$, in contrast with **QCalc**$|_{noise}/\!\sim$, where we have only the scalar $\tilde{1}$ (as 0 is not a scalar in **QCalc**$|_{noise}$). Thus, **QCalc**$|_{noise}/\!\sim$ is a deterministic process theory, and so every process has a valid operational interpretation. The states for a classical system are those of the form:



as the zero state

 ,

is no longer part of the theory. Therefore, every classical state can simply be thought of as a probability distribution.

Proposition 2.18. **QCalc**$|_{noise}/\!\sim$ *is time-symmetric.*

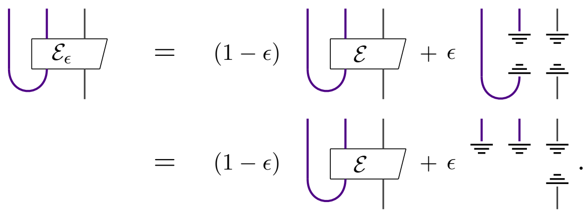
This is because it obtains a dagger from the Hermitian adjoint of **QCalc**. However, it is not time-neutral as it does not have cups & caps. It does not even possess identities or swaps. Indeed, cups & caps, identities, and swaps in **QCalc** are not processes of the form of Eq. (2.14).

Remark 2.19. As discussed above, in **QCalc**$|_{noise}$ we do not have identity processes. From a categorical point of view, this would mean that we do not have identity morphisms. Hence, this theory is a process theory but it is *not* a symmetric monoidal category, demonstrating that the framework of process theories is a more general approach to describe physical theories.

We can however freely add all of these *wiring processes* back into the theory without changing anything that we have discussed so far. We denote this process theory, constructed by appending wiring processes to $\mathbf{QCalc}|_{noise}/\!\sim$, as **QNeut**. More specifically, **QNeut** is the subtheory of $\mathbf{QCalc}/\!\sim$ generated by the noisy processes, caps, cups, swaps and identities.

Theorem 2.20. **QNeut** *is a deterministic and time-neutral process theory.*

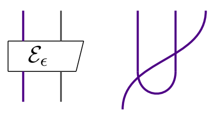
Proof. Time neutrality follows immediately from the existence of cups and caps that we just added. We note that the set of noisy processes (i.e., of the form of Eq. (2.14)) are closed under composition with wiring processes, e.g.:



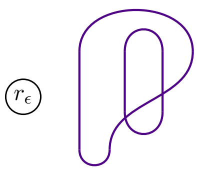
(2.15)

(2.16)

If we have some generic diagram, we can therefore always absorb these wiring processes into noisy processes unless the wiring processes are disconnected such as below:



A general scalar (i.e., closed diagram) can therefore be written as some scalar r_ϵ in $\mathbf{QCalc}|_{noise}$ composed in parallel with some closed wiring. For example:



A closed wiring as above is simply the parallel composition of closed loops and since closed loops are equal to the dimension of the system, this is a strictly positive number w. Hence, $r_\epsilon w > 0$ and therefore when we quotient we obtain the scalar $\tilde{1}$. Thus, there is a unique scalar in the theory. □

We have constructed **QNeut**, a time-neutral and at the same time deterministic version of quantum theory. Although we have focused on quantum theory, we conjecture that the philosophy of this construction is applicable even in theories beyond quantum. In particular, the key construction of quotieting by non-zero scalars makes sense for any process theory in which the scalars are $\mathbb{R}^+$. This includes arbitrary generalised probabilistic theories (GPTs) [Har01a, Bar07], operational probabilistic theories (OPTs) [CDP10, CDP11, DCP17], or categorical probabilistic theories [GS17]. We suggest that the specification of the properties that a probabilistic theory **G** needs to satisfy to be time-neutral, is a direction for future research.

Chapter 3

An operadic approach to process theories

As we have already noted in Chapters 1 and 2 process theories can be viewed as symmetric monoidal categories. (Depending on the type of process theory, the corresponding category might have an additional structure such as a dagger or cups and caps). The categorical perspective on process theories is extremely useful, as it provides a formal connection between process theories and a well-established branch of mathematical research, allowing one to use all of the examples, results, and concepts therein. However, this view is not necessarily natural for a few reasons outlined below.

- Firstly, we have to introduce a trivial system to our process theory that corresponds to the monoidal unit for the SMC. This is unnatural because there are sensible process theories in which we do not need a trivial system. One such example is the process theory of unitary transformations of sets of qubits which would be relevant for the study of the circuit model of quantum computation.

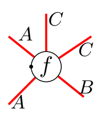
- Secondly, we have to turn certain types of wirings, such as those that correspond to the identity and swap morphisms within the SMC, into processes. This is also unnatural since we can think of a process theory in which it does not make sense to consider these as processes. The process theory of noisy quantum operations, which are the operations implemented in the lab, is a relevant example since it does not allow for the application of an identity process.

- Thirdly, as we discuss below, simple changes within the definition of a process theory do not necessarily correspond to simple changes in the respective categorical language.

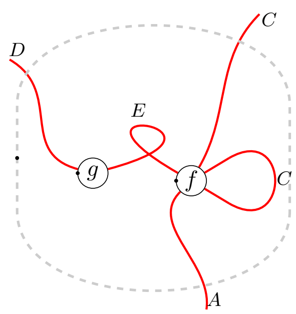
In practice, none of the first two issues serves as a great impediment to viewing process theories as SMCs. For example, in the SMC corresponding to unitary operations on qubits, we would gain a trivial object that we would never use, yet its existence does not constitute a problem. However, the third case does constitute a problem since it hinders the ability to describe process theories within a well-studied mathematical framework.

Suppose then that we want to make a fairly subtle modification to our notion of a process theory, such that processes no longer have an input-output distinction. These can be thought of as time-neutral process theories.

Definition 3.1. A time-neutral theory is defined by a collection of processes, such as



which is closed under wirings. For example,



corresponds to another process in the theory. Moreover, two diagrams are equal if they have the same connectivity.

The correspondence between process theories and SMCs no longer holds for time-neutral process theories. Integral to the notion of a morphism in a category is the specification of its domain and codomain. However, in a time neutral process theory there is no meaningful way to divide up the systems associated with a process into inputs and outputs, and thus no meaningful way to specify the domain and codomain of the morphism that we would associate with the process. One can get around this – to some extent – by working with compact closed SMCs. Nevertheless, as we will see this is not a satisfactory resolution.

The purpose of this chapter is to show that there is an alternative way to connect process theories to well-studied mathematics, which avoids all of the aforementioned problems. The crux of this is the recent work of Patterson et al., [PSV21], in which the authors present an equivalence between symmetric monoidal categories and the algebras of a particular kind of operad, namely an *acyclic wiring operad*. This equivalence, together with the correspondence between process theories and SMCs, implies that we can view process theories as certain kinds of operad algebras.

We conjecture that the operadic formalism captures more faithfully the notion of a process theory and thus can be considered more fundamental. Furthermore, we argue that it is more flexible when it comes to defining new kinds of process theories such as time-neutral process theories.

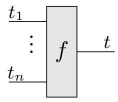
3.1 Operad basics

In this section, we provide a brief introduction to the operadic language using an intuitive graphical representation. We then discuss how process theories can be re-expressed in this language and showcase the advantages of this perspective.

Operads, much like categories, consist of a collection of objects, a collection of morphisms, and a means of morphism composition obeying the relevant associativity and identity laws. They differ from categories in the sense that their morphisms can have multiple inputs instead of a single input.

Definition 3.2. An operad $\mathcal{O}$ consists of:

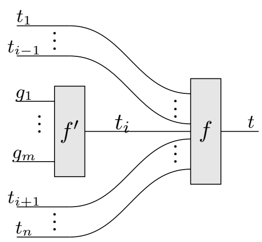
- A collection $|\mathcal{O}|$ of objects, t.
- A set of operations $\mathcal{O}(t_1, ..., t_n; t)$, for each $(t_1, .., t_n; t)$. We think of an operation $f \in \mathcal{O}(t_1, ..., t_n; t)$ as receiving the tuple of objects $(t_1, ..., t_n)$ as an input and providing the object t as an output. We diagrammatically represent this as:



- A composition function

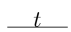
$$\circ_i : \mathcal{O}(g_1, ..., g_m; t_i) \times \mathcal{O}(t_1, ..., t_n; t) \to \mathcal{O}(t_1, ..., t_{i-1}, g_1, ..., g_m, t_{i+1,...,t_n;t})$$

called *substitution*. For instance, for $f \in \mathcal{O}(t_1, ..., t_n; t)$ and $f' \in \mathcal{O}(g_1, ..., g_m; t_i)$ substitution is the following diagram:



.

- An identity operation $id_t \in \mathcal{O}(t; t)$ for each $t \in |\mathcal{O}|$, which is represented as a single wire:

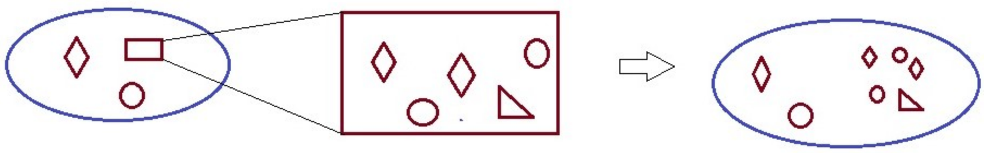


.

The composition functions are associative with units the identity operations.

For this work, we will follow the interpretation of operads of Ref. [FS18] as being abstract theories of composition. That is, the operations are thought of as describing different ways to compose small things into bigger things: Each operation in $\mathcal{O}(t_1, ..., t_n; t)$ tells us a way in which the objects $t_1, ..., t_n$ can be combined into an object t. There may well be times in which there is a unique way that they can be combined, i.e. $\mathcal{O}(t_1, ..., t_n; t)$ would be a singleton set, and there may well be times in which there is no way that they can be combined, i.e. $\mathcal{O}(t_1, ..., t_n; t)$ would be the empty set.

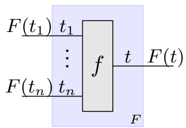
From this perspective, operation composition is very natural: Consider the operation g in $\mathcal{O}(g_1, ..., g_m; t_i)$, i.e. g takes m objects as inputs and gives the object t_i as an output. Suppose that we plug its output t_i to the i_{th} input of an operation f in $\mathcal{O}(t_1, ..., t_n; t)$. The composition of g and f results in an operation $f \circ_i g$ in $\mathcal{O}(t_1, ..., t_{i-1}, g_1, ...g_m, t_{i+1} ..., t_n; t)$, the substitution function, that provides us with the object t. Below we provide an intuitive picture of the way composition works. The output t_i of an operation g is the big red square, whereas the i_{th} input of an operation f is the small red square in the blue circle. The substitution function behaves then as advertised, providing the object t, i.e. the blue circle in the right hand side.



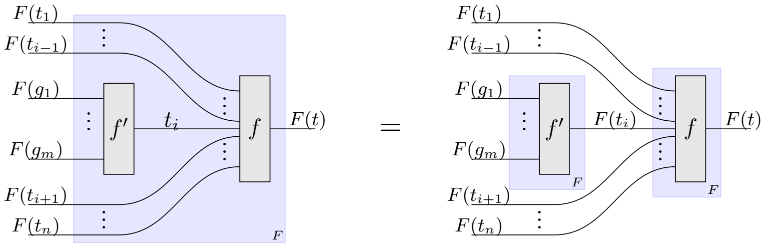
An example of operads relevant to this work is the class of operads $\mathcal{O}_\mathcal{C}$ that follow from symmetric monoidal categories $\mathcal{C}$. In particular, any symmetric monoidal category $\mathcal{C}$ defines an operad $\mathcal{O}_\mathcal{C}$ by restricting $\mathcal{C}$ to having only morphisms with a single output. That said, morphisms $C_1 \otimes C_2 \otimes ... \otimes C_n \to D$ in $\mathcal{C}$ can be viewed as operad operations $\mathcal{O}_\mathcal{C}(C_1, ..., C_n; D)$. Representative examples are the operad Set which has sets as objects and functions from the cartesian product of the input sets to the output set as operad operations as well as the operad Vect_K which has vector spaces as objects and linear maps from the tensor product of the input spaces to the output space as operations.

Given two operads $\mathcal{O}$ and $\mathcal{O}'$, we can define a functor between these.

Definition 3.3. An operad functor $F : \mathcal{O} \to \mathcal{O}'$ maps the objects of $t \in |\mathcal{O}|$ to the objects of $F(t) \in |\mathcal{O}'|$ and the operad operations $o \in \mathcal{O}(t_1, ..., t_n; t)$ to $F(o) \in \mathcal{O}'(F(t_1), ..., F(t_n); F(t))$ such that composition and identities are preserved. We diagrammatically denote these morphisms by shaded regions:

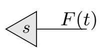


The condition that the functor preserves the operadic composition translates diagrammatically as:



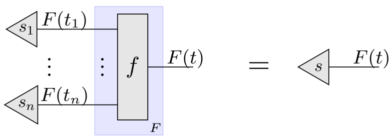
Definition 3.4. An operad algebra for some operad $\mathcal{O}$ is an operad functor $F : \mathcal{O} \to \mathrm{Set}$.

Conceptually, again following Ref [FS18], the functor F provides a concrete instantiation of the abstract notion of composition provided by $\mathcal{O}$. That is, to each object $t \in \mathcal{O}$ there is some associated set $F(t)$, which we think of as describing the set of possible ways that t can be formed. Then, to each operad operation $f \in \mathcal{O}(t_1, ..., t_n; t)$ telling us how $t_1, ..., t_n$ are to be combined to make t, F assigns a function $F(t_1) \times ... \times F(t_n) \to F(t)$. This function indicates for each way that $t_1, .., t_n$ can be to make t, how the output t will be. Diagrammatically, we represent the elements of the set $F(t)$ (i.e. the possible ways that t can be) as:

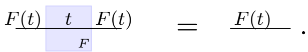


.

Then, we can represent the action of the function on the elements of the sets via



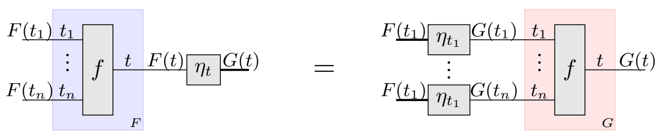
where $s = F(f)(s_1 \times \cdots \times s_n)$. Functoriality of F ensures that this is a sensible interpretation. For example, in the case of the "do nothing" operation, which is represented by the identity function, we have that:



Realizing operad algebras as concrete instantiations of abstract theories of composition will be central to this chapter. In particular, this perspective places process theories in a new light since they will be viewed as concrete instantiations of an abstract theory of how boxes can be wired together. More specifically, the abstract theory will have the form of a particular kind of operad, a 'wiring operad', which when varying its precise definition will lead to different kinds of operad algebras and consequently process theories.

We end this section by defining natural transformations between operad functors.

Definition 3.5. A natural transformation η : $F : \mathcal{O} \to \mathcal{O}'$ $\implies$ $G : \mathcal{O} \to \mathcal{O}'$ can be defined as a collection of operad operations in $\mathcal{O}'$ indexed by the objects in $\mathcal{O}$. We denote these as η_t and require that they satisfy:



for all operations $f \in \mathcal{O}$.

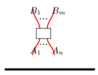
These will be useful later on for relating the operad algebras of a wiring operad with the operad algebras of another.

3.2 Wiring Operads and Process Theories

We begin by defining the wiring operad which captures the notion of composition in standard process theories.

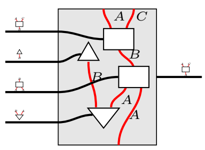
Definition 3.6. An acyclic wiring operad $\mathfrak{W}^A$ consists of:

- Boxes as objects. For example:



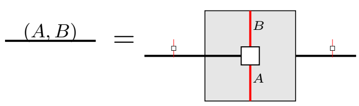
  These can be expressed symbolically as a pair of ordered lists $((A_1, .., A_n), (B_1, ..., B_m))$, where $(A_1, ..., A_n)$ are the inputs and $(B_1, ..., B_m)$ are the outputs.

- Acyclic wirings of boxes as operations. For example, the wiring

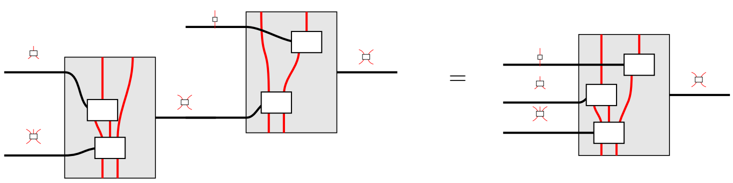


  takes four boxes as input and wires them together to form a new box as an output, namely the box $((A), (A, C))$.

- The identity operation for every box, which is the trivial wiring. For instance, the identity for a box (A, B) is given by:



- Operation composition, which is given by diagram substitution. For example,



  That said, the wiring diagram on the left is substituted into the bottom box of the wiring diagram on the right. In this way diagram substitution provides us with a more detailed wiring diagram than the one we had in the beginning.

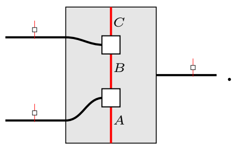
- Finally, there is a special kind of box, the trivial box symbolized as ‘( )’. If we compose any box (A, B) with the trivial box we are left with the same box (A, B).

The acyclic wiring operad $\mathfrak{W}^A$ allows us to study the properties of wirings of abstract boxes. However, typically we are not interested in abstract boxes, but an actual realization of those. In particular, we are concerned with boxes that describe physical processes. The corresponding algebras $F : \mathfrak{W}^A \to \mathrm{Set}$, implement exactly that: They turn the abstract boxes and wirings of $\mathfrak{W}^A$ into actual processes and their composition. More specifically, it was shown in Ref. [PSV21]that there is an equivalence between operad algebras for $\mathfrak{W}^A$ and symmetric monoidal categories. In the following we provide an intuition for this result, aided by the diagrammatic notation that we have set up.

To present the connection between operad algebras and process theories, it is useful to mention two particular wirings, that is sequential and parallel wiring. The former is the operation

$$\mathsf{seq}_{A,B,C} : (A, B), (B, C) \to (A, C)$$

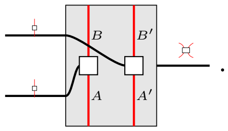
that takes boxes (A, B) and (B, C) as inputs and produces the box (A, C) as an output:



The latter, is the operation

$$\mathsf{par}_{A,A',B,B'} : (A, B), (A', B') \to ((A, A'), (B, B'))$$

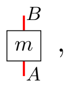
that takes boxes (A, B) and (A', B') as inputs and produces the box $((A, A'), (B, B'))$ as an output:



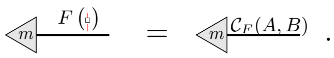
We are now in position to realize how $F : \mathfrak{W}^A \to \mathsf{Set}$ encodes all the information that constitutes a symmetric monoidal category (SMC) $\mathcal{C}_F$: on boxes (A, B), F assigns a set $F(A, B)$ which we will take to be the homset of $\mathcal{C}_F$ with objects $A, B, \ldots$ That is,

$$\mathcal{C}_F(A, B) := F(A, B).$$

More specifically, the morphisms in $\mathcal{C}_F$, which would process theoretically be denoted by

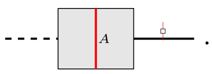


correspond to the elements of $F(A, B)$ that have the following operadic representation:



The information about various elements of $\mathcal{C}_F$, i.e. identity, composition, tensor product, trivial system and symmetry will be extracted as we apply F to suitable wiring diagrams of $\mathfrak{W}^A$.

