Meaning and Duality

From Categorical Logic to Quantum Physics

Yoshihiro Maruyama

Wolfson College

University of Oxford

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[A]dvances forged by category theorists will be of value to [..] philosophy, lending precise form [..] to ancient philosophical distinctions such as [..] objective vs. subjective, being vs. becoming, space vs. quantity [..] (William Lawvere, 1992)

Myth and art, language and science, are in this sense shapings toward being; they are not simple copies of an already present reality, but they rather present the great lines of direction of spiritual development, of the ideal process, in which reality constitutes itself for us as one and many – as a manifold of forms, which are nonetheless finally held together by a unity of meaning. (Ernst Cassirer, 1923; translation by Michael Friedman in his *A Parting of the Ways*)

To my family.

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Abstract

Duality abounds in science, both pure and applied: in mathematics there is duality between space and algebra; in physics there is duality between states and observables; in information science there is duality between systems and properties/behaviours; in logic there is duality between models and theories; and even in philosophy there is duality between realism and antirealism (Dummett), or substance metaphysics and process/function metaphysics (Whitehead/Cassirer). They look akin at a level of abstraction, and yet, taking a closer look at subtleties involved, one may find a parting of the ways. The present thesis builds upon category theory and universal algebra to explicate and articulate the dynamics of duality, the mechanism of how duality emerges, changes, and breaks. A generic duality theory is first pursued in the abstract via categorical topology and algebra, and subtler duality theories are then explored to the end of shedding light on more nuanced facets of concrete dualities such as coalgebraic dualities. Each duality theory thus developed is cashed out to uncover formerly unknown dualities; e.g., the first duality theory tells us a dual equivalence between domains and convex structures, which remained an open problem before. Dualities are often induced by Janusian objects, which sometimes form truth value objects as in topos theory. To elucidate this link between duality and categorical logic, categorical universal logic is developed on the basis of Lawvere's hyperdoctrine and Hyland-Johnstone-Pitts' tripos, thereby expanding the realm of (first-order/higher-order) categorical logic so as to encompass, *inter alia*, classical, intuitionistic, quantum, fuzzy, relevant, and linear logics. Finally, duality meets symmetry in quantum physics, yielding a categorical understanding of the Wigner theorem, i.e., a purely coalgebraic characterisation of the quantum symmetry groupoid. In passing, operational quantum duality is systematised via Chu space theory. In light of these, duality would arguably serve as a unifying principle to overcome the fragmentation of science, and to soothe the dualistic divide of philosophy between realist and antirealist worldviews.

Contents

Chapter 1

Introduction

The major concern of the present thesis lies in elucidating the dynamics of duality through the exploration of the way how duality emerges, changes (and breaks). A direct answer to the question shall be given in the concluding chapter. In this introductory chapter let us start the discussion with a conceptual account of dualism, duality, and disduality in the abstract; disduality is basically meant to be the absence of duality, or duality-breaking. Dualism and duality are related to each other, and yet surely different (how they are different is not that obvious, though). The following account of the three concepts would be slightly lengthy, and so, **if the reader in a rush would rather like to have a quick bird's-eye view of the thesis, it would be advised to skip the entire discussion on dualism, duality, and disduality, and immediately proceed to the synopsis Section 1.4 below**, in which everything you would practically need to know about the thesis, including a rough picture of technicalities involved, is explained as briefly as possible.

1.1 Dualism

Developments of philosophy have centred around a tension between realism and antirealism (or in Gödel's tems between the "right" and the "left"; see Gödel's amusingly bold, philosophical article entitled "The modern development of the foundations of mathematics in the light of philsoophy" [113], which shall be discussed later). And the dualistic tension may be illustrated by asking the nature of a variety of fundamental concepts. What is Meaning? The realist asserts it consists in the correspondence of language to reality, whilst the antirealist contends it lies in the autonomous system or internal structure of language or linguistic practice (cf. the early Wittgenstein [279] vs. the later Wittgenstein [280]; Davidson [75] vs. Dummett [87]). What is Truth? To the realist, it is the correspondence of assertions to facts or states of affairs; to the antirealist, it has no outward reference, constituted by the internal coherency of assertions or by some sort of instrumental pragmatics (cf. Russell vs. Bradley [49]). What is Being? To the realist, it is persistent substance; to the antirealist, it emerges within an evolving process, cognition, structure, network, environment, or context (cf. Aristotle [16] vs. Cassirer/Heidegger/Whitehead [52, 131, 277]). What is Intelligence? To the realist, it is more than behavioural simulation, characterised by the intentionality of mind; to the antirealist, it is fully conferred by copycatting as in the Turing test or the Chinese Room (cf. Searle [250] vs. Turing [268]). What is Space? To the realist, it is a collection of points with no extension; to the antirealist, it is a structure of regions, relations, properties, or information (cf. Newton vs. Leibniz [269]; Cantor/Russell vs. Husserl/Whitehead [28]). To cast these instances of dualism in more general terms, dualism may be conceived of as arising between the epistemic and the ontic, or between the formal and the conceptual in Lawvere's terms [171], as in the figure below:

	Ontic	***Epistemic***	
Descartes	*Matter*	*Mind*	Cartesian Dualism
Kant	*Thing-in-itself*	*Appearance*	Idealism
Cassirer	*Substance*	*Function*	Logical Idealism
Heidegger	*Essence*	*Existence*	Analysis of *Dasein*
Whitehead	*Reality*	*Process*	Holism/Organicism
Wittgenstein	*World*	*Language*	Logical Philosophy
Searle	*Intentionality*	*Simulatability*	Philosophy of Mind
Dummett	*Truth*	*Verification*	Theory of Meaning

The best known dualism would presumably be the Cartesian dualism between mind and matter (or mind and body), in which the ontic realm of matter and the epistemic realm of mind are separated. The Kantian dualism between thing-in-itself and appearance can readily be seen as a case of the ontic-epistemic dualism. Cassirer, the logical Neo-Kantian of the Marburg School, asserted the priority of the functional over the substantival [52], having built a purely functional, genetic view of knowledge, which was mainly concerned with modern science at an early stage of his thought as in *Substance and Function* [52], and yet eventually evolved to encompass everything including humanities in his mature *Philosophy of Symbolic Form* [53]. It is an all-encompassing magnificent Philosophy of Culture [178], indeed subsuming myth, art, language, humanities, and both empirical and exact sciences. Cassirer now counts as a precursor of what is called Structural Realism [97, 165, 166]. Yet his functionalist philosophy would better be characterised as Higher-Order Structural Realism, allowing for structures of structures (or abstraction of abstraction in a certain sense), just as in category theory. His genetic view even payed due attention to the process of how structures are generated, just as in type theory. In light of this, Cassirer's philosophy may be regarded as a conceptual underpinning of the enterprise of category theory, and his dichotomy between substance and function as that of categorical duality, which constitutes the subject matter of the present thesis. Cassirer actually started his career with work on Leibniz and his relationalism [51], at which we shall have a glance in the following.

Mathematially, the ontic-epistemic duality is best recognised in the nature of space aforementioned. There were two conceptions of space at the dawn of mathematical science: the Newtonian realist conception of absolute space and the Leibnizian antirealist conception of relational space [269]. Hundreds of years later, Whitehead [277] recognised a similar tension between point-free space and point-set space (as part of his inquiry into *Process and Reality*), advocating the latter point-free conception, which may also be found in the phenomenology of Brentano and Husserl as well (see [28]). On the one hand, points are recognition-transcendent entities (just like prime ideals/filters, maximally consistent theories, or what Hilbert called ideal elements; recall that the algebraic geometer indeed identify points with prime ideals, which, in general, only exist with the help of the axiom of choice or some indeterministic principle like that). On the other, regions (or any other aforementioned entities) are more recognition-friendly and epistemically better grounded (just as point-free topology can be developed constructively or even predicatively in the formal topology style). In general, realism and antirealism are grounded upon the ontic and the epistemic, respectively.

As shown in the figure, quite some major philosophers, whether in the analytic or continental tradition, have wondered about versions of the ontic-epistemic dualism. The fundamental problem of such a dualism is to account for how the two different realms can interact with each other; in the particular case of the Cartesian dualism, it boils down to explicating how the mind can know about the material world when they are totally (and so causally) separated. How can they causally interact at all when they are causally separated? It appears impossible; this is the typical way the philosopher gets troubled by dualism in accounting for the ontic-epistemic interaction.

Philosophy of mathematics faces an instance of the interaction problem as well. If the realm of mathematical objects and the realm of human existence are totally separated (in particular causally separated), how can human beings get epistemic access to mathematical entities? If mathematical objects exist in a Platonic universe, as in Gödel's realist philosophy for example, it seems quite hard to account for how it is possible to have a causal connection between humans in the ordinary universe and mathematical entities in the Platonic universe when the two universes are causally disconnected (this sort of problem is known as the Benacerraf's dilemma [31]). Yet if mathematical objects exist in the mind, as in Brouwer's antirealist philosophy (he counts as an antirealist at least in the sense of Dummett [87]), the account of interaction between the ontic and the epistemic is much easier, since the ontic is, just by assumption, reduced to the epistemic in this case; by contrast, then, it gets harder to account for the existence and objectivity of mathematical entities, especially transfinite ones. For instance, how can humans mentally construct far-reaching transfinite entities and does everyone's mental construction really yield the same results for sure? (To remedy the existence problem, Brouwer actually endorsed arguably non-constructive principles. Otherwise a continuum can be countable as in recursive mathematics in the Russian tradition; recall that the number of computable reals are only countably many.) Summing up, realist ontology makes it difficult to account for the possibility of epistemic access to mathematical objects; conversely, antirealist epistemology yields ontological difficulties in constructively justifying their existence and objectivity.

A moral drawn from the above discussion is that there is a trade-off between realism and antirealism: straightforward realist ontology leads to involved epistemology with the urgent issue of epistemic access to entities unable to exist in our ordinary, tangible universe; and straightforward antirealist epistemology to involved ontology with the compelling problem of securing their existence and objectivity. In general, realism gets troubled by our accessibility to abstract entities; antirealism faces a challenge of warranting their existence and objectivity. Put simply, an easier ontology of something often makes its epistemology more difficult, and vice versa. Something seems reversed between realist/ontological and antirealist/epistemological worldviews. And this is where the idea of duality between realism and antirealism comes into the play.

1.2 Duality

Everything, from Truth and Meaning to Being and Mind, has dual facets as aforementioned. Conceptually, duality theory, in turn, is an attempt to unite them together with the ultimate aim of showing that they are the two sides of one and the same coin.

Put another way, duality allows two things opposed in dualism to be reconciled and united as just two different appearances of one and the same fundamental reality; in this sense, duality is a sort of monism established on the top of dualism (cf. Hegelian dialectics as in Lawvere's philosophy of mathematics). In Dummett's philosophy on the theory of meaning, for example, he makes a binary opposition between realism and antirealism (cf. Platonism and Intuitionism/Constructivism); what is at stake there is basically the legitimacy of recognition-transcendent truth conditions, which is allowed in realism, but not in antirealism. As shall be discussed later (and also in the author's philosophically oriented articles [199, 201]), however, the realist and antirealist conceptions of meaning may be reconciled and united as sharing the same sort of structure, even if they are literally opposed. In view of Dummett's constitution thesis, according to which the content of metaphysical (anti)realism is constituted by semantic (anti)realism [87, 206], this would arguably count as a unification of metaphysics as well as the theory of meaning. Duality thus conceived is a constructive canon to deconstruct dualism as it were.

Whilst having posed the Cartesian dualism as aforementioned, Descartes also developed analytic geometry, which is in a sense a precursor of duality between algebra and geometry, even though he might not have been aware that systems of equations are dual to spaces of their zero loci (logically paraphrasing, this amounts to the fact that systems of axioms are dual to spaces of their models; and such correspondence between logic and algebraic geometry can be made precise in duality theory). Notwithstanding that Galois theory may be seen as an instance of duality, Galois himself would not have been aware of the essentially categorical duality underpinning his theory, either. It would, then, be Riemann who first discovered duality between geometry and algebra in a mathematically substantial form; indeed he showed how to reconstruct (what are now called) Riemann surfaces from function fields, and vice versa, thereby establishing the (dual) equivalence between them. Even earlier than Riemann, however, Dedekind-Weber and Kronecker (mathematically) gave a purely algebraic conception of space, a sort of precursor of what is now called point-free geometry (philosophically, it would date back to Leibniz as aforementioned).

The history of duality in mathematical form, thus, goes back to the late 19th century (it ought to be noted here that duality in mathematical form basically means categorically representable duality in the present thesis, and so duality in projective geometry, for example, does count as an origin of duality in this sense; it would not be categorically representable, at least to my knowledge). Duality then flourished in the early 20th century, as exemplified by Hilbert's, Stone's, Gelfand's, and Pontryagin's dualities (Hilbert's Nullstellensatz is essentially a dual equivalence between finitely generated reduced k-algebras and varieties over k for an algebraically closed field k). The discovery of dualities was thereafter followed by applications to functional analysis, general topology, and universal algebra on the one hand, and to algebraic geometry, representation theory, and number theory on the other (interestingly, the Pontryagin duality plays a vital rôle in number theory as exemplified by André Weil's *Basic Number Theory* [274]). And it was eventually accompanied by the rise of categorical language in the late 20th century, which, in particular, allowed one to identify a universal form of duality (before category theory it was only vaguely understood what exactly different dualities have in common, and no one was able to spell out what presicely duality is). Today there are a great variety of dualities found across quite different kinds of science as in the following figure (and even in engineering such as optimisation, linear programming, control theory, and electrical circuit theory; duality thus goes far beyond pure mathematics, and it may sometimes be of genuine practical use as in those engineering theories):

	Ontic	**Epistemic**	
Complex Geometry	*Complex Surface*	*Function Field*	Riemann
Algebraic Geometry	*Variety/Scheme*	*k-Algebra/Ring*	Hilbert-Grothendieck
Galois Theory	*(Profinite) G-Set*	*Algebra Extension*	Galois
Representation Th.	*Compact Group*	*Representations*	Pontryagin-Tannaka
Anabelian Geometry	*Elliptic Curve*	*Frobenioid*	Mochizuki
Topology	*Topological Space*	*Algebra of Opens*	Isbell-Papert
Convex Geometry	*Convex Space*	*Semantic Domain*	Maruyama
Logic	*Space of Models*	*Algebra of Theories*	Gödel-Stone
Computer Science	*System*	*Observable Properties*	Abramsky-Smyth
System Scicence	*Controllability*	*Observability*	Kalman
Quantum Physics	*State Space*	*Alg. of Observables*	von Neumann
General Relativity	*Spacetime Manifold*	*Vector Field*	Weyl

These dualities are diverse at first sight, and yet tightly intertwined with each other in their conceptual structures. To pursue links between different dualities is indeed one of the principal aims of duality theory. Since duality, as opposed to dualism and disduality, is the major concern of the thesis, more specialised discussions on some part of the picture above shall be continued and elaborated in the outline of the thesis below. It is particularly notable that the physics duality between states and observables quite resembles the computer science duality between systems and observable properties. The duality-theoretical correspondence between logic and algebraic geometry shall be explained later in more detail; here just note that the Stone duality is concerned with equivalence between syntax and semantics, and it is actually a strengthened version of the Gödel's completeness theorem (to reinforce this point, it is named Gödel-Stone in the picture; technically, the injectivity of an evaluation map in the Stone duality exactly amounts to completeness, whereas the surjectivity, though more involved to prove, is a pure surplus from the completeness point of view). The computer science duality above is, in its mathematical substance, a form of Stone duality, and even the physics duality between states and observables may be formulated in the Stone duality style. Quite some part of the duality picture above, therefore, boils down to Stone duality, which is the central topic of study of the present thesis.

Duality is even crucial for Hilbert's programme, as Coquand et al. [69] assert:

> A partial realisation of Hilbert's programme has recently proved successful in commutative algebra [..] One of the key tools is Joyal's point-free version of the Zariski spectrum as a distributive lattice [..]

In [69] they contrive a constructive version of Grothendieck's schemes by replacing their base spaces with point-free ones through the Stone duality for distributive lattices. From a categorical point of view, we could say that the spectrum functor $\mathrm{Spec} : \mathbf{Alg}^{\mathrm{op}} \to \mathbf{Spa}$ from an algebraic category $\mathbf{Alg}$ to a topological category $\mathbf{Spa}$ amounts to the introduction of ideal elements in Hilbert's sense, and its adjoint functor the elimination of them. Duality, therefore, has contributed to Hilbert's programme and constructivism. The point-free Tychonoff theorem is constructive; this is classic. Yet the state-of-the-art goes far beyond it, encompassing not just general topology but also some mainstream mathematics such as Grothendieck's scheme theory.

1.3 Disduality

The absence of duality, what is named disduality in the present thesis, is just as interesting on its own right as the presence of duality. According to the discussion so far, duality is about the relationships between the epistemic and the ontic. What is disduality then? In a nutshell, disduality is about an excess of the epistemic or the ontic; the duality correspondence collapses when either of the ontic and the epistemic is excessive. To articulate what is really meant here, let us focus upon two cases of disduality in the following: one is caused by incompleteness and the other by non-commutativity as in quantum theory. The former shall give a case of the excess of the ontic, and the latter a case of the excess of the epistemic.

As mentioned above, completeness may be seen as a form of duality between theories and models. What Gödel's first incompleteness theorem tells us is that there are not enough formal theories to characterise the truths of intended model(s) concerned, or to put it differently, there are some models which are unable to be axiomatised via formal theories, where theories are, of course, assumed to be finitary (or recursively axiomatisable) and stronger than the Robinson arithemetic (the technical statement of this is that the set of stronger-than-Robinson truths is not recursively enumerable). If you allow for infinitary theories, you can nonetheless obtain a complete characterisation, for example, of arithmetical truths, and yet this is not acceptable from an epistemological point of view, such as Hilbert's finitism. This is a case of disduality due to the excess of the ontic. We now turn to the other kind of disduality.

Let us have a look at a case of the excess of the epistemic. There is some sort of incompleteness in quantum algebra. The Gelfand duality tells us there is a dual equivalence between (possibly nonunital) commutative C^*-algebras and locally compact Hausdorff spaces. There have been different attempts to generalise it so as to include non-commutative algebras, in particular algebras of observables in quantum theory, and yet, as long as the duals of non-commutative algebras are purely topological, this is actually impossible (see [34]). The duality between space and algebra, thus, does not extend to the quantum realm of non-commutativity. This is indeed a case of disduality due to the excess of the epistemic: there are too many non-commutative algebras, compared to the available amount of topological spaces. The disduality may be remedied to extend the notion of space so as to include, for example, sheaves of algebras in addition to topological spaces *per se* (just as Grothendieck indeed did in his scheme-theoretical duality); in such a case, however, both sides of duality get more or less algebraic (the same may be said about the Tannaka duality for noncommutative compact groups, in which case duals are categories of representations, and so fairly algebraic).

There is another thought on the notion of disduality. No canonical agreement exists on what duality means in the first place even among category theorists: for example, some say duality is dual equivalence, whilst others say it is dual adjunction in general. A weaker notion of duality could count as a kind of (weaker) disduality. This sort of phenomena, however, shall be discussed in detail in the chapter of categorical duality, in which we see how far dual adjunctions are from, and yet how they (technically always but practically sometimes) transform into, dual equivalences.

This completes the lengthy discussion on the trichotomy of dualism, duality, and disduality. More on disduality and how to remedy it shall be addressed in the final concluding remarks, and in the following we shall concentrate on duality *per se*.

1.4 Synopsis

Category theory today has found widespread applications in diverse disciplines of science beyond mathematics; it would now be more like foundations of science in general than foundations of mathematics in particular[1]. The present thesis focuses upon categorical duality (and its applications), which surely appears beyond mathematics as discussed above. Having a look at the above picture of dualities, the physics duality between states and observables is, in a way, akin to the informational duality between systems and observable properties/behaviours. How could we, then, shed light on structural analogies and disanalogies between diverse dualities? What is the generic structure or architecture of duality in the first place? Such questions propel our investigation. In the following let us give a bird's-eye view of how the thesis proceeds whilst articulating the idea of duality.

The present thesis consists of four major chapters, excluding the introductory and concluding sections, and of three appendices as well. It would be convenient for the reader to have a rough, overall picture of them before the following, more detailed account:

- Chapter 2 is devoted to categorical duality theory in the abstract, forging forward the theory of dualities between point-set and point-free spaces by virtue of categorical topology and algebra. A consequence of the theory thus developed is domain-convexity duality, which solves an open problem in Bart Jacobs' duality theory for algebras of the distribution monad.

- Chapter 3 sheds light on subtler facets of duality by means of universal algebra and its extension, especially in logical contexts. We extend what is called natural duality theory so as to encompass coalgebraic dualities and non-Hausdorff dualities. The focus of this chapter is entirely devoted to finitary Stone-type dualities whilst the last chapter subsumes infinitary Stone-type dualities as well.

[1]This is the thing category theory has aimed at since its early days. Granted that quite some category theorists had more or less foundationalist doctrines, nevertheless, it would be appropriate to think of category theory as local relative foundations rather than global absolute foundations, which is what set theory is about in the nature of the universe or the cumulative hierarchy of sets, just as base change is a fundamental idea of category theory. Set theory can support a multiverse view as shown in recent developments, and yet category theory intrinsically does so, I would say. Note also that the practice of mathematics is concerned with different combinations of set-theoretical and category-theoretical ideas just as the thesis does, and the binary opposition between set theory and category theory are not very constructive or fruitful in practice, apart from philosophical issues.

- Chapter 4 develops the concepts of full Lambek hyperdoctrine, full Lambek tripos, and categorical universal logic in order to give uniform categorical semantics for a broad variety of both first-order and higher-order logics, thereby explicating and articulating the universal nature of categorical logic in the Lawverian tradition of hyperdoctrine (or fibration for logic).

- Chapter 5 focuses upon duality and symmetry in quantum physics. We first develop Chu duality theory encompassing operational quantum duality and the T_1-type duality of algebraic varieties. We then show, through duality theory, that the quantum symmetry groupoid embeds into the category of Born coalgebras, i.e., symmetries are maps preserving the dynamics of measurements.

- Chapter 6 concludes the thesis with succinct answers to our first questions: how does duality emerge, change, and break? Appendices outline non-commutative duality theory on the basis of Grothendieck-style scheme theory, and also provide additional materials on duality via fuzzy topology and on artificial intelligence logics, which came out as a spin-off from the duality-theoretic work.

Our pursuit of duality theory starts in Chapter 2 with the elucidation of duality between point-set space and point-free space, which would arguably be regarded as reconciling the conflict between the Newtonian asbolute and Leibnizian relative conceptions of space. Our technical developments utilise and expand concepts from categorical topology and algebra, and we thereby contrive a moderately general theory of dualities between algebraic, point-free spaces and set-theoretical, point-set spaces, which encompasses infinitary Stone dualities, such as the well-known duality between frames (aka. locales) and topological spaces, and a duality between σ-complete Boolean algebras and measurable spaces, as well as the classic finitary Stone, Gelfand, and Pontryagin dualities. Among different applications of our theory, we focus upon domain-convexity duality in particular: from the theory we derive a duality between Scott's continuous lattices and convexity spaces, and exploit the resulting insights to intrinsically identify the dual equivalence part of a dual adjunction for algebras of the distribution monad; the dual adjunction was discovered by Bart Jacobs, but with no characterisation of the induced equivalence, which we do give here. We finally place categorical duality in the broader context of the philosophy of space, and thereby shed light on philosophical underpinnings of duality from an angle different from the above picture of dualities.

In Chapter 3, then, we investigate into a subtler mechanism of duality in the context of natural duality theory, presumably the most successful theory of dualities for finitely generated (quasi)varieties (of algebras). The theory of natural dualities is a general theory of Stone-Priestley-type categorical dualities based upon the machinery of universal algebra. This chapter consists of two parts, the natural duality theory of modal or coalgebraic dualities and the natural duality theory of intuitionistic or non-Hausdorff dualities. The duality theory in Chapter 2 is more general than theories in Chapter 3, and yet the latter gets deeper into the mechanism of duality at the cost of generality. In general, there is some sort of trade-off between generality and particularity: if you want to account for so many different kinds of dualities, your theory becomes "thinner" in some sense; if you limit your focus further, your theory can shed light on subtler facets invisible within more general theories, that is to say, your theory gets "thicker", thus rendering the mechanism of duality more transparent.

Stone-Priestley-type dualities play a fundamental rôle in recent developments of coalgebraic logic. At the same time, however, natural duality theory has not subsumed important dualities in coalgebraic logic, including Jónsson-Tarski's topological duality and Abramsky-Kupke-Kurz-Venema's coalgebraic duality for the class of all modal algebras. By means of introducing a new notion of $\mathbb{ISP}_{\mathbb{M}}$, in this chapter, we extend the theory of natural dualities so as to encompass Jónsson-Tarski duality and Abramsky-Kupke-Kurz-Venema duality. The main results here are topological and coalgebraic dualities for $\mathbb{ISP}_{\mathbb{M}}(L)$ where L is a semi-primal algebra. These dualities are shown building upon Keimel-Werner's semi-primal duality theorem. Our general theory subsumes both Jónsson-Tarski and Abramsky-Kupke-Kurz-Venema dualities. Moreover, it provides new coalgebraic dualities for algebras of many-valued modal logics and certain insights into a category-equivalence problem for categories of algebras involved. It also follows from our dualities that the corresponding categories of coalgebras have final coalgebras and cofree coalgebras. $\mathbb{ISP}_{\mathbb{M}}$ provides a natural framework for the universal algebra of modalities, and as such, for the theory of modal natural dualities.

We then change our focus to natural duality theory beyond Hausdorff spaces; natural duality theory has been limited in the realm of Hausdorff spaces thus far. We take a first step towards the non-Hausdorff theory of natural dualities, showing a Hu-style generalisation of the Stone duality for distributive lattices. Hu generalised the Stone duality for Boolean algebras to a duality for the quasi-variety generated by any primal algebra. By introducing the concept of topological dualizability, which extends the concept of primality, we generalize the Stone duality for distributive lattices to a duality for the quasi-variety generated by any topologically dualisable algebra. We also provide some applications of this duality result. And this is the end of our pursuit of duality theory *per se* in the thesis. Taking the developments of duality theory into consideration, we may conclude that what determines the structure of duality for a given variety or quasi-variety is, in a certain sense, the ratio of existing term functions over all of them.

In Chapter 4, we move on to categorical logic, some models of which may be seen arising from categorical dualities. In particular, we pursue the idea that predicate logic is a “fibred algebra” while propositional logic is a single algebra; in the context of intuitionism, this algebraic understanding of predicate logic goes back to Lawvere, in particular his concept of hyperdoctrine. Here, we aim at demonstrating that the notion of monad-relativised hyperdoctrines, which are what we call fibred algebras, yields algebraisations of a wide variety of predicate logics. More specifically, we discuss a typed, first-order version of the non-commutative Full Lambek calculus, which has extensively been studied in the past few decades, functioning as a unifying language for different sorts of logical systems (classical, intuitionistic, linear, fuzzy, relevant, etc.). Through the concept of Full Lambek hyperdoctrines, we establish both generic and set-theoretical completeness results for any extension of the base system; the latter arises from a dual adjunction, and is relevant to the tripos-to-topos construction and quantale-valued sets. Furthermore, we give a hyperdoctrinal account of Girard’s and Gödel’s translation. The hyperdoctrinal approach to universal logic is named Categorical Universal Logic in the present thesis.

We then adapt this approach for higher-order categorical substructural logics, thus extending the above first-order case so as to allow for the existence of truth value objects and function space constructions in they underlying type theory. Whereas the concept of topos is seemingly difficult to generalise beyond higher-order intuitionisitc logic and its variants, the concept of tripos or higher-order hyperdoctrine, which allows us to present all toposes via the tripos-to-topos construction (but not *vice versa*), is based on a more general, fibrational mechanism, thus looking more promising for developments of categorical universal logic. We introduce the concept of full Lambek tripos, and show that full Lambek triposes give complete semantics for higher-order full Lambek calculus. Relativising this result to different classes of additional axioms, we can obtain higher-order categorical completeness theorems for a broad variety of logical systems. The general framework thus developed allows us to give a tripos-theoretical account of Girard’s and Gödel’s translation for higher-order logic.

In this chapter we finally look into duality-theoretical models of monad-relativised hyperdoctrines in a more abstract setting, showing that there are those dual adjunctions that have inherent hyperdoctrine structures in their predicate functor parts. A general account of logical translation is given in terms of hyperdoctrinal Lawvere-Tierney topology. We systematically investigate into the categorical logics of dual adjunctions by utilising Johnstone-Dimov-Tholen's duality-theoretic framework; sample applications include duality-based models of topological geometric logic, convex geometric logic, and quantum logic. Our set-theoretical duality-based hyperdoctrines for quantum logic have both universal and existential quantifiers (and higher-order structures), giving rise to a universe of Takeuti-Ozawa's quantum sets via the tripos-to-topos construction. We finally discuss how to reconcile Birkhoff-von Neumann's quantum logic and Abramsky-Coecke's categorical quantum mechanics (which is modernised quantum logic as an antithesis to the traditional one) via categorical universal logic, thus leading to the idea that the logic of quantum mechanics is a sort of logic over type theory, namely Birkhoff-von Neumann's logic of quantum propositions (it is about "how propositions compose") over Abramsky-Coecke's logic as the type theory of quantum systems (it is about "how systems compose"), in which the theory of truths and the theory of proofs are integrated together into the one concept of hyperdoctrine with its domain and value categories accounting for proofs and truths, respectively.

In Chapter 5, we address issues in foundations of quantum mechanics, especially the concept of symmetry, which are, of course, conceptually related with the notion of duality. We pursue the principles of duality and symmetry building upon Pratt's idea of the Stone Gamut and Abramsky's representations of quantum systems. In the first part of the chapter, we first observe that the Chu space representation of quantum systems leads us to an operational form of state-observable duality, and then show via the Chu space formalism enriched with a generic concept of closure conditions that such operational dualities (which we call "T_1-type" as opposed to "sober-type") actually exist in fairly diverse contexts (topology, measurable spaces, and domain theory, to name but a few). The universal form of T_1-type dualities between point-set and point-free spaces is described in terms of Chu spaces and closure conditions. From the duality-theoretical perspective, in the second part, we improve upon Abramsky's "fibred coalgebraic" representation of quantum symmetries, thereby obtaining a finer, "purely coalgebraic" representation: our representing category is properly smaller than Abramsky's, but still large enough to accommodate the quantum symmetry groupoid. Among several features, our representation reduces Abramsky's two-step construction of his representing category into a simpler one-step one, thus rendering the Grothendieck construction redundant. Our purely coalgebraic representation stems from replacing the category of sets in Abramsky's representation with the category of closure spaces in the light of the state-observable duality telling us that closure is a right perspective on quantum state spaces.

The structure of the thesis is fairly simple as you may see in the above account: we first develop two general theories of dualities, one more general and the other less general yet more focused, and then we discuss two topics from a duality-theoretical point of view, namely categorical logic and quantum physics. The above is just a bird's-eye view of major chapters, and a more detailed account of the contents of each chapter may be found in the introductory section of each chapter. A perspective to interconnect different chapters is also given at the very beginning of each chapter.

The bulk of the thesis has already been published; the full list of publications by the author is included in the Bibliography at the end of the thesis.

Chapter 2

Categorical Duality

From a mathematical point of view, duality emerges between space and algebra, or between point-set space and point-free space; this distinction has a conceptual affinity with the traditional dichotomy between the Newtonian absolute conception of space and the Leibnizian relative conception of space. In this chapter we look into a category-theoretical mechanism underpinning duality between space and algebra. The fundamental principle of duality between space and algebra is, in our general setting, what is called the harmony condition, which allows us to derive a general dual adjunction between space and algebra. The harmony condition basically states that algebraic structures involved are coherent with topological structures involved. After developing the general theory, we solve an open problem Bart Jacobs left in his paper on dualities for algebras of the distribution monad, establishing duality between convex structures and semantic domains. In the next chapter, we shall focus upon duality for finitary algebras in particular, and yet the duality theory of this chapter encompasses a much broader class of dualities, especially infinitary ones.

2.1 Introduction to the Chapter

There are two conceptions of space: one comes from the ontic idea that the ultimate constituents of space are points with no extension; the other does not presuppose the concept of points in the first place, and starts with an epistemically more certain concept such as regions or observable properties. For instance, a topological space is an incarnation of the former idea of space, and a frame (or locale) is an embodiment of the latter. Duality often exists between point-free and point-set conceptions of space (to put it differently, between epistemology and ontology of space; see the Appendix as well), as exemplified by the well-known duality between frames and topological spaces (see, e.g., Johnstone [149]).

The most general duality theorem is this: any category $\mathbf{C}$ is dually equivalent to the opposite category $\mathbf{C}^{\mathrm{op}}$. It, of course, makes no substantial sense; however, note that it prescribes a generic form of duality in a non-obvious way (we say "non-obvious" because there may be different conceptions of a generic form of duality, some of which may not be based upon category theory at all). In this chapter, we attempt to avoid such triviality by focusing upon a more specific context: we aim at developing a moderately general theory of dualities between point-free and point-set spaces, whilst having in mind applications to domain-convexity duality, where domains are seen as point-free convex structures.

Our general theory of dualities between point-free and point-set spaces builds upon the celebrated idea of a duality induced by a Janusian (aka. schizophrenic) object: "a potential duality arises when a single object lives in two different categories" (Lawvere's words quoted in Barr et al. [24]). Note that in this chapter we mean by dualities dual adjunctions in general; dual equivalences are understood as special cases. There are different theories of dualities based upon the same idea (see, e.g., [24, 63, 81, 149, 232]); some of them use universal algebra, whilst others are categorical. Our theory is in between universal algebra and category theory (although we use categorical terminology, nevertheless, everything can be recasted in terms of universal algebra and of general point-set spaces introduced in Chapter 5). More detailed comparison with related work is given below.

Our duality theory allows us to derive a number of concrete dualities, including infinitary Stone dualities, such as the aforementioned duality in point-free topology, and a certain duality between σ-complete Boolean algebras and measurable spaces, as well as the classic finitary Stone, Gelfand, and Pontryagin dualities (since the Pontryagin duality is a self-duality, how to treat it is slightly different from how to do the others as noted below).

In this chapter we focus, *inter alia*, upon dualities between point-free and point-set convex structures. On the one hand, we consider Scott's continuous lattices to represent point-free convex structures for certain reasons explained later, in Subsection 2.3.1. On the other hand, there are two kinds of point-set convex structures: i.e., convexity spaces (see van de Vel [270] or Coppel [71]; the definition is given in Preliminaries in Section 2.2) and algebras of the distribution monad (aka. barycentric algebras; see Fritz [102]).

Our general theory tells us that there is a duality between continuous lattices and convexity spaces. In contrast, Jacobs [146] shows a dual adjunction between preframes and algebras of the distribution monad, which can be reformulated as a dual adjunction between continuous lattices and algebras of the distribution monad. Although Jacobs [146] left it open to identify intrinsically the induced dual equivalence, in this chapter, we give an intrinsic characterisation of the dual equivalence part of the dual adjunction for algebras of the distribution monad.

Technical Summary. In our duality theory, we mainly rely upon concrete category theory as in Adámek et al. [12], especially concepts from categorical topology (see also [13, 48]) and categorical algebra (in particular the theory of monads).

We start with a category **C** monadic over **Set** (which is equivalent to possibly infinitary varieties in terms of universal algebra) and with a topological category **D** of certain type (for topological categories, see [12]), and then assume that there is a Janusian object Ω living in both **C** and **D**. In passing, we introduce the new concept of classical topological axiom, and use it to identify a certain class of those full subcategories of a functor-structured category that represent categories of point-set spaces. Under the assumption of what we call the harmony condition, which basically means that algebraic operations are continuous in a suitable sense, we finally show that there is a dual adjunction between **C** and **D**, given by homming into Ω.