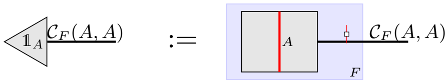
To obtain the identity in $\mathcal{C}_F$, we apply F to the wiring diagram $\mathsf{u}_A : () \to (A, A)$ in $\mathfrak{W}^A$, which is represented as



That said, it is an operad operation with the trivial box as input, represented with a dashed edge, and a box (A, A) as output.

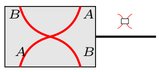
Remark 3.7. In what follows, we omit the dashed edge whenever the trivial box is an input or output following the convention adopted in process theories regarding trivial systems.

The functor F associates the trivial box in $\mathfrak{W}^A$, i.e. '()', to the trivial object in Set, i.e. the singleton set, '$\star = \{*\}$'. The wiring diagram u_A is then mapped to the function $F(\mathsf{u}_A) : \star \to F(A, A) := \mathcal{C}_F(A, A)$:

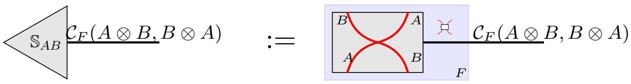


The function $F(\mathsf{u}_A)$ picks out a particular element of $\mathcal{C}_F(A, A)$, which we take to be the identity morphism for A.

Similarly, to obtain the symmetry morphism in $\mathcal{C}_F$, we apply F to the wiring diagram $\mathsf{s}_{A,B} : () \to ((A, B), (B, A))$ in $\mathfrak{W}^A$, which is represented as



That said, the wiring diagram $\mathsf{s}_{A,B}$ is mapped to the function $F(\mathsf{s}_{A,B}) : \star \to F((A, B), (B, A)) := \mathcal{C}_F(A \otimes B, B \otimes A)$:

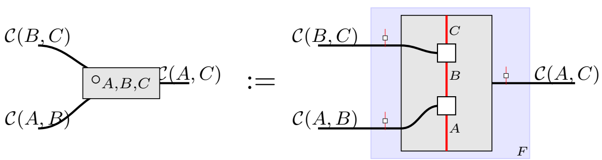


$F(\mathsf{s}_{A,B})$ then, picks out a particular element of $\mathcal{C}_F(A \otimes B, B \otimes A)$, which we will take to be the symmetry morphism.

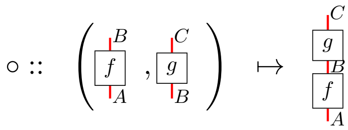
Consider now the objects A, B, C of $\mathcal{C}_F$. Sequential composition in $\mathcal{C}_F$ can be associated with a collection of functions $\circ_{A,B,C} : \mathcal{C}_F(A, B) \times \mathcal{C}_F(B, C) \to \mathcal{C}_F(A, C)$. These can be obtained from the wiring operad algebra by applying F to the wiring diagram $\mathsf{seq}_{ABC} : (A, B), (B, C) \to (A, C)$ in $\mathfrak{W}^A$, i.e.

$$\circ_{A,B,C} := F(\mathsf{seq}_{ABC}) = F(A, B) \times F(B, C) \to F(A, C)$$

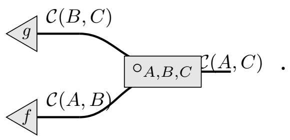
or diagrammatically:



Process theoretically we would denote the sequential composition by:



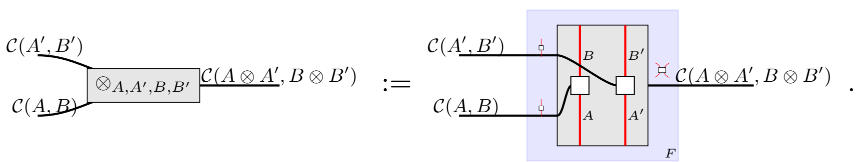
This is expressed in our operadic language as:



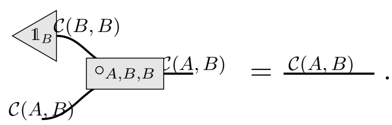
Similarly, parallel composition in $\mathcal{C}_F$ can be associated with a collection of functions $\otimes_{A,A',B,B'} : \mathcal{C}_F(A, B) \times \mathcal{C}_F(A', B') \to \mathcal{C}_F(A \otimes A', B \otimes B')$ which are obtained by applying F to the wiring diagram $\mathsf{par}_{AA'BB'} : (A, B), (A', B') \to ((A, A'), (B, B'))$ in $\mathfrak{W}^A$ for objects A, A', B, B'. That said,

$$\otimes_{A,A',B,B'} := F(\mathsf{par}_{AA'BB'}) = F(A, B) \times F(A', B') \to F(A \otimes A', B \otimes B')$$

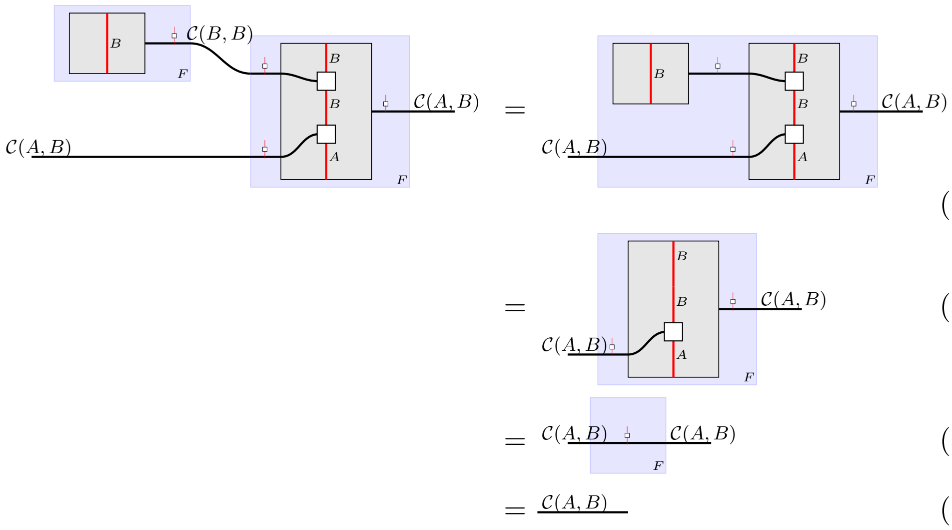
or diagrammatically:



Finally, we need to demonstrate that these definitions satisfy all of the axioms of a symmetric monoidal category. We will not provide complete proof here but instead, indicate an illustrative example with the aid of our diagrammatic representation. Specifically, we show unitality of the identity morphisms, that is the following condition:



The proof of this is straightforward. The left hand side is given by:



(3.1)

(3.2)

(3.3)

(3.4)

The first equality is given by functoriality of F, the second by the definition of composition in $\mathfrak{W}^A$, the third from the definition of the identity in $\mathfrak{W}^A$, and the fourth again from functoriality of F.

The proof technique to show the other conditions is practically identical. It follows from straightforward applications of the definitions that we have set up, namely functoriality of F and composition in the wiring operad.

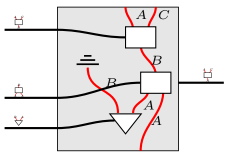
We have so far established that the algebra $F : \mathfrak{W}^A \to \mathrm{Set}$ gives rise to a SMC $\mathcal{C}_F$. The reverse also holds, i.e. given a SMC $\mathcal{C}$, we can define the algebra $F_{\mathcal{C}} : \mathfrak{W}^A \to \mathrm{Set}$ by determining the action of $F_{\mathcal{C}}$ on boxes and wiring diagrams of $\mathfrak{W}^A$. In particular, homsets $\mathcal{C}(\otimes_i A_i, \otimes_j B_j)$ are assigned to homsets $\mathfrak{W}^A((A_1, ..A_i), (B_1, ..B_j))$ of Set, that is to the action of $F_{\mathcal{C}}$ on boxes of $\mathfrak{W}^A$. Sequential composition in $\mathcal{C}$, i.e. a function of the form $\circ_{A,B,C} : \mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$, is assigned to sequential composition in Set, i.e. the function $\mathfrak{W}^A(A,B) \times \mathfrak{W}^A(B,C) \to \mathfrak{W}^A(A,C)$, which determines the action of $F_{\mathcal{C}}$ on sequential composition $\mathsf{seq}_{A,B,C} : (A,B),(B,C) \to (A,C)$ in $\mathfrak{W}^A$. The case for parallel composition and symmetry follows similarly.

Note, however, an important subtlety that we have so far glossed over. That is, there is not a unique wiring operad but one for each possible choice of labels for the inputs and outputs to boxes. The wiring operad that we get from a particular SMC will therefore be the one in which the box labels correspond to the objects of the SMC. For more details on this see Ref. [PSV21].

3.3 Causality

In this section, we ask ourselves how we should modify $\mathfrak{W}^A$, so that the corresponding operad algebra will necessarily give rise to a causal SMC $\mathcal{C}$. [1]. For this purpose, we define a new kind of wiring operad, namely a *causal wiring operad* $\mathfrak{W}^{\stackrel{=}{\top}}$ by relaxing the constraints on the wirings in $\mathfrak{W}^A$, and imposing conditions that the new wirings must satisfy.

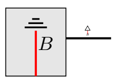
To understand the first step, note that in $\mathfrak{W}^A$ every wire must begin on some box which could be an input box or the output box. Similarly, every wire must end on some box, which again could be an input box or the output box. To incorporate causality we break this symmetry by allowing for wires which do not end on any box, as in the following case:



(3.5)

We use the ground symbol to indicate the termination of a wire that does not end on any box. We think of this as *discarding* the system B.

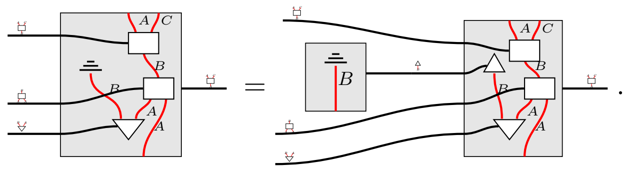
The wiring diagrams in $\mathfrak{W}^{\stackrel{=}{\top}}$ can be created from the same wirings as $\mathfrak{W}^A$, with an additional wiring diagram, $\stackrel{=}{\top}_B : () \to (B, ())$, depicted as:



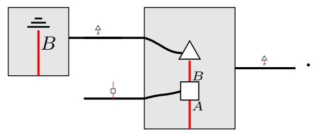
.

[1]That is, one in which the monoidal unit is terminal

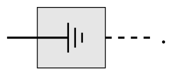
For example, we can construct Eq. (3.5) via:



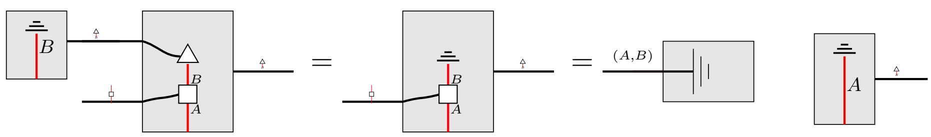
For $\bar{\top}_B$ to fully capture the notion of causality, we must impose an extra condition: If we 'discard' the output of a process, we may as well have directly discarded the input. An instantiation of this condition with diagrams is below:



Nonetheless, to be able to define a causal SMC, we must introduce a way to also discard the objects of $\mathfrak{W}^{\bar{\top}}$. Since these are boxes, the analog of the process theoretic discarding effect in $\mathfrak{W}^{\bar{\top}}$ should accept a box as an input and give the trivial box as an output. That said, it should be a map of the form $_{(A,B)}\dashv\!\Vdash$: $(A, B) \to ()$. We call the diagram '$\dashv\!\Vdash$' *operadic discarding* and we represent it diagrammatically as

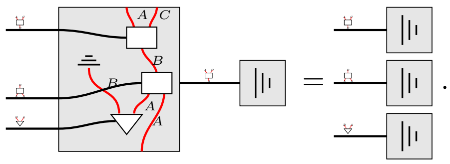


The dashed edge indicates that the output of the operadic discarding is the trivial system. As we have already noted, we omit the dashed edge by the process-theoretic convention regarding trivial systems. Having said that, the causality condition takes the following form:


(3.6)

The condition (3.6) manifests the interplay between operadic and physical discarding operations in a way that is analogous to the ignorability condition of Ref. [SSS20, Eq. 96]. In that case, the interplay is between ignoring causal and inferential systems. More generally, it states that if we discard the output of an operation, then the nature of the operation is irrelevant and we may as well have directly discarded the input.

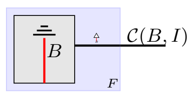
The operadic discarding must satisfy one further condition. That is, if we are not interested in the output box from some wiring, then we should be equivalently disinterested in the input boxes. For example:



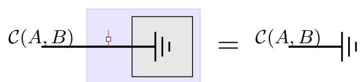
$$ \tag{3.7} $$

Definition 3.8. We call a wiring operad $\mathfrak{W}^A$ endowed with a discarding diagram and an operadic discarding operation such that conditions (3.6) and (3.7) are satisfied, a causal wiring operad, or in short $\mathfrak{W}^{\stackrel{-}{\top}}$.

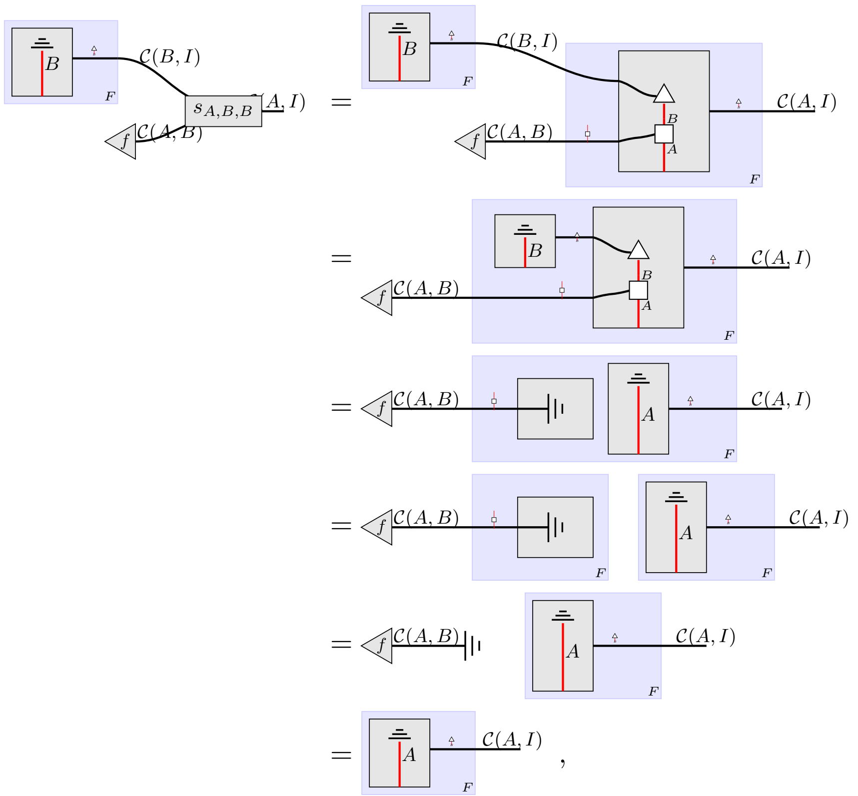
We are now in a position to show that an algebra of a causal wiring operad, $F : \mathfrak{W}^{\stackrel{-}{\top}} \to \mathsf{Set}$ gives rise to a causal SMC $\mathcal{C}_F$. To begin, we can construct the SMC in exactly the same way as for the wiring operad $\mathfrak{W}^A$. However, we now need to specify additional data for the SMC by considering the action of F on the discarding diagram $\stackrel{-}{\top}_B : () \to (B, ())$. This gives a function $F(\stackrel{-}{\top}_B) : \star \to F(B, ()) = \mathcal{C}(B, I)$. We interpret its image to be the discarding effect in $\mathcal{C}_F$, since $F(\stackrel{-}{\top}_B)$ picks out a morphism in the homset $\mathcal{C}(B, I)$. Diagrammatically it is depicted as:



Moreover, the action of F on the operadic discarding '$_{(A,B)}\!\dashv\!\Vdash$' is $F(_{(A,B)}\!\dashv\!\Vdash) : F(A, B) \to \star$. Note that for any set X, there is a unique function to the singleton set $\star$. Hence, the action of F on operadic discarding is uniquely fixed. These unique functions act as operadic discarding maps for the operad Set. That said,



Below we prove that the resulting process theory is causal, i.e. if we compose the output of any process with the discarding effect then we obtain the discading effect on the input. Indeed, we have that



(3.8)

(3.9)

(3.10)

(3.11)

(3.12)

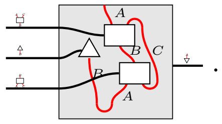
(3.13)

for all morphisms $f \in \mathcal{C}(A,B)$.

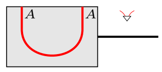
To sum up, we have extended the wiring operad $\mathfrak{W}^A$ to allow for wirings in which systems do not end on a box, and thus be discarded. We additionally introduced a new type of discarding, operadic discarding, and ensured that it interacts nicely with the discarding diagram. Consequently, the operad $\mathfrak{W}^{\bar{\top}}$ gives rise to a causal process theory or SMC $\mathcal{C}$, which is the kind of process theory we typically use to describe physical theories.

3.4 Cups and caps

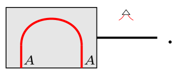
In this section, we extend $\mathfrak{W}^A$ with wiring diagrams in which inputs and outputs can be freely connected. This is in contrast with $\mathfrak{W}^A$, where we can only connect outputs to inputs in an acyclic way. For example, we now permit wiring diagrams such as:



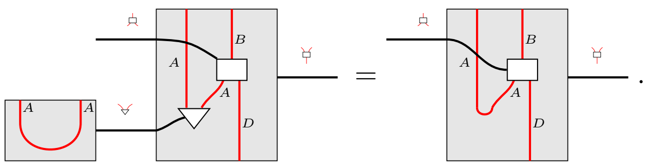
We can construct any such wiring from the wirings in $\mathfrak{W}^A$ together with two additional core wirings: That is, the wiring diagram $\mathsf{cup} : () \to ((), (A, A))$, which is diagrammatically represented as



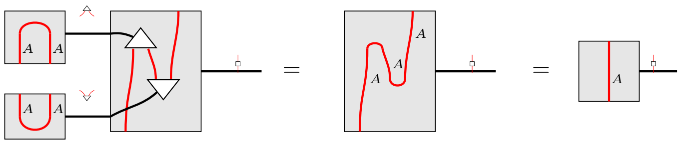
and the wiring diagram $\mathsf{cap} : () \to ((A, A), ())$,which is diagrammatically represented as



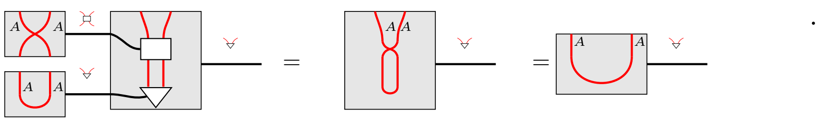
The diagrams cup and cap allow us to freely interchange inputs with outputs of boxes. For instance,



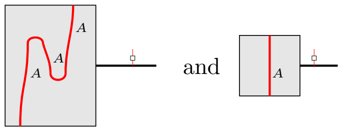
i.e. the input A of the box $((A, D), B)$ is turned into output by applying a cup. The diagrams cup and cap satisfy the following conditions, called *closure* and *commutativity* respectively:



and



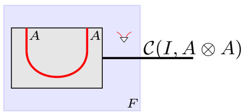
The commutativity condition for the cap follows similarly. Note that closure and commutativity are not conditions that we impose ourselves. They simply follow from the definition of the operad. The first equality in both is due to the definition of composition as diagram substitution, and the second equality is a tautology. For example,



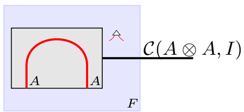
describe the same wiring. The reason for picking out these conditions, however, will be clear when we consider the operad algebras for this operad.

Definition 3.9. A wiring operad endowed with cups and caps satisfying the closure and commutativity conditions will be called a cyclic wiring operad, denoted as $\mathfrak{W}^C$.

The algebras of the cyclic wiring operad $F : \mathfrak{W}^C \to \mathrm{Set}$ correspond to compact closed categories. That said, in addition to the data specified by $\mathfrak{W}^A$, the functor F picks out special elements of the homsets $\mathcal{C}(I, A \otimes A)$ and $\mathcal{C}(A \otimes A, I)$. This is achieved via the functions $F(\mathsf{cup}) : \star \to F((), (A, A))$ which is diagrammatically drawn as

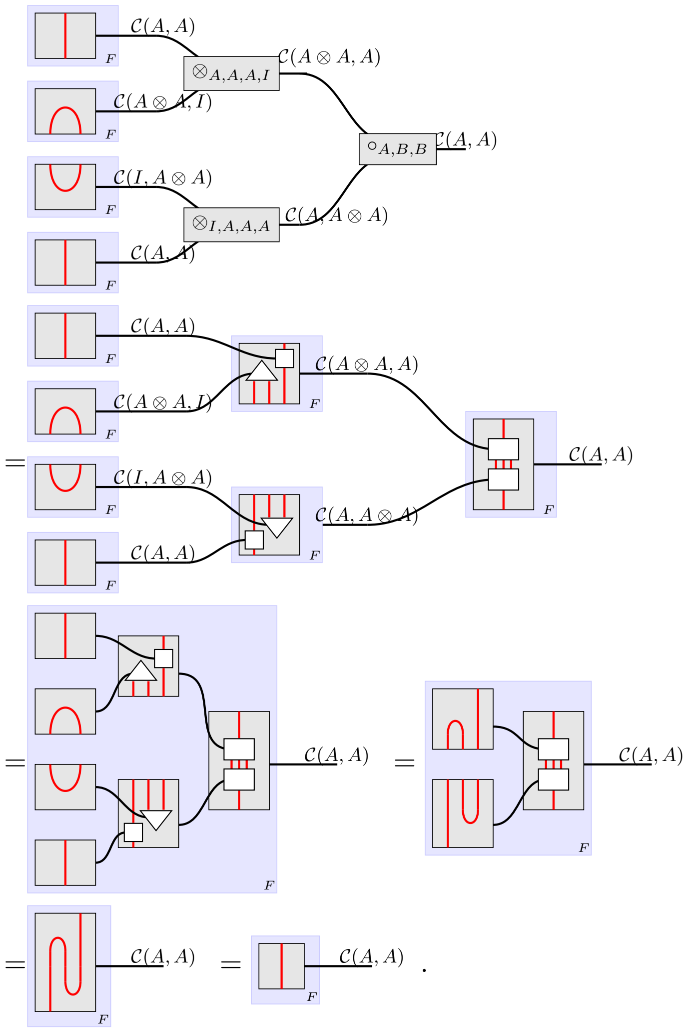


and $F(\mathsf{cap}) : \star \to F((A, A), ())$ which is diagrammatically drawn as



.

These special morphisms in $\mathcal{C}(I, A \otimes A)$ and $\mathcal{C}(A \otimes A, I)$ can easily be shown to satisfy the conditions required to define a compact closed symmetric monoidal category. For example:



(3.14)

(3.15)

(3.16)

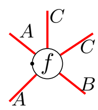
(3.17)

One may be tempted to ask whether we can define a wiring operad that allows for both discarding as well as cups and caps. This, however, quickly runs into difficulties since the cap must be discarding, and thus the resulting theory trivializes. In this sense, these two extensions of $\mathfrak{W}^A$ are incompatible with one another.

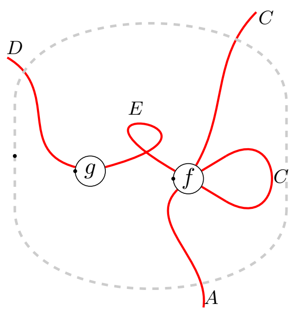
3.5 The wiring operad of dots and time neutral process theories

In the previous section, we demonstrated how an operad algebra for the cyclic wiring operad $\mathfrak{W}^C$ corresponds to a process theory with compact structure. As discussed in Ref. [SSCng], however, these are not truly time neutral theories as we still have a distinction between input and output systems. To describe theories that do not have this distinction we must revisit the basics of the definition of the wiring operad. In this section, we demonstrate how to eliminate the distinction between inputs and outputs of the operad operations and how the associated operad algebras can be thought of as entirely time-neutral process theories. We moreover establish their connection with process theories that possess compact structure.

Intuitively, a time-neutral theory is defined by a collection of processes, such as



which is closed under wirings. For example,

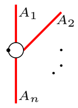


corresponds to another process in the theory. Moreover, two diagrams are equal if they have the same connectivity. Those processes, represented as circles, come with an associated list of systems such that there is no separation of this list into input and output systems – all systems are on a completely equal footing.

We will now see how to define an operad that faithfully captures this intuition by switching from a wiring operad in which objects are 'boxes' to a wiring operad in which objects are 'circles'.

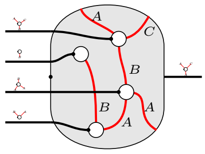
Definition 3.10. The wiring operad of dots, $\mathfrak{W}^D$ consists of

- dots as objects, expressed symbolically as lists $(A_1, ...A_n)$. They are represented as



where the black dot on the L.H.S of the circle separates the start from the end of the list.

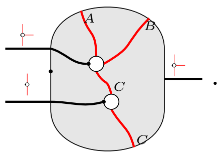
- wiring diagrams as morphisms. For example:



Along these lines we define the analogue of sequential composition. For example, the composition of the third element of (A, B, C) with the first element of (C, C) is

$$
{}_3\mathsf{wire}_1 : (A, B, C), (C, C) \to (A, B, C) \tag{3.18}
$$

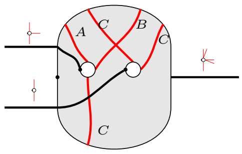
and has a corresponding diagrammatic representation as



Similarly, we define the analogue of parallel composition of the objects (A, B, C) and (C, C) as

$$
\mathsf{par} : (A, B, C), (C, C) \to (A, C, B, C, C) \tag{3.19}
$$

The systems in the codomain of par are "paired up" according to their planar orientation, i.e. the first system of the first dot is paired up with the first system of the second dot and so on. This can be diagrammatically represented as



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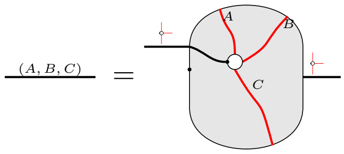
- an identity morphism that accepts a dot as an input, for example (A, B, C) and provides the same dot as an output. It is represented as



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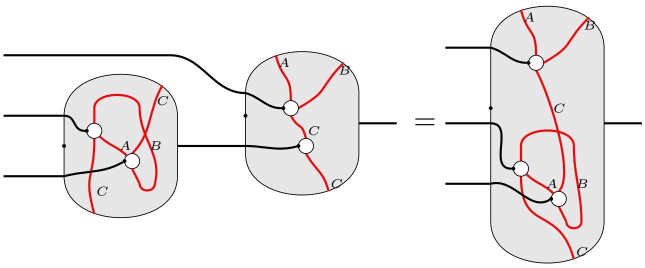
- a means of composition, which is diagram substitution. For instance,



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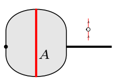
In analogy with the wiring operad $\mathfrak{W}^A$, there are diagrams in $\mathfrak{W}^D$ that remind us of core operations in process theories. A relevant paradigm is the single wire $u_A : () \to (A, A)$ represented as



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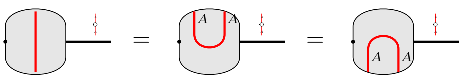
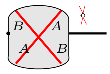
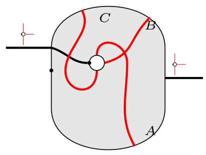
It can also be drawn in other equivalent ways which are reminiscent of the cups and caps in the cyclic wiring operad $\mathfrak{W}^C$:



The above equality is an echo of time neutrality and constitutes a hint to the connection with compact closed categories. Another example, is the symmetry operation $s_{tn} : () \to ((A,B),(B,A))$ represented as



Moreover, there are also wirings, such as $\pi : (A,B,C) \to (C,B,A)$, that simply permute the ordering of the systems:



.

A time neutral theory can be formally defined as an algebra for $\mathfrak{W}^D$, i.e. a functor $G : \mathfrak{W}^D \to \text{Set}$. On circles (A,B,C), G assigns a set, which we interpret as the set of time neutral processes with systems A, B, C denoted as N_{ABC}. On wiring diagrams, F assigns composition functions. For instance, the morphism ${}_3\mathsf{wire}_1 : (A,B,C),(C,C) \to (A,B,C)$ corresponding to sequential composition in $\mathfrak{W}^D$ is assigned to the function

$$G({}_3\mathsf{wire}_1) : N_{ABC} \times N_{CC} \to N_{ABC} \tag{3.20}$$

which wires the time neutral processes together over C. The parallel composition follows similarly from the function

$$G(\mathsf{par}) : N_{ABC} \times N_{CC} \to N_{ACBCC}.$$

To the diagram $u_A : () \to (A,A)$ in $\mathfrak{W}^D$, G maps the function $G(u_A) : \star \to N_{AA}$ from the trivial system in Set, to the set of time neutral processes that involve two copies of the same system A. This can be thought of as picking out an "identity" process from N_{AA}. Functorialiy of G and the definition of the wiring operad $\mathfrak{W}^D$ ensure that all of the composition functions will interact in exactly the way we would expect.

3.6 Relating time neutral theories and compact closed SMCs

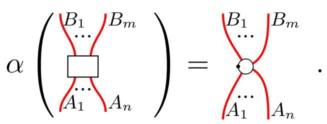
In this section we demonstrate a connection between time neutral process theories and compact closed SMCs (viewed as algebras of $\mathfrak{WD}^C$), to acquire more intuition about time neutral process theories.

More specifically, we define a pair of operad functors α and β which allows us to turn algebras for $\mathfrak{WD}^D$ into algebras for $\mathfrak{WD}^C$ and vice versa:

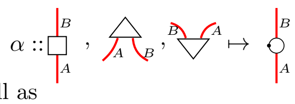
$$\begin{tikzcd}
\mathfrak{WD}^C \arrow[r, bend left, "\alpha"] \arrow[d, "F"'] & \mathfrak{WD}^D \arrow[l, bend left, "\beta"] \arrow[d, "G"] \\
\mathsf{Set} \arrow[r, phantom, "="] & \mathsf{Set}
\end{tikzcd}$$

In particular, the functor α allows us to map a time neutral theory G to an associated process theory with compact structure $F = G \circ \alpha$, whereas the functor β allows us to map a process theory with compact structure F to a time neutral theory $G = F \circ \beta$.

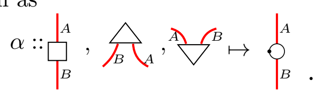
To begin we define the functor $\alpha : \mathfrak{WD}^C \to \mathfrak{WD}^D$. Its action on objects is given by:



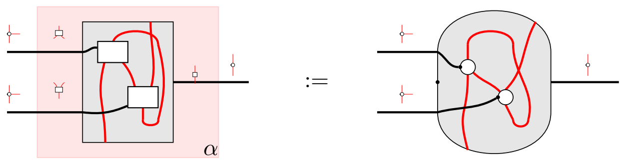
Note that this is not an injective mapping as it forgets the distinction between inputs and outputs. However, it preserves the planar orientation of systems. For example



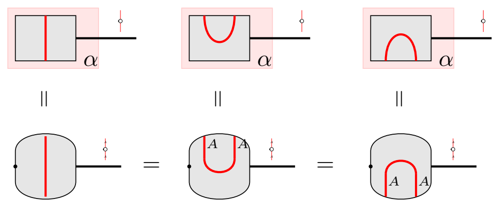
as well as



The action of α on the wirings of $\mathfrak{WD}^C$ simply gives the wiring in $\mathfrak{WD}^D$ which has the same connectivity. For example:

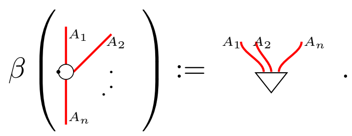


This implies that α acts on cups, caps, and identity wirings in $\mathfrak{W}^C$, as follows:

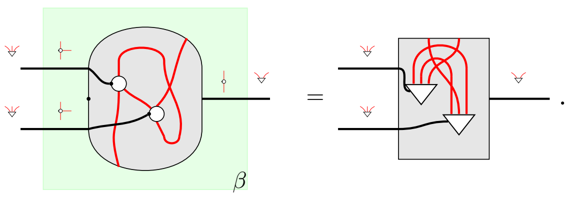


The above equality indicates that cups and caps are simply redundant once time neutrality is present.

The functor β is not quite so obvious to define. Moving from $\mathfrak{W}^C$ to $\mathfrak{W}^D$ amounted to forgetting structure, that is, forgetting the input-output distinction. To go back from $\mathfrak{W}^D$ to $\mathfrak{W}^C$ we are therefore forced to artificially reintroduce this distinction. Indeed, there is a degree of arbitrariness as to how we should approach the problem. Here we will work with the convention that all of the systems are assigned to be outputs. That is, the action of β on objects is:



The action of β then maps wirings of dots to wirings of states while preserving the connectivity. For example,

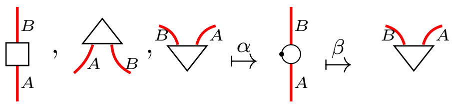


Given functors α and β, we can construct $\mathfrak{W}^C$ operad algebras from $\mathfrak{W}^D$ operad algebras and vice versa. We now ask ourselves if we end up with the same operad algebra starting from $\mathfrak{W}^C$(resp.$\mathfrak{W}^D$) to $\mathfrak{W}^D$ (resp. $\mathfrak{W}^C$) and going back again. In other words, we want to understand the two possible ways the functors α and β compose. The first possibility, namely the composite $\alpha \circ \beta$ is straightforward. Specifically, we argue that

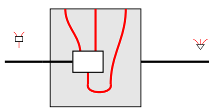
$$\alpha \circ \beta = I_{\mathfrak{W}^D}.$$

This is not surprising since β artificially adds in extra structure and then α forgets about it. This implies that if we map some operad algebra for $\mathfrak{W}^C$ to an algebra for $\mathfrak{W}^D$ and back again, we end up with the same operad algebra that we started with.

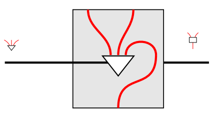
However,the second composite, $\beta \circ \alpha$ is not so simple, since $\beta \circ \alpha \neq I_{\mathfrak{W}^C}$. The composite $\beta \circ \alpha$ it is not even injective on objects. For instance,



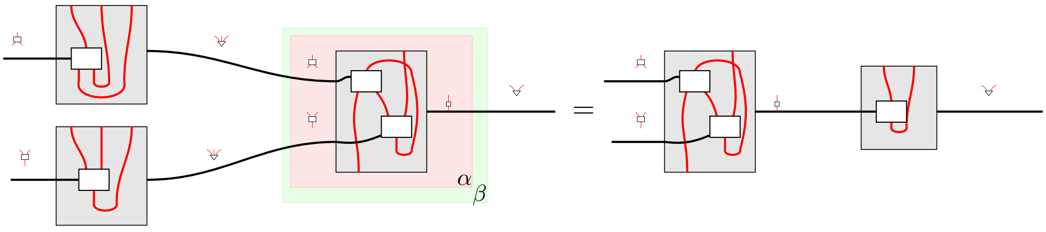
However, we will demonstrate that there is a natural isomorphism $\eta : I_{\mathfrak{W}^C} \to \beta \circ \alpha$. According to Def. 3.5, to define η we must determine a particular family of operations in $\mathfrak{W}^C$ indexed by the objects in $\mathfrak{W}^C$. We take these to be the operations that map an input box to an output state, in a way that preserves the planar ordering of the systems. For example:



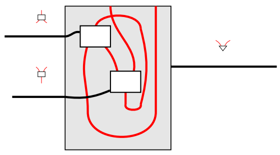
Note that such operations are invertible,i.e. the above operation has the following inverse:



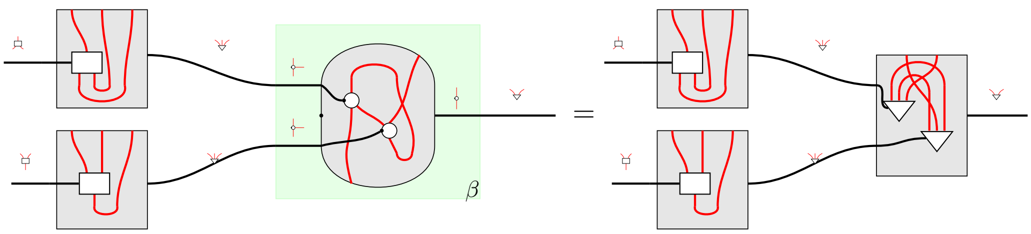
To show that this family of operations does define a natural transformation, we must show that the following holds for all operations in $\mathfrak{W}^C$:



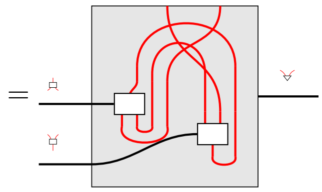
Indeed, the RHS can be rewritten to



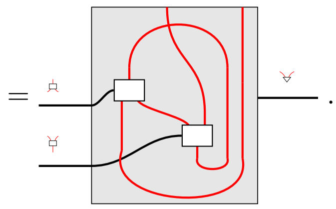
and similarly so can the LHS:



(3.21)



(3.22)



(3.23)

The natural isomorphism $\eta : \mathbb{1}_{W^C} \to \beta \circ \alpha$ implies that any algebra $F : W^C \to \mathbf{Set}$ is naturally isomorphic to the algebra $F \circ \beta \circ \alpha$. That is, any compact closed category can be thought of as one that originated from a time neutral process theory.

Chapter 4

Causal process theories in discretised spacetimes: Fields

In the previous chapters we examined the concept of time symmetry within process theories and provided a framework for time neutral process theories. Furthermore, we studied how time symmetry and time neutrality interplay with causality, which is a necessary ingredient for our theories to be considered physical.

In all these approaches the notions of time and space are implicit. For instance, the causality condition translates as 'the *future* can not signal to the *past*', while the non-signalling condition indicates that there can be no communication between two parties that are *spacelike separated*. In this chapter we provide a framework that includes spacetime explicitly within process theories. In particular, we establish a link with mathematical approaches for quantum field theories that exist in the literature, where quantum field theories are perceived as certain kinds of process theories.

Quantum field theories depend heavily on topology and geometry. Recently, however this traditional approach has seen a shift towards more abstract algebraic perpectives, which identify spacetime structure with *causal order*.

Given a Lorentzian manifold M, we can define the *causal order* between its events by setting $x \leq_M y$ iff x *causally precedes* y in M, i.e. iff there exists a future-directed causal curve from x to y.[1] In that case, we say that y(resp-x) lies in the *causal future* (resp. causal past) of x (resp-y). If it happens that $x, y \in M$ have the same exact causal future (resp. past), then they are necessarily identical and we say that M is *future- (resp. past-) distinguishing.* [2].

The identification of spacetime structure with causal order originates in a much-celebrated result by Malament [Mal77], itself based on previous work by Kronheimer, Penrose, Hawking, King and McCarthy [KP67, HKM76]. In [Mal77]it is shown that continuous timelike curves determine spacetime topology, a result that characterises causal structure as a more primitive notion.