The dual adjunction formally restricts to a dual equivalence via the standard method of taking those objects of **C** and **D** that are fixed under the unit and counit of the adjunction; to put it intuitively, the objects whose double duals are isomorphic to themselves. At the same time, however, it is often highly non-trivial to identify intrinsically the dual equivalence part of a dual adjunction in a concrete situation, as Porst-Tholen [232] remark, "This can be a very hard problem, and this is where categorical guidance comes to an end."[1]

Using specialised, context-dependent methods, rather than the generic one mentioned above, we give intrinsic characterisations of dual equivalences induced by the dual adjunction for convexity spaces, and by the dual adjunction for algebras of the distribution monad. The concept of polytopes plays a crucial rôle in the characterisations, and in understanding how semilattices involve convex structures.

Comparison with Related Work. Our general theory of dualities may be compared with other duality theories as follows. Clark-Davey's theory of natural dualities [63] is based upon the same idea of a duality induced by a Janusian object. However,

[1]To exemplify what is meant here, consider the dual adjunction between frames and spaces, which restricts to a dual equivalence between the frames and spaces whose double duals are isomorphic to themselves; this is trivial. Nevertheless, it is not trivial at all to notice that those frames are exactly the frames with enough points (i.e., spatial frames), and those spaces are precisely the spaces in which any non-empty irreducible closed set is the closure of a unique point (i.e., sober spaces).

our theory is more comprehensive than natural duality theory, in that whilst natural duality theory specialises in dualities for finitary algebras, our theory is intended to encompass infinitary algebras as well (e.g., frames, σ-complete Boolean algebras, and continuous lattices). Our theory thus encompasses both finitary and infinitary Stone-type dualities.

Johnstone's general concrete duality [149, VI.4] and Dimov-Tholen's natural dual adjunction [81, 232] are more akin to ours.[2] A crucial difference is, however, that we stick to the practice of Stone-type dualities as far as possible. In their theories, there is no concrete concern with how to equip the "spectrum" of an "algebra" with a "topology" or how to equip the (collection of) "functions" on a "space" with an "algebraic" structure.

We consider that the processes of algebraisation and topologisation are essential in the practice of Stone-type dualities. In particular, algebraisation and topologisation are strikingly different processes in practice; in spite of this, the two processes are treated in their theories as being in parallel and symmetric at a level of abstraction, which looks like an excessive abstraction from our perspective of the practice of duality.

We put a strong emphasis on the asymmetry between the two processes of algebraisation and topologisation in the practice of Stone-type dualities, and thus aim at simulating the processes within our theory, thereby representing the practice of duality in an adequate manner. In order to achieve this goal, our theory cannot and should not be so general as to symmetrise the asymmetry; this is the reason why we call our theory "moderately" general.

In comparison with Chu duality theory in Chapter 5, which discusses a theory of T_1-type dualities based upon Chu spaces and a generic concept of closure conditions, this chapter aims at a theory of sober-type dualities; an example of duality of T_1-type is a (not very well known) duality between T_1 spaces and coatomistic frames (a subtlety is frame morphisms must be "maximal" to dualise continuous maps; see [193]). Sober-type dualities are based upon prime spectra, whilst T_1-type dualities are based upon maximal spectra. Affine varieties (except singletons) in $\mathbb{C}^n$ with Zariski topologies are non-sober T_1 spaces; they are homeomorphic to the maximal spectra

[2] Johnstone's dual adjunction (Lemma VI.4.2) seems to be not very rigorous because he dares to say "we choose not to involve ourselves in giving a precise meaning to the word 'commute' in the last sentence" (p. 254), and the dual adjunction result actually relies upon the assumption of that commutativity. In this chapter, we precisely formulate the concept of commutativity as what we call the harmony condition.

of their coordinate rings. Both the former and the latter theories can be applied to different sorts of spaces, yielding T_1-type and sober-type dualities respectively.

Jacobs [146] and Maruyama [189] independently unveiled (different) dualities for convexity (convexity algebras in [189] are replaced in this chapter by continuous lattices), and this chapter is meant to elucidate a precise link between them, which remained unclear so far. The two dualities turn out to be essentially the same in spite of their rather different outlooks.

2.2 Duality Theory via Categorical Topology and Algebra

After preliminaries, we first review categorical topology, and then get into a general theory of dualities based upon the concepts of monad, functor-(co)structured category, and topological (co)axiom. Among other things, we introduce the new concept of classical topological axiom with the aim of treating different sorts of point-set spaces in a unified way.

Preliminaries. For a category $\mathbf{C}$ and a faithful functor $U : \mathbf{C} \to \mathbf{Set}$, a tuple $(\mathbf{C}, U)$ is called a concrete category, where $\mathbf{Set}$ denotes the category of sets and functions. U is called the underlying functor of the concrete category. For simplicity, we often omit and make implicit the functor U of a concrete category $(\mathbf{C}, U)$. We can also define the notion of a concrete category over a general category. For a category $\mathbf{C}$ and a faithful functor $U : \mathbf{C} \to \mathbf{D}$, $(\mathbf{C}, U)$ is called a concrete category over $\mathbf{D}$. A concrete category (over $\mathbf{Set}$) in this chapter is called a construct in [12]. $\mathbf{Top}$ denotes the category of topological spaces and continuous functions. $\mathbf{Conv}$ denotes the category of convexity spaces and convexity preserving maps, where a convexity space is a tuple $(X, \mathcal{C})$ such that X is a set and $\mathcal{C}$ is a subset of the powerset of X that is closed under directed unions and arbitrary intersections; a convexity-preserving map is such that the inverse image of any convex set under it is again convex (see van de Vel [270] and Coppel [71], which develop substantial amount of convex geometry based upon this general concept of convexity space). $\mathbf{Meas}$ denotes the category of measurable spaces and measurable functions. $\mathbf{Frm}$ denotes the category of frames and their homomorphisms. $\mathbf{ContLat}$ denotes the category of continuous lattices and their homomorphisms (i.e., maps preserving directed joins and arbitrary meets). $\mathbf{BA}_\sigma$ denotes the category of σ-complete Boolean algebras with σ-distributivity and their homomorphisms, where σ-distributivity means that countable joins distribute over countable meets. $\mathcal{Q} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ denotes the contravariant powerset functor.

2.2.1 A Categorical Conception of Point-Set Spaces

Here we introduce a general concept of space that encompasses topological spaces, convexity spaces, and measurable spaces. Our duality theory shall be developed based upon that concept of (generalised) space. For the fundamentals of functor-(co)structured category and topological (co)axiom, we refer to Adámek et al. [12]. We first review the notion of functor-structured category. Let $(\mathbf{C}, U : \mathbf{C} \to \mathbf{Set})$ be a concrete category in the following.

Definition 2.2.1 ([12]). *A category* $\mathbf{Spa}(U)$ *is defined as follows.*

1. *An object of* $\mathbf{Spa}(U)$ *is a tuple* $(C, \mathcal{O})$ *where* $C \in \mathbf{C}$ *and* $\mathcal{O} \subset U(C)$.
2. *An arrow of* $\mathbf{Spa}(U)$ *from* $(C, \mathcal{O})$ *to* $(C', \mathcal{O}')$ *is an arrow* $f : C \to C'$ *of* $\mathbf{C}$ *such that* $U(f)[\mathcal{O}] \subset \mathcal{O}'$.

A category of the form $\mathbf{Spa}(U)$ *is called a functor-structured category. A category of the form* $(\mathbf{Spa}(U))^{\mathrm{op}}$ *is called a functor-costructured category.*

We consider $\mathbf{Spa}(U)$ as a concrete category equipped with a faithful functor $U \circ F : \mathbf{Spa}(U) \to \mathbf{Set}$ where $F : \mathbf{Spa}(U) \to \mathbf{C}$ is the forgetful functor that maps $(C, \mathcal{O})$ to C.

Then we can show the following (for the definition of topological category, see [12]; although there are different notions of a topological category, we follow the terminology of [12]).

Proposition 2.2.2 ([12]). *Both a functor-structured category* $\mathbf{Spa}(U)$ *and a functor-costructured category* $(\mathbf{Spa}(U))^{\mathrm{op}}$ *are topological.*

The concept of topological (co)axiom is defined as follows.

Definition 2.2.3 ([12]). *A topological axiom in* $(\mathbf{C}, U)$ *is defined as an arrow* $p : C \to C'$ *of* $\mathbf{C}$ *such that*

1. $U(C) = U(C')$;
2. $U(p) : U(C) \to U(C)$ *is the identity morphism on* $U(C)$.

An object C *of* $\mathbf{C}$ *satisfies a topological axiom* $p : D \to D'$ *in* $(\mathbf{C}, U)$ *iff, for any arrow* $f : D \to C$ *of* $\mathbf{C}$, *there is an arrow* $f' : D' \to C$ *of* $\mathbf{C}$ *such that* $U(f) = U(f')$.

A topological coaxiom is defined as a topological axiom with the following concept of satisfaction. An object C *of* $\mathbf{C}$ *satisfies a topological coaxiom* $p : D' \to D$ *in* $(\mathbf{C}, U)$ *iff, for any arrow* $f : C \to D$ *of* $\mathbf{C}$, *there is an arrow* $f : C \to D'$ *of* $\mathbf{C}$ *such that* $U(f) = U(f')$.

Topological axioms and coaxioms are the same, but the corresponding notions of satisfaction are dual to each other. For examples of topological (co)axiom, we refer to [12].

Definition 2.2.4 ([12]). *Let X be a class of topological (co)axioms in a concrete category* $\mathbf{C}$. *A full subcategory* $\mathbf{D}$ *of* $\mathbf{C}$ *is definable by X in* $\mathbf{C}$ *iff the objects of* $\mathbf{D}$ *coincide with those objects of* $\mathbf{C}$ *that satisfy all the topological (co)axioms in X.*

As in the following proposition, we can show a topological analogue of the Birkhoff theorem in universal algebra (for more details, see Theorem 22.3 and Corollary 22.4 in [12]).

Proposition 2.2.5 ([12]). *Let* $\mathbf{C}$ *be a concrete category. The following are equivalent:*

1. $\mathbf{C}$ *is fibre-small and topological;*
2. $\mathbf{C}$ *is isomorphic to a subcategory of a functor-structured category that is definable by a class of topological axioms in the functor-structured category.*
3. $\mathbf{C}$ *can be embedded into a functor-structured category as a full subcategory that is closed under the formation of products, initial subobjects, and indiscrete objects.*

Now we introduce the novel concept of classical topological (co)axiom, which shall play a crucial rôle in formulating our dual adjunction theorem.

Definition 2.2.6. *A classical topological axiom in* $\mathbf{Spa}(U)$ *is defined as a topological axiom* $p : (C, \mathcal{O}) \to (C', \mathcal{O}')$ *in* $\mathbf{Spa}(U)$ *such that*

- *Any element of $\mathcal{O}' \setminus \mathcal{O}$ can be expressed as a (possibly infinitary) Boolean combination of elements of $\mathcal{O}$.*

A classical topological coaxiom in $(\mathbf{Spa}(U))^{\mathrm{op}}$ *is defined as a topological coaxiom* $p : (C, \mathcal{O}) \to (C', \mathcal{O}')$ *in* $(\mathbf{Spa}(U))^{\mathrm{op}}$ *such that*

- *Any element of $\mathcal{O} \setminus \mathcal{O}'$ can be expressed as a (possibly infinitary) Boolean combination of elements of $\mathcal{O}'$.*

Let $\mathcal{Q} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ denote the contravariant powerset functor. Any of the category **Top** of topological spaces, the category **Conv** of convexity spaces, and the category **Meas** of measurable spaces is a full subcategory of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of classical topological coaxioms as in the following proposition, which can be shown just by spelling out the definitions involved.

Proposition 2.2.7. **Top** *is definable by the following class of classical topological coaxioms in* $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$*:*

$$\begin{aligned} 1_S : (S, \{\emptyset, S\}) &\rightarrow (S, \emptyset) \\ 1_S : (S, \{X, Y, X \cap Y\}) &\rightarrow (S, \{X, Y\}) \\ 1_S : (S, \mathcal{O} \cup \{\bigcup \mathcal{O}\}) &\rightarrow (S, \mathcal{O}) \end{aligned}$$

for all sets S*, all subsets* X, Y *of* S *and all subsets* $\mathcal{O}$ *of the powerset of* S*.*

Conv *is definable by the following classical topological coaxioms in* $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$*: (i)* $1_S : (S, \{\emptyset, S\}) \rightarrow (S, \emptyset)$*; (ii)* $1_S : (S, \mathcal{C} \cup \{\bigcap \mathcal{C}\}) \rightarrow (S, \mathcal{C})$*; (iii)* $1_S : (S, \mathcal{C}' \cup \{\bigcup \mathcal{C}'\}) \rightarrow (S, \mathcal{C}')$ *for all sets* S*, all subsets* $\mathcal{C}$ *of the powerset of* S*, and all those subsets* $\mathcal{C}'$ *of the powerset of* S *that are directed with respect to inclusion.*

Meas *is definable by the following classical topological coaxioms in* $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$*: (i)* $1_S : (S, \{\emptyset, S\}) \rightarrow (S, \emptyset)$*; (ii)* $1_S : (S, \{X, X^c\}) \rightarrow (S, \{X\})$*; (iii)* $1_S : (S, \mathcal{B} \cup \{\bigcup \mathcal{B}\}) \rightarrow (S, \mathcal{B})$ *for all sets* S*, all subsets* X *of* S*, and all those subsets* $\mathcal{B}$ *of the powerset of* S *that are of cardinality* $\leq \omega$*.*

In order to develop a general duality theory, thus, we shall focus upon a full subcategory **Spa** of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of classical topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$.

We call $(S, \mathcal{O}) \in \mathbf{Spa}$ a generalised space and $\mathcal{O}$ a generalised topology.

Given a subset $\mathcal{P}$ of the powerset of a set S, we can generate a topology on S from $\mathcal{P}$, which is the weakest topology containing $\mathcal{P}$. We can also do the same thing in the case of generalised topology.

Proposition 2.2.8. *For a set* S*, let* $\mathcal{P}$ *be a subset of the powerset of* S*. Then, there is a weakest generalised topology on* S *containing* $\mathcal{P}$ *in* **Spa***, i.e., there is* $(S, \mathcal{O}) \in \mathbf{Spa}$ *such that, if* $\mathcal{P} \subset \mathcal{O}'$ *for* $(S, \mathcal{O}') \in \mathbf{Spa}$*, then* $\mathcal{O} \subset \mathcal{O}'$*. We then say that* $\mathcal{O}$ *is generated in* **Spa** *by* $\mathcal{P}$*.*

Proof. Define

$$\mathcal{O} = \bigcap \{\mathcal{X} \ ; \ \mathcal{P} \subset \mathcal{X} \text{ and } (S, \mathcal{X}) \in \mathbf{Spa}\}.$$

It is sufficient to show that $\mathcal{O}$ is a generalised topology on S in **Spa**, i.e., $(S, \mathcal{O})$ satisfies the class of topological coaxioms that define **Spa**. Assume that $p : (X, \mathcal{B}') \rightarrow (X, \mathcal{B})$ is one of such coaxioms and that $f : (S, \mathcal{O}) \rightarrow (X, \mathcal{B})$ is an arrow in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$. For $B \in \mathcal{B}' \setminus \mathcal{B}$, we have $f^{-1}(B) \in \mathcal{X}$ for any $\mathcal{X}$ with $\mathcal{P} \subset \mathcal{X}$ and $(S, \mathcal{X}) \in \mathbf{Spa}$, which implies that $f^{-1}(B) \in \mathcal{O}$. □

2.2.2 Dual Adjunction via Harmony Condition

Throughout this subsection, let

- **Alg** denote a full subcategory of the Eilenberg-Moore category of a monad T on **Set**;
- **Spa** denote a full subcategory of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of classical topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$.

We aim at establishing a dual adjunction between **Alg** and **Spa** under the two assumptions:

- there is an object Ω living in both **Alg** and **Spa**, i.e., there is $\Omega \in \mathbf{Set}$ both with a structure map $h_\Omega : T(\Omega) \to \Omega$ such that $(\Omega, h_\Omega) \in \mathbf{Alg}$ and with a generalised topology $\mathcal{O}_\Omega \subset \mathcal{Q}(\Omega)$ such that $(\Omega, \mathcal{O}_\Omega) \in \mathbf{Spa}$;
- $(\mathbf{Alg}, \mathbf{Spa}, \Omega)$ satisfies the harmony condition in Definition 2.2.9 below.

Ω is intuitively a set of truth values, and shall work as a so-called dualising object (some standard references such as Johnstone [149] calls it a "schizophrenic" object following Simmons' terminology as mentioned in [149], and yet we do not use the term "schizophrenic" because it has a different technical meaning in a certain context). We simply write Ω instead of (Ω, h_Ω) or $(\Omega, \mathcal{O}_\Omega)$ when there is no confusion.

The harmony condition intuitively means that the algebraic structure of **Alg** and the geometric structure of **Spa** are in harmony via Ω. The precise definition is given below.

Definition 2.2.9. $(\mathbf{Alg}, \mathbf{Spa}, \Omega)$ *is said to satisfy the harmony condition iff, for each* $S \in \mathbf{Spa}$*, there is an object*

$$(\ \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega),\ \ h_S : T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) \to \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)\ )$$

iin **Alg** *such that, for any* $s \in S$ *(let* p_s *be the corresponding projection from* $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ *to* Ω*), the following diagram commutes:*

$$\begin{tikzcd}
T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) \arrow[r, "h_S"'] \arrow[d, "T(p_s)"] & \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \arrow[d, "p_s"] \\
T(\Omega) \arrow[r, "h_\Omega"'] & \Omega
\end{tikzcd}$$

.

Remark 2.2.10. *The commutative diagram above means that the induced operations of* $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ *are defined pointwise. The harmony condition then consists of the two parts:*

(i) $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ *is closed under the pointwise operations;*

(ii) $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ *with the pointwise operations is in* $\mathbf{Alg}$*.*

Here, (ii) is not so important for the reason that we can drop condition (ii) if $\mathbf{Alg}$ *is the Eilenberg-Moore category of a monad on* $\mathbf{Set}$*, rather than a full subcategory of it; this follows from the fact that, since* $\mathbf{Alg}$ *is then closed under products and subalgebras, we have a product* Ω^S *in* $\mathbf{Alg}$*, and hence* $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ *in* $\mathbf{Alg}$ *as a subalgebra of* Ω^S *(obviously, it actually suffices to assume that* $\mathbf{Alg}$ *is a quasi-variety or an implicational full subcategory of the Eilenberg-Moore category of a monad on* $\mathbf{Set}$ *in the sense of [12]). Regarding* $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ *as the collection of generalised continuous functions on* S*, (i) above means that the continuous functions are closed under the algebraic operations defined pointwise, which is the most important part of the harmony condition, and after which the "harmony" condition is named.*

We assume the harmony condition in the following part of this subsection.

The geometric structure of $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ for $A \in \mathbf{Alg}$ can be provided as follows. By Proposition 2.2.8, equip $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ with the generalised topology generated (in $\mathbf{Spa}$) by

$$\{\langle a \rangle_O \ ; \ a \in A \text{ and } O \in \mathcal{O}_\Omega\}$$

where

$$\langle a \rangle_O := \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega) \ ; \ v(a) \in O\}.$$

The algebraic structure of $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ is provided by h_S above.

The induced contravariant Hom-functors $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega) : \mathbf{Alg} \to \mathbf{Spa}$ and $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega) : \mathbf{Spa} \to \mathbf{Alg}$ can be shown to be well defined and form a dual adjunction between categories $\mathbf{Alg}$ and $\mathbf{Spa}$, i.e., we have the following dual adjunction theorem (for the reader's convenience we make the aforementioned assumptions explicit):

Theorem 2.2.11. *Suppose:*

- $\mathbf{Alg}$ *denote a full subcategory of the Eilenberg-Moore category of a monad* T *on* $\mathbf{Set}$*;*

- $\mathbf{Spa}$ *denote a full subcategory of* $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ *that is definable by a class of classical topological coaxioms in* $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$*.*

Assume that the harmony condition above holds. There is, then, a dual adjunction between **Alg** *and* **Spa***, given by contravariant functros* $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega)$ *and* $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega)$*. To be precise,* $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega)$ *is left adjoint to* $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega)^{\mathrm{op}}$*.*

A proof of the theorem is given soon after the following remark.

Remark 2.2.12. *The theorem encompasses the well-known dual adjunction between frames and topological spaces; in this case,* Ω *is the two element frame with the Sierpinski topology, and the harmony condition boils down to the obvious fact that the collection of open sets is closed under the operations of arbitrary unions and finite intersections. The frame-space duality is thus an immediate corollary of the theorem above; this exhibits a sharp contrast to those general theories of dualities that require substantial work in deriving concrete results. Our theory is for duality in context, contrived to be effective in concrete situations.*

In a similar way, we can derive a dual adjunction between σ*-complete Boolean algebras and measurable spaces by letting* Ω *be the two element algebra with the discrete topology (in fact, any algebra with the discrete topology works as* Ω*), where* σ*-complete Boolean algebras may be seen as point-free measurable spaces. In Section 2.3, we discuss in detail a dual adjunction between continuous lattices and convexity spaces.*

The most plain case is the dual adjunction between **Set** *and* **Set***, induced by the two element set as a dualising object* Ω *(any set actually works); the harmony condition is nothing in this case. The discrete Stone adjunction between Boolean algebras and* **Set** *is well known. The theorem above gives us a vast generalisation of it: there is a dual adjunction between any algebraic category (or variety in terms of universal algebra) and* **Set***, induced by any* $\Omega \in \mathbf{Alg}$*; the harmony condition is nothing in this case as well, thanks to the discrete nature of* **Set** *(i.e., the set of all functions* $f : S \to \Omega$ *are closed under arbitrary operations on it).*

Furthermore, the theorem above encompasses the topological Stone adjunction between Boolean algebras and topological spaces, its diverse extensions for distributive lattices, MV*-algebras (*$[0, 1]$ *works as a dualising object in this case), and algebras of substructural logics, and the Gelfand adjunction between commutative* C^**-algebras with units* 1 *and topological spaces; note that the category of commutative* C^**-algebras with* 1 *is monadic over* **Set** *(see [225]). Any dual adjunction automatically cuts down to a dual equivalence as explained below, and the method can be applied to all the dual adjunctions mentioned above in order to obtain dual equivalences (still it often is not that easy to give intrinsic characterisations of the resulting dual equivalences as discussed above).*

Let us think of the Pontryagin self-duality for locally compact Abelian groups. Although for simplicity we did not assume a topological structure on **Alg** and an algebraic structure on **Spa** in our set-up, this gets relevant in order to treat the Pontryagin duality within our framework. It is indeed straightforward: we start with topological **Alg** and algebraic **Spa**, and assume two harmony conditions; and the following proof can easily be adapted to that situation (just repeat the same arguments for the additional structures on **Alg** and **Spa**).

Proof of Dual Adjunction Theorem

We first show that the two Hom-functors are well defined.

Lemma 2.2.13. *The contravariant functor* $\mathrm{Hom}_{\mathbf{Alg}}(\text{-},\Omega) : \mathbf{Alg} \to \mathbf{Spa}$ *is well defined.*

Proof. The object part is well defined by Proposition 2.2.8. We show that the arrow part is well defined. Let $f : A \to A'$ be an arrow in **Alg**. We prove that $\mathrm{Hom}_{\mathbf{Alg}}(f,\Omega) : \mathrm{Hom}_{\mathbf{Alg}}(A',\Omega) \to \mathrm{Hom}_{\mathbf{Alg}}(A,\Omega)$ is an arrow in **Spa**. For $a \in A$ and $O \in \mathcal{O}_\Omega$, we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Alg}}(f,\Omega)^{-1}(\langle a\rangle_O) &= \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A',\Omega)\ ;\ \mathrm{Hom}_{\mathbf{Alg}}(f,\Omega)(v) \in \langle a\rangle_O\} \\
&= \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A',\Omega)\ ;\ v \circ f(a) \in O\} \\
&= \langle f(a)\rangle_O.
\end{aligned}$$

Since **Spa** is definable by a class of Boolean topological coaxioms and since Boolean set operations are preserved by the inverse image function f^{-1}, this implies that $\mathrm{Hom}_{\mathbf{Alg}}(f,\Omega)$ is an arrow in **Spa**. $\square$

Lemma 2.2.14. *The contravariant functor* $\mathrm{Hom}_{\mathbf{Spa}}(\text{-},\Omega) : \mathbf{Spa} \to \mathbf{Alg}$ *is well defined.*

Proof. The object part is well defined by the harmony condition (or can be verified as in (i) or (ii) in Remark 2.2.10 if we employ either of the other two definitions of **Alg**).

We show that the arrow part is well defined. Let $f : S \to S'$ be an arrow in **Spa**. We prove that $\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega) : \mathrm{Hom}_{\mathbf{Spa}}(S',\Omega) \to \mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)$ is an arrow in **Alg**, i.e., the following diagram commutes:

$$\begin{tikzcd}
T(\mathrm{Hom}_{\mathbf{Spa}}(S', \Omega)) \arrow[r, "h_{S'}"] \arrow[d, "T(\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega))"'] & \mathrm{Hom}_{\mathbf{Spa}}(S', \Omega) \arrow[d, "\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega)"] \\
T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) \arrow[r, "h_S"'] & \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)
\end{tikzcd}$$

By the harmony condition applied to S (or the commutativity of the lower square in the figure below), this is equivalent to the commutativity of the outermost square in the following diagram for any $s \in S$:

$$\begin{tikzcd}
T(\mathrm{Hom}_{\mathbf{Spa}}(S', \Omega)) \arrow[r, "h_{S'}"] \arrow[d, "T(\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega))"'] & \mathrm{Hom}_{\mathbf{Spa}}(S', \Omega) \arrow[d, "\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega)"] \\
T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) \arrow[r, "h_S"'] \arrow[d, "T(p_s)"'] & \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \arrow[d, "p_s"] \\
T(\Omega) \arrow[r, "h_\Omega"'] & \Omega
\end{tikzcd}$$

where recall that p_s denotes the corresponding projection. By the harmony condition applied to S', we have: for any $s' \in S'$,

$$h_\Omega \circ T(p_{s'}) = p_{s'} \circ h_{S'}.$$

By taking $s' = f(s)$ in this equation, we have

$$h_\Omega \circ T(p_{f(s)}) = p_{f(s)} \circ h_{S'}.$$

It is straightforward to verify that $p_{f(s)} = p_s \circ \mathrm{Hom}_{\mathbf{Spa}}(f, \Omega)$. Thus we obtain

$$h_\Omega \circ T(p_s \circ \mathrm{Hom}_{\mathbf{Spa}}(f, \Omega)) = p_s \circ \mathrm{Hom}_{\mathbf{Spa}}(f, \Omega) \circ h_{S'}.$$

Since T is a functor, this yields the commutativity of the outermost square above. Hence, the arrow part is well defined. □

Now we define two natural transformations in order to show the dual adjunction.

Definition 2.2.15. *Natural transformations*

$$\Phi : 1_{\mathbf{Alg}} \to \mathrm{Hom}_{\mathbf{Spa}}(\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega), \Omega)$$

and

$$\Psi : 1_{\mathbf{Spa}} \to \mathrm{Hom}_{\mathbf{Alg}}(\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega), \Omega)$$

are defined as follows. For $A \in \mathbf{Alg}$*, define* Φ_A *by* $\Phi_A(a)(v) = v(a)$ *where* $a \in A$ *and* $v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$*. For* $S \in \mathbf{Spa}$*, define* Ψ_S *by* $\Psi_S(x)(f) = f(x)$ *where* $x \in S$ *and* $f \in \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$*.*

We have to show that Φ and Ψ are well defined.

Lemma 2.2.16. *For* $A \in \mathbf{Alg}$ *and* $a \in A$*,* $\Phi_A(a)$ *is an arrow in* $\mathbf{Spa}$*.*

Proof. For $O \in \mathcal{O}_\Omega$, we have

$$\Phi_A(a)^{-1}(O) = \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)\ ;\ \Phi_A(a)(v) \in O\} = \langle a \rangle_O.$$

Thus, $\Phi_A(a)$ is an arrow in $\mathbf{Spa}$. $\square$

Lemma 2.2.17. *For* $S \in \mathbf{Spa}$ *and* $x \in S$*,* $\Psi_S(x)$ *is an arrow in* $\mathbf{Alg}$*.*

Proof. This lemma follows immediately from the fact that $p_x = \Psi_S(x)$ and the harmony condition applied to $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$. $\square$

We also have to show that Φ_A is an arrow in $\mathbf{Alg}$ and that Ψ_S is an arrow in $\mathbf{Spa}$.

Lemma 2.2.18. *For* $A \in \mathbf{Alg}$*,* Φ_A *is an arrow in* $\mathbf{Alg}$*.*

Proof. Let $h_A : T(A) \to A$ denote the structure map of A. For the simplicity of description, let $\mathrm{H}(A)$ denote $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ and $\mathrm{H}\circ\mathrm{H}(A)$ denote $\mathrm{Hom}_{\mathbf{Spa}}(\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega), \Omega)$. In order to show the commutativity of the upper square in the diagram below, it is sufficient to prove that the outermost square is commutative for any $v \in \mathrm{H}(A)$, since the lower square is commutative because of the harmony condition applied to $\mathrm{H} \circ \mathrm{H}(A)$.

$$\begin{tikzcd}
T(A) \arrow[r, "h_A"] \arrow[d, "T(\Phi_A)"'] & A \arrow[d, "\Phi_A"] \\
T(\mathrm{H} \circ \mathrm{H}(A)) \arrow[r, "h_{\mathrm{H}(A)}"'] \arrow[d, "T(p_v)"'] & \mathrm{H} \circ \mathrm{H}(A) \arrow[d, "p_v"] \\
T(\Omega) \arrow[r, "h_\Omega"'] & \Omega
\end{tikzcd}$$

It is straightforward to verify that $p_v \circ \Phi_A = v$. Then, it suffices to show that $v \circ h_A = h_\Omega \circ T(v)$. This is nothing but the fact that $v \in \mathrm{H}(A)$. $\square$

Lemma 2.2.19. *For $S \in \mathbf{Spa}$, Ψ_S is an arrow in* **Spa**.

Proof. For $f \in \mathrm{Hom}_{\mathbf{Alg}}(\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega), \Omega)$ and $O \in \mathcal{O}_\Omega$, we have

$$\Psi_S^{-1}(\langle f \rangle_O) = \{x \in S\ ;\ \Psi_S(x) \in \langle f \rangle_O\} = f^{-1}(O).$$

Since **Spa** is definable by a class of Boolean topological coaxioms and since Boolean set operations are preserved by the inverse image function Ψ_S^{-1}, this implies that Ψ_S is an arrow in **Spa**. $\square$

Now it is straightforward to verify that Φ and Ψ are actually natural transformations.

We finally give a proof of the dual adjunction theorem, Theorem 2.2.11: $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega)$ is left adjoint to $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega)^{\mathrm{op}}$ with Φ the unit and Ψ^{op} the counit of the adjunction.

Proof. Let $A \in \mathbf{Alg}$ and $S \in \mathbf{Spa}$. It is enough to show that, for any $f : A \to \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ in **Alg**, there is a unique $g : S \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ in **Spa** such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H} \circ \mathrm{H}(A) \arrow[r, "H(g)"] & H(S) \\
A \arrow[u, "\Phi_A"] \arrow[ur, "f"'] &
\end{tikzcd}$$

where $\mathrm{H}(S)$ denotes $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$, $\mathrm{H}(g)$ denotes $\mathrm{Hom}_{\mathbf{Spa}}(g, \Omega)$, $\mathrm{H}(A)$ denotes $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ and $\mathrm{H} \circ \mathrm{H}(A)$ denotes $\mathrm{Hom}_{\mathbf{Spa}}(\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega), \Omega)$. We first show that such g exists. Define $g : S \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ by $g(x)(a) = \Psi_S(x)(f(a))$ where $x \in S$ and $a \in A$. Then we have

$$\begin{aligned}
(\mathrm{Hom}_{\mathbf{Spa}}(g, \Omega) \circ \Phi_A(a))(x) &= (\Phi_A(a) \circ g)(x) = g(x)(a) \\
&= \Psi_S(x)(f(a)) = f(a)(x).
\end{aligned}$$

Thus, the above diagram commutes for this g. It remains to show that g is an arrow in **Spa**. For $a \in A$ and $O \in \mathcal{O}_\Omega$, we have

$$\begin{aligned}
g^{-1}(\langle a \rangle_O) &= \{x \in S\ ;\ g(x) \in \langle a \rangle_O\} \\
&= \{x \in S\ ;\ g(x)(a) \in O\} \\
&= \{x \in S\ ;\ f(a)(x) \in O\} \\
&= f(a)^{-1}(O).
\end{aligned}$$

Since $f(a) \in \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ and since **Spa** is definable by a class of Boolean topological coaxioms, this implies that g is an arrow in **Spa**.

Finally, in order to show the uniqueness of such g, we assume that $g' : S \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ in **Spa** makes the above diagram commute. Then we have

$$f(a)(x) = (\mathrm{Hom}_{\mathbf{Spa}}(g', \Omega) \circ \Phi_A(a))(x) = (\Phi_A(a) \circ g')(x) = g'(x)(a).$$

Since we also have $f(a)(x) = g(x)(a)$, it follows that $g = g'$. This completes the proof. □

Deriving Equivalence from Adjunction

We briefly review standard methods to derive a (dual) equivalence from a (dual) adjunction. Assume that $F : \mathbf{C} \to \mathbf{D}$ is left adjoint to $G : \mathbf{D} \to \mathbf{C}$ with Φ and Ψ the unit and the counit of the adjunction, respectively.

Definition 2.2.20. $\mathrm{Fix}(\mathbf{C})$ *is a full subcategory of* $\mathbf{C}$ *such that* $C \in \mathrm{Fix}(\mathbf{C})$ *iff* Φ_C *is an isomorphism in* $\mathbf{C}$*.* $\mathrm{Fix}(\mathbf{D})$ *is a full subcategory of* $\mathbf{D}$ *such that* $D \in \mathrm{Fix}(\mathbf{D})$ *iff* Ψ_D *is an isomorphism in* $\mathbf{D}$*.*

Proposition 2.2.21. $\mathrm{Fix}(\mathbf{C})$ *and* $\mathrm{Fix}(\mathbf{D})$ *are categorically equivalent. Moreover, this equivalence is the maximal one that can be derived from the adjunction between* $\mathbf{C}$ *and* $\mathbf{D}$*.*

If we require a condition about the original adjunction, we have another way to describe $\mathrm{Fix}(\mathbf{C})$ and $\mathrm{Fix}(\mathbf{D})$. We first introduce the following notations.

Definition 2.2.22. $\mathrm{Img}(\mathbf{C})$ *is a full subcategory of* $\mathbf{C}$ *such that* $C \in \mathrm{Img}(\mathbf{C})$ *iff* $C \simeq G(D)$ *for some* $D \in \mathbf{D}$*.* $\mathrm{Img}(\mathbf{D})$ *is a full subcategory of* $\mathbf{D}$ *such that* $D \in \mathrm{Img}(\mathbf{D})$ *iff* $D \simeq F(C)$ *for some* $C \in \mathbf{C}$*.*

Proposition 2.2.23. *Assume that* $F(C) \in \mathrm{Fix}(\mathbf{D})$ *for any* $C \in \mathbf{C}$ *and that* $G(D) \in \mathrm{Fix}(\mathbf{C})$ *for any* $D \in \mathbf{D}$*. It then holds that* $\mathrm{Img}(\mathbf{C}) = \mathrm{Fix}(\mathbf{C})$ *and* $\mathrm{Img}(\mathbf{D}) = \mathrm{Fix}(\mathbf{D})$*. Hence,* $\mathrm{Img}(\mathbf{C})$ *and* $\mathrm{Img}(\mathbf{D})$ *are categorically equivalent.*

Note that the above assumption is satisfied in the case of the duality between spatial frames and sober topological spaces, and also in the case of a duality between spatial continuous lattices and sober convexity spaces, which is presented in the next section.

This Fix construction may be seen as being left adjoint to the obvious forgetful functor from the category of dual equivalences to the category of dual adjunctions.

The former is defined as the category of pairs $(\mathbf{C}, \mathbf{D})$ equipped with two functors giving a dual equivalence between categories $\mathbf{C}$ and $\mathbf{D}$, and the latter as the category of pairs $(\mathbf{C}, \mathbf{D})$ equipped with two functors giving a dual adjunction between $\mathbf{C}$ and $\mathbf{D}$. Morphisms in these categories are required to respect functors giving dual equivalences dual adjunctions. And then there is an adjunction between the category of dual equivalences and the category of dual adjunctions, given by the forgetful functor and the Fix functor, which is a sort of "free duality" functor.