Theorem 4.1. *Let M and M' be two Lorentzian manifolds, both manifolds being future-and-past–distinguishing. If $f : M \to M'$ preserves the causal order, then it also preserves continuous timelike curves.*

The result by Malament identifies future-and-past–distinguishing manifolds with their causal order. However, it does not provide any information about which partial orders arise as causal orders on manifolds. This lack of criteria governing the connection of causal order and topology is the motivation behind the work of [MP10, MP12], which aims to relate causal orders with partial orders and domain theory. However, the characterisation of which partial orders arise as the causal orders of Lorentzian manifolds remains an open question.

A different approach to the order-theoretic study of spacetime is given by the *causal sets* research programme (cf. [BLMS87, Bom06]). A *causal set* is a poset which is *locally finite*, i.e. such that for every $x, y \in C$ the subset $\{ z \in C \mid x \leq z \leq y \}$ is finite. Given a fixed causal set the question whether there exists an embedding into a Lorentzian manifold is central to the causal set programme and to the best of our knowledge one that is yet to be answered [Bom06, Sur19, WH20].

[1] A Lorentzian manifold is a pair (M, g), where M is a differentiable manifold and g is a Lorentzian metric.

[2] The requirement for a manifold M to be future- and past- distinguishing is essentially one of well-behaviour, e.g. excluding causal violations such as closed timelike curves (all points of which necessarily have the same causal past and future).

When it comes to incorporating quantum fields into the spacetimes, there are three significant directions of focus: algebraic approaches, topological approaches and quantum cellular automata.

The algebraic approaches take a functorial view of quantum fields, studying the *local* structure of fields through the assignment of algebras of observables—usually C*-algebras or von Neumann algebras—over spacetime regions. Prominent examples include *Algebraic Quantum Field Theory* (AQFT) [HK64, HM06] and the topos-theoretic programmes [HLS09, DI08]. The association of spacetime regions with observable algebras is implemented via special functors, called presheaves, in a way that respects causality and locality constraints imposed by space-time topology. We will take a deeper look at this approach in Section 4.4.

The topological approaches focus instead on *global* aspects of relativistic quantum fields, foregoing any possibility of studying local structure by requiring that field theories be *topological*, i.e. invariant under large scale deformations of spacetime. The resulting *Topological Quantum Field Theories* (TQFTs) [L$^+$09, Ati88, Wit88] have achieved enormous success in fields such as condensed matter theory and quantum error correction. Like AQFT, TQFTs have a categorical formulation as functors from a category of spacetime "pieces" to categories of Vector spaces and algebras. The difference is in the *nature* of those "pieces": in AQFT a spacetime is given and the order structure of its regions is considered; in TQFT, on the other hand, (equivalence classes of) basic topological manifolds are given, which can be combined together to form myriad different spacetimes.

The approaches based on Quantum Cellular Automata (QCA) [DP16, Arr19, vN66], finally, attempt to tame the issues with the formulation of quantum field theory by positing that full-fledged quantum fields in spacetime can be understood as the continuous limit of much-more-manageable theories, dealing with quantum fields living on discrete lattices and subject to discrete time evolution (known as Quantum Cellular Automata).

In this chapter, we propose to use tools from category theory to unify key aspects of the approaches above under a single generalised framework. Specifically, our work is part of an effort to gain an operational, process-theoretic understanding of the relationship between quantum theory and Relativistic causality [Coe16, CL13, KU17, PGC19]. Our key contribution, across the next four sections, will be the formulation of a functorial and theory-independent notion of field theory based solely on the order-theoretic structure of causality. To exemplify the flexibility of our construction, in Section 4.4 we will build a strong connection to Algebraic Quantum Field Theory, based on a sheaf-theoretic formulation of states over regions. In Section 4.5, finally, we will formulate a notion of cellular automaton which encompasses and greatly generalises notions of QCA from existing literature.

4.1 Causal orders

In this work, we will consider partial orders as an abstract model of spacetimes. For the remainder of this chapter, we will use the term *causal orders* to mean partial orders, highlighting the relationship to spacetimes.

Definition 4.2. By a *causal order* we mean a poset $\Omega = (|\Omega|, \leq)$, i.e. a set $|\Omega|$ equipped with a partial order $\leq$ on it. We refer to the elements of Ω as *events*. Given two events $x, y \in \Omega$ we say that x *causally precedes* y (equivalently that y *causally follows* x) iff $x \leq y$. We say that x and y are *causally related* iff $x \leq y$ or $y \leq x$. A *causal sub-order* Ω' of a causal order Ω is a subset $|\Omega'| \subseteq |\Omega|$ endowed with the structure of a poset by restriction. [3]

In what follows we demonstrate familiar concepts from Relativity on partial orders.

4.1.1 Causal Paths

Definition 4.3. Let Ω be a causal order and let $x, y \in \Omega$ be two events. A *causal path* from x to y is a maximal totally ordered subset $\gamma \subseteq \Omega$ such that $x = \min \gamma$ and $y = \max \gamma$. Maximality of the subset $\gamma \subseteq \Omega$ here means that there is no total

[3] I.e. such that for all $x, y \in |\Omega'|$ we have that $x \leq y$ in Ω' if and only if $x \leq y$ in Ω.

order $\gamma' \subseteq \Omega$ strictly containing gamma and such that $x = \min \gamma'$ and $y = \max \gamma'$. We write $\gamma : x \rightsquigarrow y$ to denote that γ is a causal path from x to y.

The causal diamond from x to y in a causal order Ω is the union of all causal paths $x \rightsquigarrow y$ in Ω. Furthermore, causal paths in Ω can be naturally organised into a category as follows:

- the objects are the events $x \in \Omega$;
- the morphisms from x to y are the paths $x \rightsquigarrow y$;
- the identity morphism on x is the singleton path $\{x\} : x \rightsquigarrow x$;
- composition of two paths $\gamma : x \rightsquigarrow y$ and $\xi : y \rightsquigarrow z$ is the set-theoretic union of the subsets $\gamma, \xi \subseteq \Omega$:

$$\xi \circ \gamma := (\xi \cup \gamma) : x \rightsquigarrow z \tag{4.1}$$

Definition 4.4. Let Ω be a causal order and let $x \in \Omega$ be an event. The *causal future* $J^+(x)$ of x is the set of all events y which causally follow it:

$$J^+(x) := \{\, y \in \Omega \,|\, x \leq y \,\} \tag{4.2}$$

Similarly, the *causal past* $J^-(x)$ of x is the set of all events y which causally precede it:

$$J^-(x) := \{\, y \in \Omega \,|\, y \leq x \,\} \tag{4.3}$$

We also define causal future and past for arbitrary subsets $A \subseteq \Omega$ by union:

$$J^+(A) := \bigcup_{x \in A} J^+(x) \qquad\qquad J^-(A) := \bigcup_{x \in A} J^-(x) \tag{4.4}$$

Remark 4.5. A causal order Ω is automatically future-and-past–distinguishing. To see this, assume that $J^+(x) = J^+(y)$ for some $x, y \in \Omega$: then both $x \in J^+(x) = J^+(y)$, implying $y \leq x$, and $y \in J^+(y) = J^+(x)$, implying $x \leq y$, so that $x = y$

by antisymmetry of the partial order $\leq$. The assumption that $J^{-}(x) = J^{-}(y)$ analogously implies that $x = y$.

Definition 4.6. Let Ω be a causal order and let $x \in \Omega$ be an event. By a causal path $\gamma : x \rightsquigarrow +\infty$ (resp. $\gamma : -\infty \rightsquigarrow x$) we denote a maximal totally ordered subset $\gamma \subseteq \Omega$ such that $x = \min \gamma$ (resp. $x = \max \gamma$). If Ω has a global maximum (resp. global minimum), then we denote it by $+\infty$ (resp. $-\infty$) for consistency with our previous definition of causal paths, otherwise the symbol $+\infty$ (resp. $-\infty$) is never used to denote an actual element of C.

The causal future (resp. causal past) of an event x is the union of all causal paths $x \rightsquigarrow +\infty$ (resp. $-\infty \rightsquigarrow x$).

4.1.2 Space-like slices

Space-like slices generalize the concept of spacelike hypersurfaces in Relativity and are the main focus of this work.

Definition 4.7. Let Ω be a causal order. We say that two events x, y are *space-like separated* if they are not causally related, i.e. if neither $x \leq y$ nor $y \leq x$. Consequently, we define a *(space-like) slice* Σ in Ω to be a subset $\Sigma \subseteq \Omega$ such that any two distinct $x, y \in \Sigma$ are space-like separated.

Definition 4.8. Let Ω be a causal order and let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a collection of subsets of Ω. We say that the subsets in $\mathcal{A}$ are *space-like separated* if the following conditions holds for all distinct $A, B \in \mathcal{A}$:

$$A \cap \left(J^{+}(B) \cup J^{-}(B)\right) = \emptyset \tag{4.5}$$

In particular, a space-like slice is the union of a collection of space-like separated singleton subsets.

Slices have *domains of dependence*, that is collection of events in spacetime causally influenced by events in the slice. Below we provide the relevant definition

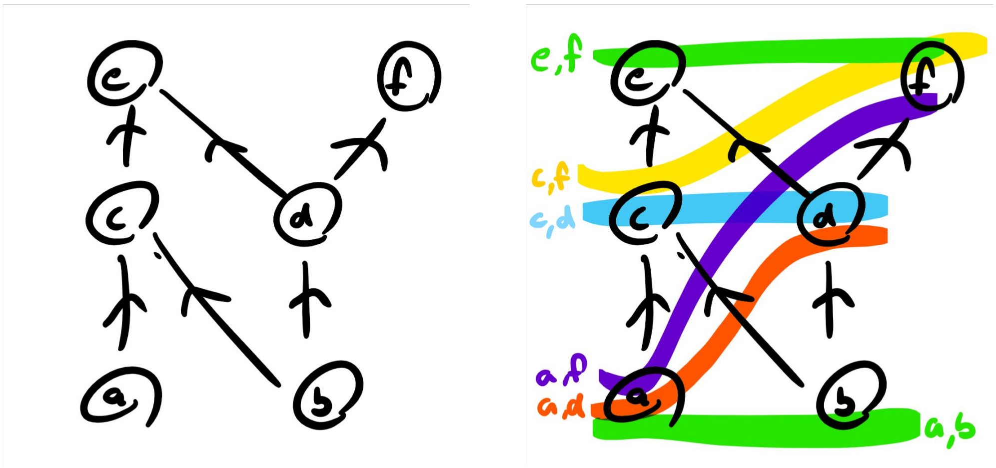


Figure 4.1: Left: the Hasse diagram for a causal order on 6 events $\{a, b, c, d, e, f\}$. Right: the maximal slices for the causal order highlighted (all other slices can be obtained as subsets of the maximal slices)[GSC21]

for any subset $A \subseteq \Omega$ (with A being a slice is regarded as a special case).

Definition 4.9. Let Ω be a causal order and let $A \subseteq \Omega$ be any subset. The *future domain of dependence* $D^+(A)$ of A is the subset of all events $x \in \Omega$ which "necessarily causally follow A", in the sense that every causal path $-\infty \rightsquigarrow x$ intersects A:

$$D^+(A) := \{\, x \in \Omega \,|\, \forall \gamma : -\infty \rightsquigarrow x.\, \gamma \cap A \neq \emptyset \,\} \tag{4.6}$$

The *past domain of dependence* $D^-(A)$ of A is the subset of all events $x \in \Omega$ which "necessarily causally precede A", in the sense that every causal path $x \rightsquigarrow +\infty$ intersects A:

$$D^-(A) := \{\, x \in \Omega \,|\, \forall \gamma : x \rightsquigarrow +\infty.\, \gamma \cap A \neq \emptyset \,\} \tag{4.7}$$

The domains of dependence of a subset A are related to its past and future by the following two Propositions.

Proposition 4.10. *Let Ω be a causal order and let $A \subseteq \Omega$ be any subset. Then $D^+(A) \subseteq J^+(A)$ and $D^-(A) \subseteq J^-(A)$.*

Proof. Let $x \in D^+(A)$ be any event in the future domain of dependence of A. The set of causal paths $-\infty \rightsquigarrow x$ is necessarily non-empty, because there must be at least one such path extending the singleton path $\{x\} : x \rightsquigarrow x$. Let $\gamma : -\infty \rightsquigarrow x$ be one such path. Because $x \in D^+(A)$, γ must intersect A at some point $y \leq x$, and we define $\gamma' := \gamma \cap J^+(\{y\}) \neq \emptyset$. By definition, $y = \min \gamma'$. Because $J^+(\{y\})$ is upward-closed, $x = \max \gamma'$ and $\gamma' : y \rightsquigarrow x$ is such that $\gamma' \subseteq J^+(\{y\}) \subseteq J^+(A)$, so we conclude that $x \in J^+(A)$. The proof that $D^-(A) \subseteq J^-(A)$ is analogous.

□ □

Proposition 4.11. *Let Ω be a causal order and let $A \subseteq \Omega$ be any subset. If $B \subseteq D^+(A)$ then $J^+(B) \subseteq J^+(A)$ and $J^-(B) \subseteq J^-(A) \cup J^+(A)$. Dually, if $B \subseteq D^-(A)$ then $J^-(B) \subseteq J^-(A)$ and $J^+(B) \subseteq J^-(A) \cup J^+(A)$.*

Proof. Without loss of generality, assume $B \subseteq D^+(A)$—the case $B \subseteq D^-(A)$ is proven analogously. From Proposition 4.10 we have that $B \subseteq D^+(A) \subseteq J^+(A)$, so we conclude that $J^+(B) \subseteq J^+(A)$ by upward-closure of $J^+(A)$. Now consider $x \in J^-(B)$. Let $\gamma : x \rightsquigarrow y$ be any path with $y \in B$ and let $\gamma' : -\infty \rightsquigarrow y$ be any path extending γ. Because $B \subseteq D^+(A)$, the intersection $\gamma' \cap A$ contains at least some point z. Because γ' is totally ordered, we have two possible cases: $z \leq x$ and $z \geq x$. If $z \leq x$, then $\gamma' \cap J^+(\{z\}) \cap J^-(\{x\}) : z \rightsquigarrow x$ shows that $x \in J^+(A)$. If $z \geq x$, then $\gamma' \cap J^-(\{z\}) \cap J^+(\{x\}) : x \rightsquigarrow z$ shows that $x \in J^-(A)$. □ □

Slices can be organized into a category denoted by $\mathrm{Slices}(\Omega)$.

Definition 4.12. Let Ω be a causal order. The *category of all slices* on Ω, denoted by $\mathrm{Slices}(\Omega)$, consists of the following data:

- Objects of $\mathrm{Slices}(\Omega)$ are the (space-like) slices of Ω. In what follows, we use the terms space-like slices and slices interchangeably.
- The category is a poset and the unique morphism from a space-like slice Σ to another space-like slice Γ is denoted $\Sigma \twoheadrightarrow \Gamma$ if it exists. Specifically, we say that $\Sigma \twoheadrightarrow \Gamma$ if and only if $\Gamma \subseteq D^+(\Sigma)$, i.e. iff Γ lies entirely into the future domain of dependence of Σ.

- The monoidal product on objects $\Sigma \otimes \Gamma$ is only defined when Σ and Γ are space-like separated, in which case it is the disjoint union $\Sigma \sqcup \Gamma$.
- The unit for the monoidal product is the empty space-like slice $\emptyset \subseteq \Omega$.
- The partial monoidal product on objects extends to morphisms because whenever $\Sigma' \subseteq D^+(\Sigma)$ and $\Gamma' \subseteq D^+(\Gamma)$—i.e. whenever $\Sigma \twoheadrightarrow \Sigma'$ and $\Gamma \twoheadrightarrow \Gamma'$—we necessarily have:

$$\Sigma' \sqcup \Gamma' \subseteq D^+(\Sigma) \cup D^+(\Gamma) \subseteq D^+(\Sigma \sqcup \Gamma), \text{ i.e. } \Sigma \otimes \Gamma \twoheadrightarrow \Sigma' \otimes \Gamma' \tag{4.8}$$

The order relation $\Sigma \twoheadrightarrow \Gamma$ on slices guarantees that the field state local to the the codomain slice Γ will be entirely determined by evolution and marginalisation of the field state on the domain slice Σ. As for marginalisation, the definition is such that any sub-slice $\Sigma' \subseteq \Sigma$ necessarily satisfies $\Sigma \twoheadrightarrow \Sigma'$. That is, by marginalisation/discarding on the field state on Σ, we can obtain the field state on Σ'. In Subsection 4.3.1 we will elaborate more on the connection to marginalisation.

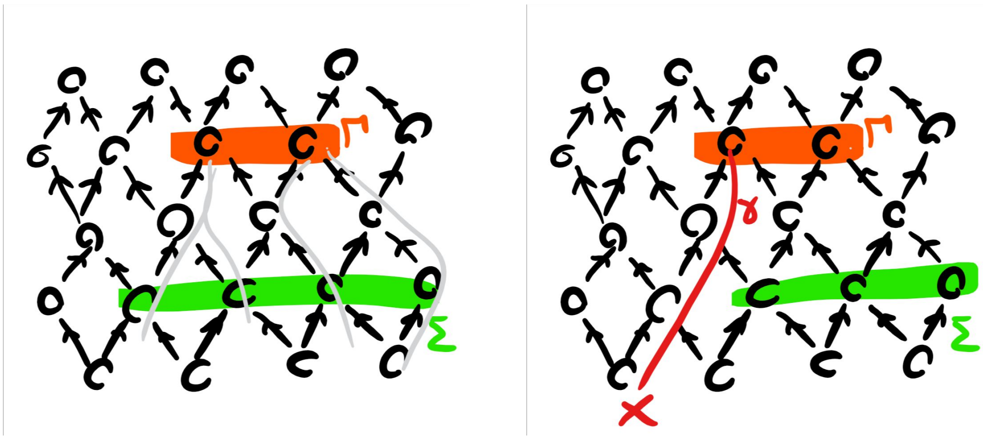


Figure 4.2: Left: two slices Σ, Γ such that $\Sigma \twoheadrightarrow \Gamma$. Right: two slices Σ, Γ such that $\Sigma \not\twoheadrightarrow \Gamma$, highlighting a past-directed path γ starting from an event of Γ and not intersecting Σ at any point[GSC21].

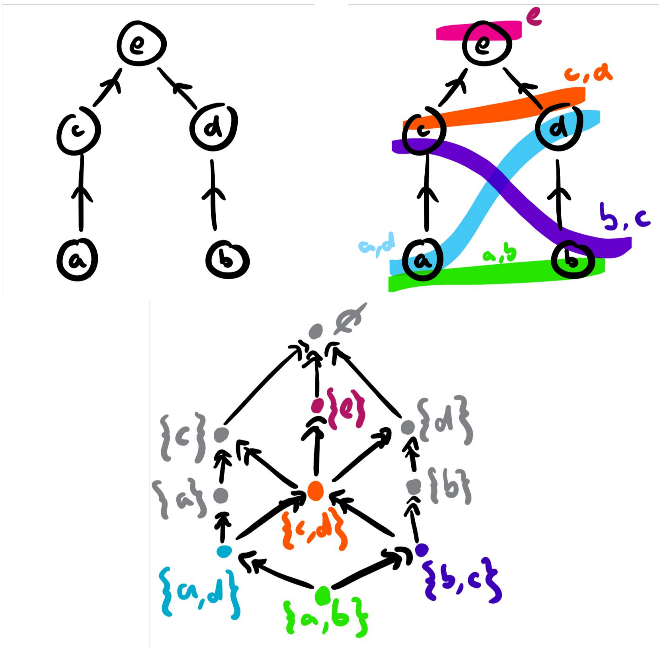


Figure 4.3: Left: the Hasse diagram for a causal order. Centre: the maximal slices for the causal order highlighted. Right: the category of all slices for the causal order[GSC21].

4.1.3 Diamonds and Regions

Causal diamonds and their set-theoretic unions thereof hold a special status in Relativity as they generate the topology of Lorentzian manifolds. Definitions 4.13 and 4.14 provide the order-theoretic incarnation of such requirement.

Definition 4.13. Let Ω be a causal order. If x, y are two events in Ω, the *causal diamond* from x to y in Ω is the causal sub-order $(\Diamond_{x,y}, \leq) \hookrightarrow \Omega$ defined as follows:

$$\Diamond_{x,y} := \{\, z \in \Omega \,|\, x \leq z \leq y \,\} = \bigcup_{\gamma : x \rightsquigarrow y} \gamma \tag{4.9}$$

Definition 4.14. Let Ω be a causal order. A *region* in Ω is a causal sub-order $(R, \leq) \hookrightarrow \Omega$ such that for all events $x, y \in R$ the causal diamond from x to y in Ω is a subset of R (i.e. R contains all paths $\gamma : x \rightsquigarrow y$ in Ω).

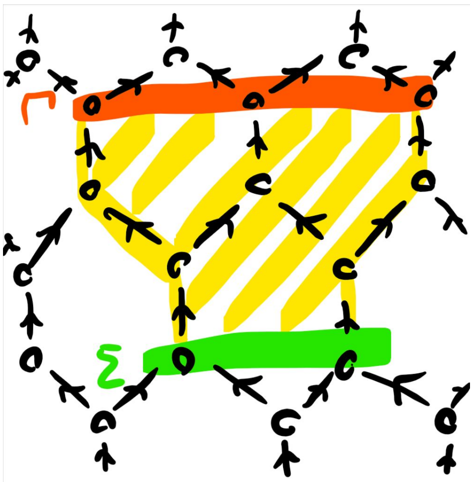
We could have equivalently stated 4.14 as saying that regions in Ω are all the possibly unions of causal diamonds in Ω (including the empty one). A special case of region of particular interest is the region *between* two slices $\Sigma \twoheadrightarrow \Gamma$.

Definition 4.15. Let Ω be a causal order and consider two slices $\Sigma \twoheadrightarrow \Gamma$. We define the *region between* Σ *and* Γ as follows:

$$\Diamond_{\Sigma,\Gamma} := \bigcup_{x\in\Sigma} \bigcup_{y\in\Gamma} \Diamond_{x,y} \tag{4.10}$$

In particular, a causal diamond $\Diamond_{x,y}$ is the region between the slices $\{x\}$ and $\{y\}$. More generally, a region between slices Σ and Γ is the intersection $\Diamond_{\Sigma,\Gamma} = J^+(\Sigma) \cap J^-(\Gamma)$ of their future and past respectively.

The slices Σ and Γ bounding the region $\Diamond_{\Sigma,\Gamma}$ can be obtained respectively as the sets of its minima $\Sigma = \min \Diamond_{\Sigma,\Gamma}$ and of its maxima $\Gamma = \max \Diamond_{\Sigma,\Gamma}$. As a special case, a slice Σ is the region between Σ and Σ. Conversely, every closed bounded region R—and in particular every finite region—is in the form $R = \Diamond_{\min R, \max R}$.



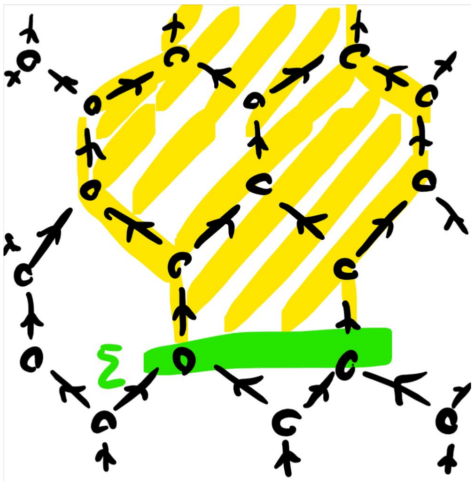


Figure 4.4: Left: the region between two slices on the honeycomb lattice. Right: an unbounded (necessarily infinite) region on the honeycomb lattice[GSC21].

4.2 Categories of slices

The category Slices(Ω) might contain exotic slices for physical fields to be defined over. To overcome this issue we define the category of slices $\mathcal{C}$, which is a subcategory of Slices (Ω) obeying certain requirements.

Definition 4.16. Let Ω be a causal order. A *category of slices* on Ω is the full sub-category $\mathcal{C}$ of Slices (Ω) defined by a given set obj $(\mathcal{C})$ of slices chosen in such a way that the following three conditions hold.

(1) For any two events $x, y \in \Omega$ with $x \leq y$, there exist slices $\Sigma, \Gamma \in$ obj $(\mathcal{C})$ such that $x \in \Sigma$, $y \in \Gamma$ and $\Sigma \twoheadrightarrow \Gamma$.

(2) If Σ, Γ and Δ are three slices in $\mathcal{C}$, then the restriction $(\Delta \cap \Diamond_{\Sigma,\Gamma})$ of Δ to the region $\Diamond_{\Sigma,\Gamma}$ is also a slice in $\mathcal{C}$.

(3) The category of slices $\mathcal{C}$ is a partially monoidal subcategory of Slices (Ω). In particular, $\emptyset \in$ obj $(\mathcal{C})$ and whenever $\Sigma \otimes \Gamma$ exists in $\mathcal{C}$ for some $\Sigma, \Gamma \in$ obj $(\mathcal{C})$ then $\Sigma \otimes \Gamma$ also exists in Slices (Ω). (Associativity and unitality of $\otimes$ are strict in $\mathcal{C}$ as they are in Slices (Ω).)

As an example of particularly well-behaved slices, we define a notion of *Cauchy slices*—akin to that of Cauchy surfaces from Relativity.

Definition 4.17. A slice Σ on Ω is a *Cauchy slice* if every causal path $\gamma : -\infty \rightsquigarrow +\infty$ in Ω intersects Σ at some (necessarily unique) event. Cauchy slices are in particular maximal slices. A *category of Cauchy slices* on Ω is a category $\mathcal{C}$ of slices on Ω such that every slice $\Sigma \in$ obj $(\mathcal{C})$ is a subset $\Sigma \subseteq \Gamma$ of some Cauchy slice $\Gamma \in$ obj $(\mathcal{C})$.

Proposition 4.18. *A* foliation *on a causal order Ω is a set $\mathcal{F}$ of Cauchy slices on Ω such that:*

(1) the slices in $\mathcal{F}$ are totally ordered according to $\twoheadrightarrow$;

(2) every event $x \in \Omega$ is contained in some slice $\Sigma \in \mathcal{F}$;

(3) the slices in $\mathcal{F}$ are pairwise disjoint.

If $\mathcal{F}$ is a foliation, write CauchySlices $(\mathcal{F})$ *for the full sub-category of* Slices (Ω) *generated by all slices which are subsets of some Cauchy slice in $\mathcal{F}$. Then* CauchySlices $(\mathcal{F})$ *is a category of Cauchy slices on Ω.*

Proof. Let $\mathrm{CauchySlices}(\mathcal{F})$ denote the full sub-category of $\mathrm{Slices}(\Omega)$ generated by all slices which are subsets of some Cauchy slice in $\mathcal{F}$.

For any two events $x \leq y$ in Ω, let $\Sigma, \Gamma \in \mathrm{obj}(\mathrm{CauchySlices}(\mathcal{F}))$ be two Cauchy slices such that $x \in \Sigma$ and $y \in \Gamma$, the existence of such slices guaranteed by the definition of foliation. Because the foliation is totally ordered, we have that $\Sigma \twoheadrightarrow \Gamma$ or $\Gamma \twoheadrightarrow \Sigma$ (or both, if $\Sigma = \Gamma$ and $x = y$). If $x = y$, either works, while if $x < y$ then necessarily $\Sigma \twoheadrightarrow \Gamma$. Either way, condition (1) for $\mathrm{CauchySlices}(\mathcal{F})$ to be a category of slices is satisfied.

Let Σ', Γ' and Δ' be three slices, respectively contained in three Cauchy slices Σ, Γ and Δ inside the foliation. Because of total ordering and disjointness of slices in $\mathcal{F}$, the only instance in which $\Delta \cap \Diamond_{\Sigma,\Gamma} \neq \emptyset$ is when $\Sigma \twoheadrightarrow \Delta \twoheadrightarrow \Gamma$. In this case, $\Delta \cap \Diamond_{\Sigma,\Gamma} = \Delta \in \mathrm{obj}(\mathrm{CauchySlices}(\mathcal{F}))$. Otherwise, $\Delta \cap \Diamond_{\Sigma,\Gamma} = \emptyset \in \mathrm{obj}(\mathrm{CauchySlices}(\mathcal{F}))$. Either way, condition (2) for $\mathrm{CauchySlices}(\mathcal{F})$ to be a category of slices is satisfied when Σ, Γ and Δ are Cauchy slices. This result immediately generalises to Σ', Γ' and Δ': we have that $\Delta' \cap \Diamond_{\Sigma',\Gamma'} \subseteq \Delta \cap \Diamond_{\Sigma,\Gamma} \subseteq \Delta$, so that $\Delta' \cap \Diamond_{\Sigma',\Gamma'} \in \mathrm{obj}(\mathrm{CauchySlices}(\mathcal{F}))$ and condition (2) for $\mathrm{CauchySlices}(\mathcal{F})$ to be a category of slices is satisfied.

Finally, if Σ, Γ are two slices such that $\Sigma \otimes \Gamma$ is defined in $\mathrm{CauchySlices}(\mathcal{F})$, then Σ, Γ are necessarily disjoint subsets of the same Cauchy slice Δ. It is then immediate to conclude that condition (3) for $\mathrm{CauchySlices}(\mathcal{F})$ to be a category of slices is satisfied.

□

4.3 Causal field theories

In the previous section we have presented the analogue of several concepts from Relativity in the context of causal orders. In this section, we endow our causal orders with fields that live in a symmetric monoidal category.

Examples of symmetric monoidal categories that could model quantum fields vary depending on the context. In particular, if the context is finite dimensional, quantum fields can live in the category CPM[Fhilb] of finite dimensional Hilbert spaces and completely positive maps. If the context is infinite-dimensional, e.g. in the case of AQFT [HM06, HLS09], the categories usually considered for quantum fields are the category Hilb of Hilbert spaces and bounded linear maps, the category C*alg of C*-algebras and its subcategories W*alg of W*-algebras (sometimes known as "abstract" von Neumann algebras) and vNA of (concrete) von Neumann algebras.

In the framework we present here any symmetric monoidal category can be considered as a suitable category for quantum fields.

Definition 4.19. Let Ω be a causal order. A *causal field theory* Ψ on Ω is a monoidal functor $\Psi : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ of slices on Ω to some symmetric monoidal category $\mathcal{D}$, which we refer to as the *field category.*

The functor Ψ encodes the following physical information: To each spacelike slice Σ, Ψ associates the space of fields over that slice, $\Psi(\Sigma)$.

Remark 4.20. If Σ is finite and the singleton slices $\{x\}$ for the individual events $x \in \Sigma$ are all in the chosen category $\mathcal{C}$ of slices, then the action of Ψ on Σ always factorises into the tensor product of its action on the individual events:

$$\Psi(\Sigma) = \bigotimes_{x \in \Sigma} \Psi(\{x\}) \tag{4.11}$$

To each morphism $\Sigma \twoheadrightarrow \Gamma$, Ψ associates the morphism $\Psi(\Sigma) \to \Psi(\Gamma)$. This action of Ψ on morphisms specifies how the field evolves from Σ to Γ. In particular, this defines the map sending a field state $|\phi\rangle$ over the initial slice $\Psi(\Sigma)$ to the evolved field state $\Psi\left(\Sigma \twoheadrightarrow \Gamma\right)|\phi\rangle$ over the final slice $\Psi(\Gamma)$. It also explains why we chose the morphisms in Slices$\,(\Omega)$ the way we did: $\Sigma \twoheadrightarrow \Gamma$ if and only if the field data on Σ is sufficient to derive the field data on Γ. This identification of functorial action with field evolution is the core idea of our work.

The functor Ψ is monoidal meaning that the union of disjoint slices corresponds to the monoidal product of spaces of fields, i.e. the tensor product, when working in the familiar linear settings of Hilbert spaces, C*-algebras, on the individual slices.

Functoriality and monoidality on morphisms manifest the *principle of locality* in field theories. Let $\Sigma \twoheadrightarrow \Sigma'$ and $\Gamma \twoheadrightarrow \Gamma'$, where Σ and Γ is a pair of space-like separated slices and Σ' and Γ' another pair of space-like separated slices. Consider the field evolution between the two disjoint unions of slices:

$$\Psi\big((\Sigma \otimes \Gamma) \twoheadrightarrow (\Sigma' \otimes \Gamma')\big) : \Psi(\Sigma) \otimes \Psi(\Gamma) \to \Psi(\Sigma') \otimes \Psi(\Gamma') \tag{4.12}$$

Due to monoidality on morphisms the field evolution above factors as the product of the individual field evolutions $\Psi(\Sigma) \to \Psi(\Sigma')$ and $\Psi(\Gamma) \to \Psi(\Gamma')$:

$$\Psi\big((\Sigma \otimes \Gamma) \twoheadrightarrow (\Sigma' \otimes \Gamma')\big) = \Psi(\Sigma \twoheadrightarrow \Sigma') \otimes \Psi(\Gamma \twoheadrightarrow \Gamma') \tag{4.13}$$

We prove this by considering the following proposition.

Proposition 4.21. *Let Ω be a causal order. If Σ and Γ are space-like separated slices in Ω and $\Sigma \twoheadrightarrow \Sigma'$, then Σ' and Γ are also space-like separated slices.*

Proof. If Σ and Γ are space-like separated, then $\Gamma \cap (J^+(\Sigma) \cup J^-(\Sigma)) = \emptyset$. Because $\Sigma \twoheadrightarrow \Sigma'$, furthermore, Proposition 4.11 tells us that $J^+(\Sigma') \cup J^-(\Sigma') \subseteq J^+(\Sigma) \cup J^-(\Sigma)$. We conclude that $\Gamma \cap (J^+(\Sigma') \cup J^-(\Sigma')) = \emptyset$, i.e. that Σ' and Γ are also space-like separated. $\square$

Proposition 4.21 above together with monoidality on morphisms imply that whenever the entire region between Σ and Σ' on one side and the entire region between Γ and Γ' on the other side are space-like separated, any causal field evolution from $\Sigma \otimes \Gamma$ to $\Sigma' \otimes \Gamma'$ would be expected to factor. This is the analogue of the *clustering* principle in standard field theory which refers to the independence of the local processes to space-like separated environments.

Remark 4.22. Please note that the principle of locality obtained above only implies that the *evolution* of fields must factorise over space-like separated regions.

This imposes no constraints on the field state, which can be any state of the space of fields. In particular, if the field category has entanglement (e.g. categories of Hilbert spaces with the usual tensor product) then the field state can entangle space-like separated regions, while field evolution cannot.

4.3.1 Causality and no-signalling

Any category $\mathcal{C}$ of slices is a subcategory of Slices(Ω) and thus contains the empty slice (the monoidal unit). Therefore, we can define the following family of effects

$$\dot{\dagger}_\Sigma := \Psi(\Sigma \twoheadrightarrow \emptyset) \tag{4.14}$$

which respect the monoidal structure:

$$\dot{\dagger}_{\Sigma\otimes\Gamma} = \Psi\big((\Sigma\otimes\Gamma) \twoheadrightarrow \emptyset\big) = \Psi\big((\Sigma \twoheadrightarrow \emptyset)\otimes(\Gamma \twoheadrightarrow \emptyset)\big) = \Psi(\Sigma \twoheadrightarrow \emptyset)\otimes\Psi(\Gamma \twoheadrightarrow \emptyset) = \dot{\dagger}_\Sigma\otimes\dot{\dagger}_\Gamma \tag{4.15}$$

By functoriality we furthermore have that

$$\dot{\dagger}_\Gamma \circ \Psi(\Sigma \twoheadrightarrow \Gamma) = \Psi(\Gamma \twoheadrightarrow \emptyset) \circ \Psi(\Sigma \twoheadrightarrow \Gamma) = \Psi(\Sigma \twoheadrightarrow \emptyset) = \dot{\dagger}_\Sigma \tag{4.16}$$

Therefore, the family of effects act as *discarding* effects. This is a manifestation of non-signalling in the evolution of fields: The field state over a given slice Σ does not depend on the field state over slices which are in the future of Σ or are space-like separated from Σ.

This emergence of causality and no-signalling from functoriality is in fact a consequence of a breaking of time symmetry which happened in the very definition of the ordering between slices. Indeed, consider the "time-reversed" causal order Ω^{rev}, obtained by reversing all causal relations in Ω (i.e. $y \leq x$ in Ω^{rev} if and only if $x \leq y$ in Ω). The slices for Ω^{rev} are exactly the slices for Ω, i.e. the categories of all slices Slices(Ω^{rev}) and Slices(Ω) have the same objects. If time symmetry were to hold, we would expect the arrows in Slices(Ω^{rev}) to be exactly the reverse of the arrows in Slices(Ω). However, the conditions defining the arrows in both categories are as follows:

- $\Sigma \twoheadrightarrow \Gamma$ in $\mathrm{Slices}(\Omega)$ iff $\Gamma \subseteq D^+(\Sigma)$ in Ω;
- $\Gamma \twoheadrightarrow \Sigma$ in $\mathrm{Slices}(\Omega^{rev})$ iff $\Sigma \subseteq D^+(\Gamma)$ in Ω^{rev}, i.e. iff $\Sigma \subseteq D^-(\Gamma)$ in Ω.

The two conditions that $\Gamma \subseteq D^+(\Sigma)$ and $\Sigma \subseteq D^-(\Gamma)$, both in Ω, are not in general equivalent: this shows that time symmetry is broken by our definition of the relationship between slices, ultimately leading to the emergence of causality and no-signalling constraints on functorial evolution of quantum fields.

4.4 Connection with Algebraic Quantum Field Theory

The framework of causal field theories has many similarities with Topological Quantum Field Theories (TQFTs), which can be considered as an axiomatization of the Schrodinger picture of quantum mechanics. In particular, TQFTs are defined as functors that assign states to space and linear maps to spacetimes. The main difference with the causal field theories we defined above is that TQFTs consider field evolution over *arbitrary spacetimes*, while causal field theories consider field evolution over a *single given* spacetime.

An alternative approach in axiomatizing quantum field theories is known as Algebraic Quantum Field Theory (AQFT), which can be considered as the axiomatization of the Heisenberg picture of quantum mechanics, and thus as a dual approach to TQFTs. AQFTs are again defined as functors, known as presheaves, which encapsulate the relationship between fields and the topology of spacetime. In particular, they map each Minkowski spacetime region (causal diamond) to an algebra of observables, a C*-algebra, localized in that region. Locality and causality in this context manifest themselves through the requirement that algebras of observables in spacelike separated regions commute (in that case, local effects cannot be entangling over space-like separated regions).

The difference between TQFT and AQFT lies in the duality between a *compositional* and a *decompositional* perspective.

In *compositionality*, larger objects are created by *composing* together given elementary building blocks in all possible ways: this is the approach behind an ever growing zoo of process theories (e.g. see [CK17] and references therein)and the philosophy of TQFTs. In *decompositionality*, on the other hand, larger objects are given as a whole and subsequently decomposed into smaller constituents, with composition of the latter constrained by the context in which they live: this approach, based on partially monoidal structure, was recently introduced by [Gog19] as a way to talk about compositionality in physical theories where a universe is fixed beforehand. This is the approach behind AQFT. While TQFTs are compositional [Koc03, L^+09], causal field theories are more naturally understood from the decompositional perspective.