2.3 Domain-Convexity Duality

In this section, we apply the general theory to obtain a dual adjunction between continuous lattices and convexity spaces, and then refine the dual adjunction into a dual equivalence between algebraic lattices and sober convexity spaces, which allows us to characterise the dual equivalence part of Jacobs' dual adjunction for algebras of the distribution monad, with the help of the notion of idempotency for those algebras.

2.3.1 Convexity-Theoretical Duality for Scott's Continuous Domains

The concept of a continuous lattice is usually defined in terms of way-below relations: i.e., a continuous lattice is a complete lattice in which any element can be expressed as the join of those elements that are way-below it. From our perspective of duality between point-free and point-set spaces, another characterisation of continuous lattices is helpful:

Proposition 2.3.1 (Theorem I-2.7 in [109])). *A poset is a continuous lattice iff it satisfies the following: (i) it has directed joins including* 0*; (ii) it has arbitrary meets including* 1*; (iii) arbitrary meets distribute over directed joins.*

The proposition above suggests that continuous lattices may be considered to be point-free convexity spaces; recall that a convexity space is a tuple $(S, \mathcal{C})$ where S is a set, and $\mathcal{C}$ is a subset of $\mathcal{P}(S)$ that is closed under directed unions and arbitrary intersections; $\mathcal{C}$ is called the convexity of the space. Many theorems in convex geometry such as Helly-type theorems (see [124]) can be treated in terms of convexity spaces with suitable conditions (see [71, 270]).

Just as a frame is a point-free abstraction of a topological space, so a continuous lattice is a point-free abstraction of a convexity space; this is what the proposition above tells us. Note that item 1 above is mathematically redundant, but suggests the definition of a homomorphism, which preserves directed joins and arbitrary meets.

This idea in turn suggests that there is a duality between **ContLat** and **Conv**. To apply our duality theory, recall that the continuous lattices are the algebras of the filter monad on **Set** (see [109]), and that the convexity spaces can be expressed as a full subcategory of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of classical topological coaxioms.

We can see $\mathbf{2}$ (i.e., $\{0,1\}$) as a continuous lattice by its natural ordering $0 < 1$ and also as a convexity space by equipping it with the Sierpinski convexity $\{\emptyset, \{1\}, \mathbf{2}\}$. In order to show that homming into $\mathbf{2}$ gives us a dual adjunction between continuous lattices and convexity spaces, it suffices to verify the harmony condition. It is immediate because the harmony condition in this case boils down to the fact that $\mathrm{Hom}_{\mathbf{Conv}}(S, \mathbf{2})$, which can be seen as the set of convex sets in S, forms a continuous lattice. We then obtain the following theorem.

Theorem 2.3.2. *There is a dual adjunction between* **ContLat** *and* **Conv***, given by contravariant functors* $\mathrm{Hom}_{\mathbf{Conv}}(\text{-}, \mathbf{2})$ *and* $\mathrm{Hom}_{\mathbf{ContLat}}(\text{-}, \mathbf{2})$*.*

We can formally refine the dual adjunction into a dual equivalence in the canonical way as already discussed. It is non-trivial, however, to find an intrinsic description of the induced dual equivalence. We shall achieve it in the following. We omit the proofs which are essentially the same as the corresponding proofs in Maruyama [189]; note that it causes no essential change in proofs to replace convexity algebras in [189] by continuous lattices.

$\mathrm{Hom}_{\mathbf{Conv}}(X, \mathbf{2})$ can be seen as the collection of convex sets in X, so we write $\mathrm{Conv}(\text{-})$ for $\mathrm{Hom}_{\mathbf{Conv}}(\text{-}, \mathbf{2})$. Likewise, we write $\mathrm{Spec}(\text{-})$ for $\mathrm{Hom}_{\mathbf{ContLat}}(\text{-}, \mathbf{2})$, for the reason that $\mathrm{Hom}_{\mathbf{ContLat}}(L, \mathbf{2})$ can be seen as the collection of Scott-open meet-complete filters of L where meet-completeness is defined as closedness under arbitrary meets.

Definition 2.3.3. *We denote by* $\Phi : \mathrm{Id}_{\mathbf{ContLat}} \to \mathrm{Conv} \circ \mathrm{Spec}$ *and* $\Psi : \mathrm{Id}_{\mathbf{Conv}} \to \mathrm{Spec} \circ \mathrm{Conv}$ *the unit and counit of the dual adjunction between* **ContLat** *and* **Conv***, respectively.*

The question is when the unit Φ and the counit Ψ give isomorphisms.

We define the notion of spatiality of continuous lattices as the existence of enough Scott-open meet-complete filters:

Definition 2.3.4. *A continuous lattice L is spatial iff, for any $a, b \in L$ with $a \not\leq b$, there is a Scott-open meet-complete filter P of L such that $a \in P$ and $b \notin P$.*

The following proposition is crucial.

Proposition 2.3.5. *A continuous lattice L is spatial iff $\Phi_L : L \to \mathrm{Conv} \circ \mathrm{Spec}(L)$ is an isomorphism.*

Spatiality is characterised as algebraicity.

Proposition 2.3.6. *Let L be a continuous lattice. Then, L is spatial iff L is algebraic (i.e., every element can be expressed as the join of a directed set of compact elements).*

Sober convexity spaces are defined in terms of polytopes, which make sense in general convexity spaces as follows.

Definition 2.3.7. *The convex hull* $\mathrm{ch}(Y)$ *of a subset Y of a convexity space $(X, \mathcal{C})$ is defined as*

$$\bigcap \{Z \mid Z \in \mathcal{C} \text{ and } Y \subset Z\}.$$

Then, a polytope in a convexity space is defined as the convex hull of a set of finitely many points in it.

A convex set C in $\mathcal{C}$ is said to be directed-irreducible iff if

$$C = \bigcup_{i \in I} C_i$$

for a directed subset $\{C_i \,;\, i \in I\}$ of $\mathcal{C}$ then there exists $i \in I$ such that $C = C_i$.

Proposition 2.3.8. *A convex subset of a convexity space is directed-irreducible iff it is a polytope.*

Polytopes form a canonical basis for any convexity (if we assume the axiom of choice):

Proposition 2.3.9. *Any convex set in a convexity space can be expressed as the union of a directed set of polytopes.*

Sobriety is defined as follows (polytopes may be replaced with directed-irreducible sets).

Definition 2.3.10. *A convexity space is sober iff every polytope in it is the convex hull of a unique point.*

In contrast to the first impression, sobriety is a natural concept. Let us see examples.

Definition 2.3.11. *Given a convexity space S, we can equip the set of polytopes in S with the ideal convexity: i.e., a convex set is an ideal of the lattice of polytopes.*

The space of polytopes is then sober, and gives the soberification of the original space. The space of polytopes corresponds to the space of irreducible varieties in algebraic geometry; to put it differently, "a unique point" above plays, in convex geometry, the rôle of "a generic point" in terms of algebraic geometry.

In algebraic geometry, we soberify a variety by adding irreducible varieties as additional generic points (in other words, the prime spectrum of the coordinate ring of a variety gives the soberification). In convex geometry, we soberify a space by adding polytopes as generic points (in other words, the prime spectrum of the lattice of convex sets gives the soberification). Here recall that polytopes can be characterised by directed-irreducibility.

Proposition 2.3.12. *For a convexity space S, S is sober iff Ψ_S is an isomorphism in* **Conv**.

Let **SobConv** denote the category of sober convexity spaces and convexity preserving maps, **AlgLat** the category of algebraic lattices and homomorphisms, and **SpaContLat** the category of spatial continuous lattices and homomorphisms. We finally obtain the following.

Theorem 2.3.13. **AlgLat** (= **SpaContLat**) *and* **SobConv** *are dually equivalent.*

2.3.2 Jacobs Duality for Algebras of the Distribution Monad

Let $\mathcal{D} : \mathbf{Set} \to \mathbf{Set}$ be the distribution monad on **Set**. The object part is defined by:

$$\mathcal{D}(X) := \left\{ f : X \to [0,1] \mid \sum_{x \in X} f(x) = 1 \text{ and } f \text{ has a finite support} \right\}.$$

The arrow part is defined by:

$$\mathcal{D}(f : X \to Y)(g : X \to [0,1])(y) = \sum_{f(x)=y} g(x).$$

As in [102, 146], algebras of $\mathcal{D}$ can concretely be described as barycentric algebras, which are basically sets with convex combination operations; the precise definition is given below.

Jacobs [146] shows a dual adjunction between preframes and algebras of $\mathcal{D}$. We first observe that we can restrict the category of preframes into the category of continuous lattices, since the dual of an algebra of $\mathcal{D}$ (i.e., $\mathrm{PF}(X)$ below) is actually a continuous lattice. And then we characterise the induced dual equivalence via the concept of idempotent algebras of $\mathcal{D}$.

Definition 2.3.14. *A $\mathcal{D}$-algebra (aka. barycentric algebra) is a set X with a ternary function*

$$\langle -,-,- \rangle : [0,1] \times X \times X \to X$$

such that

1. $\langle r, x, x \rangle = x$;
2. $\langle 0, x, y \rangle = y$;
3. $\langle r, x, y \rangle = \langle 1-r, y, x \rangle$;
4. $\langle r, x, \langle s, y, z \rangle \rangle = \langle r + (1-r)s, \langle r/(r+(1-r)s), x, y \rangle, z \rangle$.

Morphisms of $\mathcal{D}$-algebras are affine maps, i.e., maps f preserving $\langle -,-,- \rangle$ in the following way:

$$f(\langle r, x, y \rangle) = \langle r, f(x), f(y) \rangle.$$

$\mathbf{Alg}(\mathcal{D})$ *denotes the category of $\mathcal{D}$-algebras and affine maps.*

$\mathbf{Alg}(\mathcal{D})$ in the sense above is equivalent to the Eilenberg-Moore category of the distribution monad (see [146, 102]).

Semilattices can be regarded as $\mathcal{D}$-algebras in a canonical way.

Proposition 2.3.15. *Any meet-semilattice L forms a $\mathcal{D}$-algebra: define $\langle -,-,- \rangle : [0,1] \times L \times L \to L$ by*

$$\langle r, x, y \rangle = x \wedge y$$

if $r \in (0,1)$; otherwise, define $\langle r, x, y \rangle = x$ if $r = 1$, and $\langle r, x, y \rangle = y$ if $r = 0$. Similarly, any join-semilattice forms a $\mathcal{D}$-algebra (by replacing $\wedge$ above with $\vee$).

In the following, we suppose any semilattice is equipped with the convex structure defined in the proposition above. We review the following concepts from Jacobs [146].

Definition 2.3.16. *For a $\mathcal{D}$-algebra $(X, \langle -,-,- \rangle)$, a subset Y of X is defined as*

- *a subalgebra iff $y_1, y_2 \in Y$ implies that for any $r \in [0,1]$, $\langle r, y_1, y_2 \rangle \in Y$;*

- *a filter iff* $\langle r, x_1, x_2\rangle \in Y$ *and* $r \neq 0, 1$ *together imply both* $x_1 \in Y$ *and* $x_2 \in Y$*;*
- *a prime filter iff it is both a subalgebra and a filter.*

Let us define a contravariant functor

$$\mathrm{PF}(\text{-}) : \mathbf{Alg}(\mathcal{D})^{\mathrm{op}} \to \mathbf{ContLat}.$$

For a $\mathcal{D}$-algebra X, $\mathrm{PF}(X)$ is the lattice of prime filters of X. For an affine map f, we let $\mathrm{PF}(f) = f^{-1}$.

We define a contravariant functor

$$\mathrm{Sp}(\text{-}) : \mathbf{ContLat}^{\mathrm{op}} \to \mathbf{Alg}(\mathcal{D})$$

as follows. For a continuous lattice L, define $\mathrm{Sp}(L)$ as the set of Scott-open meet-complete filters of L, equipped with a meet-semilattice structure by finite intersections, and hence with a $\mathcal{D}$-algebra structure (see Proposition 2.3.15). For a homomorphism f, we let $\mathrm{Sp}(f) = f^{-1}$.

Since $\mathrm{PF}(X)$ always forms a continuous lattice, the methods of Jacobs [146] completely work in the present situation, thus yielding the following dual adjunction theorem.

Theorem 2.3.17. *There is a dual adjunction between* $\mathbf{ContLat}$ *and* $\mathbf{Alg}(\mathcal{D})$*, given by* $\mathrm{Sp} : \mathbf{ContLat}^{\mathrm{op}} \to \mathbf{Alg}(\mathcal{D})$ *and* $\mathrm{PF} : \mathbf{Alg}(\mathcal{D})^{\mathrm{op}} \to \mathbf{ContLat}$.

In the following, we aim at identifying the dual equivalence induced by the dual adjunction above. Towards this end, we introduce the concept of idempotent $\mathcal{D}$-algebras.

Definition 2.3.18. *A* $\mathcal{D}$*-algebra* $(X, \langle \text{-}, \text{-}, \text{-}\rangle)$ *is idempotent iff for any* $x, y \in X$*, and for any* $r, s \in (0, 1)$ *(i.e., the open unit interval),*

$$\langle r, x, y\rangle = \langle s, x, y\rangle.$$

It is straightforward to see the following.

Proposition 2.3.19. *Any meet-semilattice and join-semilattice form an idempotent* $\mathcal{D}$*-algebra. In particular,* $\mathrm{Sp}(L)$ *is an idempotent* $\mathcal{D}$*-algebra.*

Proposition 2.3.20. *For a* $\mathcal{D}$*-algebra* X*,* $\mathrm{PF}(X)$ *is an algebraic lattice.*

Proof. This follows from the fact that $\mathrm{PF}(X)$ is a subalgebra of the powerset algebraic lattice $\mathcal{P}(X)$ with respect to directed unions and arbitrary intersections, and that the class of all algebraic lattices is closed under subalgebras. $\square$

Proposition 2.3.21. *If L is an algebraic lattice, then L is isomorphic to* $\mathrm{PF} \circ \mathrm{Sp}(L)$.

Proof. Firstly, $\mathrm{Sp}(L)$ can be regarded as $\mathrm{Hom}_{\mathbf{ContLat}}(L, \mathbf{2})$ by identifying subsets with their characteristic functions. Likewise, $\mathrm{PF} \circ \mathrm{Sp}(L)$ can be seen as $\mathrm{Hom}_{\mathbf{Conv}}(\mathrm{Sp}(L), \mathbf{2})$. Then, the previously obtained duality between algebraic lattices and sober convexity spaces immediately tells us that L is indeed isomorphic to $\mathrm{PF} \circ \mathrm{Sp}(L)$. $\square$

Proposition 2.3.22. *If X is an idempotent $\mathcal{D}$-algebra, then X is isomorphic to* $\mathrm{Sp} \circ \mathrm{PF}(X)$.

Proof. For $x, y \in X$, define $x \wedge y$ by $\langle 1/2, x, y \rangle$. By idempotency, X with $\wedge$ forms a meet-semilattice. Since $\mathrm{Sp} \circ \mathrm{PF}(X)$ is an idempotent $\mathcal{D}$-algebra by Proposition 2.3.19, $\mathrm{Sp} \circ \mathrm{PF}(X)$ also forms a meet-semilattice in the same way. It holds that if the meet-semilattices of two idempotent $\mathcal{D}$-algebras are isomorphic, then the original $\mathcal{D}$-algebras are isomorphic as well. Thus, it suffices to prove that X is isomorphic to $\mathrm{Sp} \circ \mathrm{PF}(X)$ as a meet-semilattice.

Now, X can in turn be equipped with the ideal convexity: the convex sets are defined as the ideals of X. Then, X is a sober convexity space (with the convex sets of X forming an algebraic lattice). The polytopes of X, denoted $\mathrm{Poly}(X)$, form a join-semilattice: for two polytopes $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ with X, Y finite, their join is defined as $\mathrm{ch}(X \cup Y)$, where recall ch(-) denotes the convex hull operation. And then $\mathrm{Poly}(X)^{\mathrm{op}}$, the order dual of the polytope join-semilattice $\mathrm{Poly}(X)$, is actually isomorphic to X as a meet-semilattice; this holds for any meet-semilattice X by an equivalence between the categories of join-semilattices and of sober convexity spaces as remarked in Maruyama [189]. Since $\mathrm{PF}(X)$ is the lattice of ideals of X, it turns out that $\mathrm{Sp} \circ \mathrm{PF}(X)$ is the meet-semilattice of compact (aka. directed-irreducible) elements of the ideal lattice, which is precisely $\mathrm{Poly}(X)^{\mathrm{op}}$ (see Proposition 2.3.8); recall $\mathrm{Poly}(X)^{\mathrm{op}}$ is isomorphic to X, and the proof is done. $\square$

Let $\mathbf{IdemAlg}(\mathcal{D})$ denote the category of idempotent $\mathcal{D}$-algebras. Propositions 2.3.21 and 2.3.22 above finally give us the following theorem identifying the dual equivalence part of the dual adjunction between **ContLat** and $\mathbf{Alg}(\mathcal{D})$.

Theorem 2.3.23. *The dual adjunction between* **ContLat** *and* $\mathbf{Alg}(\mathcal{D})$ *restricts to a dual equivalence between* **AlgLat** *and* $\mathbf{IdemAlg}(\mathcal{D})$. *This is the largest dual equivalence induced by the dual adjunction.*

Since the converses of Propositions 2.3.21 and 2.3.22 hold by Propositions 2.3.19 and 2.3.20 respectively, the theorem above gives the maximal dual equivalence that can result from restricting the dual adjunction between **ContLat** and $\mathbf{Alg}(\mathcal{D})$. Note that the duality above is closely related to the classic Hofmann-Mislove-Stralka duality [135]; indeed, the duality above reveals a convexity-theoretical aspect of the Hofmann-Mislove-Stralka duality.

Summing up, we have obtained the following dualities in this section:

$$\mathbf{IdemAlg}(\mathcal{D}) \simeq \mathbf{AlgLat}^{\mathrm{op}} \simeq \mathbf{SobConv}.$$

It thus follows that $\mathbf{IdemAlg}(\mathcal{D}) \simeq \mathbf{SobConv}$; behind the equivalence, we actually have an adjunction between $\mathbf{IdemAlg}(\mathcal{D})$ and **Conv**. And the functor from **Conv** to $\mathbf{IdemAlg}(\mathcal{D})$ has clear meaning in terms of polytopes: it maps a convexity space S to the $\mathcal{D}$-algebra of $\mathrm{Poly}(S)^{\mathrm{op}}$, i.e., the order dual of the polytope join-semilattice of S. These domain-convexity dualities tell us that domain theory and convex geometry are naturally intertwined in the (sometimes beautiful, sometimes insane) universe of mathematics.

2.4 Categorical Duality as Philosophy of Space

We finally speculate about categorical duality in a broader context, and attempt to elucidate conceptual foundations of categorical duality, especially in relation to the philosophy of space; this section gives another philosophical perspective on duality as a supplement to our earlier discussion in the introductory chapter. Although we change some formulations, nonetheless, what is meant is essentially the same.

Let us start, again, with the following picture of categorical dualities in diverse disciplines, almost all of which may be conceived of as arising between the epistemic and the ontic. The concept of duality between ontology and epistemology, we think, yields a unifying perspective on categorical dualities in wide-ranging fields; it is like "duality between the conceptual and the formal" in Lawvere's terms in his seminal hyperdoctrine paper "Adjointness in Foundations" (more links with other thinkers shall be pursued afterwards, including those with Gödel's distinction between the "right" and the "left", and Wittgenstein's philosophy of space).

	Ontic	Epistemic	Duality
Logic	Model	Theory	Gödel-Stone
Alg. Logic	Alg. Semantics	Logical System	Tarski
Complex Geom.	Riemann Surface	Alg. Func. Field	Riemann
Classical Alg. Geom.	Variety over k	k-Algebra	Hilbert
Modern Alg. Geom.	Scheme	Ring	Grothendieck
Representation Th.	Group	Representation	Pontryagin
Topology	Point	Open Set	Isbell-Papert
Convex Geometry	Point	Convex Set	Maruyama
Galois Theory	(Profinite) G-set	Algebra Extension	Galois
Program Semantics	Denotation	Observable Prop.	Abramsky
System Science	Computer System	Its Behaviour	Coalg—Alg
General Relativity	Spacetime Manifold	Field	Weyl
Quantum Physics	State	Observable	Gelfand

The aforementioned duality between denotations and observable properties of programs basically amounts to domain-theoretical variants of infinitary Stone dualities, such as the Isbell-Papert one. The duality theory of this chapter is relevant to finitary and infinitary Stone, Gelfand, Pontryagin, and even Hilbert dualities (because Hilbert and Stone dualities are closely related as discussed below). The above duality between computer systems and their behaviours boils down to algebra-coalgebra duality in mathematical terms. The most basic case is the Abramksy duality between modal algebras and coalgebras of the Vietoris endofunctor on the category of Stone spaces, which was later rediscovered and explicated by Kupke-Kurz-Venema. A universal-algebraic general theory of such algebra-coalgebra dualities is developed in the next chapter.

Some of the dualities above are tightly intertwined as a matter of fact. The Stone duality for classical logic is precisely equivalent to a Hilbert duality for geometry over $\mathbb{GF}(2)$ (i.e., the prime field of two elements). Furthermore, logical completeness for classical logic corresponds to Nullstellensatz for geometry over $\mathbb{GF}(2)$, in a mathematically rigorous manner; note that this is different from the model-theoretic correspondence between logic and algebraic geometry. Here it should be noted that logical completeness tells us a poset duality between models and theories, and Nullstellensatz a poset duality between affine varieties and radical ideals, which can be upgraded into the corresponding categorical dualities, namely the Stone duality and the Hilbert duality, respectively (note that the Stone duality is a generalisation of completeness, namely the syntax-semantics equivalence, in a mathematically precise sense). In this sense, completeness and Nullstellensatz may be said to be "predualities." The correspondence between logic and algebraic geometry may be summarised as follows:

	Logic	Algebraic Geometry
Algebra	Formulae	Polynomials
Spectrum	Models	Variety
Poset Duality	Completeness	Nullstellensatz
Categorical Duality	Stone Duality	Hilbert Duality

The author's recent investigation shows that this correspondence between logic and algebraic geometry extends to $\mathbb{GF}(p^n)$-valued logic and geometry over $\mathbb{GF}(p^n)$ where p is a prime number, and n is an integer more than 1 (and $\mathbb{GF}(p^n)$ is the Galois field of order p^n).

The concept of space has undergone a revolution in the modernisation of mathematics, shifting the emphasis from underlying point-set spaces to algebraic structures upon them, to put it more concretely, from topological spaces to locales (or formal topology as predicate locales), toposes, schemes (i.e., sheaves of rings), and non-commutative point-free spaces (such as C^* and von Neumann algebras). Categorical duality has supported and eased this shift from point-set to point-free space, since it basically tells us the algebraic point-free structure on a point-set space keeps the same amount of information as the original point-set space, allowing us to recover the points as the spectrum of the algebraic structure.

Having seen different categorical dualities seemingly share certain conceptual essence, it would be natural to ask where the (mathematical) origin of those dualities lie, even though there may be no single origin, and the concept of origin *per se* may be misguided. Since duality allows us to regard algebra itself as (point-free) space, another relevant question is where the origin of the shift from point-set to point-free space is.

The first mathematician who elucidated the point could be Riemann, who proved (what is now called) a Riemann surface can be recovered from its function field. At the same time, however, we may think of several serious contenders, especially Kronecker and Dedekind-Weber on the one hand, who are considered (e.g., by Harold Edwards) to be precursors of arithmetic geometry, and Brouwer on the other. Whilst it seems Riemann did not take algebra itself to be space, Kronecker and Dedekind-Weber indeed algebraised complex geometry (e.g., the Riemann-Roch theorem), considering algebraic function fields *per se* to be (equivalents of) spaces (and uncovering a grand link with algebraic number theory, the crucial analogy between algebraic number fields and function fields). Brouwer, even though coming into the scene later than them, vigorously formulated and articulated, in terms of so-called spreads and choice sequences, the notion of continuums that does not presuppose point, transforming a bare, speculative idea into a full-fledged, mathematically substantial enterprise.

Comparable shifts seem to have been caused in philosophy as well. Whitehead's process philosophy puts more emphasis on dynamic *processes* than static *substances*. His philosophy of space is, in its spirit, very akin to the idea of point-free topology:

> Whitehead's basic thought was that we obtain the abstract idea of a spatial point by considering the limit of a real-life series of volumes extending over each other, for example in much the same way that we might consider a nested series of Russian dolls or a nested series of pots and pans. However, it would be a mistake to think of a spatial point as being anything more than an abstraction.

This is from Irvine's *Stanford Encyclopedia of Philosophy* article on Whitehead, and may indeed be read as a brilliant illustration of the idea of prime ideals (or filter) as points in duality theory and algebraic geometry: recall that the open neigbourhoods of a point in a topological space form a completely prime filter of its open set locale, with the complement yielding a prime ideal. Although Whitehead's philosophy of space tends to be discussed in the context of mereology, which is sort of peculiar mathematics, nevertheless, it is indeed highly relevant to the core idea of modern geometry in mainstream mathematics; the idea of points as prime ideals is particularly important in algebraic and non-commutative geometry.

Whitehead's process philosophy would be relevant to category theory in general: for example, John Baez asserts "a category is the simplest framework where we can talk about systems (objects) and processes (morphisms)" in his paper "Physics, Topology, Logic and Computation: A Rosetta Stone." Abramsky-Coecke's categorical quantum mechanics follows a similar line of idea, regarding a †-compact category as a "universe" of quantum processes expressed in an intuitively meaningful graphical language; this is Bob Coecke's quantum picturalism. At the same time, however, we must be aware of the possibility that formalisation distorts or misses a crucial point of an original philosophical idea. Indeed, Whitehead's concept of process would ultimately be unformalisable by its nature. This remark is applicable throughout the whole discussion here, and we have to be cautious of distortion via formalisation, a common mistake the mathematician or logician tends to make.

Yet another point-free philosopher of space is Wittgenstein: "What makes it apparent that space is not a collection of points, but the realization of a law?" (*Philosophical Remarks*, p. 216). Wittgenstein's intensional view on space is a compelling consequence of his persistent disagreement with the set-theoretical extensional view of mathematics:

> Mathematics is ridden through and through with the pernicious idioms of set theory. One example of this is the way people speak of a line as composed of points. A line is a law and isn't composed of anything at all (*Philosophical Grammar*, p. 211).

What does he mean by "law"? Brouwer defined his concept of a spread as a certain law to approximate a "point", and this could possibly be a particular case of Brouwer's influence on Wittgenstein's philosophy (more detailed discussion may be found in the author's paper [197]).

Where is the philosophical origin of such a mode of thinking? Just as remarked in the case of the mathematical origin, there may be no single origin, and it might even be wrong to seek an origin at all. Certain postmodern philosophers assert that the idea of the original tends to be invented through a number of copies: after all, there may only be copies having no origin or essence in common. Anyway, we could just envisage a bunch of family-resemblant copies (possibly sharing no genuine feature in common at all) in the form of a series of dichotomies:

Cassirer Shift	Substance	Function
Whitehead Shift	Material	Process
Brouwer Shift	Point	Choice Sequence
Wittgenstein Shift	Tractatus	Investigations
Bohr Shift	Classical Realism	Complementarity
Gödel Shift	Right	Left
Lawvere Duality	Conceptual	Formal
Granger Duality	Object	Operation
Zeno Paradox	Continuous	Discrete
Aristotle	Matter	Form
Natural Philosophy	Newton	Leibniz
Kant	Thing Itself	Appearance
Phenomenology	Object	Subject
Theory of Meaning	Davidson	Dummett

"Shift" means that each thinker emphasises in his dichotomy the shift from a concept on the left-hand side to that on the right-hand side. Cassirer could possibly be a philosophical origin of the modernist shift discussed so far. In contrast with "shift", "duality" does not imply anything on which concept is prior to the other; rather, it does suggest equivalence between two views concerned. Finally, no uniform relationships are intended to hold between two concepts in each of the rest of dichotomies, which do not particularly focus upon shifting from one concept to the other. It would be of conceptual significance to reflect upon the table of categorical dualities in the light of these philosophical dichotomies.

Duality is more than dualism, just as categorical duality in point-free geometry starts with the dualism of space and then tells us that the two conceptions of space are equivalent via functors (in a sense reducing dualism to monism; or it could be called monism on the top of dualism). Category theory often goes beyond dualism. Other sorts of dualism include "geometry vs. algebra", and "model-theoretic vs. proof-theoretic semantics." For instance, the concept of algebras of monads even encompasses geometric structures such as topological spaces and convex structures. Categorical logic tells us model-theoretic semantics amounts to interpreting logic in set-based categories, and proof-theoretic semantics to interpreting logic in so-called syntactic categories. We may thus say category theory transcends dualism.

Gödel's shift from "right" to "left" would need to be explicated. In his "The modern development of the foundations of mathematics in the light of philosophy", Gödel says:

> [T]he development of philosophy since the Renaissance has by and large gone from right to left [...] Particularly in physics, this development has reached a peak in our own time, in that, to a large extent, the possibility of knowledge of the objectivisable states of affairs is denied, and it is asserted that we must be content to predict results of observations. This is really the end of all theoretical science in the usual sense [...]

In the physical context, thus, Gödel's "right" means the emphasis of reality, substance, and the like, and "left" something like observational phenomena. Turning into other contexts, Gödel says metaphysics is "right", and formal logic is "left" in his terminology.

We finally articulate three senses of foundations of mathematics, thereby arguing that philosophy of space counts as foundations mathematics in one of the three senses. A popular, prevailing conception of foundations of mathematics is what may be called a "Reductive Absolute Foundation", which reduces everything to one framework, giving an absolute, domain-independent context to work in. The most popular one is (currently) set-theoretical foundations, but category theory (e.g., Lawvere's ETCS and toposes) can do the job as well.

Category theory can give another sense of foundation. That is a "Structural Relative Foundation", which changes a framework according to our structural focus (and see what remains *invariant*, and what does not), and gives a relative, domain-specific context to work in: e.g., ribbon categories for foundations of knot theory and †-compact categories for foundations of quantum mechanics and information (in these two cases, certain monoidal or linear logical structures are shared and invariant). Recall Grothendieck's relative point of view, and that change of base is a fundamental idea of category theory. The reductive-structural distinction is taken from Prawitz' notions of reductive and structural proof theory.

Philosophy of space as discussed above, we believe, counts as a "Conceptual Foundation of Mathematics", which aims at elucidating the nature of fundamental concepts in mathematics, and, presumably, are compelling for the working mathematician as well (where we basically mean mathematical space rather than physical or intuitive space). In this strand, our Categorical Universal Logic proposes a logical universal concept of space to unify toposes and quantum space categories in terms of monad-relativised Lawvere hyperdoctrines, allowing us to reconcile Abramsky-Coecke's categorical quantum mechanics and Birkhoff-von Neumann's traditional quantum logic, which have (slightly misleadingly) been claimed to be in conflict with each other.

Chapter 3

Articulating Duality

Duality between algebra and space boils down to duality between theories and models in logic, which is the subject matter of this chapter. Theories may be seen as point-free spaces in the spirit of mereology or formal topology; models are prime ideals, and so models are points in the spirit of algebraic geometry. The corresponding spectrum functor takes the space of models of a given theory; the other function algebra functor takes the algebra of propositions on a given space of models (note that propositions are truth functions). Categorical duality in logic tells us the geometric meaning of logical principles; for example, the law of excluded middle amounts to zero-dimensionality in topology. Whilst the scope of the duality theory in the last chapter is fairly broad, in this chapter, we narrow the scope, and thereby attempt to shed light on subtler facets of duality. More specifically, we focus upon dualities in finitary logical contexts, especially coalgebraic dualities and non-Hausdorff dualities, which are two major extensions of the classic Stone duality for Boolean algebras. Among other things, we formulate two fundamental principles underlying those dualities, that is, the Kripke condition for coalgebraic dualities and the topological dualisability condition for non-Hausdorff dualities. This chapter is accordingly divided into two parts. Both principles allow us to construct unknown dualities, and to reconstruct known dualities including the Stone duality for distributive lattices and the Abramsky-Kupke-Kurz-Venema duality between modal algebras and Stone coalgebras.

3.1 Introduction to the First Part

By proposing a new notion of $\mathbb{ISP}_{\mathbb{M}}$ as the modalization of $\mathbb{ISP}$ in universal algebra (see, e.g., [47]), in this chapter, we attempt to extend the theory of natural dualities (see [63, 228]) so that it encompasses Jónsson-Tarski's topological duality (see [151, 37, 119, 127]) and Abramsky-Kupke-Kurz-Venema's coalgebraic duality (see [3, 159]) for the class of all modal algebras. Such dualities play a fundamental role in recent developments of coalgebraic logic (see [62]), which allows us to unify different kinds of modal logics, based on the theory of coalgebras.

A typical story in coalgebraic logic is as follows (see [163]). A dual adjunction induced by a schizophrenic object (see [232]) represents the syntax and semantics of a propositional logic (some researchers call such an adjunction a logical connection; see [164]). The Stone adjunction between Boolean algebras and sets is a typical example of this. We then fix an endofunctor on one category in the dual adjunction, which in turn induces an endofunctor on the other. The algebras and coalgebras of the endofunctors give rise to the syntax and semantics of the propositional logic equipped with modality. In particular, the standard modal logic K and Kripke semantics arise from the Stone adjunction by taking the power-set functor as an endofunctor on sets. Many modal logics such as monotone modal logic and probabilistic modal logic fall into this picture. In "good" cases, we can finally obtain duality between the corresponding algebras and coalgebras, as we are able to lead from Stone duality to Abramsky-Kupke-Kurz-Venema duality via Jónsson-Tarski duality.

In relation to this picture of coalgebraic logic, we start with Keimel-Werner's semi-primal duality theorem (see [63, Theorem 3.3.14] and [154]) in natural duality theory. Keimel-Werner's theorem is a universal-algebraic generalisation of Stone duality for Boolean algebras and can be seen as dual adjunctions induced by schizophrenic objects, representing the syntax and semantics of propositional logics. Our aim is then to establish the corresponding Jónsson-Tarski-type duality and Abramsky-Kupke-Kurz-Venema-type duality by introducing the new notion of $\mathbb{ISP}_{\mathbb{M}}$. At least under the assumption of semi-primality, categories arising from $\mathbb{ISP}_{\mathbb{M}}$ can be considered as categories of algebras for certain "free-generation" endofunctors on categories obtained via $\mathbb{ISP}$, and the duals of categories induced by $\mathbb{ISP}_{\mathbb{M}}$ can be described as categories of coalgebras for certain "Vietoris-style" endofunctors on the duals of categories corresponding to $\mathbb{ISP}$.

In the following, let us first review an aspect of natural duality theory and a certain difficulty in incorporating into it Jónsson-Tarski duality and Abramsky-Kupke-Kurz-Venema duality for the class of all modal algebras. We shall then see that the difficulty can be overcome with the help of $\mathbb{ISP}_{\mathbb{M}}$.

The theory of natural dualities by Davey et al. is a powerful general theory of Stone-Priestley-type dualities based on the machinery of universal algebra. It basically considers duality theory for $\mathbb{ISP}(M)$ where M is a finite algebra. It is useful for obtaining new dualities and actually encompasses many known dualities, including Stone duality for Boolean algebras (see [262]), Priestley duality for distributive lattices (see [239, 240]), and Cignoli duality for MV_n-algebras, i.e., algebras of Łukasiewicz n-valued logic (see [58, 215]), to name but a few (for more instances, see [63, 228]).

At the same time, however, it has not encompassed Jónsson-Tarski duality or Abramsky-Kupke-Kurz-Venema duality for the class of all modal algebras, any of which is important in coalgebraic logic. We consider that this is mainly because the class of all modal algebras cannot be expressed as $\mathbb{ISP}(M)$ for a finite algebra M, in contrast to the fact that any of the class of Boolean algebras, the class of distributive lattices, and the class of MV_n-algebras can be expressed as $\mathbb{ISP}(M)$ for a suitable finite algebra M.