To see this, we draw a connection of our causal field theories with an AQFT approach turning our functors, defined on slices, into presheafs defined on "regions" (generalising unions of causal diamonds in AQFT). However, our approach differs from the AQFT approach in a number of ways:

- We dispense of the algebras themselves. That said, our approach is independent of the specific process theory chosen for the fields.
- Instead of looking at the space of local observables/effects, we take the (equivalent) dual perspective and work with the space of local states.
- Local states can be entangling, so the formulation of locality and causality as "commutativity" is no longer applicable, even in the case where the field category is a category of C*-algebras. Instead, locality and causality arise as a consequence of factorisation of field evolution over space-like separated slices.

Firstly, we show that categories of slices can be restricted to regions provided that regions are defined in a way that respects the requirements imposed by a choice of category of slices.

Definition 4.23. A *bounded region* in a category of slices $\mathcal{C}$ on a causal order Ω is a region on Ω in the form $\Diamond_{\Sigma,\Gamma}$ for some $\Sigma, \Gamma \in \mathrm{obj}(\mathcal{C})$. Bounded regions in $\mathcal{C}$ form a poset $\mathrm{Regions}_{\mathrm{bnd}}(\mathcal{C})$ under inclusion.

Definition 4.24. A *region* in a category of slices $\mathcal{C}$ is a region R on Ω which can be obtained as a union $R = \bigcup_{\lambda\in\Lambda} \Diamond_{\Sigma_\lambda,\Gamma_\lambda}$ of a family $(\Diamond_{\Sigma_\lambda,\Gamma_\lambda})_{\lambda\in\Lambda}$, closed under finite unions, of bounded regions in $\mathcal{C}$. Regions in $\mathcal{C}$ also form a poset $\mathrm{Regions}(\mathcal{C})$ under inclusion, with $\mathrm{Regions}_{\mathrm{bnd}}(\mathcal{C})$ as a sub-poset.

Remark 4.25. A region is more general than a bounded region. For instance, a region could be defined as an infinite union of bounded regions.

We now show that if we restrict $\mathcal{C}$ to a region R in $\mathcal{C}$, then this restriction $\mathcal{C}|_R$ is a valid category of slices.

Proposition 4.26. *Let $\mathcal{C}$ be a category of slices and R be a region in it. The* restriction $\mathcal{C}|_R$ *of $\mathcal{C}$ to the region R, defined as the full sub-category of $\mathcal{C}$ spanned by the slices $\Delta \in \mathrm{obj}(\mathcal{C})$ such that $\Delta \subseteq R$, is itself a category of slices.*

Proof. If $R = \Diamond_{\Sigma,\Gamma}$ is a *bounded* region in $\mathcal{C}$, then the statement is an immediate consequence of requirement (2) for categories of slices. Now assume that $R = \bigcup_{\lambda\in\Lambda} \Diamond_{\Sigma_\lambda,\Gamma_\lambda}$ is a union of bounded regions in $\mathcal{C}$.

If $x \leq y$ are two events in R, then it must be that $x \in \Diamond_{\Sigma_{\lambda_x},\Gamma_{\lambda_x}}$ and $y \in \Diamond_{\Sigma_{\lambda_y},\Gamma_{\lambda_y}}$ for some $\lambda_x, \lambda_y \in \Lambda$: closure under finite union of the family $(\Diamond_{\Sigma_\lambda,\Gamma_\lambda})_{\lambda\in\Lambda}$ then guarantees that there exists some $\lambda_{x,y} \in \Lambda$ with $x, y \in \Diamond_{\Sigma_{\lambda_{x,y}},\Gamma_{\lambda_{x,y}}}$. Because $\mathcal{C}$ is a category of slices, we can find two slices $\Delta_x \twoheadrightarrow \Delta_y$ in $\mathcal{C}$ such that $x \in \Delta_x$ and $y \in \Delta_y$. Then the restrictions $(\Delta_x \cap \Diamond_{\Sigma_{\lambda_{x,y}},\Gamma_{\lambda_{x,y}}})$ and $(\Delta_y \cap \Diamond_{\Sigma_{\lambda_{x,y}},\Gamma_{\lambda_{x,y}}})$ satisfy requirement (1) for $\mathcal{C}|_R$ to be a category of slices.

If Σ, Γ and Δ are three slices in $\mathcal{C}|_R$, then in particular the diamond $\Diamond_{\Sigma,\Gamma}$ is a subset of R (the latter is a region) and so is the intersection $\Delta \cap \Diamond_{\Sigma,\Gamma}$, which exists in $\mathcal{C}$ because the latter is a category of slices. Hence requirement (2) for $\mathcal{C}|_R$ to be a category of slices is satisfied.

Requirement (3) for $\mathcal{C}|_R$ to be a category of slices is satisfied, because if $\Sigma, \Gamma \subseteq R$ then also $\Sigma \otimes \Gamma \subseteq R$ whenever the latter is defined.

□

Therefore, given a causal field theory $\Psi : \mathcal{C} \to \mathcal{D}$, the restrictions $\Psi|_R : \mathcal{C}|_R \to \mathcal{D}$ are again causal field theories. To establish a connection with AQFT we need two more ingredients: the assignment of a space of states $\text{States}_\Psi(R)$ over a region R and the definition of restrictions $\text{States}_\Psi(R) \to \text{States}_\Psi(R')$ between spaces of states associated with inclusions $R' \subseteq R$ of regions.

Definition 4.27. Given a region R in a category of slices $\mathcal{C}$, the *space of states* $\text{States}_\Psi(R)$ over the region is defined to be the set comprising all families ρ of states over the slices in $\mathcal{C}|_R$

$$\rho \in \prod_{\Delta \in \text{obj}(\mathcal{C}|_R)} \text{States}_{\mathcal{D}}(\Psi(\Delta)) \tag{4.17}$$

such that for all $\Delta, \Delta' \in \text{obj}(\mathcal{C}|_R)$ with $\Delta \twoheadrightarrow \Delta'$ the following condition is satisfied:

$$\Psi(\Delta \twoheadrightarrow \Delta') \circ \rho_\Delta = \rho_{\Delta'} \tag{4.18}$$

By $\text{States}_{\mathcal{D}}(\Psi(\Delta))$ we have denoted the states on the object $\Psi(\Delta)$ of the symmetric monoidal category $\mathcal{D}$, i.e. the homset $\text{Hom}_{\mathcal{D}}[I, \Psi(\Delta)]$ where I is the monoidal unit of $\mathcal{D}$.

Proposition 4.28. *Given a causal field theory* $\Psi : \mathcal{C} \to \mathcal{D}$*, we can construct a presheaf* $\text{States}_\Psi : \text{Regions}(\mathcal{C})^{op} \to \text{Set}$ *by associating each region* $R \in \text{obj}(\text{Regions}(\mathcal{C}))$ *to the space of states* $\text{States}_\Psi(R)$ *over the region, and each inclusion* $i : R' \subseteq R$ *to the restriction function* $\text{States}_\Psi(R) \to \text{States}_\Psi(R')$ *defined by sending a family* $\rho \in \text{States}_\Psi(R)$ *to the family* $\text{States}_\Psi(i)(\rho) \in \text{States}_\Psi(R')$ *given as follows:*

$$\text{States}_\Psi(i)(\rho)_{\Delta'} = \rho_{i(\Delta')} \tag{4.19}$$

We refer to States_Ψ *as the* presheaf of states *over regions of* $\mathcal{C}$.

Proof. The only thing to show is functoriality of States_Ψ. If $i = \mathrm{id}_R : R \subseteq R$ is the identity on a region R, then we have:

$$\mathrm{States}_\Psi(i)(\rho)_\Delta = \rho_{i(\Delta)} = \rho_\Delta \tag{4.20}$$

i.e. $\mathrm{States}_\Psi(i) = \mathrm{id}_{\mathrm{States}_\Psi(R)}$ is the identity on the space of states over the region. If now $j : R'' \subseteq R'$ and $i : R' \subseteq R$, then $i \circ j : R'' \subseteq R$ and we have:

$$\mathrm{States}_\Psi(j)\left(\mathrm{States}_\Psi(i)(\rho)\right)_{\Delta''} = \mathrm{States}_\Psi(i)(\rho)_{j(\Delta'')} = \rho_{i(j(\Delta''))} = \mathrm{States}_\Psi(i \circ j)(\rho)_{\Delta''} \tag{4.21}$$

Hence States_Ψ is a presheaf $\mathrm{States}_\Psi : \mathrm{Regions}\,(\mathcal{C})^{op} \to \mathrm{Set}$ □

Definition 4.29. A *global state* ρ for a causal field theory $\Psi : \mathcal{C} \to \mathcal{D}$ is a global compatible family for States_Ψ, i.e. a family $\rho = \left(\rho^{(R)}\right)_{R \in \mathrm{Regions}(\mathcal{C})}$ such that $\mathrm{States}_\Psi(i)(\rho^{(R)}) = \rho^{(R')}$ for all inclusions $i : R' \subseteq R$ in $\mathrm{Regions}\,(\mathcal{C})$. We refer to the set of all global states as the *space of global states.*

Definition 4.27 provides us with additional information regarding the specific structure of regions, which is redundant. However, under certain circumstances an equivalent description of the space of states over regions can be given.

To start with, consider two slices $\Sigma \twoheadrightarrow \Gamma$ and note that the state on any slice $\Delta \subseteq \Diamond_{\Sigma,\Gamma}$ in a bounded region $\Diamond_{\Sigma,\Gamma}$ is uniquely determined by applying $\Psi(\Sigma \twoheadrightarrow \Delta)$ to the state on Σ:

$$\rho_\Delta = \Psi(\Sigma \twoheadrightarrow \Delta)(\rho_\Sigma) \tag{4.22}$$

This is, for example, the case for all bounded regions between Cauchy slices in a category of slices $\mathrm{CauchySlices}\,(\mathcal{F})$ generated by some foliation $\mathcal{F}$. If the foliation $\mathcal{F}$ has a minimum Σ_0—an *initial Cauchy slice*—then any global state $\rho \in \mathrm{States}_\Psi\,(\Omega)$ is entirely determined by its component ρ_{Σ_0} over the initial slice Σ_0:

$$\rho_\Delta = \Psi(\Sigma_0 \twoheadrightarrow \Delta) \circ \rho_{\Sigma_0} \tag{4.23}$$

for any $\Delta \in \mathcal{F}$ and any region R in CauchySlices$(\mathcal{F})$ such that $\Delta \subseteq R$.

Inspired by Relativity, we would like the state on *any* Cauchy slice in the foliation to determine the global state, not only that on an initial Cauchy slice. For this to happen, we need to strengthen our requirements on the causal field theory, which needs to be *causally reversible*.

Definition 4.30. Let Ω be any causal order. By the *causal reverse* of Ω we mean the causal order Ω^{rev} on the same events as Ω and such that $x \leq y$ in Ω^{rev} if and only if $x \geq y$ in Ω.

Definition 4.31. A category of slices $\mathcal{C}$ on a causal order Ω is said to be *causally reversible* if the full sub-category of Slices(Ω^{rev}) spanned by obj$(\mathcal{C})$ is a category of slices on the causal reverse Ω^{rev}. If this is the case, we write $\mathcal{C}^{rev}$ for said category of slices over Ω^{rev} and refer to it as the *causal reverse* of $\mathcal{C}$. We write $\overset{rev}{\twoheadrightarrow}$ for the morphisms of $\mathcal{C}^{rev}$.

Definition 4.32. Let $\Psi : \mathcal{C} \to \mathcal{D}$ be a causal field theory on a causal order Ω. If $\mathcal{C}$ is causally reversible, a *causal reversal* of Ψ is a causal field theory $\Phi : \mathcal{C}^{rev} \to \mathcal{D}$ such that:

(1) the functors Ψ and Φ agree on objects, i.e. for all $\Sigma \in$ obj$(\mathcal{C})$ we have that $\Psi(\Sigma) = \Phi(\Sigma)$;

(2) whenever we have two chains of alternating morphisms in $\mathcal{C}$ and $\mathcal{C}^{rev}$ which start and end at the same slices Σ, Γ, say in the form

$$\begin{aligned} &\Sigma \twoheadrightarrow \Delta_1 \overset{rev}{\twoheadrightarrow} \Delta_2 \twoheadrightarrow ...\Delta_{2n} \twoheadrightarrow \Gamma \\ &\Sigma \twoheadrightarrow \Delta_1' \overset{rev}{\twoheadrightarrow} \Delta_2' \twoheadrightarrow ...\Delta_{2m}' \twoheadrightarrow \Gamma \end{aligned} \tag{4.24}$$

for some $n, m \geq 0$, the composition of the images of the morphisms under Ψ

and Φ always yield the same morphism $\Psi(\Sigma) \to \Psi(\Gamma)$:

$$\begin{aligned} &\Psi(\Delta_{2n} \twoheadrightarrow \Gamma) \circ ... \circ \Phi(\Delta_1 \overset{rev}{\twoheadrightarrow} \Delta_2) \circ \Psi(\Sigma \twoheadrightarrow \Delta_1) \\ =&\Psi(\Delta'_{2m} \twoheadrightarrow \Gamma) \circ ... \circ \Phi(\Delta'_1 \overset{rev}{\twoheadrightarrow} \Delta'_2) \circ \Psi(\Sigma \twoheadrightarrow \Delta'_1) \end{aligned} \tag{4.25}$$

We say that $\Psi : \mathcal{C} \to \mathcal{D}$ is *causally reversible*—or simply *reversible*—if $\mathcal{C}$ is causally reversible and Ψ admits a causal reversal.

Proposition 4.33. *Let* CauchySlices$(\mathcal{F})$ *be the category of slices on a causal order* Ω *generated by some foliation* $\mathcal{F}$*. Then* CauchySlices$(\mathcal{F})$ *is always causally reversible and for any two Cauchy slices* Δ, Σ *we have that* $\Delta \twoheadrightarrow \Sigma$ *if and only if* $\Sigma \overset{rev}{\twoheadrightarrow} \Delta$*. Furthermore, if a causal field theory* Ψ : CauchySlices$(\mathcal{F}) \to \mathcal{D}$ *is reversible, then a global state* ρ *is entirely determined by the state* ρ_Σ *on* any *Cauchy slice* $\Sigma \in \mathcal{F}$ *as follows:*

$$\rho_\Delta = \begin{cases} \Psi(\Sigma \twoheadrightarrow \Delta) \circ \rho_\Sigma & \text{if } \Sigma \twoheadrightarrow \Delta \\ \Phi(\Sigma \overset{rev}{\twoheadrightarrow} \Delta) \circ \rho_\Sigma & \text{if } \Delta \twoheadrightarrow \Sigma \end{cases} \tag{4.26}$$

where Φ : CauchySlices$(\mathcal{F})^{rev} \to \mathcal{D}$ *is any causal reversal of* Ψ*.*

Proof. The main observation behind this result is as follows: if Σ, Δ are two Cauchy slices, then the conditions $\Delta \subseteq D^+(\Sigma)$ and $\Sigma \subseteq D^-(\Delta)$ are equivalent. Hence CauchySlices$(\mathcal{F})$ is always causally reversible and $\Delta \twoheadrightarrow \Sigma$ if and only if $\Sigma \overset{rev}{\twoheadrightarrow} \Delta$ for any two Cauchy slices Δ, Σ.

Now let Ψ be causally reversible, let $\Sigma \in \mathcal{F}$ be a Cauchy slice in the foliation and consider any global state ρ. If $\Sigma \twoheadrightarrow \Delta$ for some other Cauchy slice $\Delta \in \mathcal{F}$, then the definition of a global state implies that $\rho_\Delta = \Psi(\Sigma \twoheadrightarrow \Delta) \circ \rho_\Sigma$. If instead $\Delta \twoheadrightarrow \Sigma$, then $\Sigma \overset{rev}{\twoheadrightarrow} \Delta$ and the definition of a global state implies that $\rho_\Sigma = \Psi(\Delta \twoheadrightarrow \Sigma) \circ \rho_\Delta$. But the definition of a causal reverse also implies that:

$$\Phi(\Sigma \overset{rev}{\twoheadrightarrow} \Delta) \circ \rho_\Sigma = \Phi(\Sigma \overset{rev}{\twoheadrightarrow} \Delta) \circ \Psi(\Delta \twoheadrightarrow \Sigma) \circ \rho_\Delta = \Psi(\Sigma \twoheadrightarrow \Sigma) \circ \rho_\Delta = \mathrm{id}_{\Psi(\Sigma)} \circ \rho_\Delta = \rho_\Delta \tag{4.27}$$

Hence the value ρ_Σ completely determines the global state ρ (since the value on all other slices in $\mathrm{CauchySlices}(\mathcal{F})$ is determined by restriction from the value on a corresponding Cauchy slice). $\square$

4.5 Connection to quantum cellular automata

The idea of a cellular automaton was first introduced by von Neumann, aimed at designing a self replicating machine [vN66]. A *Cellular Automaton* (CA) over some finite alphabet A has its state stored as a d-dimensional lattice of values in A, i.e. as a function $\psi : \mathbb{Z}^d \to A$. The state is updated at discrete time steps, each step updated as $\psi^{(t+1)} := F(\psi^{(t)})$ according to some fixed function $F : (\mathbb{Z}^d \to A) \to (\mathbb{Z}^d \to A)$. The function F acts *locally* and *homogeneously*: there is some fixed finite subset $\mathcal{N} \subset \mathbb{Z}^d$ (typically a neighbourhood of $\underline{0} \in \mathbb{Z}^d$) and some function $f : \mathcal{N} \to A$ such that the value of each lattice site $\underline{x}$ at time step $t+1$ only depends on the finitely many values in the subset $\underline{x} + \mathcal{N}$ at time t:

$$F(\psi) := \underline{x} \mapsto f(\psi|_{\underline{x}+\mathcal{N}}) \tag{4.28}$$

A *Quantum Cellular Automaton* (QCA) is a generalization of a CA where the lattice states $\psi : \mathbb{Z}^d \to A$ are replaced by (pure) states in the tensor product of Hilbert spaces $\bigotimes_{\underline{x}\in\mathbb{Z}^d} \mathcal{H}_x$ (all $\mathcal{H}_{\underline{x}}$ finite-dimensional and isomorphic) and the function F is replaced by a unitary $U : \bigotimes_{\underline{x}\in\mathbb{Z}^d} \mathcal{H}_{\underline{x}} \to \bigotimes_{\underline{x}\in\mathbb{Z}^d} \mathcal{H}_{\underline{x}}$, with requirements of locality and homogeneity.

Remark 4.34. There are several slightly different formulations of the infinite tensor product above that can be used, each with its own advantages and disadvantages: though it is not going to be a concern for this work, the authors are partial to the construction by von Neumann [vN39].

An early formulation of the notion of QCA is due to Richard Feynman, in the context of simulations of physics using quantum computers [Fey82]. More recent work on quantum information and quantum causality has shown that the evolution of certain free quantum fields can be recovered as the continuous limit of certain quantum cellular automata (cf. [DP16, Arr19] and references therein). In the final section of this work, we show that our framework is well-suited to capture notions of QCA such as those appearing in the literature. Specifically, our construction encompasses and greatly generalises that presented in [Arr19].

4.5.1 Causal cellular automata

The first requirement in the definition of a QCA is that of *homogeneity*—called "translation invariance" in [Arr19]—i.e. the requirement that the automaton act the same way at all points of spacetime. Because presentations of QCAs are usually given in terms of discrete updates of states on a lattice by means of a unitary U, only the requirement of homogeneity *in space* is usually mentioned. However, such presentations also have homogeneity in time as an implicit requirement, namely in the assumption that the same unitary U be used to update the state at all times.