We should note here that, given a modal algebra, the Boolean operations of the function algebra on its spectrum (i.e., the space of prime filters) can be defined pointwise, while only the modal operation of the function algebra cannot be defined pointwise (recall that it is defined depending on the canonical relation induced by the modal operation of the original modal algebra). In a nutshell, modality is not a pointwise operation unlike the other Boolean operations. For the very reason, the class of all modal algebras cannot be expressed as $\mathbb{ISP}(M)$ (all the operations of $A \in \mathbb{ISP}(M)$ are pointwise by definition), and we have to pay a special attention to modality when developing natural duality theory for algebras with modal operations. We remark that the same thing can be said also for the implication operation of a Heyting algebra, which is not pointwise (on the spectrum of the Heyting algebra). And this actually tells us a duality-theoretic reason why Gödel failed to capture intuitionistic logic as a many-valued logic (broadly speaking, $\mathbb{ISP}(M)$ amounts to algebras of M-valued logic).

In this chapter, we introduce a new notion of $\mathbb{ISP}_{\mathbb{M}}$ in order to extend the theory of natural dualities so that it encompasses Jónsson-Tarski's topological duality and Abramsky-Kupke-Kurz-Venema's coalgebraic duality. It is crucial here that the class of all modal algebras coincides with $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ for the two-element Boolean algebra $\mathbf{2}$. Moreover, we have the following facts: for $\mathbf{n}$ defined in Definition 3.2.9 below, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ coincides with the class of all algebras of Łukasiewicz n-valued modal logic (for this logic, see, e.g., [43, 128, 265]); a similar thing holds also for algebras of a version of Fitting's many-valued modal logic (for this logic, see, e.g., [93, 94, 186, 187, 190]). Thus, the notion of $\mathbb{ISP}_{\mathbb{M}}$ seems to be natural and useful for our goal.

Our main results (Theorem 3.3.24 and Theorem 3.4.11) are topological and coalgebraic dualities for $\mathbb{ISP}_{\mathbb{M}}(L)$ where L is a semi-primal algebra with a bounded lattice reduct. Our results encompass both Jónsson-Tarski and Abramsky-Kupke-Kurz-Venema dualities as the case $L = \mathbf{2}$. They also encompass topological dualities in [265, 187, 190] for algebras of many-valued modal logics. Our dualities are developed based on Keimel-Werner's semi-primal duality theorem in the theory of natural dualities, and may be considered as modalized extensions of the semi-primal duality theorem on algebras with bounded lattice reducts. As applications, we obtain new coalgebraic dualities for algebras of Łukasiewicz n-valued modal logic and for algebras of a version of Fitting's many-valued modal logic. With the help of the duality results, we can also show the existence of final coalgebras and cofree coalgebras in the cateogries of coalgebras involved. Note that final coalgebras are significantly used for the semantics of programming languages (see [267]). We finally provide a duality-based criterion for the equivalence of categories of algebras concerned.

Several authors have developed duality theories for those classes of modal algebras (in the wider sense) that can be expressed as $\mathbb{ISP}(M)$ for finite algebras M. (see, e.g., [241, 264]). However, they do not encompass Jónsson-Tarski duality or Abramsky-Kupke-Kurz-Venema duality for the class of all modal algebras, since the class of all modal algebras cannot be expressed as $\mathbb{ISP}(M)$ for a finite algebra M. By modalizing the notion of $\mathbb{ISP}$, this chapter makes it possible to incorporate both Jónsson-Tarski and Abramsky-Kupke-Kurz-Venema dualities into the theory of natural dualities.

As a (rough) historical note, we remark that the duality of modal algebras and coalgebras for the Vietoris functor (or Stone coalgebras) was essentially discovered by Abramsky, and his relevant talk was given at the 1988 British Colloquium on Theoretical Computer Science as mentioned in [3]. The paper version [3] of the 1988 talk, however, had remained unpublished until 2005. On the other hand, in 2003, Kupke, Kurz, and Venema published their paper [159] providing a detailed description of the duality. Their work was done independently of Abramsky's. Taking all this into consideration, we call the duality "Abramsky-Kupke-Kurz-Venema duality" in this chapter. It could also be called just "Abramsky duality" (especially if a shorter term is preferred). At the same time, we emphasize that Esakia first mentioned the use of Vietoris spaces in the context of non-classical logics in his paper [89], as early as in 1974.

The first part of the chapter is organized as follows. In Section 3.2, we introduce the notions of $\mathbb{ISP}_{\mathbb{M}}$ and Kripke condition. The Kripke condition may be considered as completeness in logical terms and plays an important role in our duality theory. In Section 3.3, we show the first result, i.e., a topological duality for $\mathbb{ISP}_{\mathbb{M}}(L)$. In Section 3.4, we show the second result, i.e., a coalgebraic duality for $\mathbb{ISP}_{\mathbb{M}}(L)$, which implies coalgebraic dualities for algebras of Łukasiewicz n-valued modal logic and for algebras of a version of Fitting's many-valued modal logic. As an application of our dualities, we obtain a result on the equivalence of categories of algebras involved. It also follows from our dualities that the corresponding categories of coalgebras have cofree coalgebras and final coalgebras. Finally, we conclude the first part of the chapter by discussing several future directions of research, including a coalgebraic extension of the notion of $\mathbb{ISP}_{\mathbb{M}}$, and by comparing natural duality theory with categorical duality theory.

3.2 The Notion of $\mathbb{ISP}_{\mathbb{M}}$ and the Kripke Condition

For universal algebra and lattice theory, we refer the reader to [47, 72]. For category theory, we refer to [12], which contains categorical universal algebra and categorical universal topology (especially, categorical Birkhoff theorems and its topological analogues).

Throughout this chapter, let L denote a finite algebra with a bounded lattice reduct (it is natural from a logical point of view to suppose the existence of a bounded lattice reduct, since most logics are equipped with the lattice connectives $\wedge$ and $\vee$ and the truth constants 0 and 1). Let $\mathbf{2}$ denote the two-element Boolean algebra.

From a logical point of view, we may see L as an algebra of truth values. Since the lattice reduct of L turns out to be a complete Heyting algebra (note that any finite distributive lattice is a Heyting algebra), the lattice reduct of L is actually a so-called truth-value object Ω in an elementary topos. The case $L = \mathbf{2}$ amounts to classical logic, and $\mathbb{ISP}(\mathbf{2})$ coincides with the class of all Boolean algebras.

We define the notion of modal power as follows. For a set S, L^S denotes the set of all functions from S to L. A Kripke frame is defined as a tuple (S, R) such that S is a non-empty set and R is a binary relation on S.

Definition 3.2.1. *For a Kripke frame (S, R), the modal power of L with respect to (S, R) is defined as $L^S \in \mathbb{ISP}(L)$ equipped with a unary operation $\Box_R$ on L^S defined by*

$$(\Box_R f)(w) = \bigwedge \{f(w') \; ; \; wRw'\}$$

where $f \in L^S$ and $w \in S$. Then, a modal power of L is defined as the modal power of L with respect to (S, R) for some Kripke frame (S, R). (To be precise about the order of quantifiers, this means that, for any modal power A of L, there is some Kripke frame (S, R) such that A is a modal power of L with respect to (S, R).)

For a Kripke frame (S, R), let $L^{(S,R)}$ denote the modal power of L with respect to (S, R).

The notion of $\mathbb{ISP}_{\mathbb{M}}$ is then defined as follows.

Definition 3.2.2. $\mathbb{ISP}_{\mathbb{M}}(L)$ *denotes the class of all isomorphic copies of subalgebras of modal powers of L.*

We often denote by $(A, \Box)$ an element of $\mathbb{ISP}_{\mathbb{M}}(L)$. Note that $\Box(x \wedge y) = \Box x \wedge \Box y$ for $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$ and $x, y \in A$.

Definition 3.2.3. $\mathbf{ISP}(L)$ *denotes the category of algebras in $\mathbb{ISP}(L)$ and homomorphisms where a homomorphism is defined as a function which preserves all the operations of L.*

$\mathbf{ISP_M}(L)$ *denotes the category of algebras in $\mathbb{ISP}_{\mathbb{M}}(L)$ and modal homomorphisms where a modal homomorphism is defined as a function which preserves $\Box$ and all the operations of L.*

The modalization of $\mathbb{ISP}$ preserves the closedness under $\mathbb{I}$, $\mathbb{S}$, and $\mathbb{P}$ as follows.

Proposition 3.2.4. $\mathbb{ISP}_{\mathbb{M}}(L)$ *is closed under $\mathbb{I}$, $\mathbb{S}$, and $\mathbb{P}$.*

Proof. It is clear that $\mathbb{ISP}_{\mathbb{M}}(L)$ is closed under $\mathbb{I}$ and $\mathbb{S}$. In order to show that it is closed under direct products, let I be a set and $(A_i, \Box_i) \in \mathbb{ISP}_{\mathbb{M}}(L)$ for $i \in I$. Then it follows that for each $i \in I$ there is a Kripke frame (S_i, R_i) such that $(A_i, \Box_i)$ is embedded into $L^{(S_i,R_i)}$, i.e., the modal power of L with respect to (S_i, R_i). Define a Kripke frame (S, R) by

$$S = \coprod_{i \in I} S_i \text{ and } R = \coprod_{i \in I} R_i.$$

We claim that $\prod_{i \in I}(A_i, \Box_i)$ can be embedded into $L^{(S,R)}$. To show this, we define a function

$$e : \prod_{i \in I}(A_i, \Box_i) \to L^{(S,R)}$$

as follows. Given $x \in S$ and $f_i : A_i \to L$ for $i \in I$, define $(e((f_i)_{i \in I}))(x) = f_k(x)$ where k is the unique $j \in I$ such that $x \in S_j$. Let $\Box$ denote the modal operation of $\prod_{i \in I}(A_i, \Box_i)$. Note that $\Box$ is defined pointwise. We show that $e(\Box(f_i)_{i \in I}) = \Box_R e((f_i)_{i \in I})$. Let $x \in S$. It follows from the definition of (S, R) that if $x \in S_k$ for $k \in I$ then

$$(\Box_R e((f_i)_{i \in I}))(x) = \bigwedge\{e((f_i)_{i \in I})(y) \; ; \; xR_k y\} = (\Box_k f_k)(x).$$

It also holds that if $x \in S_k$ then

$$e(\Box(f_i)_{i\in I})(x) = e((\Box_i f_i)_{i\in I})(x) = (\Box_k f_k)(x).$$

Thus, we have shown that e preserves $\Box$. It is straightforward to see that e also preserves the other operations of $\prod_{i\in I}(A_i, \Box_i)$. Hence, $\mathbb{ISP}_{\mathbb{M}}(L)$ is closed under direct products. $\Box$

According to the theory of free algebras in universal algebra, the above proposition gives us the following.

Corollary 3.2.5. $\mathbb{ISP}_{\mathbb{M}}(L)$ *has free algebras.*

Given $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$, we define the corresponding canonical relation $R_\Box$ on $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$.

Definition 3.2.6. *For* $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$*, we define a binary relation* $R_\Box$ *on* $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ *as follows: For* $v, u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$*,* $vR_\Box u$ *iff the following holds:*

$$\forall a \in L \ \forall x \in A \ (v(\Box x) \geq a \text{ implies } u(x) \geq a).$$

Definition 3.2.7. $\mathbb{ISP}_{\mathbb{M}}(L)$ *(or L) satisfies the Kripke condition iff, for any* $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$*, any* $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$*, and any* $x \in A$*, the following holds:*

$$v(\Box x) = \bigwedge\{u(x) \,;\, vR_\Box u\}.$$

The Kripke condition may be considered as completeness in logical terms.

In this chapter, the Kripke condition can be seen as a condition on L rather than $\mathbb{ISP}_{\mathbb{M}}(L)$, since we concentrate on "normal" modal logic induced by L. If we also consider other types of modal logics, however, it seems that $\mathbb{ISP}_{\mathbb{M}}$ is not a unique way to generate the corresponding classes of modal algebras (in the wider sense). In that case, the Kripke condition depends on the way of generating modal algebras as well as the basic structure L.

The notions of $\mathbb{ISP}_{\mathbb{M}}$ and Kripke condition are motivated by Proposition 3.2.8 and Proposition 3.2.10 below.

Proposition 3.2.8. $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ *coincides with the class of all modal algebras and satisfies the Kripke condition.*

Proof. By Jónsson-Tarski representation (see, e.g., [37, Theorem 5.43]), any modal algebra can be embedded into a modal power of $\mathbf{2}$. It is straightforward to see that any $A \in \mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ is a modal algebra. Thus, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ coincides with the class of all modal algebras. It follows from Proposition 3.3.14 below that $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ satisfies the Kripke condition (a direct proof of this fact can also be given in a similar way to the completeness proof of classical modal logic K). $\square$

The algebra $\mathbf{n}$ of truth values in Łukasiewicz n-valued logic is defined as follows (see, e.g., [125]):

Definition 3.2.9. *Let* $\mathbf{n}$ *denote* $\{0, 1/(n-1), ..., (n-2)/(n-1), 1\}$ *equipped with the operations* $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1)$ *defined by*

$$
\begin{aligned}
x \wedge y &= \min(x, y); \\
x \vee y &= \max(x, y); \\
x * y &= \max(0, x + y - 1); \\
x \wp y &= \min(1, x + y); \\
x \rightarrow y &= \min(1, 1 - (x - y)); \\
x^{\perp} &= 1 - x.
\end{aligned}
$$

An $\mathcal{MMV}_n$-algebra introduced in [265, Definition 3.1] is an algebra of Łukasiewicz n-valued modal logic. We then have the following.

Proposition 3.2.10. $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ *coincides with the class of all* $\mathcal{MMV}_n$*-algebras and satisfies the Kripke condition.*

Proof. By Teheux representation following from [265, Theorem 4.11], any $\mathcal{MMV}_n$-algebra can be embedded into a modal power of $\mathbf{n}$. It is straightforward to see that any $A \in \mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ is an $\mathcal{MMV}_n$-algebra. Thus, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ coincides with the class of all modal algebras. It follows from Proposition 3.3.14 below that $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ satisfies the Kripke condition (or this also follows from the completeness of Łukasiewicz n-valued modal logic). $\square$

A similar proposition can be shown also for L-**ML**-algebras, which are algebras of a version of Fitting's many-valued modal logic (see [186, 187, 190]).

Thus, the notion of $\mathbb{ISP}_{\mathbb{M}}$ seems to be natural and useful.

3.3 Modal Semi-Primal Duality

In this section and the next section, we assume that L is semi-primal. A semi-primal algebra is a useful concept in universal algebra and is defined as follows.

Definition 3.3.1. *Let A be an algebra (in the sense of universal algebra) and n a positive integer. A function $f : A^n \to A$ is called conservative iff, for any $a_1, ..., a_n \in A$, $f(a_1, ..., a_n)$ is in the subalgebra of A generated by $\{a_1, ..., a_n\}$.*

A semi-primal algebra is a finite algebra A such that, for any positive integer n, every conservative function $f : A^n \to A$ is a term function of A. (Note that a term function is called a polynomial in some literature.)

Intuitively, we may say that a conservative function on an algebra is a function preserving the subalgebra structure of the algebra. For characterizations of semi-primality and term-definable operations on semi-primal algebras, we refer the reader to [231, 73].

We remark that, under the assumption of the semi-primality of L, $\mathbf{ISP_M}(L)$ actually forms a variety (or a monadic category in categorical terms), which shall be shown in subsequent work on the finite axiomatizability of $\mathbb{ISP}(L)$ and $\mathbb{ISP}_{\mathbb{M}}(L)$.

Now it is straightforward to verify the following lemmas by cheking that each function is conservative.

Lemma 3.3.2. *For a semi-primal algebra L, define a function $q : L^4 \to L$ by*

$$q(x, y, z, w) = \begin{cases} w & \text{if } x \neq y \\ z & \text{if } x = y \end{cases}$$

where $x, y, z, w \in L$. Then, $q : L^4 \to L$ is a term function of L.

The function $q : L^4 \to L$ is called the quaternary discriminator.

Lemma 3.3.3. *Let L a semi-primal algebra with a bounded lattice reduct, and $a \in L$. Define a function $\mathrm{T}_a : L \to L$ by*

$$\mathrm{T}_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

where $x \in L$. Then, T_a is a term function of L.

To verify the proposition above, note that any subalgebra of L contains constants 0 and 1 by the definition of a subalgebra.

From a logical point of view, $\mathrm{T}_a(p)$ intuitively means that the truth value of a proposition p is exactly a for an element a of the algebra L of truth values, which may be seen as a truth-value object in a topos, since the lattice reduct of L is a complete Heyting algebra.

Lemma 3.3.4. *Let L be a semi-primal algebra with a bounded lattice reduct, and $a \in L$. Define a function $\mathrm{U}_a : L \to L$ by*

$$\mathrm{U}_a(x) = \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{if } x \ngeq a \end{cases}$$

where $x \in L$. Then, U_a is a term function of L.

We can also define the function $\mathrm{U}_a : L \to L$ by using T_a in the following way:

$$\mathrm{U}_a(x) = \bigvee \{\mathrm{T}_b(x) \,;\, a \leq b \text{ and } b \in L\}.$$

It is straightforward to see that U_a and $\wedge$ are commutative, i.e.,

$$\mathrm{U}_a(x \wedge y) = \mathrm{U}_a(x) \wedge \mathrm{U}_a(y).$$

Moreover, $\Box$ and U_a are commutative, i.e.,

$$\Box \mathrm{U}_a(x) = \mathrm{U}_a(\Box x)$$

for any $x \in A$ where $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$. This can be verified using the fact that U_a and $\wedge$ are commutative (note that $\Box$ is defined via $\wedge$). We also remark that $\mathrm{U}_1(x) = \mathrm{T}_1(x)$.

Lemma 3.3.5. *Let L be a semi-primal algebra with a bounded lattice reduct, and $a \in L$. Define a function $(\text{-}) \to (\text{-}) : L^2 \to L$ by*

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x \nleq y \end{cases}$$

where $x, y \in L$. Then, $\to$ is a term function of L.

The function $(\text{-}) \to (\text{-}) : L^2 \to L$ can also be defined by $x \to y = q(x \wedge y, x, 1, y)$.

We can apply Keimel-Werner's semi-primal duality theorem [63, Theorem 3.3.14] to obtain a topological duality for $\mathbb{ISP}(L)$, which is explained in the following subsection. We shall later build a duality theory for $\mathbb{ISP}_{\mathbb{M}}(L)$ based on the semi-primal duality theorem.

Semi-Primal Duality for $\mathbb{ISP}(L)$

Let $\mathrm{SubAlg}(L)$ denote the set of all subalgebras of L. For a Boolean space S, let $\mathrm{SubSp}(S)$ denote the set of all closed subspaces of S, where a Boolean space means a zero-dimensional compact Hausdorff space.

Definition 3.3.6. *We define a category $\mathbf{BS}_L$. An object in $\mathbf{BS}_L$ is a tuple (S, α) such that S is a Boolean space and that a function $\alpha : \mathrm{SubAlg}(L) \to \mathrm{SubSp}(S)$ satisfies:*

1. *$S = \alpha(L)$;*
2. *if $L_3 = L_1 \cap L_2$ for $L_1, L_2, L_3 \in \mathrm{SubAlg}(L)$, then $\alpha(L_3) = \alpha(L_1) \cap \alpha(L_2)$.*

An arrow $f : (S, \alpha) \to (S', \beta)$ in L-BS is a continuous map $f : S \to S'$ that satisfies the condition that, for any $M \in \mathrm{SubAlg}(L)$, if $x \in \alpha(M)$ then $f(x) \in \beta(M)$. We call a map satisfying the condition a subspace-preserving map.

Having an object in $\mathbf{BS}_L$ is equivalent to having a meet-preserving function $\alpha : \mathrm{SubAlg}(L) \to \mathrm{SubSp}(S)$. This provides another definition of an object in $\mathbf{BS}_L$ as a $\mathrm{SubAlg}(L)$-indexed family of Boolean spaces satisfying the condition of meet-preservation.

Note also that the condition 2 above implies that, if $L_1 \subset L_2$ for $L_1, L_2 \in \mathrm{SubAlg}(L)$, then $\alpha(L_1) \subset \alpha(L_2)$.

We equip L and its subalgebras with the discrete topologies. Define $\alpha_L : \mathrm{SubAlg}(L) \to \mathrm{SubSp}(L)$ by $\alpha_L(M) = M$ for $M \in \mathrm{SubAlg}(L)$. Then, (L, α_L) is an object in $\mathbf{BS}_L$.

For $A \in \mathbb{ISP}(L)$, we equip $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ with the topology generated by $\{\langle x \rangle \ ; \ x \in A\}$ where

$$\langle x \rangle = \{v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L) \ ; \ v(x) = 1\}$$

for $x \in A$. Note that for $x \in A$, $\langle x \rangle$ is clopen, since $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L) \setminus \langle x \rangle = \langle \mathrm{T}_1(x) \to 0 \rangle$ by Lemma 3.3.3 and Lemma 3.3.5.

Definition 3.3.7. *We define a contravariant functor $\mathrm{Spec} : \mathbf{ISP}(L) \to \mathbf{BS}_L$. For an object A in $\mathbf{ISP}(L)$, let*

$$\mathrm{Spec}(A) = (\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L), \alpha_A)$$

where α_A is defined by

$$\alpha_A(M) = \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M)$$

for $M \in \mathrm{SubAlg}(L)$. For an arrow $f : A \to B$ in $\mathbf{ISP}(L)$, $\mathrm{Spec}(f)$ is defined by $\mathrm{Spec}(f)(v) = v \circ f$ for $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(B, L)$.

The functor Spec can be defined also for $\mathbf{ISP_M}(L)$ (by considering modality-free reducts). The domain of Spec is defined to be $\mathbf{ISP}(L)$ just because it is an ingredient of duality between $\mathbf{ISP}(L)$ and $\mathbf{BS}_L$.

Definition 3.3.8. *We define a contravariant functor* $\mathrm{Cont} : \mathbf{BS}_L \to \mathbf{ISP}(L)$. *For an object* (S, α) *in* $\mathbf{BS}_L$, *define* $\mathrm{Cont}(S, \alpha)$ *as*

$$\mathrm{Hom}_{\mathbf{BS}_L}((S, \alpha), (L, \alpha_L))$$

equipped with the pointwise operations. For an arrow $f : (S_1, \alpha_1) \to (S_2, \alpha_2)$ *in* $\mathbf{BS}_L$, $\mathrm{Cont}(f)$ *is defined by* $\mathrm{Cont}(f)(g) = g \circ f$ *for* $g \in \mathrm{Cont}(S_2, \alpha_2)$.

Later we shall extend Spec and Cont to the modal setting (RSpec and MCont respectively).

By Keimel-Werner's semi-primal duality theorem [63, Theorem 3.3.14], we obtain the following.

Theorem 3.3.9. $\mathbf{ISP}(L)$ *and* $\mathbf{BS}_L$ *are dually equivalent via* Spec *and* Cont.

The semi-primal duality theorem is essentially due to [154].

Based on the above duality, we shall show that $\mathbf{ISP_M}(L)$ is dually equivalent to $\mathbf{RBS}_L$, which is defined in Definition 3.3.15 below. In order to prove this duality, we first verify the Kripke condition for $\mathbb{ISP}_\mathbb{M}(L)$ in the next subsection.

The Verification of the Kripke Condition

In order to show that $\mathbb{ISP}_\mathbb{M}(L)$ satisfies the Kripke condition, we use the prime filter theorem for Boolean algebras (see, e.g., [149]). We first introduce the notion of the Boolean core $\mathcal{B}(A)$ of $A \in \mathbb{ISP}(L)$.

Definition 3.3.10. *For* $A \in \mathbb{ISP}(L)$, *define*

$$\mathcal{B}(A) = \{x \in A\,;\ \mathrm{T}_1(x) = x\}.$$

Note that $\mathrm{T}_a(x), \mathrm{U}_a(x) \in \mathcal{B}(A)$ for any $x \in A$ and $a \in L$.

Lemma 3.3.11. *For* $A \in \mathbb{ISP}(L)$, $(\mathcal{B}(A), \wedge, \vee, \mathrm{T}_0, 0, 1)$ *forms a Boolean algebra.*

Proof. This follows from the the two facts that $(\mathcal{B}(A), \wedge, \vee, \mathrm{T}_0, 0, 1)$ is a subalgebra of a direct power of $(\mathcal{B}(L), \wedge, \vee, \mathrm{T}_0, 0, 1)$ and that $(\mathcal{B}(L), \wedge, \vee, \mathrm{T}_0, 0, 1)$ is the two-element Boolean algebra (note that T_0 is the complement operation). $\square$

Lemma 3.3.12. *For $A \in \mathbb{ISP}(L)$, let P be a prime filter of a Boolean algebra $\mathcal{B}(A)$. Define $v_P : A \to L$ by*

$$v_P(x) = a \Leftrightarrow \mathrm{T}_a(x) \in P.$$

Then, v_P is an element of $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$*.*

Proof. Since $\bigvee_{a \in L} \mathrm{T}_a(x) = 1$ for $x \in A$ and since $\mathrm{T}_a(x) \wedge \mathrm{T}_b(x) = 0$ for $a, b \in L$ with $a \neq b$, v_P is well defined as a function from A to L. Let $t : A^n \to A$ be an n-ary operation of A. Let $x_i \in A$ and $a_i = v_P(x_i)$ for $i = 1, ..., n$. Then it follows by definition that

$$\mathrm{T}_{a_1}(x_1) \wedge ... \wedge \mathrm{T}_{a_n}(x_n) \in P.$$

It is straightforward to show the following inequality (note that it is enough to verify the inequality in L):

$$\mathrm{T}_{a_1}(x_1) \wedge ... \wedge \mathrm{T}_{a_n}(x_n) \leq \mathrm{T}_{t(a_1,...,a_n)}(t(x_1, ..., x_n)).$$

Thus we have $\mathrm{T}_{t(a_1,...,a_n)}(t(x_1, ..., x_n)) \in P$, which implies that

$$v_P(t(x_1, ..., x_n)) = t(a_1, ..., a_n) = t(v_P(x_1), ..., v_P(x_n)).$$

This completes the proof. $\square$

The following lemma is crucial for the verification of the Kripke condition.

Lemma 3.3.13. *Let $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$, $x \in A$, $a \in L$, and $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$. Then the following holds:*

$$v(\Box x) \geq a \text{ iff for any } u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L),\ vR_{\Box}u \text{ implies } u(x) \geq a.$$

Proof. It is easily verified that the left-hand side implies the right-hand side. We show the converse by proving the contrapositive. Assume that $v(\Box x) \not\geq a$. This means that $v(\mathrm{U}_a(\Box x)) \neq 1$. Let

$$X = \{\mathrm{U}_b(y)\ ;\ v(\mathrm{U}_b(\Box y)) = 1\}.$$

Note that $X \subset \mathcal{B}(A)$. Let F be the filter of $\mathcal{B}(A)$ generated by X.

We claim that $\mathrm{U}_a(x) \notin F$. Suppose for contradiction that $\mathrm{U}_a(x) \in F$. Then there is $\varphi \in A$ such that $\varphi \leq \mathrm{U}_a(x)$ and φ is constructed from $\wedge$ and elements of X. Since the equation $\mathrm{U}_b(y \wedge y') = \mathrm{U}_b(y) \wedge \mathrm{U}_b(y')$ holds in general, we may assume that for some $\{\mathrm{U}_b(x_b)\ ;\ b \in L\} \subset X$,

$$\varphi = \bigwedge\{\mathrm{U}_b(x_b)\ ;\ b \in L\}.$$

By $\varphi \leq \mathrm{U}_a(x)$, it follows from the definition of modal power that $\Box\varphi \leq \Box\mathrm{U}_a(x)$. We also have

$$\Box\varphi = \bigwedge\{\mathrm{U}_b(\Box x_b)\,;\, b \in L\}.$$

Since $\mathrm{U}_b(x_b) \in X$, we have $v(\mathrm{U}_b(\Box x_b)) = 1$ for any $b \in L$, whence it follows that $v(\Box\varphi) = 1$. Thus, we have

$$v(\mathrm{U}_a(\Box x)) = v(\Box\mathrm{U}_a(x)) = 1,$$

which is a contradiction. Hence, we have $\mathrm{U}_a(x) \notin F$.

By the prime filter theorem for Boolean algebras, there is a prime filter P of $\mathcal{B}(A)$ such that $F \subset P$ and $\mathrm{U}_a(x) \notin P$. Define $v_P : A \to L$ as in Lemma 3.3.12 and then we have

$$v_P \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L).$$

Since $\mathrm{U}_a(x) \notin P$ and since $\mathrm{T}_1(\mathrm{U}_a(x)) = \mathrm{U}_a(x)$, it follows that

$$v_P(\mathrm{U}_a(x)) \neq 1, \text{ i.e., } v_P(x) \not\geq a.$$

To complete the proof, it remains to show that $vR_\Box v_P$. By using $X \subset P$, this follows from the fact that $v_P(\mathrm{U}_b(y)) = 1$ for any $\mathrm{U}_b(y) \in X$ (i.e., $v(\Box y) \geq b$ implies $v_P(y) \geq b$). $\Box$

By the above lemma we obtain the following proposition.

Proposition 3.3.14. *$\mathbb{ISP}_{\mathbb{M}}(L)$ satisfies the Kripke condition, i.e., for any $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$, any $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$, and any $x \in A$, the following holds:*

$$v(\Box x) = \bigwedge\{u(x)\,;\, vR_\Box u\}.$$

The above proposition plays an important role in establishing our duality result.

Category $\mathbf{RBS}_L$

For a Kripke frame (S, R) and $X \subset S$, define $R^{-1}[X] = \{w \in S\,;\, \exists w' \in X\ wRw'\}$. For $w \in S$, define $R[w] = \{w' \in S\,;\, wRw'\}$.

Definition 3.3.15. *We define a category $\mathbf{RBS}_L$. An object in $\mathbf{RBS}_L$ is a triple (S, α, R) such that (S, α) is an object in $\mathbf{BS}_L$ and that a binary relation R on S satisfies:*

1. *$R[w]$ is closed in S for any $w \in S$;*

2. if $X \subset S$ is clopen in S, then $R^{-1}[X]$ is clopen in S;

3. for any $M \in \mathrm{SubAlg}(L)$, if $w \in \alpha(M)$ then $R[w] \subset \alpha(M)$.

An arrow $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ in $\mathbf{RBS}_L$ is defined as an arrow $f : (S_1, \alpha_1) \to (S_2, \alpha_2)$ in $\mathbf{BS}_L$ which satisfies:

4. if wR_1w' then $f(w)R_2f(w')$;

5. if $f(w_1)R_2w_2$ then there is $w' \in S_1$ such that w_1R_1w' and $f(w') = w_2$.

In order to show a dual equivalence between the categories $\mathbf{ISP_M}(L)$ and $\mathbf{RBS}_L$, we introduce functors RSpec and MCont in the next subsection.

Functors RSpec and MCont

Definition 3.3.16. *We define a contravariant functor* $\mathrm{RSpec} : \mathbf{ISP_M}(L) \to \mathbf{RBS}_L$*. For an object $(A, \Box)$ in $\mathbf{ISP_M}(L)$, let*

$$\mathrm{RSpec}(A) = (\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L), \alpha_L, R_\Box)$$

where $R_\Box$ is defined in Definition 3.2.6. For an arrow $f : A \to B$ in $\mathbf{ISP_M}(L)$, define $\mathrm{RSpec}(f)$ *by*

$$\mathrm{RSpec}(f)(v) = v \circ f$$

for $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(B, L)$.

The well-definedness of RSpec is shown by the following two lemmas.

Lemma 3.3.17. *Let $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$. Then,* $\mathrm{RSpec}(A)$ *is an object in* $\mathbf{RBS}_L$*.*

Proof. By Theorem 3.3.9, $\mathrm{RSpec}(A)$ without $R_\Box$ is an object in $\mathbf{BS}_L$.

We first show that $\mathrm{RSpec}(A)$ satisfies item 1 in Definition 3.3.15. Let $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$. Assume $u \notin R_\Box[v]$ for $u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$. It suffices to show that there is an open subset O of $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ such that

$$u \in O \text{ and } R_\Box[v] \cap O = \emptyset.$$

Since $u \notin R_\Box[v]$, there is $x_0 \in A$ such that $\exists a \in L$ $(v(\Box x_0) \geq a$ and $u(x_0) \not\geq a)$. Then it follows from Lemma C.2.7 and Lemma 3.3.5 that

$$u \in \langle \mathrm{U}_a(x_0) \to 0 \rangle \text{ and } R_\Box[v] \cap \langle \mathrm{U}_a(x_0) \to 0 \rangle = \emptyset.$$

We next show that $\mathrm{RSpec}(A)$ satisfies item 2 in Definition 3.3.15. Since $R_{\Box}^{-1}$ preserves unions of sets and since $\{\langle x\rangle\ ;\ x \in A\}$ forms a base of the topology of $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ (note that it is closed under finite intersections), it suffices to show that $R_{\Box}^{-1}(\langle x\rangle)$ is clopen in S for any $x \in A$. We claim that

$$R_{\Box}^{-1}(\langle x\rangle) = \langle \neg\Box\neg \mathrm{T}_1(x)\rangle$$

where $\neg\varphi$ is the abbreviation of $\varphi \to 0$. Note that the right-hand side is clopen. To show the claim, we first assume $v \in \langle \neg\Box\neg \mathrm{T}_1(x)\rangle$. By Lemma 3.3.5, we have $v(\Box\neg \mathrm{T}_1(x)) = 0$. Then it follows from the Kripke condition that

$$0 = v(\Box\neg \mathrm{T}_1(x)) = \bigwedge\{u(\neg \mathrm{T}_1(x))\ ;\ vR_{\Box}u\}.$$

Since $u(\neg \mathrm{T}_1(x))$ is either 0 or 1 by Lemma 3.3.3 and Lemma 3.3.5, there is $u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ with $vR_{\Box}u$ such that $u(\neg \mathrm{T}_1(x)) = 0$. Then we have $u \in \langle x\rangle$. Therefore we conclude $v \in R_{\Box}^{-1}(\langle x\rangle)$. The converse is similarly proved by using the Kripke condition.

We finally show that $\mathrm{RSpec}(A)$ satisfies item 3 in Definition 3.3.15. Assume for contradiction that $u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M)$ and $R_{\Box}[u] \setminus \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M) \neq \emptyset$ for $M \in \mathrm{SubAlg}(L)$. Then there is $v \in R_{\Box}[u] \setminus \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M)$, which means that $uR_{\Box}v$ and there is $z_0 \in A$ such that $v(z_0) \notin M$. Let $a = v(z_0)$. Then we have the following: for any $w \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$,

$$w(\mathrm{T}_a(z_0) \to z_0) = \begin{cases} 1 & \text{if } w(z_0) \neq a \\ a & \text{if } w(z_0) = a. \end{cases}$$

Thus it follows from the Kripke condition and $uR_{\Box}v$ that

$$u(\Box(\mathrm{T}_a(z_0) \to z_0)) = \bigwedge\{w(\mathrm{T}_a(z_0) \to z_0)\ ;\ uR_{\Box}w\} = a.$$

This contradicts $u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M)$, since $a \notin M$. Thus, $\mathrm{RSpec}(A)$ satisfies item 3. $\Box$

The following lemma is shown using the prime filter theorem for Boolean algebras.

Lemma 3.3.18. *For $(A_1, \Box_1), (A_2, \Box_2) \in \mathbb{ISP}_{\mathbb{M}}(L)$, let f be a modal homomorphism from $(A_1, \Box_1)$ to $(A_2, \Box_2)$. Then,* $\mathrm{RSpec}(f)$ *is an arrow in $\mathbf{RBS}_L$.*

Proof. By Theorem 3.3.9, $\mathrm{RSpec}(f)$ is an arrow in $\mathbf{BS}_L$. Thus it remains to show that $\mathrm{RSpec}(f)$ satisfies items 4 and 5 in Definition 3.3.15. We first verify item 4. For $v, u \in \mathrm{RSpec}(A_2)$, assume $(v, u) \in R_{\Box_2}$. Then it suffices to show that

$$(v \circ f, u \circ f) \in R_{\Box_1}.$$

To show this, suppose that $v \circ f(\Box_1 x) \geq a$ for $x \in A_1$ and $a \in L$. Then we have $v(\Box_2 f(x)) \geq a$. It follows from assumption that $u(f(x)) \geq a$. Thus we have $(v \circ f, u \circ f) \in R_{\Box_1}$.

We next verify item 5. Assume that $(\mathrm{RSpec}(f)(v), u) \in R_{\Box_1}$ for $v \in \mathrm{RSpec}(A_2)$ and $u \in \mathrm{RSpec}(A_1)$. Define

$$\begin{aligned} X_v &= \{\mathrm{U}_a(x) \ ; \ v(\Box_2 \mathrm{U}_a(x)) = 1\}; \\ X_u &= \{f(\mathrm{U}_a(x)) \ ; \ u(\mathrm{U}_a(x)) = 1\}. \end{aligned}$$

Let $X = X_v \cup X_u$. We claim that X has the finite intersection property. Suppose for contradiction that X does not have the finite intersection property. Then, since by $\mathrm{U}_a(x) = \mathrm{U}_1(\mathrm{U}_a(x))$ we have

$$X_v = \{\mathrm{U}_1(x) \ ; \ v(\Box_2 \mathrm{U}_1(x)) = 1\} \text{ and } X_u = \{f(\mathrm{U}_1(x)) \ ; \ u(\mathrm{U}_1(x)) = 1\}$$

and since U_1 distributes over $\wedge$, there are $\mathrm{U}_1(x), f(\mathrm{U}_1(y)) \in A_2$ such that $v(\Box_2 \mathrm{U}_1(x)) = 1$, $u(\mathrm{U}_1(y)) = 1$, and $\mathrm{U}_1(x) \leq \neg f(\mathrm{U}_1(y))$ where $\neg\varphi$ is the abbreviation of $\varphi \to 0$. Then we have

$$\Box_2 \mathrm{U}_1(x) \leq \Box_2 \neg f(\mathrm{U}_1(y)) = f(\Box_2 \neg \mathrm{U}_1(y)).$$

It follows from $v(\Box_2 \mathrm{U}_1(x)) = 1$ that

$$v(f(\Box_2 \neg \mathrm{U}_1(y))) = 1, \text{ i.e., } (\mathrm{RSpec}(f)(v))(\Box_2 \neg \mathrm{U}_1(y)) = 1.$$

By assumption, we have $u(\neg \mathrm{U}_1(y)) = 1$, which contradicts $u(\mathrm{U}_1(y)) = 1$. Thus X has the finite intersection property. By the prime filter theorem for Boolean algebras, there is a prime filter P of $\mathcal{B}(A_2)$ such that $X \subset P$. Define $v_P : A_2 \to L$ as in Lemma 3.3.12 and then we have

$$v_P \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A_2, L).$$

It follows from $X_v \subset P$ that $v R_{\Box_2} v_P$. It follows from $X_u \subset P$ that $\mathrm{RSpec}(f)(v_P) = u$. This completes the proof. $\Box$

Thus we have shown that RSpec is well defined.

Definition 3.3.19. *We define a contravariant functor* $\mathrm{MCont} : \mathbf{RBS}_L \to \mathbf{ISP}_{\mathbf{M}}(L)$. *For an object* (S, α, R) *in* $\mathbf{RBS}_L$, *define*

$$\mathrm{MCont}(S, \alpha, R) = (\mathrm{Cont}(S, \alpha), \Box_R)$$

(for the definition of $\Box_R$*, see Definition 3.2.1). For an arrow* $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ *in* $\mathbf{RBS}_L$*, define* $\mathrm{MCont}(f)$ *by*

$$\mathrm{MCont}(f)(g) = g \circ f$$

for $g \in \mathrm{Cont}(S_2, \alpha_2)$*.*

The well-definedness of MCont is shown by the following two lemmas.

Lemma 3.3.20. *Let* (S, α, R) *be an object in* $\mathbf{RBS}_L$*. Then,* $\mathrm{MCont}(S, \alpha, R)$ *is in* $\mathbb{ISP}_{\mathbb{M}}(L)$*.*

Proof. By Theorem 3.3.9, $\mathrm{MCont}(S, \alpha, R)$ without $\Box$ is in $\mathbb{ISP}(L)$. We first verify that $\Box_R$ is well defined on $\mathrm{MCont}(S, \alpha, R)$, i.e., if $f \in \mathrm{MCont}(S, \alpha, R)$ then $\Box_R f \in \mathrm{MCont}(S, \alpha, R)$. Let $f \in \mathrm{MCont}(S, \alpha, R)$. We then have the following: For $a \in L$,

$$(\Box_R f)^{-1}(a) = R^{-1}[(\mathrm{T}_a(f))^{-1}(1)] \cap (S \setminus R^{-1}[(\mathrm{U}_a(f))^{-1}(0)]),$$

where note that $w \in R^{-1}[(\mathrm{T}_a(f))^{-1}(1)]$ means that there is $w' \in S$ such that wRw' and $f(w') = a$; and $w \in S \setminus R^{-1}[(\mathrm{U}_a(f))^{-1}(0)]$ means that there is no $w \in S$ such that wRw' and $f(w') \not\geq a$. Since

$$R^{-1}[(\mathrm{T}_a(f))^{-1}(1)] \cap (S \setminus R^{-1}[(\mathrm{U}_a(f))^{-1}(0)])$$

is clopen in S, $\Box_R f$ is a continuous map from S to L. It follows from the condition 3 in Definition 3.3.15 that $\Box_R f$ is subspace-preserving. Thus we have $\Box_R f \in \mathrm{MCont}(S, \alpha, R)$, whence $\Box_R$ is well defined. It follows from the definition of $\Box_R$ that $\mathrm{MCont}(S, \alpha, R)$ is a subalgebra of a modal power L^S of L, whence we have $\mathrm{MCont}(S, \alpha, R) \in \mathbb{ISP}_{\mathbb{M}}(L)$. $\Box$

Lemma 3.3.21. *Let* $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ *be an arrow in* $\mathbf{RBS}_L$*. Then,* $\mathrm{MCont}(f)$ *is a modal homomorphism.*

Proof. By Theorem 3.3.9, $\mathrm{MCont}(f)$ is an arrow in $\mathbf{ISP}(L)$. It suffices to show that $\mathrm{MCont}(f)(\Box g_2) = \Box(\mathrm{MCont}(f)(g_2))$ for $g_2 \in \mathrm{Cont}(S_2, \alpha_2)$. Let $w_1 \in S_1$. Then, we have

$$(\mathrm{MCont}(f)(\Box g_2))(w_1) = \Box g_2 \circ f(w_1) = \bigwedge\{g_2(w_2)\ ;\ f(w_1)R_2w_2\}.$$

Let a denote the rightmost side of the above equation. We also have

$$(\Box(\mathrm{MCont}(f)(g_2)))(w_1) = (\Box(g_2 \circ f))(w_1) = \bigwedge\{g_2(f(w'))\ ;\ w_1R_1w'\}.$$

Let b denote the rightmost side of the above equation. Since f satisfies item 4 in Definition 3.3.15, we have $a \leq b$. Since f satisfies item 5 in Definition 3.3.15, we have $a \geq b$. Hence we have $a = b$. $\Box$

Thus we have shown that MCont is well defined.

Topological Duality for $\mathbb{ISP}_{\mathbb{M}}(L)$

In this subsection, we show a topological duality for $\mathbb{ISP}_{\mathbb{M}}(L)$, thus generalising Jónsson-Tarski duality for modal algebras from the viewpoint of universal algebra.

Theorem 3.3.22. *Let L be a semi-primal algebra with a bounded lattice reduct, and $A \in \mathbb{ISP}_{\mathbb{M}}(L)$. Then, A is isomorphic to* $\mathrm{MCont} \circ \mathrm{RSpec}(A)$ *in the category* $\mathbf{ISP_M}(L)$.

Proof. Define $\varepsilon_A : A \to \mathrm{MCont} \circ \mathrm{RSpec}(A)$ by

$$\varepsilon_A(x)(v) = v(x)$$

for $x \in A$ and $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$. It follows from Theorem 3.3.9 that ε_A is an isomorphism in $\mathbf{ISP}(L)$. Thus it remains to show that ε_A preserves $\Box$, i.e., $\varepsilon_A(\Box x) = \Box_{R_\Box}\varepsilon_A(x)$ for $x \in A$. For $v \in \mathrm{RSpec}(A)$, we have the following:

$$\begin{aligned}
(\Box_{R_\Box}\varepsilon_A(x))(v) &= \bigwedge\{\varepsilon_A(x)(u)\,;\, vR_\Box u\} \\
&= \bigwedge\{u(x)\,;\, vR_\Box u\} \\
&= v(\Box x) \quad \text{(by the Kripke condition)} \\
&= \varepsilon_A(\Box x)(v).
\end{aligned}$$

This completes the proof. ∎

Theorem 3.3.23. *Let (S, α, R) be an object in $\mathbf{RBS}_L$. Then, (S, α, R) is isomorphic to* $\mathrm{RSpec} \circ \mathrm{MCont}(S, \alpha, R)$ *in the category* $\mathbf{RBS}_L$.

Proof. Define $\eta_{(S,\alpha,R)} : (S, \alpha, R) \to \mathrm{RSpec} \circ \mathrm{MCont}(S, \alpha, R)$ by

$$\eta_{(S,\alpha,R)}(x)(f) = f(x)$$

for $x \in S$ and $f \in \mathrm{Cont}(S, \alpha)$. By Theorem 3.3.9, $\eta_{(S,\alpha,R)}$ is an isomorphism in the category $\mathbf{BS}_L$. Below, we denote $\eta_{(S,\alpha,R)}$ by η_S. We first show that, for any $w, w' \in S$, wRw' iff $\eta_S(w)R_{\Box_R}\eta_S(w')$. Recall that the right-hand side holds iff the following condition holds: $\forall a \in L\, \forall f \in \mathrm{Cont}(S, \alpha)$ $(\eta_S(w)(\Box_R f) \geq a$ implies $\eta_S(w')(f) \geq a)$.

Assume that wRw'. We verify the above condition. Let $a \in L$ and $f \in \mathrm{Cont}(S, \alpha)$. Assume $\eta_S(w)(\Box_R f) \geq a$. Since

$$a \leq \eta_S(w)(\Box_R f) = (\Box_R f)(w) = \bigwedge\{f(z)\,;\, wRz\},$$

we have $\eta_S(w')(f) = f(w') \geq a$.

The converse is shown as follows. To prove the contrapositive, assume that $(w, w') \notin R$. It follows from Definition 3.3.15 that there is a clopen subset O of S such that $w' \in O$ and $R[w] \cap O = \emptyset$. Define $f : S \to L$ by $f(w) = 0$ for $w \in O$ and $f(w) = 1$ for $w \notin O$. Then we have $f \in \mathrm{Cont}(S, \alpha)$, $(\Box_R f)(w) = 1$, and $f(w') \neq 1$. Thus we have

$$\eta_S(w)(\Box_R f) \geq 1 \text{ and } \eta_S(w')(f) \not\geq 1,$$

whence the above condition does not hold.

It remains to show that η_S and η_S^{-1} satisfy item 5 in Definition 3.3.15. This follows immediately from the facts that wRw' iff $\eta_S(w) R_{\Box_R} \eta_S(w')$ and that η_S is bijective. $\Box$

Finally we obtain the modal semi-primal duality theorem.

Theorem 3.3.24. *Let L be a semi-primal algebra with a bounded lattice reduct. The induced categories $\mathbf{ISP_M}(L)$ and $\mathbf{RBS}_L$ are dually equivalent via the functors* RSpec *and* MCont *defined above.*

Proof. Let Id_1 denote the identity functor on $\mathbf{ISP_M}(L)$ and Id_2 denote the identity functor on $\mathbf{RBS}_L$. It is sufficient to show that there are natural isomorphisms $\varepsilon : \mathrm{Id}_1 \to \mathrm{MCont} \circ \mathrm{RSpec}$ and $\eta : \mathrm{Id}_2 \to \mathrm{RSpec} \circ \mathrm{MCont}$. For an L-$\mathbf{ML}$-algebra A, define ε_A as in the proof of Theorem B.6.11. For an object (S, α, R) in $\mathbf{RBS}_L$, define $\eta_{(S,\alpha,R)}$ as in the proof of Theorem B.6.12. Then it is straightforward to verify that η and ε are natural transformations. It follows from Theorem B.6.11 and Theorem B.6.12 that η and ε are natural isomorphisms. $\Box$

The original Jónsson-Tarski duality can be recovered by letting L be the two-element Boolean algebra in the above theorem.

We have extended Keimel-Werner's semi-primal duality without modality:

$$\mathbf{ISP}(L) \simeq \mathbf{BS}_L{}^{\mathrm{op}}$$

to the duality with modality:

$$\mathbf{ISP_M}(L) \simeq \mathbf{RBS}_L{}^{\mathrm{op}}.$$

This was accomplished via the new notion of $\mathbb{ISP}_{\mathbb{M}}$, without which it would be difficult to obtain such a modalized analogue of the semi-primal duality theorem.

In the next section, we shall show how to describe the category $\mathbf{RBS}_L$ in terms of the theory of coalgebras, thus obtaining a coalgebraic description of the duality $\mathbf{ISP_M}(L) \simeq \mathbf{RBS}_L{}^{\mathrm{op}}$.

3.4 Coalgebraic Duality and its Applications

Let us recall the definitions of coalgebra and its morphism (for the basics of coalgebras, we refer the reader to [11, 148]).

Definition 3.4.1. *Let* $\mathbf{C}$ *be a category and* T *an endofunctor on* $\mathbf{C}$*. A* T*-coalgebra is defined as a tuple* (C, δ) *for an object* C *in* $\mathbf{C}$ *and an arrow* $\delta : C \to \mathrm{T}(C)$ *in* $\mathbf{C}$*. For* T*-coalgebras* (C_1, δ_1) *and* (C_2, δ_2)*, a* T*-coalgebra morphism from* (C_1, δ_1) *to* (C_2, δ_2) *is defined as an arrow* $f : C_1 \to C_2$ *in* $\mathbf{C}$ *that satisfies* $\delta_2 \circ f = \mathrm{T}(f) \circ \delta_1$*. Then,* $\mathbf{Coalg}(\mathrm{T})$ *denotes the category of* T*-coalgebras and* T*-coalgebra morphisms.*

Let us recall the definition of Vietoris topology.

Definition 3.4.2. *Let* S *be a topological space,* $\mathcal{O}_S$ *the set of all open subsets of* S*, and* $\mathcal{C}_S$ *the set of all closed subsets of* S*. For a subset* U *of* S*, define*

$$B_S(U) = \{F \in \mathcal{C}_S \, ; \, F \subset U\} \text{ and } D_S(U) = \{F \in \mathcal{C}_S \, ; \, F \cap U \neq \emptyset\}.$$

The Vietoris space $\mathrm{V}(S)$ *of* S *is defined as a topological space whose underlying set is* $\mathcal{C}_S$ *and whose topology is generated by*

$$\{B_S(U) \, ; \, U \in \mathcal{O}_S\} \cup \{D_S(U) \, ; \, U \in \mathcal{O}_S\}.$$

Then we have the following proposition (see [204]).

Proposition 3.4.3. *If* S *is a Boolean space, then* $\mathrm{V}(S)$ *is a Boolean space whose topology is generated by the following set of clopen subsets of* $\mathrm{V}(S)$*:*

$$\{B_S(U) \, ; \, U \in \mathcal{O}_S \cap \mathcal{C}_S\} \cup \{D_S(U) \, ; \, U \in \mathcal{O}_S \cap \mathcal{C}_S\}.$$

We now introduce the concept of L-Vietoris functor.

Definition 3.4.4. *We define the* L*-Vietoris functor* $\mathrm{V}_L : \mathbf{BS}_L \to \mathbf{BS}_L$ *as follows. For an object* (S, α) *in* $\mathbf{BS}_L$*, define*

$$\mathrm{V}_L(S, \alpha) = (\mathrm{V}(S), \mathrm{V} \circ \alpha),$$

where, for $M \in \mathrm{SubAlg}(L)$*,* $\mathrm{V} \circ \alpha(M)$ $(= \mathrm{V}(\alpha(M)))$ *is the Vietoris space of a subspace* $\alpha(M)$ *of* S*. For an arrow* $f : (S, \alpha) \to (S', \alpha')$ *in* $\mathbf{BS}_L$*,* $\mathrm{V}_L(f) : (\mathrm{V}(S), \mathrm{V} \circ \alpha) \to (\mathrm{V}(S'), \mathrm{V} \circ \alpha')$ *is defined by*

$$\mathrm{V}_L(f)(F) = f(F) \ (= \{f(x) \, ; \, x \in F\})$$

for $F \in \mathrm{V}(S)$*.*

The well-definedness of the L-Vietoris functor is shown by the following two lemmas. We use the notations of Definition 3.4.2 in the following proofs of them.

Lemma 3.4.5. *Let* (S, α) *be an object in* $\mathbf{BS}_L$. *Then,* $\mathrm{V}_L(S, \alpha)$ *is an object in* $\mathbf{BS}_L$.

Proof. By Proposition 3.4.3, $\mathrm{V}(S)$ is a Boolean space.

We show that for $M \in \mathrm{SubAlg}(L)$, $\mathrm{V} \circ \alpha(M)$ is a closed subspace of $\mathrm{V}(S)$. Since an element of $\mathrm{V} \circ \alpha(M)$ is of the form $F \cap \alpha(M)$ for $F \in \mathcal{C}_S$ and since by $\alpha(M) \in \mathcal{C}_S$ we have $F \cap \alpha(M) \in \mathcal{C}_S$ for $F \in \mathcal{C}_S$, $\mathrm{V} \circ \alpha(M)$ is a subset of $\mathrm{V}(S)$. Since for $U \in \mathcal{O}_S$ we have both

$$B_S(U) \cap \mathrm{V} \circ \alpha(M) = \{F \in \mathrm{V} \circ \alpha(M) \,;\, F \subset U\} = B_{\alpha(M)}(U \cap \alpha(M))$$

and

$$D_S(U) \cap \mathrm{V} \circ \alpha(M) = \{F \in \mathrm{V} \circ \alpha(M) \,;\, F \cap U \neq \emptyset\} = D_{\alpha(M)}(U \cap \alpha(M)),$$

$\mathrm{V} \circ \alpha(M)$ is a subspace of $\mathrm{V}(S)$. In order to show that $\mathrm{V} \circ \alpha(M)$ is closed in $\mathrm{V}(S)$, assume that $F \in \mathrm{V}(S)$ and $F \notin \mathrm{V} \circ \alpha(M)$. Then, there is $x \in F$ such that $x \notin \alpha(M)$. Since $\alpha(M)$ is closed in S, $D_S(S \setminus \alpha(M))$ is open in $\mathrm{V}(S)$. Moreover, we have

$$F \in D_S(S \setminus \alpha(M)) \text{ and } \mathrm{V} \circ \alpha(M) \cap D_S(S \setminus \alpha(M)) = \emptyset.$$

Hence, $\mathrm{V} \circ \alpha(M)$ is closed in $\mathrm{V}(S)$.

We next show that $\mathrm{V} \circ \alpha$ satisfies the three conditions in Definition 3.3.6. It follows from $\alpha(L) = S$ that $\mathrm{V} \circ \alpha(L) = \mathrm{V}(S)$. If $L_1 \subset L_2$ for $L_1, L_2 \in \mathrm{SubAlg}(L)$, then $\alpha(L_1) \subset \alpha(L_2)$ and, since $\alpha(L_1)$ is closed in $\alpha(L_2)$, we have $\mathrm{V} \circ \alpha(L_1) \subset \mathrm{V} \circ \alpha(L_2)$. Assume that $L_1 = L_2 \cap L_3$ for $L_1, L_2, L_3 \in \mathrm{SubAlg}(L)$. Then, we have

$$\mathrm{V} \circ \alpha(L_1) = \mathrm{V} \circ \alpha(L_2 \cap L_3) = \mathrm{V}(\alpha(L_2) \cap \alpha(L_3)).$$

An element of $\mathrm{V}(\alpha(L_2) \cap \alpha(L_3))$ is of the form

$$F \cap \alpha(L_2) \cap \alpha(L_3)$$

for $F \in \mathcal{C}_S$. An element of $\mathrm{V}(\alpha(L_2)) \cap \mathrm{V}(\alpha(L_3))$ is of the form

$$(F_1 \cap \alpha(L_2)) \cap (F_2 \cap \alpha(L_3))$$

for $F_1, F_2 \in \mathcal{C}_S$, which follows from the fact that for $X \subset S$ we have

$$\exists F_1, F_2 \in \mathcal{C}_S \; X = F_1 \cap \alpha(L_2) = F_2 \cap \alpha(L_3) \Leftrightarrow \exists F_1, F_2 \in \mathcal{C}_S \; X = (F_1 \cap \alpha(L_2)) \cap (F_2 \cap \alpha(L_3)).$$

Hence we have $\mathrm{V}(\alpha(L_2) \cap \alpha(L_3)) = \mathrm{V}(\alpha(L_2)) \cap \mathrm{V}(\alpha(L_3))$ and so $\mathrm{V} \circ \alpha(L_1) = \mathrm{V} \circ \alpha(L_2) \cap \mathrm{V} \circ \alpha(L_3)$. □

Lemma 3.4.6. *Let $f : (S, \alpha) \to (S', \alpha')$ be an arrow in $\mathbf{BS}_L$. Then, $\mathrm{V}_L(f)$ is an arrow in $\mathbf{BS}_L$.*

Proof. Since f is a continuous map between Boolean spaces, it follows from [88, Theorem 3.1.8] that $\mathrm{V}_L(f)$ maps a closed subset of S to a closed subset of S'. In order to show that $\mathrm{V}_L(f)$ is continuous, let $U \in \mathcal{O}_{S'}$. Then we have

$$\mathrm{V}_L(f)^{-1}(B_{S'}(U)) = \{F \in \mathcal{C}_S \,;\, f(F) \subset U\} = \{F \in \mathcal{C}_S \,;\, F \subset f^{-1}(U)\} = B_S(f^{-1}(U))$$

and also

$$\mathrm{V}_L(f)^{-1}(D_{S'}(U)) = \{F \in \mathcal{C}_S; f(F) \cap U \neq \emptyset\} = \{F \in \mathcal{C}_S; F \cap f^{-1}(U) \neq \emptyset\} = B_S(f^{-1}(U)).$$

Thus, $\mathrm{V}_L(f)$ is continuous. It remains to show that $\mathrm{V}_L(f)$ is subspace-preserving. Assume that $F \in \mathrm{V} \circ \alpha(M)$ for $M \in \mathrm{SubAlg}(L)$. Then we have $F \subset \alpha(M)$. Since f is subspace-preserving, we have $f(F) \subset \alpha'(M)$. Thus it follows that

$$\mathrm{V}_L(f)(F) = f(F) \subset \alpha'(M).$$

Hence we have $\mathrm{V}_L(f)(F) \in \mathrm{V} \circ \alpha'(M)$. □

In order to show that $\mathbf{Coalg}(\mathrm{V}_L)$ is isomorphic to $\mathbf{RBS}_L$, we introduce two functors RC and CR between the two categories.

Definition 3.4.7. *A functor* $\mathrm{RC} : \mathbf{RBS}_L \to \mathbf{Coalg}(\mathrm{V}_L)$ *is defined as follows. For an object (S, α, R) in $\mathbf{RBS}_L$,* $\mathrm{RC}(S, \alpha, R)$ *is defined as a V_L-coalgebra*

$$((S, \alpha), R[\text{-}])$$

where $R[\text{-}] : (S, \alpha) \to \mathrm{V}_L(S, \alpha)$ is defined by $R[x] = \{y \in S \,;\, xRy\}$ for $x \in S$. For an arrow f in $\mathbf{RBS}_L$, define $\mathrm{RC}(f) = f$.

In the above definition, $\mathrm{RC}(S, \alpha, R)$ is a V_L-coalgebra, since $R[\text{-}] : (S, \alpha) \to \mathrm{V}_L(S, \alpha)$ is an arrow in $\mathbf{BS}_L$ by items 1, 2 and 3 in Definition 3.3.15 and by Proposition 3.4.3. It is straightforward to verify that $\mathrm{RC}(f)$ is an arrow in $\mathbf{Coalg}(\mathrm{V}_L)$ for an arrow f in $\mathbf{RBS}_L$. Thus, the functor RC is well defined.

Definition 3.4.8. *A functor* $\mathrm{CR} : \mathbf{Coalg}(\mathrm{V}_L) \to \mathbf{RBS}_L$ *is defined as follows. For an object $((S, \alpha), \delta)$ in $\mathbf{Coalg}(\mathrm{V}_L)$, define*

$$\mathrm{CR}((S, \alpha), \delta) = (S, \alpha, R_\delta)$$

where a binary relation R_δ on S is defined by

$$xR_\delta y \Leftrightarrow y \in \delta(x)$$

for $x, y \in S$. For an arrow f in $\mathbf{Coalg}(\mathrm{V}_L)$, define $\mathrm{CR}(f) = f$.

The well-definedness of the functor CR is shown by the following lemma.

Lemma 3.4.9. *For an object* $((S, \alpha), \delta)$ *in* $\mathbf{Coalg}(\mathrm{V}_L)$*,* $\mathrm{CR}((S, \alpha), \delta)$ *is an object in* $\mathbf{RBS}_L$*.*

Proof. It suffices to show that (S, α, R_δ) satisfies the three conditions in Definition 3.3.15. First, for $x \in S$ we have $R_\delta[x] = \delta(x) \in \mathrm{V}(S)$ and so $R_\delta[x]$ is a closed subset of $\mathrm{V}(S)$. Second, for a clopen subset O of S, the following holds:

$$\begin{aligned} R_\delta^{-1}[O] &= \{x \in S\,;\, \exists y \in O\ xR_\delta y\} = \{x \in S\,;\, \exists y \in O\ y \in \delta(x)\} \\ &= \{x \in S\,;\, O \cap \delta(x) \neq \emptyset\} = \{x \in S\,;\, \delta(x) \in D_S(O)\} \\ &= \delta^{-1}(D_S(O)). \end{aligned}$$

Since O is clopen in S, $D_S(O)$ is clopen in $\mathrm{V}(S)$ by Proposition 3.4.3. Thus, since δ is continuous, $R_\delta^{-1}[O]$ is clopen. Since S is a Boolean space, this implies that R_δ is a continuous map from S to $\mathrm{V}(S)$. Third, if $x \in \alpha(M)$ for $M \in \mathrm{SubAlg}(L)$, then we have $R_\delta[x] = \delta(x) \in \mathrm{V} \circ \alpha(M)$ (recall that δ is subspace-preserving by definition). This completes the proof. $\square$

It is straightforward to verify that $\mathrm{CR}(f)$ is an arrow in $\mathbf{RBS}_L$ for an arrow f in $\mathbf{Coalg}(\mathrm{V}_L)$.

Thus we obtain the following theorem.

Theorem 3.4.10. *Let L be a semi-primal algebra with a bounded lattice reduct. The induced categories* $\mathbf{Coalg}(\mathrm{V}_L)$ *and* $\mathbf{RBS}_L$ *are isomorphic via the functors* CR *and* RC*.*

Proof. Clearly we have $\mathrm{CR} \circ \mathrm{RC}(f) = f$ for an arrow f in $\mathbf{RBS}_L$ and $\mathrm{RC} \circ \mathrm{CR}(f) = f$ for an arrow f in $\mathbf{Coalg}(\mathrm{V}_L)$. Let (S, α, R) be an object in $\mathbf{RBS}_L$. Then we have:

$$xR_{R[\text{-}]}y \;\Leftrightarrow\; y \in R[x] \;\Leftrightarrow\; xRy.$$

Thus, (S, α, R) is exactly the same as $\mathrm{CR} \circ \mathrm{RC}(S, \alpha, R)$. Let $((S, \alpha), \delta)$ be an object in $\mathbf{Coalg}(\mathrm{V}_L)$. Then we have:

$$y \in \delta_{R_\delta}(x) \;\Leftrightarrow\; xR_\delta y \;\Leftrightarrow\; y \in \delta(x).$$

Thus, $((S, \alpha), \delta)$ is exactly the same as $\mathrm{RC} \circ \mathrm{CR}((S, \alpha), \delta)$. $\square$

By Theorem 3.3.24 and Theorem 3.4.10, we obtain the following coalgebraic duality theorem, which generalises Abramsky-Kupke-Kurz-Venema duality for modal algebras from the viewpoint of universal algebra.

Theorem 3.4.11. *Let L be a semi-primal algebra with a bounded lattice reduct. The induced categories* $\mathbf{ISP_M}(L)$ *and* $\mathbf{Coalg}(\mathrm{V}_L)$ *are dually equivalent.*

Thus, the modal semi-primal duality $\mathbf{ISP_M}(L) \simeq \mathbf{RBS}_L{}^{\mathrm{op}}$ can be described in terms of the theory of coalgebras. Abramsky-Kupke-Kurz-Venema duality can be recovered by letting L be the two-element Boolean algebra in the above theorem.

Since $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ coincides with the class of all $\mathcal{MMV}_n$-algebras and since $\mathbf{n}$ forms a semi-primal algebra with a lattice reduct, the above theorem gives us a coalgebraic duality for $\mathcal{MMV}_n$-algebras (i.e., algebras of Łukasiewicz n-valued modal logic):

Corollary 3.4.12. *The category of $\mathcal{MMV}_n$-algebras and their homomorphisms is dually equivalent to* $\mathbf{Coalg}(\mathrm{V}_{\mathbf{n}})$.

In a similar way, we obtain a coalgebraic duality for L-**ML**-algebras (i.e., algebras of a version of Fitting's many-valued modal logic). We remark that [188, Lemma 2.6] is useful when proving that $\mathbf{n}$ is semi-primal (this can be shown in a similar way to [187, Lemma 2.3] via [188, Lemma 2.6]).

With the help of Corollary 3.2.5 and Theorem 3.4.11, we obtain the following.

Corollary 3.4.13. $\mathbf{Coalg}(\mathrm{V}_L)$ *has cofree coalgebras.*

Since $\mathbf{ISP_M}(L)$ has the initial algebra, we obtain the final coalgebra theorem for V_L.

Corollary 3.4.14. *The endofunctor V_L has a final coalgebra.*

If L is not only semi-primal but also primal (for its definition, see [63]), then by Hu theorem (see [63, Theorem 4.1.1]) $\mathbf{ISP}(L)$ is dually equivalent to the category of Boolean spaces (i.e., $\mathbf{BS}_2$), whence $\mathbf{ISP_M}(L)$ is dually equivalent to the category of descriptive general frames (i.e., $\mathbf{RBS}_2$) and is also dually equivalent to the category of Stone coalgebras (i.e., $\mathbf{Coalg}(\mathrm{V}_2)$).

This implies that if L and L' are primal then the categories $\mathbf{ISP_M}(L)$ and $\mathbf{ISP_M}(L')$ are equivalent. More generally, since the definition of $\mathbf{RBS}_L$ depends only on the order structure of subalgebras of L, Theorem 3.3.24 gives us the following.

Corollary 3.4.15. *If L and L' are semi-primal algebras with lattice reducts and if* $\mathrm{SubAlg}(L)$ *and* $\mathrm{SubAlg}(L')$ *are order isomorphic, then the categories* $\mathbf{ISP_M}(L)$ *and* $\mathbf{ISP_M}(L')$ *are equivalent.*

Similarly, if L and L' are semi-primal and if $\mathrm{SubAlg}(L)$ and $\mathrm{SubAlg}(L')$ are order isomorphic, then the categories $\mathbf{Coalg}(\mathrm{V}_L)$ and $\mathbf{Coalg}(\mathrm{V}_{L'})$ are equivalent. Note that if L and L' are primal, then $\mathrm{SubAlg}(L)$ and $\mathrm{SubAlg}(L')$ are always order isomorphic.

Summing up the first part, we have introduced the new notion of $\mathbb{ISP}_{\mathbb{M}}$ and extended the theory of natural dualities so as to encompass Jónsson-Tarski duality and Abramsky-Kupke-Kurz-Venema duality for the class of all modal algebras, which are becoming more and more important in coalgebraic logic. Whereas $\mathbb{ISP}(M)$ cannot be the class of all modal algebras, crucially, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ coincides with the class of all modal algebras, and furthermore, there are similar facts for algebras of many-valued modal logics. $\mathbb{ISP}_{\mathbb{M}}$ thus provides a natural framework for the universal algebra of modalities, and as such, for the theory of modal natural dualities. From a technical point of view, our starting point was Keimel-Werner's semi-primal duality for $\mathbb{ISP}(L)$ in natural duality theory. Having shifted our focus from $\mathbb{ISP}(L)$ to $\mathbb{ISP}_{\mathbb{M}}(L)$, we verified the Kripke condition for $\mathbb{ISP}_{\mathbb{M}}(L)$ where L is a semi-primal algebra with a bounded lattice reduct. The Kripke condition is completeness in logical terms, and we needed a weaker form of the axiom of choice for the verification of it. As main results, we have obtained topological and coalgebraic dualities for $\mathbb{ISP}_{\mathbb{M}}(L)$ with three kinds of applications of them: coalgebraic dualities for many-valued modal logics; the existence of a final coalgebra and cofree coalgebras in $\mathbf{Coalg}(\mathrm{V}_L)$; and a criterion for the equivalence of categories of the form $\mathbf{ISP_M}(L)$.

3.5 Introduction to the Second Part

Stone Duality for Boolean algebras ([262]) is one of the most important results in algebraic logic and has been generalised in various directions (see, e.g., [63, 107, 109, 123, 149]). Moreover, Stone-type dualities have been applied to diverse areas of research, including program semantics, non-classical logics and pointfree geometry (see, e.g., [2, 40, 45, 108, 149]). Stone-type dualities naturally connect logic, algebra and geometry, and therefore, for example, we can understand the geometric meanings of logics and their properties by Stone-type dualities.

In this chapter, we show a general duality theorem which extends Stone Duality for distributive lattices ([149, 261]) from the viewpoint of universal algebra. Our generalisation is made based on the following slogan: The coincidence of term functions and continuous maps provides a Stone-type duality. The slogan is just a slogan and inadequate in some respects, but, as shown in this chapter, it provides useful insights into duality theory.

Let us give more details on the above slogan. Given a finite algebra M equipped with a topology, we consider a Stone-type duality for $\mathbb{ISP}(M)$ (i.e., the quasi-variety generated by M). Let $\mathrm{TermFunc}_n(M)$ denote the set of all n-ary term functions of M and $\mathrm{Cont}_n(M)$ the set of all continuous maps from M^n to M. Then, M is said to be topologically dualizable with respect to the topology iff the following holds:

$$\forall n \in \omega \;\; \mathrm{Cont}_n(M) = \mathrm{TermFunc}_n(M).$$

The precise meaning of the slogan is that if M is topologically dualizable with respect to the topology, then a Stone-type duality holds for $\mathbb{ISP}(M)$. Actually, this might not necessarily hold. However, as we shall see, the slogan works well at least for some types of finite topological algebras M.

Hu [139, 140] generalised Stone Duality for Boolean algebras to a duality for $\mathbb{ISP}(M)$ where M is a primal algebra (see Definition 3.6.4 and Theorem 3.6.6), which states that, for any primal algebra M, the category of algebras in $\mathbb{ISP}(M)$ and homomorphisms is dually equivalent to that of zero-dimensional compact Hausdorff spaces and continuous maps. Now the following holds (Proposition 3.6.5): For a finite algebra M, M is primal iff M is topologically dualizable with respect to the discrete topology. Therefore, the notion of topological dualisability extends that of being primal and Hu's duality theorem provides an example in which the slogan works well.

Stone Duality for distributive lattices states that the category of distributive lattices and homomorphisms is dually equivalent to that of coherent spaces and continuous proper maps (for definitions, see Preliminaries above). By prime filter theorem for distributive lattices, the class of distributive lattices coincides with $\mathbb{ISP}(\mathbf{2}_d)$, where $\mathbf{2}_d$ denotes the two-element distributive lattice. Moreover, $\mathbf{2}_d$ $(= \{0,1\})$ is topologically dualizable with respect to the Alexandrov topology $\{\emptyset, \{1\}, \{0,1\}\}$ (which follows from Proposition 3.7.41). Hence, Stone Duality for distributive lattices provides another example in which the slogan works well.