Instead of updating the state time-step by time-step in a compositional fashion, our formulation of quantum cellular automata will see the entirety of spacetime at once, with states over slices and regions recovered in a decompositional approach. Nevertheless, the requirement of homogeneity for a QCA can still be formulated as a requirement of invariance under certain symmetries of spacetime, so we begin by formulating such a notion of invariance for causal field theories.

Definition 4.35. A *symmetry* on a causal order Ω is an action of a group G on Ω by automorphisms of causal orders, i.e. a group homomorphism $G \to \mathrm{Aut}_{\mathrm{CausOrd}}[\Omega]$. If $\mathcal{C}$ is a category of slices on Ω, a *symmetry* on $\mathcal{C}$ is a symmetry on Ω which extends to an action on $\mathcal{C}$ by partially monoidal functors, i.e. one such that the following conditions are satisfied:

(1) for all $g \in G$, if $\Sigma \in \mathrm{obj}(\mathcal{C})$ then $g(\Sigma) \in \mathrm{obj}(\mathcal{C})$;

(2) for all $g \in G$ and all $\Sigma, \Gamma \in \mathrm{obj}(\mathcal{C})$, if $\Sigma \twoheadrightarrow \Gamma$ then $g(\Sigma) \twoheadrightarrow g(\Gamma)$;

(3) for all $g \in G$ and all $\Sigma, \Gamma \in \mathrm{obj}(\mathcal{C})$, if $\Sigma \otimes \Gamma$ is defined in $\mathcal{C}$ then $g(\Sigma \otimes \Gamma) =$

$g(\Sigma) \otimes g(\Gamma)$ is also defined in $\mathcal{C}$.

Note, for all $g \in G$, that $g(\emptyset) = \emptyset$ and that $g(\Sigma)$ is automatically a slice whenever Σ is a slice.

Definition 4.36. Let $\mathcal{C}$ is a category of slices with a symmetry action of a group G. A *G-invariant* (or simply *symmetry-invariant*) causal field theory on $\mathcal{C}$ is a causal field theory $\Psi : \mathcal{C} \to \mathcal{D}$ equipped with a family of natural isomorphisms $\Psi \overset{\alpha_g}{\Rightarrow} \Psi \circ g$ such that $\alpha_{h \cdot g} = \alpha_h g \circ \alpha_g$, where we have again identified elements $g \in G$ with their action as partially monoidal functors $g : \mathcal{C} \to \mathcal{C}$.

Remark 4.37. The spirit behind the definition of symmetry-invariant causal field theories is that the functors Ψ (sending slices $\mapsto$ fields) and $\Psi \circ g$ (sending slices $\mapsto$ g-translated slices $\mapsto$ fields) should be the same. However, we have remarked when first defining causal field theories that—be it for ease of physical interpretation or for conformity with existing literature on causal categories—it may sometimes be desirable that the images $\Psi(\Sigma)$ of different slices be different. Not being able to impose the equality $\Psi = \Psi \circ g$ in such a setting, the next best thing is to ask for natural isomorphism $\Psi \cong \Psi \circ g$.

Because we are dealing with symmetries, however, it is sensible to require for the natural isomorphisms α_g themselves to respect the group structure. Again the first instinct might be to require something in the form $\alpha_{h \cdot g} = \alpha_h \circ \alpha_g$, but this expressions does not type-check: we have a natural transformation $\alpha_{h \cdot g} : \Psi \Rightarrow \Psi \circ h \circ g$, a natural transformation $\alpha_g : \Psi \Rightarrow \Psi \circ g$ and a natural transformation $\alpha_h : \Psi \Rightarrow \Psi \circ h$. In order to compose α_h and α_g we instead have to take the action of α_h *translated* to $\Psi \circ g$:

$$\alpha_h g : \Psi \circ g \Rightarrow (\Psi \circ h) \circ g \tag{4.29}$$

Explicitly, the natural transformation $\alpha_h g$ is defined by $(\alpha_h g)(\Sigma) := \alpha_h(g(\Sigma))$.

The second requirement in the definition of a QCA is that of *locality* (or *causality*). When quantum cellular automata are considered in a relativistic context—

e.g. as discrete models of quantum field theories—the requirement of locality is meant to capture the idea that the action of the automaton should respect the causal structure of spacetime (so that the state on a point $\underline{x}$ at time $t+\Delta t$ should not depend on the state at the previous time t on points $\underline{y}$ which are "too far away", i.e. such that $(\underline{x}, t+\Delta t)$ and $(\underline{y}, t)$ are space-like separated).

In [Arr19], the requirement of locality is formulated as the requirement that the output state of the automaton over a point $\underline{x}$ of the lattice at time $t+1$ only depend on the state over a finite neighbourhood $\underline{x}+\mathcal{N}$ at time t: this is both in terms of local state (causality) and in the stronger sense that the field evolution should factor into a product of local maps (localisability). In our framework, on the other hand, causality and localisability are both automatically enforced: the field evolution always factors over space-like separated regions, as a consequence of monoidality, and the local state over a slice never depends on the state on any other slice which is space-like separated from it (as a consequence of factorisation).

Remark 4.38. The causal order Ω which captures the causality requirement from [Arr19] with finite neighbourhood $\mathcal{N} \subset \mathbb{Z}^d$ can be constructed by endowing the set $|\Omega| := \mathbb{Z}^d \times \mathbb{Z}$ with the reflexive-transitive closure of the relation $(\underline{y}, t) \leq (\underline{x}, t+1)$ for all times $t \in \mathbb{Z}$, for all points of the lattice $\underline{x} \in \mathbb{Z}^d$ and for all points $\underline{y} \in \underline{x}+\mathcal{N}$ in the neighbourhood of $\underline{x}$.

The third and final requirement in the definition of a QCA is that of *unitarity*. In our framework, this is a problem for two (mostly unrelated) reasons.

- Our formulation of causal field theories aims to be agnostic to the choice of process theory. On the other hand, unitarity is a strongly quantum-like feature, the formulation of which would require a significant amount of additional structure on the field category.

- The usual formulation of quantum cellular automata only considers global evolution, never directly dealing with restrictions—situations e.g. in which the state is evolved unitarily but part of the output state is discarded as environment. Our framework instead treats such restrictions as an integral part of evolution.

Luckily, unitarity *per se* is not necessary from an abstract foundational standpoint: the real feature of interest is *reversibility*, a feature of causal field theories which we have already explored. For the sake of generality, we will not include reversibility in the definition below, leaving it as an explicit desideratum.

Remark 4.39. In categories of Hilbert spaces and completely positive maps, it is legitimate to imagine that causality and reversibility would jointly imply that the cellular automata also be unitary. This is indeed the case under the conditions of Proposition 4.33: because the state on any Cauchy slice automatically determines the state on all the other slices—and because that state on a single slice is arbitrary—evolution between Cauchy slices must be unitary.

Definition 4.40. A *Causal Cellular Automaton* (CCA) consists of the following ingredients.

(1) A foliation $\mathcal{F}$ on a causal order Ω.

(2) A category of Cauchy slices $\mathcal{C}$ such that each slice in $\mathcal{C}$ is a subset of some Cauchy slice in $\mathcal{F}$. [4]

(3) A symmetry action of a group G on $\mathcal{C}$, inducing—via the G-action on Ω—a transitive action of G on the Cauchy slices in the foliation $\mathcal{F}$.

(4) A G-invariant causal field theory $\Psi : \mathrm{CauchySlices}(\mathcal{F}) \to \mathcal{D}$.

A *reversible* CCA is one where the causal field theory Ψ is reversible.

Definition 4.40 is much more general than the definition of QCA from [Arr19] and hence captures more sophisticated examples. However, its ingredients are directly analogous to those appearing in that definition of a QCA.

[4]Each Cauchy slice Σ in $\mathcal{F}$ is then automatically the union of all slices $\Delta \in \mathrm{obj}(\mathcal{C})$ such that $\Delta \subseteq \Sigma$.

- The foliation $\mathcal{F}$ on Ω generalises the discrete time steps in the definition of a QCA.
- The slices in $\mathcal{C}$ generalise the equal-time hyper-surfaces which support the state of a QCA at fixed time.
- The symmetry action of G on $\mathrm{CauchySlices}\,(\mathcal{F})$ and its transitivity on the foliation $\mathcal{F}$ generalise homogeneity in both space and in time of the lattices supporting a QCA.
- The G-invariance of the causal field theory Ψ generalises both the translation symmetry in space and the time-translation symmetry of a QCA.

4.5.2 Partitioned causal cellular automata

We now proceed to construct a large family of examples of CCAs based on the *partitioned QCAs* of [Arr19]. In doing so, we generalise the scattering unitaries to arbitrary processes and allow for the definition of state restriction to non-Cauchy equal-time surfaces. We refer to the resulting CCA as *partitioned CCA*.

4.5.2.1 Causal order

As our causal order Ω we consider the following subset of $(1 + d)$-dimensional Minkowski spacetime (setting the constant c for the speed of light to $c = \sqrt{d}$):

$$\Omega := \left\{ (t, \underline{x}) \,\middle|\, t \in \mathbb{Z}, \underline{x} \in (t, ..., t) + 2\mathbb{Z}^d \right\} \tag{4.30}$$

where $(t, ..., t) + 2\mathbb{Z}^d$ is the set of all $\underline{x} \in \mathbb{Z}^d$ such that $x_i = t \pmod 2$. For $d = 1$ we get the $(1 + 1)$-dimensional diamond lattice discussed before. In general, the immediate causal predecessors of a point $(t, \underline{x})$ are the following 2^d points:

$$(t - 1, \underline{x} - \mathcal{N}) = \{\, (t - 1, \underline{x} - \underline{\delta}) \,|\, \underline{\delta} \in \mathcal{N} \,\} \tag{4.31}$$

where we defined the "neighbourhood" $\mathcal{N} := \{\pm 1\}^d$. Similarly, the immediate successors of $(t, \underline{x})$ are the following 2^d points:

$$(t+1, \underline{x} + \mathcal{N}) = \{\, (t-1, \underline{x} + \underline{\delta}) \,|\, \underline{\delta} \in \mathcal{N} \,\} \tag{4.32}$$

4.5.2.2 Foliation and category of slices

The causal order Ω admits a foliation $\mathcal{F}$ where each slice is a constant-time Cauchy slice Σ_t for some $t \in \mathbb{Z}$:

$$\Sigma_t := \left\{ (t, \underline{x}) \,\middle|\, \underline{x} \in (t, ..., t) + 2\mathbb{Z}^d \right\} \tag{4.33}$$

A suitable category of slices $\mathcal{C}$ to associate to this foliation is given by taking as slices all the finite sets $\Sigma_{t,\mathcal{X}} \subset \Sigma_t$ of events having the same time coordinate t:

$$\Sigma_{t,\mathcal{X}} = \{\, (t, \underline{x}) \,|\, \underline{x} \in \mathcal{X} \,\} \tag{4.34}$$

where $\mathcal{X} \subset (t, ..., t) + 2\mathbb{Z}^d$ is some finite subset. The morphisms $\twoheadrightarrow$ of $\mathcal{C}$ are given as follows for $k \geq 0$:

$$\Sigma_{t,\mathcal{X}} \twoheadrightarrow \Sigma_{t+k,\mathcal{Y}} \qquad \text{if and only if} \qquad \bigcup_{\underline{y} \in \mathcal{Y}} \left(\left(t, \underline{y} + \mathcal{N}^{(k)} \right) \right) \subseteq \mathcal{X} \tag{4.35}$$

where the "iterated neighbourhood" $\mathcal{N}^{(k)}$ is defined as $\mathcal{N} + ... + \mathcal{N}$ by adding together $k \geq 0$ copies of $\mathcal{N}$ (and we set $\mathcal{N}^{(0)} := \{0\}$). Explicitly we have:

$$\mathcal{N}^{(k)} := \begin{cases} \{-k, -k+2, ... -1, +1, ..., k-2, k\} & \text{if } k \text{ odd} \\ \{-k, -k+2, ... -2, 0, +2, ..., k-2, k\} & \text{if } k \text{ even} \end{cases} \tag{4.36}$$

It is easy to check (by a $t \mapsto -t$ symmetry argument) that $\mathcal{C}$ is reversible.

4.5.2.3 Symmetry

The category $\mathcal{C}$ admits a symmetry action of the group $G := \mathbb{Z}^{\mathcal{N}} \cong \mathbb{Z}^{2^d}$. We index the coordinates of vectors in $\mathbb{Z}^{\mathcal{N}}$ by the 2^d points $\underline{\delta} \in \mathcal{N} = \{\pm 1\}^d$. We denote by $\tau_{\underline{\delta}}$ the vector in $\mathbb{Z}^{\mathcal{N}}$ which is 1 at the coordinate labelled by $\underline{\delta}$ and 0 at all other coordinates. The action is then specified by setting:

$$\tau_{\underline{\delta}}(t, \underline{x}) := (t+1, \underline{x} - \underline{\delta}) \tag{4.37}$$

That is, the 2^d generators of $\mathbb{Z}^{\mathcal{N}}$ send a generic event $(t, \underline{x})$ to each of its 2^d immediate causal successors in Ω, one for each possible choice of sign ± 1 along each of the d directions of the space lattice $\mathbb{Z}^d$. [5] Each generator $\tau_{\underline{\delta}}$ for the symmetry action sends a Cauchy slice Σ_t in the foliation to the next Cauchy slice Σ_{t+1}, so the action of G on the foliation is transitive.

4.5.2.4 Causal field theory - field over slices

As our field category we consider a generic causal process theory $\mathcal{D}$, i.e. a symmetric monoidal category equipped with a family of *discarding maps* $\bar{\top}_{\mathcal{H}} : \mathcal{H} \to I$ for all objects $\mathcal{H} \in \mathrm{obj}(\mathcal{D})$, respecting the tensor product $\otimes$ and tensor unit I of $\mathcal{D}$: $\bar{\top}_{\mathcal{H}\otimes\mathcal{K}} = \bar{\top}_{\mathcal{H}} \otimes \bar{\top}_{\mathcal{K}}$ and $\bar{\top}_I = 1$. Discarding maps generalise the partial trace of quantum theory: normalised states $\rho : I \to \mathcal{H}$—generalising density matrices—are defined to be those such that $\bar{\top}_{\mathcal{H}} \circ \rho = 1$ and normalised morphisms $U : \mathcal{H} \to \mathcal{K}$—generalising CPTP maps—are defined to be those such that $\bar{\top}_{\mathcal{K}} \circ U = \bar{\top}_{\mathcal{H}}$. See e.g. [GS17, CL13, CK17] for more information.

To create a G-invariant causal field theory Ψ, we consider some object $\mathcal{H} \in \mathrm{obj}(\mathcal{D})$ together with some endomorphism $U : \mathcal{H}^{\otimes 2^d} \to \mathcal{H}^{\otimes 2^d}$, which we will refer to as the *scattering map*. For reasons that will soon become clear, it is more convenient to index the factors of $\mathcal{H}^{\otimes 2^d}$ by the 2^d points in the neighbourhood $\mathcal{N}$, hence writing $U : \mathcal{H}^{\otimes \mathcal{N}} \to \mathcal{H}^{\otimes \mathcal{N}}$.

[5] The reason for the negative sign in $\underline{x} - \underline{\delta}$ is that $\mathcal{N}$ was originally defined to be the neighbourhood *in the past*.

We define the action of Ψ on the slices in $\mathcal{C}$ as follows:

$$\Psi(\Sigma_{t,\mathcal{X}}) := \left(\mathcal{H}^{\otimes\mathcal{N}}\right)^{\otimes\mathcal{X}} = \mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{X})} \tag{4.38}$$

The tensor product is well-defined in all symmetric monoidal categories, since $\mathcal{X}$ is always finite. Physically, the field takes values in a copy of $\mathcal{H}^{\otimes\mathcal{N}}$ over each event $(t,\underline{x})$ of spacetime, each individual $\mathcal{H}$ factor of $\mathcal{H}^{\otimes\mathcal{N}}$ encoding the contribution to the field state at $(t,\underline{x})$ from the field state at each of its immediate causal predecessors in $(t-1,\underline{x}+\mathcal{N})$.

4.5.2.5 Causal field theory - restriction and evolution

From their definition in Equation 4.35, it is easy to see that morphisms $\Sigma_{t,\mathcal{X}_0} \twoheadrightarrow \Sigma_{t+k,\mathcal{X}_k}$ on $\mathcal{C}$ can always be factored in the following way:

$$\Sigma_{t,\mathcal{X}_0} \twoheadrightarrow \Sigma_{t,\mathcal{Y}_0} \twoheadrightarrow \Sigma_{t+1,\mathcal{X}_1} \twoheadrightarrow \Sigma_{t+1,\mathcal{Y}_1} \twoheadrightarrow \ldots \twoheadrightarrow \Sigma_{t+k,\mathcal{X}_k} \tag{4.39}$$

where $\mathcal{Y}_i \subseteq \mathcal{X}_i$ for all $i = 0,...,k-1$ and the following holds for each $i = 1,...,k$:

$$\mathcal{Y}_{i-1} = \bigcup_{\underline{x}\in\mathcal{X}_i} \{\, (t+i-1, \underline{x}+\underline{\delta}) \,|\, \underline{\delta} \in \mathcal{N} \,\} \tag{4.40}$$

This means that we only need to care about the action of Ψ on two kinds of morphisms:

- the *restrictions* $\Sigma_{t,\mathcal{X}} \twoheadrightarrow \Sigma_{t,\mathcal{Y}}$, where $\mathcal{Y} \subseteq \mathcal{X}$;
- the *1-step evolutions* $\Sigma_{t,\mathcal{Y}} \twoheadrightarrow \Sigma_{t+1,\mathcal{X}}$, where $\mathcal{Y} = \bigcup_{\underline{x}\in\mathcal{X}} \{\, (t, \underline{x}+\underline{\delta}) \,|\, \underline{\delta} \in \mathcal{N} \,\}$.

The existence of the factorisation above can be proven by induction, observing that any morphism $\Sigma_{t,\mathcal{X}_0} \twoheadrightarrow \Sigma_{t+1,\mathcal{X}_1}$ factors into the product:

$$\left(\Sigma_{t,\mathcal{Y}_0} \twoheadrightarrow \Sigma_{t+1,\mathcal{X}_1}\right) \otimes \left(\Sigma_{t,\mathcal{X}_0\backslash\mathcal{Y}_0} \twoheadrightarrow \emptyset\right) \tag{4.41}$$

where $\mathcal{Y}_0$ is defined as before so that $\Sigma_{t,\mathcal{Y}_0}$ is exactly the set of immediate causal predecessors of the codomain $\Sigma_{t+1,\mathcal{X}_1}$.