In this chapter, we generalise Stone Duality for distributive lattices via the concept of topological dualisability as follows. Let L be a finite algebra which has a bounded join-semilattice reduct. We equip L with the Alexandrov topology with respect to the partial order induced by the join-semilattice reduct. Then we can show the following (Theorem 5.2.13): If L is topologically dualizable with respect to the Alexandrov topology, then the category of algebras in $\mathbb{ISP}(L)$ and homomorphisms is dually equivalent to that of coherent spaces and continuous proper maps. This is a universal algebraic generalisation of Stone Duality for distributive lattices, since $\mathbb{ISP}(\mathbf{2}_d)$ coincides with the class of distributive lattices.

Thus we can confirm the applicability of the slogan and the usefulness of the concept of topological dualisability. It is expected that more dualities can be developed by using the concept of topological dualisability.

The duality between the category of distributive lattices and that of coherent spaces restricts to the duality between the category of Heyting algebras and that of $\mathbf{2}_d$-Heyting spaces (see Definition 3.7.33), which we call Stone Duality for Heyting algebras (for a related duality, see [89]). By assuming that L has a binary operation $*$, we introduce the concept of the $*$-residuated quasi-variety generated by L, which is denoted by $\mathbb{IRSP}(L)$. Then, by applying the above duality (Theorem 5.2.13), it is straightforward to obtain a duality for $\mathbb{IRSP}(L)$, where L is assumed to be $*$-residuated and be topologically dualizable with respect to the Alexandrov topology. This is a universal algebraic generalisation of Stone Duality for Heyting algebras, since $\mathbb{IRSP}(\mathbf{2}_d)$ coincides with the class of Heyting algebras.

The concept of $\mathbb{IRSP}$(-) provides a general framework to discuss "Heyting-type" or "residuated" algebras (see Definition 3.7.30), where note that the concept of residuation plays a significant role in algebraic logic (see, e.g., [104, 220]). Especially, $\mathbb{IRSP}$(-) is useful for developing general duality theory for such kind of algebras. The reason for this may be explained as follows. When studying duality theory for a class of algebras, it is often useful to have a *single* algebra by which the class of algebras is "generated", where such single algebra (equipped with some topology) is sometimes called schizophrenic object or dualizing object in a broader sense (see [149]). For any Heyting algebra A, $\mathbb{ISP}(A)$ does not coincide with the class of Heyting algebras, but $\mathbb{IRSP}(\mathbf{2}_d)$ coincides with the class of Heyting algebras. Thus, $\mathbb{IRSP}$(-), rather than $\mathbb{ISP}$(-), is appropriate in the context of general theory of dualities for residuated algebras.

By applying the above duality theorems, we obtain Stone-type dualities for n-valued distributive lattices and n-valued Heyting algebras, which are naturally defined based on an Łukasiewicz n-valued algebra $\{0, 1/(n-1), 2/(n-1), ..., 1\}$. As classical propositional logic can be considered as the free Boolean algebra generated by propositional variables, we can define an intuitionistic Łukasiewicz n-valued logic as the free n-valued Heyting algebra generated by propositional variables.

This second part is organized as follows. We first review basic results from general topology. In Section 3.6, we introduce the concept of topological dualisability. In Section 3.7, we generalise Stone Duality for distributive lattices via the concept of topological dualisability. We then generalise Stone Duality for Heyting algebras by introducing the concept of $\mathbb{IRSP}$(-). We also obtain dualities for n-valued distributive lattices and n-valued Heyting algebras by applying the above dualities. Throughout the second part, a lattice and a semilattice mean a bounded lattice and a bounded semilattice respectively.

Preliminaries from General Topology

Let us review several concepts and results from general topology.

Definition 3.5.1. *A Boolean space is defined as a zero-dimensional compact Hausdorff space.*

Definition 3.5.2. *For topological spaces S_1 and S_2, a map $f : S_1 \to S_2$ is proper iff $f^{-1}(O)$ is a compact open subset of S_2 for any compact open subset O of S_1.*

Note that a map f between Boolean spaces is continuous iff f is continuous and proper.

Definition 3.5.3. *A non-empty closed subset A of a topological space S is irreducible iff, for any closed subsets A_1 and A_2 of S, $A = A_1 \cup A_2$ implies either $A_1 = A$ or $A_2 = A$.*

Definition 3.5.4. *A topological space S is sober iff, for any irreducible closed subset A of S, there is a unique element x of S such that $A = \overline{\{x\}}$, where $\overline{\{x\}}$ denotes the closure of $\{x\}$.*

In the following, we review basic facts on sober spaces (see [109, 149, 271]). A sober space is T_0. A Hausdorff space is sober.

Lemma 3.5.5. *Any product of sober spaces is also sober.*

Proof. See [136, Theorem 1.4] or [109, Exercise O-5.16]. □

Definition 3.5.6. *A coherent space S is defined as a compact sober space such that the set of compact open subsets of S forms an open basis of S.*

A proper map between coherent spaces is always continuous.

Definition 3.5.7. *Let S be a topological space and $\mathcal{B}$ the set of all compact open subsets of S. Then, the patch topology of S is defined as the topology generated by*

$$\mathcal{B} \cup \{S \setminus X \ ; \ X \in \mathcal{B}\}.$$

Let S^* denote the new space equipped with the patch topology.

Patch topology is a useful tool for the study of sober spaces and coherent spaces.

Lemma 3.5.8. *Let S be a coherent space. Then, S^* is a Boolean space.*

Proof. See [149, Proposition 4.2.5]. □

3.6 The Notion of Topological Dualisability

In this section, we introduce the notion of topological dualisability, which plays the most important role in this chapter.

As in universal algebra, we mean by an algebra a set M equipped with a collection of finitary operations on M (for basic concepts from universal algebra, see [47, 63, 121]). Note that a constant of M is considered as a function from M^0 to M, where M^0 is a singleton.

For an algebra M, $\mathbb{ISP}(M)$ denotes the class of all isomorphic copies of subalgebras of direct powers of M. In other words, $\mathbb{ISP}(M)$ is the quasi-variety generated by M. As usual, a homomorphism between algebras in $\mathbb{ISP}(M)$ is defined as a function which preserves all the operations of M. Note that a homomorphism preserves any term function.

Definition 3.6.1. *For an algebra M and $n \in \omega$, $\mathrm{TermFunc}_n(M)$ denotes the set of all n-ary term functions of M.*

Recall that any projection function from M^n to M is an element of $\mathrm{TermFunc}_n(M)$ by the definition of term functions.

Definition 3.6.2. *For a topological space S and $n \in \omega$, $\mathrm{Cont}_n(S)$ denotes the set of all continuous maps from S^n to S, where S^n is equipped with the product topology (S^0 is a singleton topological space).*

Then, the notion of topological dualisability is defined as follows.

Definition 3.6.3. *Let M be a finite algebra equipped with a topology. Then, M is said to be topologically dualizable with respect to the topology iff*

$$\forall n \in \omega \ \ \mathrm{Cont}_n(M) = \mathrm{TermFunc}_n(M).$$

Note that any projection function from M^n to M is continuous by the definition of the product topology.

Let us review the notion of primal algebra.

Definition 3.6.4 ([139, 140]). *A finite algebra M is primal iff* $\mathrm{TermFunc}_n(M)$ *coincides with the set of all functions from M^n to M.*

Proposition 3.6.5. *Let M be a finite algebra equipped with the discrete topology. Then, M is primal iff M is topologically dualizable with respect to the discrete topology.*

Proof. This is immediate from the fact that, since M^n is a discrete space, $\mathrm{Cont}_n(M)$ coincides with the set of all functions from M^n to M. $\square$

By the above proposition, we notice that the notion of topological dualisability is a (topological) generalisation of that of being primal.

T. K. Hu proved the following duality theorem.

Theorem 3.6.6 ([139, 140]). *Let M be a primal algebra. Then, the category of algebras in $\mathbb{ISP}(M)$ and homomorphisms is dually equivalent to the category of Boolean spaces and continuous maps.*

Let $\mathbf{2}_b$ denote the two-element Boolean algebra. Then, $\mathbf{2}_b$ is a primal algebra. $\mathbb{ISP}(\mathbf{2}_b)$ coincides with the class of Boolean algebras, which follows from ultrafilter theorem for Boolean algebras. Thus, the above theorem is a generalisation of Stone Duality for Boolean algebras.

3.7 Non-Hausdorff Stone Duality and its Applications

Throughout this chapter, let L be a finite algebra such that

- L has a join-semilattice reduct;
- L has the greatest element 1 and the least element 0 with respect to a partial order $\leq$ defined by $x \leq y \Leftrightarrow x \vee y = y$ for $x, y \in L$, where $\vee$ denotes the join operation of L.

We equip L with the Alexandrov topology with respect to $\leq$ above, i.e., the topology of L is generated by $\{\uparrow x\, ;\, x \in L\}$, where $\uparrow x = \{y \in L\, ;\, x \leq y\}$. Note that the set of all open (resp. closed) subsets of L coincides with the set of all upward (resp. downward) closed subsets of L. For a set S, L^S denotes the set of all functions from S to L. We equip L^S with the product topology.

Throughout this chapter, we additionally assume that

- L is topologically dualizable with respect to the Alexandrov topology.

In this section, we develop a topological duality for $\mathbb{ISP}(L)$, which is a generalisation of Stone Duality for distributive lattices from the viewpoint of universal algebra.

Lemma 3.7.1. *Define a function* $t_\wedge : L^2 \to L$ *by*

$$t_\wedge(x,y) = \begin{cases} 1 & \textit{if } x = y = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Then, $t_\wedge$ *is a term function of* L.

Proof. Since L is topologically dualizable with respect to the Alexandrov topology, it suffices to show that $t_\wedge$ is continuous, which is straightforward to verify. ☐

In similar ways, we obtain the following lemmas.

Lemma 3.7.2. *Let* $n \in \omega$. *Define a function* $t_\vee^n : L^n \to L$ *by*

$$t_\vee^n(x_1, ..., x_n) = \begin{cases} 1 & \textit{if } \ \exists i \in \{1, ..., n\} \ x_i = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Then, $t_\vee^n$ *is a term function of* L.

Lemma 3.7.3. *Let* $r \in L$. *Define a function* $\tau_r : L \to L$ *by*

$$\tau_r(x) = \begin{cases} 1 & \textit{if } x \geq r \\ 0 & \textit{otherwise.} \end{cases}$$

Then, τ_r *is a term function of* L.

Operations like τ_r are often useful in the context of many-valued logics (for example, see applications to many-valued algebras below and also [186, 187, 203, 265]).

Lemma 3.7.4. *Let* $r \in L$. *Define a function* $\theta_r : L \to L$ *by*

$$\theta_r(x) = \begin{cases} r & \textit{if } x = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Then, θ_r *is a term function of* L.

Note that a homomorphism preserves the operations $t_\wedge$, $t_\vee^n$, τ_r and θ_r, since they are term functions.

The Spectra of Algebras in $\mathbb{ISP}(L)$

We define the spectrum $\mathrm{Spec}(A)$ of an algebra A in $\mathbb{ISP}(L)$ as follows.

Definition 3.7.5. *For $A \in \mathbb{ISP}(L)$, $\mathrm{Spec}(A)$ denotes the set of all homomorphisms from A to L. For $a \in A$, define*

$$\langle a \rangle = \{v \in \mathrm{Spec}(A) \; ; \; v(a) = 1\}.$$

We equip $\mathrm{Spec}(A)$ with the topology generated by $\{\langle a \rangle \; ; \; a \in A\}$.

Note that, by Lemma 3.7.1, $\langle a \rangle \cap \langle b \rangle = \langle t_\wedge(a, b) \rangle$ and that, by Lemma 3.7.2, $\langle a_1 \rangle \cup ... \cup \langle a_n \rangle = \langle t_\vee^n(a_1, ..., a_n) \rangle$.

Proposition 3.7.6. *Let $A \in \mathbb{ISP}(L)$. Then, $\{\langle a \rangle \; ; \; a \in A\}$ forms an open basis of $\mathrm{Spec}(A)$.*

Proof. It suffices to show that $\{\langle a \rangle \; ; \; a \in A\}$ is closed under $\cap$. Let $a, b \in A$. Then, we have $\langle t_\wedge(a, b) \rangle = \langle a \rangle \cap \langle b \rangle$. This completes the proof. $\square$

Lemma 3.7.7. *Let $A \in \mathbb{ISP}(L)$. For $v, u \in \mathrm{Spec}(A)$, (i) $v = u$ iff (ii) $v^{-1}(\{1\}) = u^{-1}(\{1\})$.*

Proof. Clearly, (i) implies (ii). We show that (ii) implies (i). To prove the contrapositive, assume that (i) does not hold. Then, we may assume that there is $a \in A$ with $v(a) \not\leq u(a)$. Let $r = v(a)$. Then, by Lemma 3.7.3, we have $v(\tau_r(a)) = 1$ and $u(\tau_r(a)) \neq 1$, whence (ii) does not hold. $\square$

Definition 3.7.8. *Let $A \in \mathbb{ISP}(L)$ and $X \subset L^A$. For $a \in A$ and $r \in L$, define*

$$\langle a \rangle_X^r = \{f \in X \; ; \; f(a) \geq r\}.$$

Define $X^\star$ as a topological space whose underlying set is X and whose topology is generated by

$$\{\langle a \rangle_X^1 \; ; \; a \in A\} \cup \{X \setminus \langle a \rangle_X^1 \; ; \; a \in A\}.$$

Lemma 3.7.9. *Let $A \in \mathbb{ISP}(L)$. Then, $\mathrm{Spec}(A)$ ($\subset L^A$) is a subspace of L^A, i.e., the topology of $\mathrm{Spec}(A)$ coincides with the relative topology induced by L^A on a set $\mathrm{Spec}(A)$, where L^A is equipped with the product topology.*

Proof. For $a \in A$, we have

$$\langle a \rangle = \mathrm{Spec}(A) \cap \langle a \rangle^1_{L^A} .$$

Here, $\langle a \rangle^1_{L^A}$ is an open subset of L^A. Thus, since the topology of $\mathrm{Spec}(A)$ is generated by $\{\langle a \rangle \ ; \ a \in A\}$, the topology of $\mathrm{Spec}(A)$ is weaker than or equal to the relative topology induced by L^A on a set $\mathrm{Spec}(A)$.

The topology of L^A is generated by $\{\langle a \rangle^r_{L^A} \ ; \ a \in A \text{ and } r \in L\}$. By Lemma 3.7.3, we have

$$\langle a \rangle^r_{L^A} \cap \mathrm{Spec}(A) = \langle \tau_r(a) \rangle.$$

Therefore, the relative topology induced by L^A on a set $\mathrm{Spec}(A)$ is weaker than or equal to the topology of $\mathrm{Spec}(A)$. This completes the proof. □

Let L_d denote the topological space whose underlying set is L and whose topology is the discrete topology.

Lemma 3.7.10. *Let $A \in \mathbb{ISP}(L)$. Then,* $\mathrm{Spec}(A)^\star$ *is a subspace of L_d^A, i.e., the topology of* $\mathrm{Spec}(A)^\star$ *coincides with the relative topology induced by L_d^A on a set* $\mathrm{Spec}(A)$*, where L_d^A is equipped with the product topology of L_d's.*

Proof. Let $a \in L$. Let $\{r_1, ..., r_n\}$ be the set of those elements of $(\uparrow r) \setminus \{r\}$ which are minimal with respect to the restriction of $\leq$ on L to $(\uparrow r) \setminus \{r\}$ (since L is finite, $n \in \omega$). By the definition of (-)*, $\mathrm{Spec}(A) \setminus \langle \tau_{r_i}(a) \rangle$ is open in $\mathrm{Spec}(A)^\star$ for any $i \in \{1, ..., n\}$. It follows from the definition of r_i and Lemma 3.7.3 that

$$\langle \tau_r(a) \rangle \cap \left(\bigcap_{i \in \{1,...,n\}} \mathrm{Spec}(A) \setminus \langle \tau_{r_i}(a) \rangle \right) = \mathrm{Spec}(A) \cap \{f \in L^A \ ; \ f(a) = r\}.$$

Hence, $\mathrm{Spec}(A) \cap \{f \in L^A \ ; \ f(a) = r\}$ is open in $\mathrm{Spec}(A)^\star$. Since the topology of L_d^A is generated by $\{\{f \in L^A \ ; \ f(a) = r\} \ ; \ a \in A \text{ and } r \in L\}$, the topology of $\mathrm{Spec}(A)^\star$ is stronger than or equal to the topology induced by L_d^A on a set $\mathrm{Spec}(A)$. It is easily verified that the converse holds. □

Lemma 3.7.11. *Let $A \in \mathbb{ISP}(L)$. Then, (i)* $\mathrm{Spec}(A)^\star$ *is compact; (ii) $\langle a \rangle$ is a compact subset of* $\mathrm{Spec}(A)^\star$ *for any $a \in A$.*

Proof. We first show (i). By Tychonoff's theorem, L_d^A is compact. If $\mathrm{Spec}(A)^\star$ is a closed subspace of L_d^A then $\mathrm{Spec}(A)^\star$ is compact, since a closed subspace of a compact space is compact. Since $\mathrm{Spec}(A)^\star$ is a subspace of L_d^A by Lemma 3.7.10, it suffices to show that $\mathrm{Spec}(A)^\star$ is closed in L_d^A. Assume that $f \in L_d^A$ and that $f \notin \mathrm{Spec}(A)^\star$.

By $f \notin \mathrm{Spec}(A)^\star$, there are $n \in \omega$ and an n-ary operation t of A such that, for some $a_1, ..., a_n \in A$,

$$f(t(a_1, ..., a_n)) \neq t(f(a_1), ..., f(a_n)).$$

Let O be the set of those $g \in L^A$ such that $g(a_i) = f(a_i)$ for any $i \in \{1, ..., n\}$ and that $g(t(a_1, ..., a_n)) = f(t(a_1, ..., a_n))$. Then, O is an open subset of L^A_d. We also have $f \in O$ and $O \cap \mathrm{Spec}(A)^\star = \emptyset$, since any element of O is not a homomorphism. Hence, $\mathrm{Spec}(A)^\star$ is closed in L^A_d, whence we obtain (i).

Next we show (ii). Let $a \in A$. By the definition of $\mathrm{Spec}(A)^\star$, $\langle a \rangle$ is a closed subset of $\mathrm{Spec}(A)^\star$, whence $\langle a \rangle$ is a compact subset of $\mathrm{Spec}(A)^\star$. $\square$

Proposition 3.7.12. *Let $A \in \mathbb{ISP}(L)$. Then, (i)* $\mathrm{Spec}(A)$ *is compact; (ii) $\langle a \rangle$ is a compact subset of* $\mathrm{Spec}(A)$ *for any $a \in A$.*

Proof. By Lemma 3.7.11, $\mathrm{Spec}(A)^\star$ is compact. Thus, since the topology of $\mathrm{Spec}(A)$ is weaker than or equal to that of $\mathrm{Spec}(A)^\star$, $\mathrm{Spec}(A)$ is also compact. It is verified in a similar way that $\langle a \rangle$ is a compact subset of $\mathrm{Spec}(A)$. $\square$

Lemma 3.7.13. *Let $A \in \mathbb{ISP}(L)$. Then, L^A is a sober space.*

Proof. By Lemma 3.5.5, it suffices to prove that L is a sober space. Let X be an irreducible closed subset of L. Let $\{x_1, ..., x_m\}$ be the set of those $x \in X$ such that x is maximal with respect to the restriction of the partial order $\leq$ on L to X. Since L is finite, $m \in \omega$. Since a closed subset of L is downward closed with respect to $\leq$, we have

$$X = \bigcup_{i=1}^{m} \{x \in L \,;\, x \leq x_i\}.$$

Since X is irreducible and since $\{x \in L \,;\, x \leq x_i\}$ is closed in L for any $i \in \{1, ..., m\}$, we have $X = \{x \in L \,;\, x \leq x_j\}$ for some $j \in \{1, ..., m\}$. Thus, X is equal to the closure of $\{x_j\}$.

To show the uniqueness of such x_j, assume that X is equal to the closure of $\{y\}$ for $y \in L$. Then, $X = \{x \in L \,;\, x \leq y\}$. Since $X = \{x \in L \,;\, x \leq x_j\}$, we have $x_j \leq y$ and $y \leq x_j$. Hence, $x_j = y$. $\square$

Recall the definition of patch topology (Definition 3.5.7).

Lemma 3.7.14. *Let $A \in \mathbb{ISP}(L)$. Then,* $\mathrm{Spec}(A)^\star$ *is equal to* $\mathrm{Spec}(A)^*$.

Proof. In order to show that the topology of $\mathrm{Spec}(A)^\star$ is equal to that of $\mathrm{Spec}(A)^*$, it is enough to show that $\{\langle a\rangle\ ;\ a \in A\}$ coincides with the set of all compact open subsets of $\mathrm{Spec}(A)$. By (ii) of Proposition 3.7.12, $\langle a\rangle$ is a compact open subset of $\mathrm{Spec}(A)$. If O is a compact open subset of $\mathrm{Spec}(A)$, then we have $O = \bigcup_{i\in I}\langle a_i\rangle$ for some a_i's by Lemma 3.7.6 and, by compactness, there is a finite subset J of I such that $O = \bigcup_{j\in J}\langle a_j\rangle$. We may assume $J = \{1, ..., n\}$ for some $n \in \omega$. Then, by Lemma 3.7.2, we have $O = \langle t^n_\vee(a_1, ..., a_n)\rangle$. This completes the proof. □

Proposition 3.7.15. *Let* $A \in \mathbb{ISP}(L)$*. Then,* $\mathrm{Spec}(A)$ *is a sober space.*

Proof. It is known that, for a sober space S and a subspace X of S, if X^* is a closed subspace of S^*, then X is sober (see [214, 1.1 and 1.5]). Thus, by Lemma 3.7.13, it suffices to show that $\mathrm{Spec}(A)^*$ is a closed subspace of $(L^A)^*$. By Lemma 3.7.9, $\mathrm{Spec}(A)$ is a subspace of L^A. As is shown in the proof of Lemma 3.7.11, $\mathrm{Spec}(A)^\star$ is a closed subspace of L^A_d. It is verified in a similar way to the proof of Lemma 3.7.10 that the topology of L^A_d is equal to the topology of $(L^A)^*$ (i.e., the patch topology of L^A). Hence, it follows from Lemma 3.7.14 that $\mathrm{Spec}(A)^*$ is a closed subspace of $(L^A)^*$. □

By the above facts, we obtain the following proposition.

Proposition 3.7.16. *Let* $A \in \mathbb{ISP}(L)$*. Then,* $\mathrm{Spec}(A)$ *is a coherent space.*

Proof. By Proposition 3.7.15 and Proposition 3.7.12, $\mathrm{Spec}(A)$ is a compact sober space. As is shown in the proof of Lemma 3.7.14, $\{\langle a\rangle\ ;\ a \in A\}$ coincides with the set of all compact open subsets of $\mathrm{Spec}(A)$. By Proposition 3.7.6, $\{\langle a\rangle\ ;\ a \in A\}$ forms an open basis of $\mathrm{Spec}(A)$. □

Categories and Functors

In this subsection, we define categories $\mathrm{ISP}(L)$ and CohSp, and functors Spec and ContProp between those categories.

Definition 3.7.17. $\mathrm{ISP}(L)$ *denotes the category of algebras in* $\mathbb{ISP}(L)$ *and homomorphisms.*

Definition 3.7.18. CohSp *denotes the category of coherent spaces and continuous proper maps.*

Definition 3.7.19. *We define a contravariant functor* Spec : ISP(L) $\to$ CohSp *as follows. For an object A in* ISP(L)*,* Spec(A) *has already been defined in Definition B.4.9. For an arrow $f : A \to B$ in* ISP(L)*,* $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is defined by* $\mathrm{Spec}(f)(v) = v \circ f$ *for* $v \in \mathrm{Spec}(B)$*.*

The object part of the functor Spec is well-defined by Proposition 3.7.16. The arrow part of Spec is well-defined by the following lemma.

Lemma 3.7.20. *Let $f : A \to B$ be a homomorphism for $A, B \in \mathbb{ISP}(L)$. Then,* Spec(f) *is a continuous proper map.*

Proof. Let $a \in A$. Then we have

$$\mathrm{Spec}(f)^{-1}(\langle a \rangle) = \{v \in \mathrm{Spec}(B)\ ;\ v \circ f(a) = 1\} = \langle f(a) \rangle.$$

Since any compact open subset of Spec(A) is equal to $\langle a \rangle$ for some $a \in A$ (see the proof of Lemma 3.7.14), this lemma follows from Lemma 3.7.12. $\square$

Definition 3.7.21. *We define a contravariant functor* ContProp : CohSp $\to$ ISP(L) *as follows. For an object S in* CohSp*, define* ContProp(S) *as the set of all continuous proper maps from S to L endowed with the pointwise operations defined as follows: For each n-ary operation t of L and $f_1, ..., f_n \in \mathrm{ContProp}(S)$, define $t(f_1, ..., f_n) : S \to L$ by*

$$(t(f_1, ..., f_n))(x) = t(f_1(x), ..., f_n(x)).$$

For an arrow $f : S \to S'$ in CohSp*, define* $\mathrm{ContProp}(f) : \mathrm{ContProp}(S') \to \mathrm{ContProp}(S)$ *by* $\mathrm{ContProp}(f)(g) = g \circ f$ *for* $g \in \mathrm{ContProp}(S')$*.*

The functor ContProp is well-defined by the following two lemmas.

Lemma 3.7.22. *Let S be a coherent space. Then,* ContProp(S) *is in* $\mathbb{ISP}(L)$*.*

Proof. Since $\mathrm{ContProp}(S) \subset L^S$ and since the operations of ContProp(S) are defined pointwise, it suffices to show that ContProp(S) is closed under the operations of ContProp(S). Let t be an n-ary operation of L. Since L is topologically dualizable with respect to the Alexandrov topology and since L^n is a finite topological space, $t : L^n \to L$ is a continuous proper map. Let $f_1, ..., f_n \in \mathrm{ContProp}(S)$ and $r \in L$. Then we have the following:

$$\begin{aligned}
(t(f_1, ..., f_n))^{-1}(\uparrow r) &= \{x \in S\ ;\ t(f_1(x), ..., f_n(x)) \geq r\} \\
&= \{x \in S\ ;\ (f_1(x), ..., f_n(x)) \in t^{-1}(\uparrow r)\} \\
&= \{x \in S\ ;\ x \in (f_1, ..., f_n)^{-1} \circ t^{-1}(\uparrow r)\} \\
&= (f_1, ..., f_n)^{-1} \circ t^{-1}(\uparrow r),
\end{aligned}$$

where $(f_1, ..., f_n) : S \to L^n$ is defined by $(f_1, ..., f_n)(x) = (f_1(x), ..., f_n(x))$ for $x \in S$. Since $f_1, ..., f_n$ are continuous proper and since S is coherent, it is straightforward to verify that $(f_1, ..., f_n) : S \to L^n$ is continuous proper. Since t is also continuous proper and since $\uparrow r$ is compact open, $(t(f_1, ..., t_n))^{-1}(\uparrow r)$ is compact open. Since the topology of L is generated by $\{\uparrow r\ ;\ r \in L\}$ and since S is coherent, this completes the proof. $\square$

Lemma 3.7.23. *Let $f : S \to S'$ be a continuous proper map between coherent spaces S and S'. Then,* $\mathrm{ContProp}(f)$ *is a homomorphism.*

Proof. This lemma follows immediately from the fact that the operations of $\mathrm{ContProp}(S')$ are defined pointwise. $\square$

Stone-type Duality for $\mathbb{ISP}(L)$

In this subsection, we show a Stone-type duality theorem for $\mathbb{ISP}(L)$.

Theorem 3.7.24. *Let $A \in \mathbb{ISP}(L)$. Then, there is an isomorphism from A to* $\mathrm{ContProp} \circ \mathrm{Spec}(A)$.

Proof. Define $\Phi : A \to \mathrm{ContProp} \circ \mathrm{Spec}(A)$ by $\Phi(a)(v) = v(a)$ for $a \in A$ and $v \in \mathrm{Spec}(A)$. Let $r \in L$. By Lemma 3.7.3, we have

$$\Phi(a)^{-1}(\uparrow r) = \{v \in \mathrm{Spec}(A)\ ;\ v(a) \geq r\} = \langle \tau_r(a) \rangle.$$

Thus, by Lemma 3.7.12, $\Phi(a) : \mathrm{Spec}(A) \to L$ is continuous proper and so Φ is well-defined.

Let t be an n-ary operation of A for $n \in \omega$. For $a_1, ..., a_n \in A$ and $v \in \mathrm{Spec}(A)$, we have

$$\begin{aligned} \Phi(t(a_1, ..., a_n))(v) &= v(t(a_1, ..., a_n)) \\ &= t(v(a_1), ..., v(a_n)) \\ &= t(\Phi(a_1)(v), ..., \Phi(a_n)(v)) \\ &= (t(\Phi(a_1), ..., \Phi(a_n)))(v). \end{aligned}$$

Therefore, Φ is a homomorphism.

We show that Φ is injective. Let $a, b \in A$ with $a \neq b$. By $A \in \mathbb{ISP}(L)$, A is isomorphic to a subalgebra A' of L^I for some I. Thus, we may identify A with A'. Then, a and b are functions from I to L. By $a \neq b$, there is $i \in I$ such that $a(i) \neq b(i)$. Define $p_i : A \to L$ by $p_i(x) = x(i)$ for $x \in A$. Note that $p_i(a) \neq p_i(b)$.

Then, since the operations of L^I are defined pointwise, p_i is a homomorphism, which means $p_i \in \mathrm{Spec}(A)$. Moreover, we have $\Phi(a)(p_i) \neq \Phi(b)(p_i)$. Thus, Φ is injective.

Finally, we show that Φ is surjective. Let $f \in \mathrm{ContProp} \circ \mathrm{Spec}(A)$. Let $r \in L$. By Lemma 3.7.6 and the continuity of f, there are an index set K and $a_r^k \in A$ for $k \in K$ such that $f^{-1}(\uparrow r) = \bigcup_{k \in K} \langle a_r^k \rangle$. By Lemma 3.7.2 and the properness of f, there is $a_r \in A$ such that $f^{-1}(\uparrow r) = \langle a_r \rangle$. Then, we claim that

$$\Phi(\bigvee\{\theta_r(a_r) \; ; \; r \in L\}) = f,$$

where θ_r is defined in Lemma 3.7.4. In order to show this, suppose that $v \in f^{-1}(\{s\})$ for $s \in L$. Then, we have: For each $r \in L$,

$$v(\theta_r(a_r)) = \begin{cases} r & \text{if } r \leq s \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have $\Phi(\bigvee\{\theta_r(a_r) \; ; \; r \in L\})(v) = s = f(v)$. Hence the above claim holds. □

Lemma 3.7.25. *Let S be a coherent space. Assume that P_i is a compact open subset or a closed subset of S for any $i \in I$. Then, if $\{P_i \; ; \; i \in I\}$ has finite intersection property, then $\bigcap\{P_i \; ; \; i \in I\}$ is not empty.*

Proof. We use the patch topology of S. It follows from Lemma 3.5.8 that S^* is compact. By assumption, P_i is a closed subset of S^* for any $i \in I$. Thus, if $\{P_i ; i \in I\}$ has finite intersection property, then $\bigcap\{P_i \; ; \; i \in I\}$ is not empty by the compactness of S^*. □

It is straightforward to verify the following.

Lemma 3.7.26. *Let S be a coherent space and O a compact open subset of S. Define the indicator function $\mu_O : S \to L$ of O by $\mu_O(x) = 1$ for $x \in O$ and $\mu_O(x) = 0$ for $x \in S \setminus O$. Then, $\mu_O \in \mathrm{ContProp}(S)$.*

Lemma 3.7.27. *Let S be a coherent space and $v \in \mathrm{Spec} \circ \mathrm{ContProp}(S)$. Let $G = \{ f^{-1}(\{1\}) \; ; \; v(f) = 1 \}$ and $H = \{ S \setminus f^{-1}(\{1\}) \; ; \; v(f) \neq 1 \}$. Then, $G \cup H$ has finite intersection property.*

Proof. By Lemma 3.7.1, we have

$$f^{-1}(\{1\}) \cap g^{-1}(\{1\}) = (t_\wedge(f, g))^{-1}(\{1\}).$$

By Lemma 3.7.2, we also have

$$(S \setminus f^{-1}(\{1\})) \cap (S \setminus g^{-1}(\{1\})) = S \setminus (t^2_\vee(f,g))^{-1}(\{1\}).$$

Therefore, it is sufficient to show that, if $v(f) = 1$ and $v(g) \neq 1$, then $f^{-1}(\{1\}) \cap (S \setminus g^{-1}(\{1\}))$ is not empty. Suppose for contradiction that $v(f) = 1$, $v(g) \neq 1$ and $f^{-1}(\{1\}) \cap (S \setminus g^{-1}(\{1\})) = \emptyset$. Then, $f^{-1}(\{1\}) \subset g^{-1}(\{1\})$. By Lemma 3.7.3, $\tau_1(f) \leq \tau_1(g)$. By $v(\tau_1(f)) = \tau_1(v(f)) = 1$, we have $v(\tau_1(g)) = 1$ and so $\tau_1(v(g)) = 1$, which contradicts $v(g) \neq 1$. □

Theorem 3.7.28. *Let S be a coherent space. Then, there is a homeomorphism from S to* $\mathrm{Spec} \circ \mathrm{ContProp}(S)$.

Proof. Define $\Psi : S \to \mathrm{Spec} \circ \mathrm{ContProp}(S)$ by $\Psi(x)(f) = f(x)$ for $x \in S$ and $f \in \mathrm{ContProp}(S)$. Since the operations of $\mathrm{ContProp}(S)$ are defined pointwise, $\Psi(x)$ is a homomorphism and so Ψ is well-defined. We claim that Ψ is a homeomorphism. First, Ψ is continuous and proper, since we have the following for $f \in \mathrm{ContProp}(S)$:

$$\Psi^{-1}(\langle f\rangle) = \{x \in S\, ;\, \Psi(x) \in \langle f\rangle\} = f^{-1}(\{1\})$$

and since a compact open subset of $\mathrm{Spec} \circ \mathrm{ContProp}(S)$ is of the form $\langle f\rangle$ for some $f \in \mathrm{ContProp}(S)$ by Lemma 3.7.6 and Lemma 3.7.2.

Second, we show that Ψ is injective. Assume that $x, y \in S$ with $x \neq y$. Since S is a coherent space, S is T_0 and has an open basis consisting of compact open subsets of S. Thus, we may assume that there is a compact open subset O of S such that $x \in O$ and $y \notin O$. By Lemma 3.7.26, we have $\mu_O \in \mathrm{ContProp}(S)$ and $\Psi(x)(\mu_O) = 1 \neq 0 = \Psi(y)(\mu_O)$. Hence, we have $\Psi(x) \neq \Psi(y)$. Thus, Ψ is injective.

Third, we show that Ψ is surjective. Let $v \in \mathrm{Spec} \circ \mathrm{ContProp}(S)$. Let $G = \{\, f^{-1}(\{1\})\, ;\, v(f) = 1\,\}$ and $H = \{\, S \setminus f^{-1}(\{1\})\, ;\, v(f) \neq 1\,\}$. Since f is continuous proper, $f^{-1}(\{1\})$ is compact open and $S \setminus f^{-1}(\{1\})$ is closed. By Lemma 3.7.27, $G \cup H$ enjoys finite intersection property. Therefore, by Lemma C.4.7, there is $y \in S$ such that

$$y \in \bigcap(G \cup H) = (\bigcap G) \cap (\bigcap H).$$

Since $y \in \bigcap G$, if $v(f) = 1$ then $\Psi(y)(f) = f(y) = 1$. Since $y \in \bigcap H$, if $\Psi(y)(f) = f(y) = 1$ then $v(f) = 1$. Thus $v^{-1}(\{1\}) = \Psi(y)^{-1}(\{1\})$. By Lemma 3.7.7, we have $v = \Psi(y)$. Hence, Ψ is surjective.

Fourth, we show that Ψ is an open map. Let O be an open subset of S. Since S is coherent, $O = \bigcup_{i \in I} O_i$ for some compact open subsets O_i of S. By Lemma 3.7.26, $\mu_{O_i} \in \mathrm{ContProp}(S)$. We claim that

$$\Psi[O] = \bigcup \{ \langle \mu_{O_i} \rangle \; ; \; i \in I \}.$$

If $x \in O$, then $x \in O_i$ for some i, whence $\Psi(x) \in \langle \mu_{O_i} \rangle$. To show the converse, suppose $v \in \langle \mu_{O_i} \rangle$. Then $v = \Psi(y)$ for some $y \in S$, since Ψ is surjective. Since $\Psi(y) \in \langle \mu_{O_i} \rangle$, we have $\mu_{O_i}(y) = 1$. Thus, $y \in O_i$ by the definition of μ_{O_i}. Hence the claim holds.

Finally, since a continuous function maps a compact set to a compact set, Ψ^{-1} is a proper map. This completes the proof. $\square$

By the above results, we obtain the following duality theorem for $\mathbb{ISP}(L)$. Note that L is assumed to be topologically dualizable with respect to the Alexandrov topology.

Theorem 3.7.29. *The category* $\mathrm{ISP}(L)$ *is dually equivalent to the category* CohSp *via the functors* Spec *and* ContProp*.*

Proof. Let $\mathrm{Id}_{\mathrm{alg}}$ denote the identity functor on $\mathrm{ISP}(L)$ and $\mathrm{Id}_{\mathrm{sp}}$ denote the identity functor on CohSp.

Define a natural transformation $\epsilon : \mathrm{Id}_{\mathrm{alg}} \to \mathrm{ContProp} \circ \mathrm{Spec}$ by $\epsilon_A = \Phi$ for $\mathbb{ISP}(L)$, where Φ is defined in the proof of Theorem 3.7.24. It is verified by straightforward computation that ϵ is actually a natural transformation. By Theorem 3.7.24, ϵ is a natural isomorphism.

Define a natural transformation $\eta : \mathrm{Id}_{\mathrm{sp}} \to \mathrm{Spec} \circ \mathrm{ContProp}$ by $\eta_S = \Psi$ for a coherent space S, where Ψ is defined in the proof of Theorem 3.7.28. It is verified by straightforward computation that η is actually a natural transformation. By Theorem 3.7.28, η is a natural isomorphism. $\square$

Let $\mathbf{2}_d$ denote the two-element distributive lattice. Since $\mathbb{ISP}(\mathbf{2}_d)$ coincides with the class of distributive lattices and since $\mathbf{2}_d$ is topologically dualizable with respect to the Alexandrov topology, Theorem 5.2.13 is a universal algebraic generalisation of Stone Duality for distributive lattices.

In the following sections, we consider applications of Theorem 5.2.13.

Applications to Heyting-type Algebras

In this section, we consider a generalisation of Stone Duality for Heyting algebras from the viewpoint of universal algebra, which follows directly from the duality for $\mathbb{ISP}(L)$ (Theorem 5.2.13).

The notion of residuation (or relative pseudo-complement) plays an essential role in the definition of Heyting algebras. Thus, inspired by residuated lattices (see [104, 220]), we introduce the notion of being $*$-residuated for a binary operation $*$ as follows.

Definition 3.7.30. *Let A be an ordered algebra with a binary operation $*$. Then, A is $*$-residuated iff, for all $x, y \in A$, the set of $z \in A$ such that $x * z \leq y$ has a greatest element, which is denoted by $x \to y$.*

In this section, we additionally assume:

- L has a binary operation $*$;
- L is $*$-residuated.

We do not consider $\to$ as an operation of L and so a subalgebra of a direct power of L is not necessarily $*$-residuated, as a distribtuive lattice is not necessarily $\wedge$-residuated.

Definition 3.7.31. *$\mathbb{IRSP}(L)$ is defined as the class of all isomorphic copies of $*$-residuated subalgebras of direct powers of L. We call $\mathbb{IRSP}(L)$ the $*$-residuated quasi-variety generated by L.*

In this section, we show that a duality for $\mathbb{IRSP}(L)$ follows directly from the duality for $\mathbb{ISP}(L)$ (Theorem 5.2.13). This is considered as a universal algebraic extension of Stone Duality for Heyting algebras.

By letting $* = \wedge$, $\mathbb{IRSP}(\mathbf{2}_d)$ coincides with the class of Heyting algebras, since a Heyting algebra is a $\wedge$-residuated distributive lattice and since $\mathbb{ISP}(\mathbf{2}_d)$ coincides with the class of distributive lattices.

Stone-type Duality for $\mathbb{IRSP}(L)$

A homomorphism between algebras in $\mathbb{IRSP}(L)$ is defined as a function which preserves the operations of L, which may not contain the operation $\to$. As a lattice-homomorphism between Heyting algebras does not necessarily preserve the operation $\to$, a homomorphism between algebras in $\mathbb{IRSP}(L)$ does not necessarily preserve the operation $\to$.

Definition 3.7.32. $\mathrm{IRSP}(L)$ *denotes the category of algebras in* $\mathbb{IRSP}(L)$ *and homomorphisms.*

Note that $\mathrm{IRSP}(L)$ is a subcategory of $\mathrm{ISP}(L)$.

Definition 3.7.33. *A topological space S is an L-Heyting space iff S is coherent and* $\mathrm{ContProp}(S)$ *is $*$-residuated.*

Definition 3.7.34. HeytSp_L *denotes the category of L-Heyting spaces and continuous proper maps.*

Theorem 3.7.35. *Let $A \in \mathbb{IRSP}(L)$. Then, (i)* $\mathrm{ContProp} \circ \mathrm{Spec}(A)$ *is $*$-residuated; (ii) there is an isomorphism between A and* $\mathrm{ContProp} \circ \mathrm{Spec}(A)$ *which additionally preserves the operation $\rightarrow$.*

Proof. By Theorem 3.7.24, A is isomorphic to $\mathrm{ContProp} \circ \mathrm{Spec}(A)$ in the category $\mathrm{ISP}(L)$. Let $\Phi : A \rightarrow \mathrm{ContProp} \circ \mathrm{Spec}(A)$ be an isomorphism in the category $\mathrm{ISP}(L)$.

We first show (i). Let $f, g \in \mathrm{ContProp} \circ \mathrm{Spec}(A)$. Then, there are $a, b \in A$ such that $\Phi(a) = f$ and $\Phi(b) = g$. Since A is $*$-residuated, $a * (a \rightarrow b) \leq b$. Since Φ preserves $*$ and $\vee$, we have $\Phi(a) * \Phi(a \rightarrow b) \leq \Phi(b)$, i.e., $f * \Phi(a \rightarrow b) \leq g$. If $f * h \leq g$ for $h \in \mathrm{ContProp} \circ \mathrm{Spec}(A)$, then $\Phi(a) * h \leq \Phi(b)$ and so $a * \Phi^{-1}(h) \leq b$. Thus, $\Phi^{-1}(h) \leq a \rightarrow b$, whence $h \leq \Phi(a \rightarrow b)$. Hence, we have $f \rightarrow g = \Phi(a \rightarrow b) \in \mathrm{ContProp} \circ \mathrm{Spec}(A)$.

Finally we show (ii). Let $a, b \in A$. By letting $f = \Phi(a)$ and $g = \Phi(b)$ in the argument of the above paragraph, we have $\Phi(a) \rightarrow \Phi(b) = \Phi(a \rightarrow b)$. □

By Theorem 3.7.28, we obtain the following.

Theorem 3.7.36. *Let S be an L-Heyting space. Then, S is homeomorphic to* $\mathrm{Spec} \circ \mathrm{ContProp}(S)$.

Then we obtain the following duality theorem for $\mathrm{IRSP}(L)$ (i.e., the $*$-residuated quasi-variety generated by L). Note that L is assumed to be topologically dualizable with respect to the Alexandrov topology and be residuated with respect to $*$.

Theorem 3.7.37. *The category* $\mathrm{IRSP}(L)$ *is dually equivalent to the category* HeytSp_L *via the restrictions of the functors* Spec *and* ContProp.

Proof. We first need to check that the restrictions of Spec and ContProp are well-defined: For $A \in \mathbb{IRSP}(L)$, $\mathrm{Spec}(A)$ is an L-Heyting space by (i) of Theorem 3.7.35.

For an L-Heyting space S, $\mathrm{ContProp}(S)$ is in $\mathbb{IRSP}(L)$ by the definition of L-Heyting spaces.

Define two natural transformations as in the proof of Theorem 5.2.13 and then they are natural isomorphisms by Theorem 3.7.35 and Theorem 3.7.36. $\square$

By letting $* = \wedge$ and $L = \mathbf{2}_d$, we can recover Stone Duality for Heyting algebras from Theorem 3.7.37.

Applications to Many-Valued Algebras

In this section, we consider applications of the general duality theorems (Theorem 5.2.13 and Theorem 3.7.37) to many-valued logics and algebras (for basics of many-valued logics and algebras, see [120, 58, 42]).

Let $n \in \omega$ with $n > 1$ in this section.

Definition 3.7.38. *Let* $\mathbf{n}$ *denote* $\{0, 1/(n-1), 2/(n-1), ..., 1\}$. *We equip* $\mathbf{n}$ *with all constants* $(0, 1/(n-1), 2/(n-1), ..., 1)$ *and with the operations* $(\wedge, \vee, *, \wp)$ *defined as follows:*

$$\begin{aligned} x \wedge y &= \min(x, y) \\ x \vee y &= \max(x, y) \\ x * y &= \max(0, x + y - 1) \\ x \wp y &= \min(1, x + y). \end{aligned}$$

Note that $*$ and $\wp$ are defined as in Łukasiewicz n-valued logic. In the above definition, $\mathbf{n}$ is not equipped with $\neg$ or $\rightarrow$, which is because our aim here is to consider an n-valued version of distributive lattice.

The class of distributive lattices coincides with $\mathbb{ISP}(\mathbf{2})$, i.e., a distributive lattice can be defined as an isomorphic copy of a subalgebra of a powerset algebra $\mathbf{2}^X$ for a set X. Thus, it is natural to define an n-valued distributive lattice as an algebra in $\mathbb{ISP}(\mathbf{n})$, i.e., an n-valued distributive lattice is defined as an isomorphic copy of a subalgebra of an n-valued powerset algebra $\mathbf{n}^X$ for a set X:

Definition 3.7.39. *An n-valued distributive lattice is an algebra in* $\mathbb{ISP}(\mathbf{n})$.

A homomorphism of n-valued distributive lattices is a function which preserves the constants $r \in \mathbf{n}$ and the operations $(\wedge, \vee, *, \wp)$.

DLat_n *denotes the category of n-valued distributive lattices and homomorphisms of n-valued distributive lattices.*

Note that 2-valued distributive lattices coincide with distributive lattices.

By using Theorem 5.2.13, we can develop a Stone-type duality for n-valued distributive lattices as follows.

Lemma 3.7.40. *Let $r \in \mathbf{n}$. Define $\tau_r : \mathbf{n} \to \mathbf{n}$ by letting $L = \mathbf{n}$ in Lemma 3.7.3. Then, τ_r is a term function of $\mathbf{n}$.*

Proof. See [222, Section 1] (and also [265, Definition 3.7]). □

We equip $\mathbf{n}$ with the Alexandrov topology.

Proposition 3.7.41. *In fact, $\mathbf{n}$ is topologically dualizable with respect to the Alexandrov topology, i.e., $\mathrm{Cont}_m(\mathbf{n}) = \mathrm{TermFunc}_m(\mathbf{n})$ for any $m \in \omega$.*

Proof. We first show that $\mathrm{Cont}_m(\mathbf{n}) \supset \mathrm{TermFunc}_m(\mathbf{n})$ for any $m \in \omega$, i.e., any term function of $\mathbf{n}$ is continuous. Since a composition of continuous functions is also continuous, it suffices to show that the constants $r \in \mathbf{n}$ and the operations $(\wedge, \vee, *, \wp)$ are continuous. Since a function on a singleton space is always continuous, the constants $r \in \mathbf{n}$ are continuous. We show that $* : \mathbf{n}^2 \to \mathbf{n}$ is continuous. This follows from the following fact:

$$*^{-1}\left(\uparrow \frac{k}{n-1}\right) = \bigcup_{i=0}^{n-1} \left(\left(\uparrow \frac{i}{n-1}\right) \times \left(\uparrow \frac{k-i+n-1}{n-1}\right) \right),$$

where we define $(\uparrow r) = \emptyset$ for $r > 1$. It is verified in similar ways that $(\wedge, \vee, \wp)$ are continuous.

Next we show that $\mathrm{Cont}_m(\mathbf{n}) \subset \mathrm{TermFunc}_m(\mathbf{n})$ for any $m \in \omega$. Let $f \in \mathrm{Cont}_m(\mathbf{n})$ for $m \in \omega$. For $i = 1, ..., m$, let $p_i : \mathbf{n}^m \to \mathbf{n}$ be the i-th projection function from $\mathbf{n}^m$ to $\mathbf{n}$. For $r \in \mathbf{n}$ and $i = 1, ..., m$, define $s_{i,r} \in \mathbf{n}$ as the least element of $p_i(f^{-1}(\uparrow r))$. Then we claim that

$$f(x_1, ..., x_m) = \bigvee_{r \in \mathbf{n}} \left(r \wedge \bigwedge_{i=1}^{m} \tau_{s_{i,r}}(x_i) \right)$$

for any $(x_1, ..., x_m) \in \mathbf{n}^m$. To show this, suppose that $f(x_1, ..., x_m) = p$ for $p \in \mathbf{n}$. Since $f^{-1}(\uparrow r)$ is an open subset of $\mathbf{n}^m$, $p_i(f^{-1}(\uparrow r))$ is an open subset of $\mathbf{n}$ and so is upward closed. Thus, by Lemma 3.7.40, we have the following: For each $r \in \mathbf{n}$,

$$\bigwedge_{i=1}^{m} \tau_{s_{i,r}}(x_i) = \begin{cases} 1 & \text{if } (x_1, ..., x_m) \in f^{-1}(\uparrow r) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, since $(x_1, ..., x_m) \in f^{-1}(\uparrow q)$ for any $q \in \mathbf{n}$ with $q \leq p$ and since $(x_1, ..., x_m) \notin f^{-1}(\uparrow q)$ for any $q \in \mathbf{n}$ with $q > p$, we have

$$\bigvee_{r \in \mathbf{n}} \left(r \wedge \bigwedge_{i=1}^{m} \tau_{s_{i,r}}(x_i) \right) = p.$$

Hence the above claim holds. Therefore, it follows from Lemma 3.7.40 that f is a term function of $\mathbf{n}$. $\square$

By the above proposition and applying Theorem 5.2.13, we obtain the following Stone-type duality for n-valued distributive lattices.

Proposition 3.7.42. DLat_n *is dually equivalent to* CohSp.

In the following, we consider an n-valued version of Heyting algebra. Since $\mathbb{IRSP}(\mathbf{2})$ coincides with the class of Heyting algebras, it is natural to define n-valued Heyting algebras as follows.

Definition 3.7.43. *An n-valued Heyting algebra is an algebra in $\mathbb{IRSP}(\mathbf{n})$.*

An implication-free reduct homomorphism is a function which preserves the constants $r \in \mathbf{n}$ and the operations $(\wedge, \vee, *, \wp)$.

Let $\mathrm{HeytAlg}_n$ *denote the category of n-valued Heyting algebras and implication-free reduct homomorphisms.*

Note that 2-valued Heyting algebras coincide with Heyting algebras.

Definition 3.7.44. *For $r, s \in \mathbf{n}$, define $r \to_c s = \min(1, 1 - r + s)$.*

As classical propositional logic can be considered as the free Boolean algebra generated by propositional variables, we can define an intuitionistic Łukasiewicz n-valued logic $\mathrm{IŁL}_n^c$ with truth constants as the free n-valued Heyting algebra generated by propositional variables. Then, the multiplicative excluded middle $\varphi \wp (\varphi \to 0)$ is not valid in $\mathrm{IŁL}_n^c$, though it is valid in Łukasiewicz n-valued logic.

Moreover, $\mathrm{IŁL}_n^c$ plus the multiplicative excluded middle coincides with Łukasiewicz n-valued logic $\mathrm{ŁL}_n^c$ with truth constants $r \in \mathbf{n}$, which is shown by the following proposition. Note that the class of all algebras of $\mathrm{ŁL}_n^c$ coincides with $\mathbb{ISP}(\mathbf{n}_\to)$, where $\mathbf{n}_\to = (\mathbf{n}, \wedge, \vee, *, \wp, \to_c, 0, 1/(n-1), ..., 1)$.

Proposition 3.7.45. *The class of those n-valued Heyting algebras A such that $a \wp (a \to 0) = 1$ for any $a \in A$ coincides with the class of all algebras of* $\mathrm{ŁL}_n^c$.

Proof. Let A be an n-valued Heyting algebra such that $a \wp (a \to 0) = 1$ for any $a \in A$. We may consider A as a subset of $\mathbf{n}^I$ for an index set I. For $a, b \in A$, define $a \to_c b : I \to \mathbf{n}$ by $(a \to_c b)(i) = a(i) \to_c b(i)$ for $i \in I$. Since $a \wp (a \to 0) = 1$ for any $a \in A$ and since $a \to b \leq a \to_c b$ for any $a, b \in A$, we have $a \to_c 0 \in A$ for any $a \in A$, whence we have $a \to_c b = (a \to_c 0) \wp b \in A$ for any $a, b \in A$. It is straightforward to see that, for any $c \in \mathbf{n}^I$, $a * c \leq b$ iff $c \leq (a \to_c b)$. Thus, by $a \to_c b \in A$, we have $a \to_c b = a \to b$. Hence, $A \in \mathbb{ISP}(\mathbf{n}_{\to})$, i.e., A is an algebra of ŁL_n^c.

If A is an algebra of ŁL_n^c, then $A \in \mathbb{ISP}(\mathbf{n}_{\to})$ and so $A \in \mathbb{IRSP}(\mathbf{n})$. □

By applying Theorem 3.7.37, we obtain the following Stone-type duality for n-valued Heyting algebras.

Proposition 3.7.46. $\mathrm{HeytAlg}_n$ *is dually equivalent to* $\mathrm{HeytSp}_{\mathbf{n}}$.

Proof. By Proposition 3.7.41, $\mathbf{n}$ is topologically dualizable with respect to the Alexandrov topology. We also have the following fact: For $r, s, x \in \mathbf{n}$, $r*x \leq s$ iff $x \leq r \to_c s$. Thus, $\mathbf{n}$ is $*$-residuated. Hence, by Theorem 3.7.37, $\mathrm{HeytAlg}_n$ is dually equivalent to $\mathrm{HeytSp}_{\mathbf{n}}$. □

We can show the above duality also by using results in [203], where [203] considers an intuitionistic Łukasiewicz n-valued logic IŁL_n *without truth constants*. It also follows from results in [203] that IŁL_n^c is complete with respect to n-valued Kripke semantics and that a Glivenko-type theorem holds between IŁL_n^c and ŁL_n^c. Thus, it seems that n-valued Heyting algebra is actually a natural concept.

3.8 Mathematical and Philosophical Remarks

Before closing the present chapter, we briefly address several related issues, and then add some philosophical remarks.

Firstly, it would be fruitful to generalise the notion of $\mathbb{ISP}_{\mathbb{M}}$ from the viewpoint of coalgebraic logic, since a number of modal logics (e.g., monotone modal logic and graded modal logic) can be described in coalgebraic terms. This is expected to allow us to develop natural duality theory for coalgebraic modal logics. Now, how can we generalise $\mathbb{ISP}_{\mathbb{M}}$ to a coalgebraic-logical setting? Let us begin with an endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ and fix a function $\heartsuit : T(L^n) \to L$ where $n \in \omega$ and L is an algebra (possibly with some conditions on $T, \heartsuit, L$). Then, given a T-coalgebra $\delta : X \to T(X)$, we can define an n-ary modal operation on $\mathrm{Hom}_{\mathbf{Set}}(X, L)$ by

$$f \in \mathrm{Hom}_{\mathbf{Set}}(X, L)^n \mapsto \heartsuit \circ T(f) \circ \delta \in \mathrm{Hom}_{\mathbf{Set}}(X, L)$$

where f is considered as an element of $\mathrm{Hom}_{\mathbf{Set}}(X, L^n)$ via the isomorphism $\mathrm{Hom}_{\mathbf{Set}}(X, L)^n \simeq \mathrm{Hom}_{\mathbf{Set}}(X, L^n)$. We can recover $\Box_R : \mathrm{Hom}_{\mathbf{Set}}(X, L) \to \mathrm{Hom}_{\mathbf{Set}}(X, L)$ in Definition 3.2.1 by letting $T : \mathbf{Set} \to \mathbf{Set}$ be the power-set functor and $\heartsuit : T(L) \to L$ the meet operation of L. Thus, this yields an extended notion of modal power parametrized by T and $\heartsuit$, and hence a generalisation of $\mathbb{ISP}_{\mathbb{M}}(L)$ from the viewpoint of coalgebraic logic. In future work, we will attempt to develop natural duality theory for this coalgebraic generalisation of $\mathbb{ISP}_{\mathbb{M}}(L)$.

Another important direction of research would be to establish an intuitionistic analogue of the theory presented in this chapter, which involves a universal-algebraic generalisation of Esakia duality for Heyting algebras. First of all, the class of all Heyting algebras cannot be expressed as $\mathbb{ISP}(L)$ for any single algebra L. This is nothing but a duality-theoretic expression of the reason why Gödel failed to capture intuitionistic logic as a many-valued logic. Hence we have to consider a new way to generate a class of algebras. Given an intuitionistic frame $(X, \leq)$, we can define an implication operation $\to: \mathrm{Hom}_{\mathbf{Set}}(X, L)^2 \to \mathrm{Hom}_{\mathbf{Set}}(X, L)$ by

$$(f \to g)(x) = \bigwedge \{f(y) \to g(y) \ ; \ x \leq y\}.$$

In this way, we obtain the concept of an intuitionistic power of L and so the concept of $\mathbb{ISP}_{\mathbb{I}}(L)$ i.e., the class of isomorphic copies of subalgebras of intuitionistic powers of L. We can show that, for the two-element distributive lattice $\mathbf{2}$, $\mathbb{ISP}_{\mathbb{I}}(\mathbf{2})$ coincides with the class of all Heyting algebras and that, for $\mathbf{n}$ without $\to$ or $\neg$, $\mathbb{ISP}_{\mathbb{I}}(\mathbf{n})$ coincides with the class of all algebras of intuitionistic Łukasiewicz n-valued logic (which is naturally defined via n-valued Kripke semantics). In future work, we will attempt to develop natural duality theory for $\mathbb{ISP}_{\mathbb{I}}(L)$, in order to make it possible to incorporate Esakia duality for Heyting algebras into the theory of natural dualities. At the same time, however, we have to remark that there is a different perspective on intuitionistic logic, i.e., we can see it as distributive lattices with residuation or the right adjoints of meets. This point of view leads us to the notion of $\mathbb{IS}_{\mathbb{R}}\mathbb{P}$, and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathbf{2})$ coincides with the class of all Heyting algebras. Although we do not describe a precise definition here, $\mathbb{S}_{\mathbb{R}}(M)$ is the class of "residuated" subalgebras of a given ordered algebra M. Interestingly, it does not hold in general that $\mathbb{ISP}_{\mathbb{I}}(M) = \mathbb{IS}_{\mathbb{R}}\mathbb{P}(M)$. Hence, the two perspectives on intuitionistic logic (i.e., the former, Kripke-semantics-based one and the latter, residuation-based one) are really different in that sense.

While natural duality theory is based on universal algebra and general topology (possibly with relational structures), which are of set-theoretical character, we can also develop duality theory building upon category theory, especially categorical algebra and categorical topology (see [12]). Because universal algebra is well developed for finitary algebras (though not for infinitary ones), we consider that natural duality theory is suitable for "finitary Stone-type dualities", by which we mean Stone-type dualities concerning finitary operations and so compact spectrums. On the other hand, the theory of monads, which is categorical universal algebra, naturally encompass infinitary algebras such as frames (or locales) and continuous lattices (both are the Eilenberg-Moore algebras of certain monads). Accordingly, categorical duality theories (see, e.g., [191, 232]) seem suitable for "infinitary Stone-type dualities", a typical example of which is Isbell-Papert's dual adjunction between frames and topological spaces. Note that finitary Stone-type dualities often require a weaker form of the axiom of choice, whereas infinitary ones sometimes avoid such a non-deterministic principle, as is the case in Isbell-Papert duality or duality between point-free spaces and point-set spaces in general (see [189, 191]).

Categorical duality theories (including those cited above) are usually more general than natural duality theory, subsuming both finitary and infinitary ones. At the same time, however, they are less substantial than natural duality theory, especially in the sense that they often lack the "adequate" treatment of dual equivalences. Category theory can lead us to dual adjunctions in a significant way, but not to dual equivalences. Although there is a mechanical way to derive equivalences from adjunctions, it is quite trivial, and, at the moment, there appears to be no general, substantial way to do it categorically as [232, p.102] says (roughly, categories $\mathbf{A}_i$ and $\mathbf{B}_i$ below amount to trivial descriptions of a dual equivalence derived from a given dual adjunction):

> The main task for establishing a duality in a concrete situation is now to identify $\mathbf{A}_i$ and $\mathbf{B}_i$. This can be a very hard problem, and this is where categorical guidance comes to an end.

The real issue thus lies in providing substantial characterizations of $\mathbf{A}_i$ and $\mathbf{B}_i$. In contrast to this situation in categorical duality theories, natural duality theory does yield non-trivial descriptions of $\mathbf{A}_i$ and $\mathbf{B}_i$ involved, thus revealing how dual equivalences can be developed in various concrete situations. We consider that this is an important strength of natural duality theory, gained by restricting its scope more than categorical duality theories. By focusing on less general situations, natural duality theory succeeds in giving a more nuanced understanding of Stone duality.

Finally, we briefly touch upon the fundamental question: why do we study Stone duality (in a wider sense) at all? Stone-type dualities are theoretically elegant, and there would be no doubt that they are highly beneficial in practice, since they have indeed had numerous applications in logic, mathematics, and computer science. This is not what we really want to say here, however. Facing the question, we dare to say that Stone duality is duality between human knowledge and the reality of the world, or duality between epistemology and ontology, the two fundamental disciplines of philosophy. This nature of Stone duality is particularly striking in the case of duality between point-set spaces and point-free spaces, as points are ontological ingredients of the notion of space, and regions (or properties of space) are its epistemological ingredients (for more details, see [191]).

The idea of Stone duality as duality between ontology and epistemology is not merely a philosophical doctrine, but also a crucial notion lurking behind practical applications of Stone duality. For example, the main idea of [1] was to see Stone duality as duality between observable properties and denotational meanings of computational processes. Obviously, observable properties of computational processes are human knowledge in the context of computer science, and their denotational meanings are a matter of reality and not that of human knowledge (of course, computational processes are the "world" in computer science; we do not necessarily mean this real world by "world").

Duality between algebras and coalgebras, including those relevant to this chapter, may also be considered as an expression of duality between the epistemological and the ontological, via the idea of coalgebraic logic that coalgebras represent some sort of systems (e.g., computer systems) and algebras the (observable) properties of them. Here recall that usually we can only know about computer systems through their (observable) properties; evidently, the former is on the side of reality, and the latter on the side of our knowledge. Broadly speaking, most Stone-type dualities in mathematical logic are expressions of duality between syntax and semantics, which is in turn a specific kind of duality between the epistemological and the ontological.

Such a dichotomy (or duality) between epistemological and ontological things or perspectives can actually be observed in a much broader context, and so is the relation of the epistemological with the ontological. We mention only one case here. Kitaro Nishida, a philosopher of the Kyoto School, considered experience as having a person, rather than a person as having experience, saying (see [216]):

> Over time I came to realize that it is not that experience exists because there is an individual, but that an individual exists because there is experience.

That is, a person is (or at least may be identified with) a bundle of experiences, which is conceived of as being more primary than the notion of a person, in the Nishida philosophy. Its family resemblance to point-free geometry could be clarified in analogy with the leading idea of point-free geometry that a point is a bundle of shrinking regions (or certain properties of space).

Philosophical dichotomies can evolve into mathematical (categorical) dualities, as the case of point-free geometry shows. Indeed, foundational ideas of point-free geometry were first proposed by philosophers including Whitehead and Husserl, and then they were implemented in mathematical fashions, giving rise to categorical dualities between point-free spaces and point-set spaces as mentioned above. It would thus be fruitful to pursuit categorical dualities corresponding to given philosophical dichotomies, which may even have practical impacts as Stone duality was applied to computer science.

With such evidence in mind, we believe that Stone duality can form a significant theme of philosophy as well as mathematics. From the viewpoint of the history of ideas, it would also be worth noting that the 20th century was the time when the emphasis drastically shifted from the ontological to the epistemological in diverse disciplines, ranging from mathematics (e.g., non-commutative geometry), to physics (e.g., algebraic quantum field theory), to computer science (e.g., domain theory in logical form; logic in general is of such nature), and to philosophy (e.g., the theory of meaning; we wonder if we could add phenomenology here).

Chapter 4

Duality and Categorical Logic

Categorical duality often exists between the syntax and semantics of propositional logic. We can actually observe that duality for propositional logic forms a model of its predicate extension if logic is viewed as Lawvere's hyperdoctrine; this allows us to interlink duality with categorical logic. To elucidate such a link, in the present chapter, we work with a general set-up so as to integrate different categorical logics in a common setting. We show that the hyperdoctrinal conception of logic works far beyond intuitionistic logic, subsuming most standard logical systems; the hyperdoctrinal approach to universal logic is called categorical universal logic in the present thesis, which is the theory of monad-relativised hyperdoctrines or fibred universal algebras. Technically, this chapter consists of two parts: on the one had we develop categorical semantics for a wide variety of substructural logics in the spirit of categorical universal logic; on the other we give duality-theoretical methods to construct categorical models of logics within the same framework of categorical universal logic. The former is a bottom-up approach, and the latter a top-down approach to categorical universal logic. We also give a hyperdoctrinal account of logical translation, such as Gödel's and Girard's translation, which is concerned with Lawvere-Tierney topology in a general hyperdoctrinal setting. Granted that there is no agreed concept of algebras of predicate logic, hyperdoctrines or fibred universal algebras, we believe, provide a right platform for the algebraic analysis of different kinds of predicate logic. As illustrated by the hyperdoctrinal account of logical translation, the general framework developed in this chapter allows us to compare different categorical logics within the one setting.

4.1 Introduction to the First Part

Categorical logic deconstructs the traditional dichotomy between proof theory and model theory, in the sense that both of them can be represented in certain syntactic and set-theoretical categories (or hyperdoctrines in this chapter) respectively. We may thus say that categorical semantics does encompass both proof-theoretic and model-theoretic semantics, or verification-conditional and truth-conditional semantics in terms of philosophy of logic.

Categorical semantics divides into two sub-disciplines: semantics of provability (e.g., semantics via toposes or logoses) and semantics of proofs (e.g., semantics via CCC or monoidal CC). Our focus shall be upon the former with respect to logic and the latter with respect to type theory because we aim at developing categorical semantics for a broad range of logics over type theories, including classical, intuitionistic, linear, and fuzzy logics. Type theories have inherent identities of proofs (or terms), and fully admit semantics of proofs, however, logics in general do not allow semantics of proofs, due to collapsing phenomena on their identities of proofs (for the case of classical logic, refer to the Joyal lemma, e.g., in Lambek-Scott [168]).

Thus, the Curry-Howard paradigm does not make so much sense in this general context of logics over type theories, for the logics of the latter (types) may differ from the former original logics (propositions), just as Abramsky-Coecke's type theory of quantum mechanics is distinct from Birkhoff-von Neumann's logic of it. In general, we thus need to treat logic and type theory separately, and the concept of fibred universal algebras does the job, as elucidated below. Aczel's idea of logic-enriched type theory is along a similar line (see Aczel [10] and Gambino-Aczel [106]). Fibred algebras to represent logics over monoidal type theories even allow us to reconcile Birkhoff-von Neumann's cartesian logic of quantum propositions and Abramsky-Coecke's monoidal logic (or type theory) of quantum systems; this is future work, however.

Substructural logics over the Full Lambek calculus (FL for short), which encompass a wide variety of logical systems (classical, intuitionistic, linear, fuzzy, relevant, etc.), have extensively been investigated in the past few decades, especially by algebraic logicians in relation to residuated lattices[1]. Although some efforts have been made towards the algebraic treatment of logics over quantified FL (see, e.g., Ono

[1] Although the literature on logics over FL is already massive, Galatos-Jipsen-Kowalski-Ono [104] gives a fairly comprehensive overview, apart from more recent developments, such as equivalence (up to a certain point) between cut-elimination and algebraic completion (see Ciabattoni-Galatos-Terui [56]), which is only established in the propositional case so far, but could be extended to the first-order and higher-order cases via the substructural concepts of hyperdctrine and tripos (this is actually another technical motivation for our theory).

[219, 221] and references therein), however, it seems that there has so far been no adequate concept of algebraic models of them. Note that complete residuated lattices can only give complete semantics for those classes of substructural predicate logics for which completions (such as Dedekind-MacNeille's or Crawley's) of Lindenbaum-Tarski algebras work adequately (see, e.g., Ono [219, 221]); for this reason, complete residuated lattices (or quantales) cannot serve the purpose.

In the context as articulated above, we propose fibred algebras as algebraic models of predicate logic, especially substructural logics over quantified FL. Fibred algebras expand Lawvere's concept of hyperdoctrine [171]. According to Pitts' formulation [229], a hyperdoctrine is a functor (presheaf)

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$$

where **HA** is the category of heyting algebras; there are additional conditions on P (and $\mathbf{C}$) to express quantifiers and other logical concepts (for a fibrational formulation of hyperdoctrine, see Jacobs [145]; the two formulations are equivalent via the Grothendieck construction).[2] We may see a hyperdoctrine as a fibred heyting algebra $(P(C))_{C\in\mathbf{C}}$, a bunch of algebras indexed by $\mathbf{C}$.

Now, a fibred algebra is a universal algebra indexed by a category: categorically, it is a functor (presheaf)

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

(apart from logical conditions to express quantifiers and others) where T is a monad on **Set**, and $\mathbf{Alg}(T)$ is its Eilenberg-Moore algebras; note that monads on **Set** are equivalent to (possibly infinitary) varieties in terms of universal algebra (see, e.g., Adámek et al. [12]). The intuitive meaning of the base category $\mathbf{C}$ is the category of types (aka. sorts) or domains of discourse, and then $P(C)$ is the algebra of predicates on a type C. If a propositional logic L is sound and complete with respect to a variety $\mathbf{Alg}(T)$, then the corresponding fibred algebras

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

yield sound and complete semantics for the predicate logic that extends L. This may be called the thesis of completeness lifting:

- The completeness of propositional logic with respect to $\mathbf{Alg}(T)$ lifts to the completeness of predicate logic with respect to $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$.

[2]Toposes amount to higher-order hyperdoctrines via the two functors of taking subobject hyperdoctrines and of the tripos-to-topos construction (see, e.g., Frey [98]). We shall get back to this later in more detail.

This is one of the major tenets of categorical universal logic.

The present chapter is meant to demonstrate the completeness lifting thesis in the fairly general context of substructural logics over FL, hopefully bridging between algebraic logic, in which logics over FL have been studied, and categorical logic, in which hyperdoctrines have been pursued. Although the two disciplines are currently separated to the author's eyes, nevertheless, Lawvere's original ideas on categorical logic are of algebraic nature (especially, his functorial semantics directly targets universal algebra), and it would be fruitful to restore lost interactions between them. By doing so we take a first step towards categorical universal logic.

Universal logic is a recent trend with an entire journal, *Logica Universalis*, dedicated to it. It would not be a contingency with no rationale. Although there were known only a few logical systems in the early days of logic around the early 20th century, yet in the middle 20th century and from then on, numerous systems of logic have been proposed and elaborated to serve different purposes in foundations of mathematics, computer science, linguistics, and so fourth. Today, logical pluralism (e.g., Beall-Restall [45] or Shapiro [253]) is presumably more dominant than logical monism, which was prevailing during the dawn of modern logic as exemplified by Russell's and Wittgenstein's philosophy of logic. Pluralism is not just a tendency in logic, and even the disunity of science in general is (or at least was) claimed by the so-called Stanford school of philosophy (Galison, Hacking, Suppes *et. al*; see Galison-Stump [105]), forming, in a way, a "received view" in certain part of the philosophy of science. Living within this general atmosphere of pluralism pervasive in the postmodern era, we, presumably naturally, lack the unity of logic or the unified theory of diverse systems of logic; after all, disunity, as in the Stanford school philosophy, might ought to count as a positive feature of modern science in light of (possibly irreducible) diversity in the world.