On restrictions $\Sigma_{t,\mathcal{X}} \twoheadrightarrow \Sigma_{t,\mathcal{Y}}$, where $\mathcal{Y} \subseteq \mathcal{X}$, the functor Ψ is defined to act by marginalisation, discarding the field state over all those events in the larger slice $\Sigma_{t,\mathcal{X}}$ which don't belong to the smaller slice $\Sigma_{t,\mathcal{Y}}$:

$$\Psi(\Sigma_{t,\mathcal{X}} \twoheadrightarrow \Sigma_{t,\mathcal{Y}}) := \bigotimes_{\underline{x}\in\mathcal{X}} F_{\underline{x}} \quad \text{where} \quad F_{\underline{x}} := \begin{cases} \mathrm{id}_{\mathcal{H}^{\otimes\mathcal{N}}} & \text{if } \underline{x} \in \mathcal{Y} \\ \uparrow_{\mathcal{H}^{\otimes\mathcal{N}}} & \text{if } \underline{x} \notin \mathcal{Y} \end{cases} \tag{4.42}$$

On 1-step evolutions $\Sigma_{t,\mathcal{Y}} \twoheadrightarrow \Sigma_{t+1,\mathcal{X}}$, where $\mathcal{Y} = \bigcup_{\underline{x}\in\mathcal{X}} \{\, (t, \underline{x} + \underline{\delta}) \,|\, \underline{\delta} \in \mathcal{N} \,\}$, the functor Ψ is defined to act by a combination of evolution by U and marginalisation. The evolution component is simply an application of U to the state at each event of $\mathcal{Y}$:

$$U^{\otimes\mathcal{Y}} : \mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{Y})} \to \mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{Y})} \tag{4.43}$$

The marginalisation component then needs to go from the codomain $\mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{Y})}$ of the map above to the desired codomain $\mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{X})}$. To do this, we recall that the $\mathcal{H}$ factor of $\mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{X})}$ corresponding to a given $\underline{\delta} \in \mathcal{N}$ and a given $\underline{x} \in \mathcal{X}$ is intended to encode the component of the state at $(t+1, \underline{x})$ coming from $(t, \underline{x} + \underline{\delta})$. Analogously, the $\mathcal{H}$ factor of $\mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{Y})}$ corresponding to a given $\underline{\delta} \in \mathcal{N}$ and a given $\underline{y} \in \mathcal{Y}$ is intended to encode the component of the evolved state going to $(t+1, \underline{y} - \underline{\delta})$. Hence to go from $\mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{Y})}$ to $\mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{X})}$ we need to discard all factors in $\mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{Y})}$ corresponding to components of the evolved state which are not going to some $(\underline{y} - \underline{\delta}) \in \mathcal{X}$:

$$\left(\bigotimes_{(\underline{\delta},\underline{y})\in\mathcal{N}\times\mathcal{Y}} F_{\underline{\delta},\underline{y}} \right) : \mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{Y})} \to \mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{X})} \quad \text{where} \quad F_{\underline{\delta},\underline{y}} := \begin{cases} \mathrm{id}_{\mathcal{H}} & \text{if } (\underline{y} - \underline{\delta}) \in \mathcal{X} \\ \uparrow_{\mathcal{H}} & \text{if } (\underline{y} - \underline{\delta}) \notin \mathcal{X} \end{cases} \tag{4.44}$$

Putting the evolution and marginalisation components together we get the action of Ψ on 1-step evolutions:

$$\Psi\left(\Sigma_{t,\mathcal{Y}} \twoheadrightarrow \Sigma_{t+1,\mathcal{X}}\right) := \left(\left(\bigotimes_{(\underline{\delta},\underline{y})\in\mathcal{N}\times\mathcal{Y}} F_{\underline{\delta},\underline{y}}\right) \circ U^{\otimes\mathcal{Y}}\right) : \mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{Y})} \to \mathcal{H}^{\otimes(\mathcal{N}\times\mathcal{X})} \quad (4.45)$$

By construction, the above is a G-invariant causal field theory, completing the definition of our partitioned causal cellular automaton. If U is an isomorphism, the same construction on $\mathcal{C}^{rev}$ using U^{-1} provides a causal reversal for Ψ, showing that the partitioned causal cellular automata above is reversible under those circumstances. Finally, Figure 4.5 below depicts an example of action on morphisms for a $(1+1)$-dimensional partitioned causal cellular automaton.

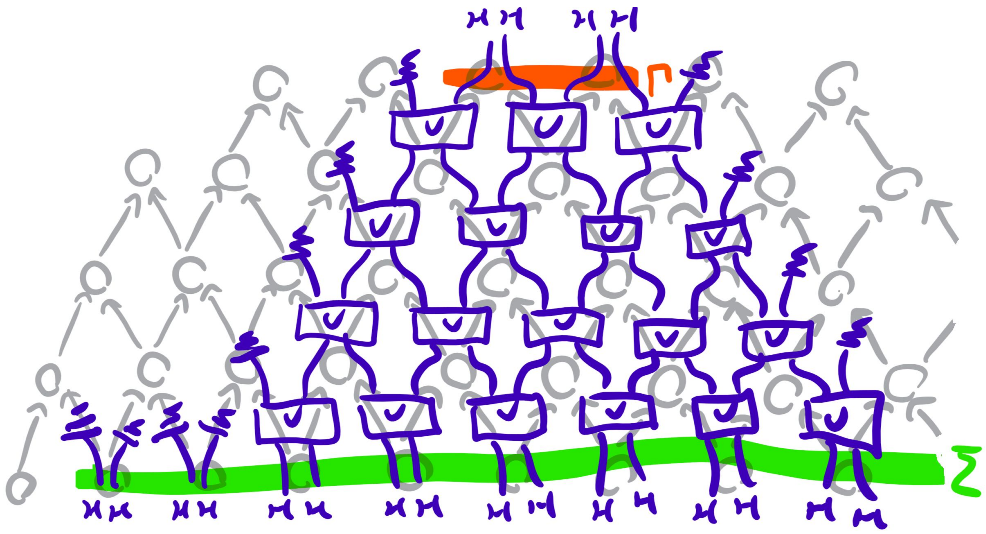


Figure 4.5: Action of a partitioned causal cellular automaton over a complicated morphism $\Sigma \twoheadrightarrow \Gamma$ in the $(1+1)$-dimensional example of the diamond lattice. Here $\mathcal{N} = \{\pm 1\}$, so each event in the causal order is associated to a copy of $\mathcal{H}^{\otimes\mathcal{N}} \cong \mathcal{H}\otimes\mathcal{H}$. The restriction action of the CCA (Equation 4.42) can be seen on the two events at the bottom left. The pure evolution action of the CCA (Equation 4.43) can be seen on the central pyramid of ten events, as the application of U without discarding. The evolution + marginalisation action of the CCA (Equation 4.45) can be seen on the eight events at the sides of the central pyramid, as the application of U followed by discarding of one of the two outputs. The input of the morphism depicted consists of eight copies of $\mathcal{H}\otimes\mathcal{H}$, one for each event of Σ, while the output of the morphism depicted consists of two copies of $\mathcal{H}\otimes\mathcal{H}$, one for each event of Γ[GSC21].

Outlook

The broad aim of this project was to study causality and the arrow of time in quantum process theories and beyond. In particular, we developed different approaches to time-symmetrize quantum theory from a process-theoretic perspective both by taking into account (retro)causality constraints and by completely removing them (Chapter 2). Along these lines we developed a toy model for particle physics by treating causal systems as particles and retrocausal systems as antiparticles.

We furthermore argued that process theories can better realized as certain kinds of operad algebras (Chapter 3). The versatility of the operadic framework allowed us to capture causal as well as time neutral process theories. To gain further intuition, we established a connection between compact closed categories and time neutral process theories. To our knowledge, this is the first application of operadic tools for the foundations of physics. The diagrammatic representation that we have established for operads and their algebras aims at making those mathematical tools accessible to a wider audience.

Finally, we created a compositional algebraic framework to study the evolution of fields, realized as causal process theories, in discretised spacetimes (Chapter 4). The highlight of this work is that the notion of the 'field' is theory-independent and thus applies to theories more general than quantum. We establish a link with algebraic quantum field theory and cellular automata. In particular, our cellular automata treat the interaction with the environment as an integral part of evolution, generalising those from existing literature, where evolution is unitary.

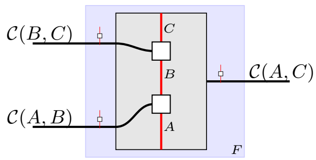
The results of this thesis suggest many possible research avenues. We will mention some of them.

In Chapter 2 we created a toy model for particle physics by perceiving particles as finite representations of a group G. In the literature, however, particles are defined as the irreducible representations of the Poincare group, which has infinite dimensionality. We would therefore like to extend our toy model to the infinite dimensional case. This will provide a new perspective to particle physics, since it will allow particles to interact with classical systems and furthermore to be in mixed states. Furthermore, we can ask whether it is possible to acquire a reconstruction of quantum theory that reproduces the process theory **QNeut**, rather than **QPhys**.

In Chapter 3 we define the operad algebras as functors $F : W \to \text{Set}$. The image of F determines the homset of the symmetric monoidal category that is constructed. However, there are many times where the process theories that we are interested are enriched. That said, processes from a particular set of inputs to outputs do not form a set, but have additional structure. We conjecture that enriched process theories could be captured by changing the codomain of F from Set to the category of interest.

A particularly useful example comes from the study of operational theories, in which we are not just interested in which processes are occuring, but in what we know about which processes are occuring. This naturally means that the hom-sets have the structure of a simplex describing the set of states of knowledge that one can have about the transformation.

In order to capture this situation we define the operad Stoch in which objects are finite sets and operations are stochastic maps between these. Then, a novel kind of operad algebra is a functor $F : W \to \mathsf{Stoch}$. On objects, it conveys the same information with the standard operad algebras. That said, it simply conveys the set of possibilities for what a box can be. However, on operations, we gain additional flexibility, since now they are mapped to stochastic processes rather than functions. For example:



is a stochastic map which tells us how to propagate our beliefs about the transformation from A to B and the transformation from B to C to a belief about how A transforms into C, given that the output of the first transformation is fed into the input of the second.

This view seems to be closely related with the framework of causal inferential theories in Ref. [SSS20], a connection we would like to explore further. In particular, one of the main research questions in Ref. [SSS20] is to understand how non-classical theories of inference could be defined to obtain a realist account of quantum theory. While this is not clear how to achieve for the full causal-inferential framework, we conjecture that it could be handled within the operadic framework by simply changing the codomain of the operad algebra. In particular, rather than taking the codomain as Stoch we can take it to be **CPM**. In that way, we are dealing with quantum states of knowledge about transformations, rather than classical.

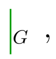
Finally, we believe that the connection with AQFT in Chapter 4 can be strengthened to the point that it will be a tool for the construction of new models. Furthermore, we would like to explore the possibilities in working in the continuum limit of QCAs and relate them to perturbative quantum field theory. Finally, it would be interesting to see how time neutral process theories behave in a spacetime context and principles like non-signalling maifest themselves in this context.

Appendix

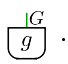
4.6 Diagrams for groups and their representations

4.6.1 Diagrams for groups

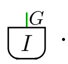
We will denote a group G by a new kind of wire,

 (4.46)

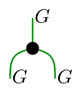
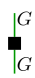
such that the elements of the group, $g \in G$ are then states of this wire,

 (4.47)

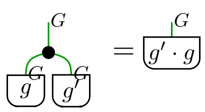
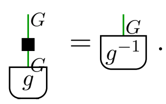
In particular, we denote the identity element of the group as

 (4.48)

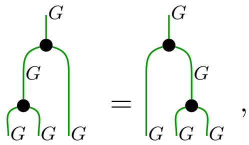
We then introduce the defining operations for a group, namely, group multiplication and group inverse as

 and  (4.49)

respectively. These can be defined by their action on the group elements via:

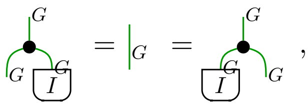
 and  (4.50)

Various important properties of groups can be elegantly captured using these diagrams. These include associativity of multiplication, i.e.



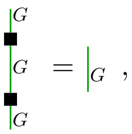
(4.51)

that the identity element is the unit, i.e.



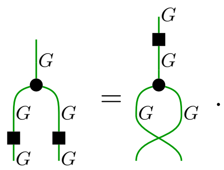
(4.52)

that the group inverse is idempotent, i.e.



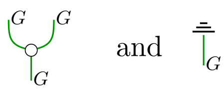
(4.53)

and that multiplication is “antisymmetric”, i.e.



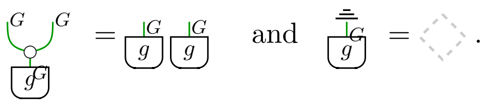
(4.54)

To capture more of the group structure diagrammatically, we introduce a copy map and a discard map for these new systems,denoted as



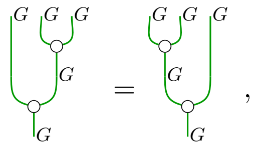
(4.55)

respectively. These too can be defined by their action on group elements via:



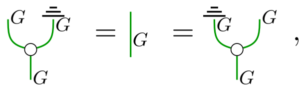
(4.56)

Important properties of these can also be captured diagrammatically such as: associativity of copy,



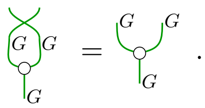
(4.57)

that the discard is the counit for copy,



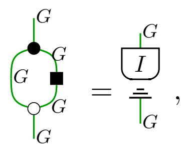
(4.58)

and that copying is symmetric



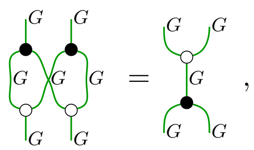
(4.59)

Finally, we can use the interaction of these new processes with the group multiplication and inverse in order to capture more structure of the group. In particular, the equation



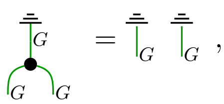
(4.60)

ensures that the group inverse behaves as expected. Moreover, one can show that copying and multiplication form a bialgebra, i.e.



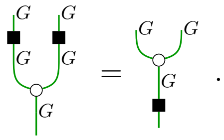
(4.61)

group multiplication is causal, i.e.



(4.62)

and that the inverse is copied by the copy operation i.e.

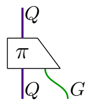


(4.63)

4.6.2 Group representations

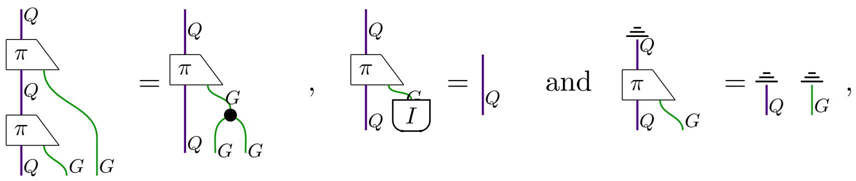
We exploit the above material to define group representations on systems in **QCalc** (or **QPhys**) as an interaction between the new systems G and systems from **QCalc**. Formally we can think of the systems G as being particular (potentially infinite dimensional) classical systems where the point distributions correspond to the group elements.

A *causal* group representation of G on some system Q in **QCalc** is a process, π, of the form



(4.64)

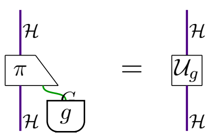
such that the following equations are satisfied:



(4.65)

The first two equations guarantee that this is a valid representation and the third guarantees that this is a causal representation. Note that if we are working with **QPhys** rather than **QCalc** then this last condition is automatically satisfied.

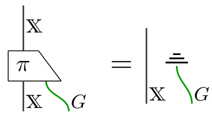
If Q is strictly quantum, that is, a system $\mathcal{H}$, then these causal representations are necessarily unitary representations. In other words, they satisfy the following equation



$:: \rho \mapsto U_g \rho U_g^\dagger$ (4.66)

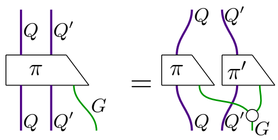
for all $g \in G$. To see this note that the axioms for group representations imply that the processes on the left hand side are necessarily reversible, and reversible CPTP maps are necessarily unitary supermaps.

For finite dimensional classical systems $\mathbb{X}$ with a non-finite group G, the only possible representation is the trivial representation, namely:



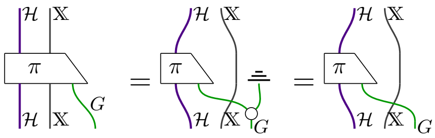
(4.67)

We can compose representations of single systems to define representations of composite systems via:



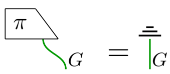
(4.68)

Thus, if we compose a quantum and a classical representation we end up with a representation:



(4.69)

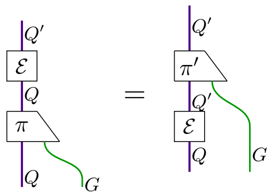
That is, the quantum part of the composite system may transform non-trivially under the action of the group, contrary to the classical part, which is left invariant. Finally, note that the representation on a trivial system is necessarily trivial, that is:



(4.70)

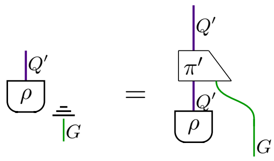
4.6.3 Intertwiners

In this section we introduce *intertwiners*. They are processes in **QCalc** (or **QPhys**) that are symmetric with respect to the group. In particular, they are characterised by the covariance condition:

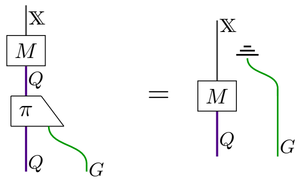


(4.71)

This implies that states and measurements that are intertwiners are invariant under the group action, namely:



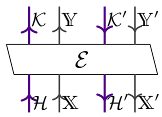
and



(4.72)

4.6.4 Dual systems and conjugate representations

Finally, we introduce dual systems and their representations. Systems in **QCalc** have duals, which we will now explicitly denote with arrows. That is, a generic process is represented as



(4.73)

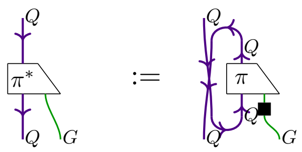
and its symbolic notation is the CP map

$$\mathcal{E} : \mathcal{B}[\mathcal{H}]\otimes\left(\bigoplus_{x\in\mathbb{X}}\mathcal{B}[\mathbb{C}]\right)\otimes\mathcal{B}[\mathcal{H}']^*\otimes\left(\bigoplus_{x'\in\mathbb{X}'}\mathcal{B}[\mathbb{C}]\right)^* \to \mathcal{B}[\mathcal{K}]\otimes\left(\bigoplus_{y\in\mathbb{Y}}\mathcal{B}[\mathbb{C}]\right)\otimes\mathcal{B}[\mathcal{K}']^*\otimes\left(\bigoplus_{y'\in\mathbb{Y}'}\mathcal{B}[\mathbb{C}]\right)^* . \tag{4.74}$$

That is, when the arrow is pointing up we use the primal vector space, and when the arrow is pointing down we use the dual vector space.

Given a representation on a system $\mathcal{Q}^{\uparrow}$ we define the conjugate representation,

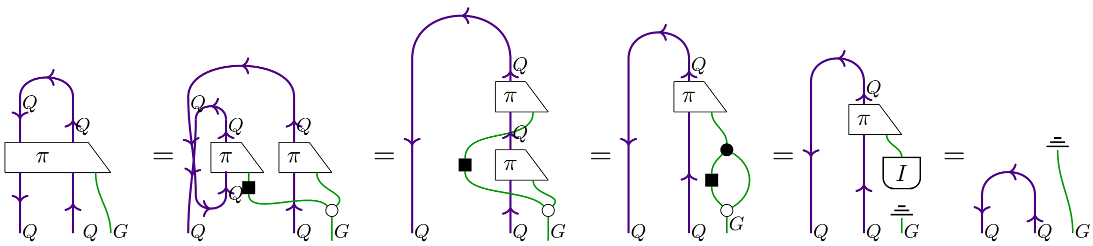
π^* on the system $\mathcal{Q}^\downarrow$ as follows:



(4.75)

Note that if $Q^\uparrow$ is classical, $\mathbb{X}^\uparrow$, then as the representations on $\mathbb{X}^\uparrow$ were necessarily trivial so is the conjugate representation on $\mathbb{X}^\downarrow$.

It is now easy to see that cups and caps are intertwiners, for composite representations on the inputs. That is,



(4.76)

In the first equality we use the assumption that the representation on the input is a composite of representations and the above definition of conjugate representation. In the second equality we are using the definition of cups and caps. In the third we use the definition of a representation. In the fourth we use one of the key results about groups, and the final stems again from the definition of a representation.

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