Yet, even if there is no grand unified theory of diverse logical systems, it would still be possible to have a unified perspective on them, just as Bourbaki's "unity of mathematics" project yielded a reasonable account of the "architecture of mathematics" in unified structural terms. What is, then, the "architecture of logic"? As Lawvere emphasises in his retrospective commentary on his seminal paper "Adjointness in Foundations", in which the concept of hyperdoctrine was introduced, the architecture of logic must be accounted for in a presentation-independet (in particular syntax-independent) manner; in fact, this was what Lawvere's hyperdoctrine was meant to do (see the reprint of [171] as noted in the references). Yet his theory was limited to the realm of intuitionistic logic (if not without reasons), and thus we aim to expand the horizon of Lawverian categorical logic so as to encompass a general class of logical systems by virtue of the generalised concept of hyperdoctrine, both of which are basically contravariant functors relating categories of types to categories of propositions on them, incarnating, in a way, what Lawvere calls "duality between the formal and the conceptual" in his aforementioned paper [171].

What if we extend all this machinery to higher-order logic? What exactly is the scope of higher-order categorical logic (in the topos-theoretic tradition)? How broadly is the methodology of topos-style semantics applicable in the realm of higher-order logics? We address these issues by conceiving of the logical essence of topos as tripos (standing for "topos-representing indexed partially ordered set"; introduced in Hyland-Johnstone-Pitts [143][3]), or higher-order hyperdoctrine, showing that (generalised) tripos semantics works for a wide variety of logical systems.

Why is tripos suitable for universal logic rather than topos *per se*? In a nutshell, a tripos is the fibrational (or indexed-categorical) concept of topos, and the fibrational mechanism allows for universality. To put it in more detail, a tripos is a functor P from one category of types C (or domains of discourse) to another category of (algebras of) propositions $P(C)$ on them, whilst a topos is merely a single category in which objects are types whose subobjects yield the algebras of propositions on them. What is at stake here is the uniformity of the two structures of types and of propositions. In the tripos-theoretical approach, we do not have to presuppose the uniformity of types and propositions, yet in the topos-theoretical approach, we have to do so, since the two structures must be harmoniously bundled into a single category. The tripos-theoretical approach can thus encompass both conceptions of logic, namely the one which harmonises the two structures and the one which does not presuppose the harmony and may apprehend the two structures on their own intrinsic traits (any "logic over type theory" or Aczel's "logic-enriched type theory" is of the latter kind). The topos-theoretical approach is thus focused on a more specific situation forcing the harmony, and for that very reason limited in its range of applicability.

Conceptually, tripos or higher-order hyperdoctrine may be seen as the logical essence of topos; the definition of a topos as a category with its subobject functor forming a higher-order hyperdoctrine is called the *logical* definition of a topos in Jacobs [145]. What is really indispensable in the categorical interpretation of logic is the hyperdoctrinal structure rather than the topos structure *per se*; indeed there

[3]There are actually several non-equivalent versions of the concept of tripos; see, e.g., Pitts' retrospective paper [230].

are some triposes which are not based on any topos and yet capable of interpreting logic (more on this below). And therefore tripos is logically more fundamental than topos. Technically, there is an adjunction between the category of triposes and that of toposes, all toposes arising as the images of some triposes via the tripos-to-topos construction, and yet not all triposes obtaining as the images (subobject fibrations) of toposes; in this way, triposes encompass toposes, being more comprehensive than toposes. Taking all this into account, we theorise universal logic in the spirit of tripos rather than topos *per se*.

Main technical contributions on first-order and higher-order categorical substructural logics are as follows: (i) first-order completeness via Full Lambek hyperdoctrines and higher-order completeness via Full Lambek triposes (they are hyperdoctrines and triposes for logics over Full Lambek calculus FL, respectively); (ii) hyperdoctrinal and tripos-theoretical formulations of Girard's translation and Gödel's translation. The completeness results are established for any axiomatic extension of FL, therefore covering a great majority of standard logical systems (classical, intuitionistic, linear, fuzzy, relevant, and so on). In passing, we also discuss the tripos-to-topos construction in the present setting. Our uniform categorical semantics for various logical systems enables us to compare different categorical logics within the one setting, and our results indeed illustrate such a comparison in terms of logical translation.

The structure of the first part of the chapter is as follows. After an introduction to typed FL with quantifiers, denoted TFL^q, we introduce the concept of Full Lambek hyperdoctrines (FL hyperdoctrine, for short), prove the corresponding first-order completeness theorem, and give hyperdoctrinal accounts of Girard's and Gödel's translation. Then we move on to higher-order categorical substructural logics. We first present the syntax of higher-order full Lambek calculus HoFL, which obtains by adapting higher-order intuitionistic logic to full Lambek calculus FL. And we introduce the concept of full Lambek tripos (FL tripos, for short), thereby showing the higher-order completeness theorem for HoFL. Finally, our general framework thus developed is applied, via the internal language of FL tripos, to the higher-order categorical analysis of Girard's and Gödel's translation.

4.2 Typed Full Lambek Calculus and Full Lambek Hyperdoctrine

In this section, we define a typed (or many-sorted) version of quantified FL as in Ono [221], which shall be called TFL^q ("T" means "typed"; "q" means "quantified"). In particular, TFL^q follows the typing style of Pitts [229].

Standard categorical logic discusses a typed version of intuitionistic (or coherent or regular) logic, as observed in Pitts [229], Lambek-Scott [168], Jacobs [145], and Johnstone [150]. Typed logic is more natural than single-sorted one from a categorical point of view, and is more expressive in general, since it can encompass various type constructors. If one prefers single-sorted logic to typed logic, the latter can be reduced to the former by allowing for one type (or sort) only.

To put it differently, typed logic is the combination of logic and type theory, and has not only a logic structure but also a type structure, and the latter itself has a rich structure as well as the former. For this reason, syntactic hyperdoctrines constructed from typed systems of logic (which are discussed in relation to completeness in the next section) are amalgamations of syntactic categories obtained from their type theories on the one hand, and Lindenbaum-Tarski algebras obtained from their logic parts on the other; in a nutshell, syntactic hyperdoctrines are type-fibred Lindenbaum-Tarski algebras.

Another merit of typed logic is that the problem of empty domains is resolved because it allows us to have explicit control on type contexts. This was discovered by Joyal, and shall be touched upon later, in more detail.

TFL^q has the following logical connectives:

$$\otimes, \wedge, \vee, \backslash, /, 1, 0, \top, \bot, \forall, \exists.$$

Note that there are two kinds of implication connectives $\backslash$ and $/$, owing to the non-commutative nature of TFL^q.

In TFL^q, every variable x comes with its type σ. That is, TFL^q has basic types, which are denoted by letters like σ, τ, and $x : \sigma$ is a formal expression meaning that a variable x is of type σ. Then, a (type) context is a finite list of type declarations on variables: $x_1 : \sigma_1, ..., x_n : \sigma_n$. A context is often denoted Γ.

Accordingly, TFL^q has typed predicate symbols (aka. predicates in context) and typed function symbols (aka. function symbols in context): $R(x_1, ..., x_n)$ $[x_1 : \sigma_1, ..., x_n : \sigma_n]$ is a formal expression meaning that R is a predicate with n variables $x_1, ..., x_n$ of types $\sigma_1, ..., \sigma_n$ respectively; likewise, $f : \tau$ $[x_1 : \sigma_1, ..., x_n : \sigma_n]$ is a formal expression meaning that f is a function symbol with n variables $x_1, ..., x_n$ of types $\sigma_1, ..., \sigma_n$ and with its values in τ. Then, formulae-in-context φ $[\Gamma]$ and terms-in-context $t : \tau$ $[\Gamma]$ are defined in the usual, inductive way. Our terminology is basically following Pitts [229].

Here we do not consider any specific type constructor, and we shall focus upon plainly typed predicate logic with no complicated type structure. Still, products (not as types but as categorical structures) shall be used in categorical semantics in the next section, to the end of interpreting predicate and function symbols (of arity greater than one).

TFL^q thus has both a type structure and a logic structure, dealing with sequents-in-contexts: $\Phi \vdash \varphi\ [\Gamma]$ where Γ is a type context, and Φ is a finite list of formulae: $\varphi_1, ..., \varphi_n$. Although it is common to write $\Gamma \mid \Phi \vdash \varphi$ rather than $\Phi \vdash \varphi\ [\Gamma]$, we employ the latter notation following Pitts [229], since TFL^q is an adaptation of Pitts' typed system for intuitionistic logic to the system of the Full Lambek calculus.

The syntax of type contexts Γ in TFL^q is the same as that of typed intuitionistic logic in Pitts [229]. Yet we note it is allowed to add a fresh $x : \sigma$ to a context Γ: e.g., $\Phi \vdash \varphi\ [\Gamma, x : \sigma]$ whenever $\Phi \vdash \varphi\ [\Gamma]$. On the other hand, it is not permitted to delete redundant variables; the reason becomes clear in later discussion on empty domains. It is allowed to change the order of contexts (e.g., $[\Gamma, \Gamma']$ into $[\Gamma', \Gamma]$). In the below, we focus upon logical rules of inference, which are most relevant part of TFL^q, being of central importance for us.

TFL^q has no structural rule other than the following cut rule

$$\frac{\Phi_1 \vdash \varphi\ [\Gamma] \quad \Phi_2, \varphi, \Phi_3 \vdash \psi\ [\Gamma]}{\Phi_2, \Phi_1, \Phi_3 \vdash \psi\ [\Gamma]}\ (cut)$$

where ψ may be empty; this is allowed in the following L (left) rules as well. As usual, we have the rule of identity

$$\frac{}{\varphi \vdash \varphi\ [\Gamma]}\ (id)$$

In the following, we list the rules of inference for the logical connectives of TFL^q. There are two kinds of conjunction in TFL^q: multiplicative or monoidal $\otimes$ and additive or cartesian $\wedge$:

$$\frac{\Phi, \varphi, \psi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \otimes \psi, \Psi \vdash \chi\ [\Gamma]}\ (\otimes L) \qquad \frac{\Phi \vdash \varphi\ [\Gamma] \quad \Psi \vdash \psi\ [\Gamma]}{\Phi, \Psi \vdash \varphi \otimes \psi\ [\Gamma]}\ (\otimes R)$$

$$\frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \wedge \psi, \Psi \vdash \chi\ [\Gamma]}\ (\wedge L_1) \qquad \frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \psi \wedge \varphi, \Psi \vdash \chi\ [\Gamma]}\ (\wedge L_2)$$

$$\frac{\Phi \vdash \varphi\ [\Gamma] \quad \Phi \vdash \psi\ [\Gamma]}{\Phi \vdash \varphi \wedge \psi\ [\Gamma]}\ (\wedge R)$$

There is only one disjunction in TFL^q, which is additive, since TFL^q is intuitionistic in the sense that only one formula is allowed to appear on the right-hand side of sequents. Nevertheles, we can treat classical logic as an axiomatic extension of TFL^q, by adding to TFL^q exchange, weakening, contraction, and the excluded middle; note that structural rules can be expressed as axioms.

$$\frac{\Phi, \varphi, \Psi \vdash \chi \ [\Gamma] \quad \Phi, \psi, \Psi \vdash \chi \ [\Gamma]}{\Phi, \varphi \vee \psi, \Psi \vdash \chi \ [\Gamma]} \ (\vee L)$$

$$\frac{\Phi \vdash \varphi \ [\Gamma]}{\Phi \vdash \varphi \vee \psi \ [\Gamma]} \ (\vee R_1) \qquad \frac{\Phi \vdash \varphi \ [\Gamma]}{\Phi \vdash \psi \vee \varphi \ [\Gamma]} \ (\vee R_2)$$

Due to non-commutativity, there are two kinds of implication in TFL^q, $\backslash$ and $/$, which are a right adjoint of $\varphi \otimes (\text{-})$ and a right adjoint of $(\text{-}) \otimes \psi$ respectively.

$$\frac{\Phi \vdash \varphi \ [\Gamma] \quad \Psi_1, \psi, \Psi_2 \vdash \chi \ [\Gamma]}{\Psi_1, \Phi, \varphi\backslash\psi, \Psi_2 \vdash \chi \ [\Gamma]} \ (\backslash L) \qquad \frac{\varphi, \Phi \vdash \psi \ [\Gamma]}{\Phi \vdash \varphi\backslash\psi \ [\Gamma]} \ (\backslash R)$$

$$\frac{\Phi \vdash \varphi \ [\Gamma] \quad \Psi_1, \psi, \Psi_2 \vdash \chi \ [\Gamma]}{\Psi_1, \psi/\varphi, \Phi, \Psi_2 \vdash \chi \ [\Gamma]} \ (/L) \qquad \frac{\Phi, \varphi \vdash \psi \ [\Gamma]}{\Phi \vdash \psi/\varphi \ [\Gamma]} \ (/R)$$

There are two kinds of truth and falsity constants, monoidal and cartesian ones.

$$\frac{\Psi_1, \Psi_2 \vdash \varphi \ [\Gamma]}{\Psi_1, 1, \Psi_2 \vdash \varphi \ [\Gamma]} \ (1L) \qquad \frac{}{\vdash 1 \ [\Gamma]} \ (1R)$$

$$\frac{}{0 \vdash \ [\Gamma]} \ (0L) \qquad \frac{\Phi \vdash \ [\Gamma]}{\Phi \vdash 0 \ [\Gamma]} \ (0R)$$

$$\frac{}{\Phi \vdash \top \ [\Gamma]} \ (\top R) \qquad \frac{}{\Phi_1, \bot, \Phi_2 \vdash \varphi \ [\Gamma]} \ (\bot L)$$

Finally, we have the following rules for quantifiers $\forall$ and $\exists$, in which type contexts explicitly change; notice that type contexts do not change in the rest of the rules presented above.

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi \ [x : \sigma, \Gamma]}{\Phi_1, \forall x\varphi, \Phi_2 \vdash \psi \ [x : \sigma, \Gamma]} \ (\forall L) \qquad \frac{\Phi \vdash \varphi \ [x : \sigma, \Gamma]}{\Phi \vdash \forall x\varphi \ [\Gamma]} \ (\forall R)$$

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi \ [x : \sigma, \Gamma]}{\Phi_1, \exists x\varphi, \Phi_2 \vdash \psi \ [\Gamma]} \ (\exists L) \qquad \frac{\Phi \vdash \varphi \ [x : \sigma, \Gamma]}{\Phi \vdash \exists x\varphi \ [x : \sigma, \Gamma]} \ (\exists R)$$

As usual, there are eigenvariable conditions on the rules above: x does not appear as a free variable in the bottom sequent of Rule $\forall R$; likewise, x does not appear as a free variable in the bottom sequent of Rule $\exists L$ (eigenvariable conditions are actually avoidable in some typed logics as done in cylindric and polyadic logics; see, e.g., Kurz and Petrisan [160]). The other two rules do not have eigenvariable conditions, and this is why contexts do not change in them.

The deducibility of sequents-in-context in TFL^q is defined in the usual way. In this chapter, we denote by FL the propositional (and hence no contextual) part of TFL^q. Note that what is called FL in the literature often lacks $\bot$ and $\top$.

As is well known, the following propositional (resp. predicate) logics can be represented as axiomatic (to be precise, axiom-schematic) extensions of FL (resp. TFL^q): classical logic, intuitionistic logic, linear logic (without exponentials), relevance logics, fuzzy logics such as Gödel-Dummett logic (see, e.g., Galatos et al. [104]). Given a set of axioms (to be precise, axiom schemata), say X, we denote by FL_X (resp. TFL^q_X) the corresponding extension of FL (resp. TFL^q) via X.

Lemma 4.2.1. *The following sequents-in-context are deducible in* TFL^q*:*

- $\varphi \otimes (\exists x \psi) \vdash \exists x(\varphi \otimes \psi)$ $[\Gamma]$ *and* $\exists x(\varphi \otimes \psi) \vdash \varphi \otimes (\exists x \psi)$ $[\Gamma]$.
- $(\exists x \psi) \otimes \varphi \vdash \exists x(\psi \otimes \varphi)$ $[\Gamma]$ *and* $\exists x(\psi \otimes \varphi) \vdash (\exists x \psi) \otimes \varphi$ $[\Gamma]$.

where it is supposed that φ *does not contain* x *as a free variable, and* Γ *contains type declarations on those free variables that appear in* φ *and* $\exists x \psi$*.*

A striking feature of typed predicate logic is that domains of discourse in semantics can be empty; they are assumed to be non-empty in the usual Tarski semantics of predicate logic. This means that a type σ can be interpreted as an initial object in a category. We therefore need no ad hoc condition on domains of discourse if we work with typed predicate logic. This resolution of the problem of empty domains is due to Joyal as noted in Marquis and Reyes [184] (see p. 58).

A proof-theoretic manifestation of this feature is that the following sequent-in-context is not necessarily deducible in TFL^q: $\forall x \varphi \vdash \exists x \varphi$ $[\,]$ where the context is empty. Nonetheless, the following is deducible in TFL^q: $\forall x \varphi \vdash \exists x \varphi$ $[x : \sigma, \Gamma]$ where Γ is an appropriate context including type declarations on free variables in φ. This means that we can prove the sequent above when a type σ is inhabited. Here, it is crucial that it is not allowed to delete redundant free variables (e.g., $[x : \sigma, \Gamma]$ cannot be reduced into $[\Gamma]$ even if x does not appear as a free variable in formulae involved); however, it is allowed to add fresh free variables to a context.

Full Lambek Hyperdoctrine

It is well known that FL algebras (defined below) provide sound and complete semantics for propositional logic FL (see, e.g., Galatos et al. [104]). In the following we show that fibred FL algebras, or FL hyperdoctrines (defined below), yield sound and complete semantics for typed (or many-sorted) predicate logic TFL^q.

We again emphasise the simple, algebro-logical idea that single algebras (symbolically, A with no indexing) correspond to propositional logic, and fibred algebras (symbolically, $(A_C)_{C\in\mathbf{C}}$ indexed by a category $\mathbf{C}$) correspond to predicate logic. As universal algebra gives foundations for algebraic propositional logic, so fibred universal algebra lays foundations for algebraic predicate logic.

Definition 4.2.2 ([104])**.** $(A,\otimes,\wedge,\vee,\backslash,/,1,0,\top,\bot)$ *is called an* FL *algebra iff*

- $(A,\otimes,1)$ *is a monoid;* 0 *is a (distinguished) element of* A*;*
- $(A,\wedge,\vee,\top,\bot)$ *is a bounded lattice, which induces a partial order* $\leq$ *on* A*;*
- *for any* $a\in A$*,* $a\backslash(\text{-}) : A\to A$ *is a right adjoint of* $a\otimes(\text{-}) : A\to A$*: i.e.,* $a\otimes b\leq c$ *iff* $b\leq a\backslash c$ *for any* $a,b,c\in A$*;*
- *for any* $b\in A$*,* $(\text{-})/b : A\to A$ *is a right adjoint of* $(\text{-})\otimes b : A\to A$*: i.e.,* $a\otimes b\leq c$ *iff* $a\leq c/b$ *for any* $a,b,c\in A$*.*

A homomorphism of FL *algebras is defined as a map preserving all the operations* $(\otimes,\wedge,\vee,\backslash,/,1,0,\top,\bot)$*.* **FL** *denotes the category of* FL *algebras and their homomorphisms.*

Although 0 is just a neutral element of A with no axiom, the rules for 0 are automatically valid by the definition of interpretations defined below.

FL is an algebraic category (i.e., a category monadic over **Set**), or a variety in terms of universal algebra, since the two adjointness conditions can be rephrased by equations (see, e.g., Galatos et al. [104]). An axiomatic extension FL_X of FL corresponds to an algebraic full subcategory (or sub-variety) of **FL**, denoted $\mathbf{FL}_X$ (algebraicity follows from definability by axioms); this is the well-known, logic-variety correspondence for logics over FL (see Galatos et al. [104]).

Definition 4.2.3. *An* FL *(Full Lambek) hyperdoctrine is an* **FL***-valued presheaf* $P : \mathbf{C}^{\mathrm{op}}\to\mathbf{FL}$ *such that* $\mathbf{C}$ *is a category with finite products, and the following conditions on quantifiers hold:*

- *For any projection* $\pi : X\times Y\to Y$ *in* $\mathbf{C}$*,* $P(\pi) : P(Y)\to P(X\times Y)$ *has a right adjoint, denoted*
$$\forall_\pi : P(X\times Y)\to P(Y).$$

And the corresponding Beck-Chevalley condition holds, i.e., the following diagram commutes for any arrow $f : Z \to Y$ in $\mathbf{C}$ ($\pi' : X \times Z \to Z$ below denotes a projection):

$$\begin{tikzcd} P(X \times Y) \arrow[r, "\forall_\pi"] \arrow[d, "P(X\times f)"'] & P(Y) \arrow[d, "P(f)"] \\ P(X \times Z) \arrow[r, "\forall_{\pi'}"'] & P(Z) \end{tikzcd}$$

- *For any projection $\pi : X \times Y \to Y$ in $\mathbf{C}$, $P(\pi) : P(Y) \to P(X \times Y)$ has a left adjoint, denoted*

$$\exists_\pi : P(X \times Y) \to P(Y).$$

The corresponding Beck-Chevalley condition holds:

$$\begin{tikzcd} P(X \times Y) \arrow[r, "\exists_\pi"] \arrow[d, "P(X\times f)"'] & P(Y) \arrow[d, "P(f)"] \\ P(X \times Z) \arrow[r, "\exists_{\pi'}"'] & P(Z) \end{tikzcd}$$

Furthermore, the Frobenius Reciprocity conditions hold: for any projection $\pi : X \times Y \to Y$ in $\mathbf{C}$, any $a \in P(Y)$, and any $b \in P(X \times Y)$,

$$\begin{aligned} a \otimes (\exists_\pi b) &= \exists_\pi(P(\pi)(a) \otimes b) \\ (\exists_\pi b) \otimes a &= \exists_\pi(b \otimes P(\pi)(a)). \end{aligned}$$

For an axiomatic extension FL_X *of* FL*, an* FL_X *hyperdoctrine is defined by restricting the value category* $\mathbf{FL}$ *into* $\mathbf{FL}_X$*. An* FL *(resp.* FL_X*) hyperdoctrine is also called a fibred* FL *(resp.* FL_X*) algebra.*

The category $\mathbf{C}$ of an FL hyperdoctrine $P : \mathbf{C} \to \mathbf{FL}$ is called its base category or type category, and P is also called its predicate functor; intuitively, $P(C)$ is the algebra of predicates on a type, or domain of discourse, C.

The logical reading of the Beck-Chevalley conditions above is that substitution commutes with quantification. Note that, in the definition above, we need two Frobenius Reciprocity conditions due to the non-commutativity of FL algebras.

An FL hyperdoctrine may be seen as an indexed category, and so as a fibration via the Grothendieck construction. Although we discuss in terms of indexed categories in this chapter, we can do the job in terms of fibrations as well. In the view of fibrations, each $P(C)$ is called a fibre of an FL hyperdoctrine P.

The FL (resp. FL_X) hyperdoctrine semantics for TFL^q (resp. TFL^q_X) is defined as follows.

Definition 4.2.4. *Fix an* FL *hyperdoctrine* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$*. An interpretation* $[\![\text{-}]\!]$ *of* TFL^q *in the* FL *hyperdoctrine* P *consists of the following:*

- *assignment of an object* $[\![\sigma]\!]$ *in* $\mathbf{C}$ *to each basic type* σ *in* TFL^q*;*
- *assignment of an arrow* $[\![f : \tau\ [\Gamma]]\!] : [\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!] \to [\![\sigma]\!]$ *in* $\mathbf{C}$ *to each typed function symbol* $f : \tau\ [\Gamma]$ *in* TFL^q *where* Γ *is supposed to be* $x_1 : \sigma_1, ..., x_n : \sigma_n$ *(note that* $[\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!]$ *makes sense because* $\mathbf{C}$ *has finite products);*
- *assignment of an element* $[\![R\ [\Gamma]]\!]$ *in* $P([\![\Gamma]\!])$*, which is an* FL *algebra, to each typed predicate symbol* $R\ [\Gamma]$ *in* TFL^q*; if the context* Γ *is* $x_1 : \sigma_1, ..., x_n : \sigma_n$*, then* $[\![\Gamma]\!]$ *denotes* $[\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!]$*.*

Then, terms are inductively interpreted in the following way:

- $[\![x : \sigma\ [\Gamma_1, x : \sigma, \Gamma_2]]\!]$ *is defined as the following projection in* $\mathbf{C}$*:*

$$\pi : [\![\Gamma_1]\!] \times [\![\sigma]\!] \times [\![\Gamma_2]\!] \to [\![\sigma]\!].$$

- $[\![f(t_1, ..., t_n) : \tau\ [\Gamma]]\!]$ *is defined as:*

$$[\![f]\!] \circ \langle [\![t_1 : \sigma_1\ [\Gamma]]\!], ..., [\![t_n : \sigma_n\ [\Gamma]]\!] \rangle$$

  where it is supposed that $f : \tau\ [x_1 : \sigma_1, ..., x_n : \sigma_n]$*, and* $t_1 : \sigma_1\ [\Gamma], ..., t_n : \sigma_n\ [\Gamma]$*. Note that* $\langle [\![t_1 : \sigma_1\ [\Gamma]]\!], ..., [\![t_n : \sigma_n\ [\Gamma]]\!] \rangle$ *above is the product (or pairing) of arrows in* $\mathbf{C}$*.*

Formuli are then interpreted inductively in the following manner:

- $[\![R(t_1, ..., t_n)\ [\Gamma]]\!]$ *is defined as*

$$P(\langle [\![t_1 : \sigma_1[\Gamma]]\!], ..., [\![t_n : \sigma_n[\Gamma]]\!] \rangle)([\![R\ [x : \sigma_1, ..., x_n : \sigma_n]]\!])$$

  where R *is a predicate symbol in context* $x_1 : \sigma_1, ..., x_n : \sigma_n$*.*

- $[\![\varphi \otimes \psi\ [\Gamma]]\!]$ *is defined as* $[\![\varphi\ [\Gamma]]\!] \otimes [\![\psi\ [\Gamma]]\!]$*. The other binary connectives* $\wedge, \vee, \backslash, /$ *are interpreted in the same way.* $[\![1\ [\Gamma]]\!]$ *is defined as the monoidal unit of* $P([\![\Gamma]\!])$*. The other constants* $0, \top, \bot$ *are interpreted in the same way.*

- $[\![\forall x\varphi\ [\Gamma]]\!]$ *is defined as*

$$\forall_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!])$$

where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ *is a projection in* $\mathbf{C}$*, and* φ *is a formula in context* $[x:\sigma,\Gamma]$*. Similarly,* $[\![\exists x\varphi\ [\Gamma]]\!]$ *is defined as*

$$\exists_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!]).$$

Finally, satisfaction of sequents is defined:

- $\varphi_1, ..., \varphi_n \vdash \psi\ [\Gamma]$ *is satisfied in an interpretation* $[\![\text{-}]\!]$ *in an* FL *hyperdoctrine* P *iff the following holds in* $P([\![\Gamma]\!])$*:*

$$[\![\varphi_1\ [\Gamma]]\!] \otimes ... \otimes [\![\varphi_n\ [\Gamma]]\!] \leq [\![\psi\ [\Gamma]]\!].$$

In case the right-hand side of a sequent is empty, $\varphi_1, ..., \varphi_n \vdash [\Gamma]$ *is satisfied in* $[\![\text{-}]\!]$ *iff* $[\![\varphi_1\ [\Gamma]]\!] \otimes ... \otimes [\![\varphi_n\ [\Gamma]]\!] \leq 0$ *in* $P([\![\Gamma]\!])$*. In case the left-hand side of a sequent is empty,* $\vdash \varphi\ [\Gamma]$ *is satisfied in* $[\![\text{-}]\!]$ *iff* $1 \leq [\![\varphi[\Gamma]]\!]$ *in* $P([\![\Gamma]\!])$*.*

An interpretation of TFL^q_X *in an* FL_X *hyperdoctrine is defined by replacing* **FL** *and* TFL^q *above with* $\mathbf{FL}_X$ *and* TFL^q_X *respectively.*

In the following, we show that the FL (resp. FL_X) hyperdoctrine semantics is sound and complete for TFL^q (resp. TFL^q_X). Let $[\![\Phi\ [\Gamma]]\!]$ denote $[\![\varphi_1\ [\Gamma]]\!] \otimes ... \otimes [\![\varphi_n\ [\Gamma]]\!]$ if Φ is $\varphi_1, ..., \varphi_n$.

Intuitively, an arrow f in $\mathbf{C}$ is a term, and $P(f)$ is a substitution operation (this is exactly true in syntactic hyperdoctrines defined later); then, the Beck-Chevalley conditions and the functoriality of P tell us that substitution commutes with all the logical operations (namely, both propositional connectives and quantifiers). From such a logical point of view, the meaning of the Beck-Chevalley conditions is crystal clear; they just say that substitution after quantification is the same as quantification after substitution.

Proposition 4.2.5. *If* $\Phi \vdash \psi\ [\Gamma]$ *is deducible in* TFL^q *(resp.* TFL^q_X*), then it is satisfied in any interpretation in any* FL *(resp.* FL_X*) hyperdoctrine.*

Proof. Fix an FL or FL_X hyperdoctrine P and an interpretation $[\![\text{-}]\!]$ in P. Initial sequents in context are satisfied because $a \leq a$ in any fibre $P(C)$. The cut rule preserves satisfaction, since tensoring preserves $\leq$ and $\leq$ has transitivity. It is easy to verify that all the rules for the logical connectives preserve satisfaction.

Let us consider universal quantifier $\forall$. To show the case of Rule $\forall R$, assume that $[\![\Phi\ [x:\sigma,\Gamma]]\!] \leq [\![\varphi\ [x:\sigma,\Gamma]]\!]$ in $P([\![\sigma]\!] \times [\![\Gamma]\!])$. It then follows that $[\![\Phi\ [x:\sigma,\Gamma]]\!] = P(\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!])([\![\Phi\ [\Gamma]]\!])$ where π is a projection in $\mathbf{C}$, and note that Φ does not include x among its free variables by the eigenvariable condition. We thus have $P(\pi)([\![\Phi\ [\Gamma]]\!]) \leq [\![\varphi\ [x:\sigma,\Gamma]]\!]$. Since $\forall_\pi : P([\![\sigma]\!] \times [\![\Gamma]\!]) \to P([\![\Gamma]\!])$ is a right adjoint of $P(\pi)$, it follows that $[\![\Phi\ [\Gamma]]\!] \leq \forall_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!]) = [\![\forall x\varphi\ [\Gamma]]\!]$. We next show the case of $\forall L$. Assume that $[\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!] \leq [\![\psi\ [x:\sigma,\Gamma]]\!]$. The adjunction condition for universal quantifier gives us $P(\pi)(\forall_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!])) \leq [\![\varphi\ [x:\sigma,\Gamma]]\!]$ where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection. Yet we also have $P(\pi)(\forall_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!])) = P(\pi)([\![\forall x\varphi\ [\Gamma]]\!]) = [\![\forall x\varphi\ [x:\sigma,\Gamma]]\!]$. Since tensoring respects $\leq$, these together imply that $[\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\forall x\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!] \leq [\![\psi\ [x:\sigma,\Gamma]]\!]$.

It remains to show the case of existential quantifier $\exists$. In order to prove that Rule $\exists L$ preserves satisfaction, assume that $[\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!] \leq [\![\psi\ [x:\sigma,\Gamma]]\!]$. This is equivalent to the following: $[\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!] \leq P(\pi)([\![\psi\ [\Gamma]]\!])$ where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection. Since $\exists_\pi : P([\![\sigma]\!] \times [\![\Gamma]\!]) \to P([\![\Gamma]\!])$ is left adjoint to $P(\pi)$, it follows that $\exists_\pi([\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!]) \leq [\![\psi\ [\Gamma]]\!]$. This is equivalent to the following: $\exists_\pi(\ P(\pi)([\![\Phi_1\ [\Gamma]]\!]) \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes P(\pi)([\![\Phi_2\ [\Gamma]]\!])\ ) \leq [\![\psi\ [\Gamma]]\!]$. Repeated applications of the two Frobenius Reciprocity conditions give us $[\![\Phi_1\ [\Gamma]]\!] \otimes \exists_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!]) \otimes [\![\Phi_2\ [\Gamma]]\!] \leq [\![\psi\ [\Gamma]]\!]$. Then we finally have the following: $[\![\Phi_1\ [\Gamma]]\!] \otimes [\![\exists x\varphi\ [\Gamma]]\!] \otimes [\![\Phi_2\ [\Gamma]]\!] \leq [\![\psi\ [\Gamma]]\!]$. To show the case of $\exists R$, assume that $[\![\Phi\ [x:\sigma,\Gamma]]\!] \leq [\![\varphi\ [x:\sigma,\Gamma]]\!]$. The adjunction condition for existential quantifier tells us that $[\![\varphi\ [x:\sigma,\Gamma]]\!] \leq P(\pi)(\exists_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!]))$ where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection. We thus have the following: $[\![\Phi\ [x:\sigma,\Gamma]]\!] \leq P(\pi)(\exists_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!])) = [\![\exists x\varphi\ [x:\sigma,\Gamma]]\!]$. This completes the proof. □

Syntactic hyperdoctrines are then defined as follows towards the goal of proving completeness. They are the categorificaton of Lindenbaum-Tarski algebras.

Definition 4.2.6. *The syntactic hyperdoctrine of* TFL^q *is defined as follows; that of* TFL^q_X *is defined by replacing* $\mathbf{FL}$ *and* TFL^q *below with* $\mathbf{FL}_X$ *and* TFL^q_X.

We first define the base category $\mathbf{C}$. *An object in* $\mathbf{C}$ *is a context* Γ *up to* α-*equivalence (i.e., the naming of variables does not matter). An arrow in* $\mathbf{C}$ *from an object* Γ *to another* Γ' *is a list of terms* $[t_1, ..., t_n]$ *(up to equivalence) such that* $t_1 : \sigma_1\ [\Gamma], ..., t_n : \sigma_n\ [\Gamma]$ *where* Γ' *is supposed to be* $x_1 : \sigma_1, ..., x_n : \sigma_n$.

The syntactic hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ *is then defined in the following way. For an object* Γ *in* $\mathbf{C}$, *let* $\mathrm{Form}_\Gamma = \{\varphi \mid \varphi$ *is a formula in context* $\Gamma\}$. *Define an* *equivalence relation* $\sim$ *on* Form_Γ *as follows: for* $\varphi, \psi \in \mathrm{Form}_\Gamma$*,* $\varphi \sim \psi$ *iff both* $\varphi \vdash \psi$ $[\Gamma]$ *and* $\psi \vdash \varphi$ $[\Gamma]$ *are deducible in* TFL^q*. We then define*

$$P(\Gamma) = \mathrm{Form}_\Gamma / \sim$$

with an FL *algebra structure induced by the logical connectives.*

The arrow part of P *is defined as follows. Let* $[t_1, ..., t_n] : \Gamma \to \Gamma'$ *be an arrow in* $\mathbf{C}$ *where* Γ' *is* $x_1 : \sigma_1, ..., x_n : \sigma_n$*. Then we define* $P([t_1, ..., t_n]) : P(\Gamma') \to P(\Gamma)$ *by*

$$P([t_1, ..., t_n])(\varphi) = \varphi[t_1/x_1, ..., t_n/x_n]$$

where it is supposed that $t_1 : \sigma_1$ $[\Gamma], ..., t_n : \sigma_n$ $[\Gamma]$*, and that* φ *is a formula in context* $x_1 : \sigma_1, ..., x_n : \sigma_n$*.*

Intuitively, $P(\Gamma)$ above is a Lindenbaum-Tarski algebra sliced with respect to each Γ. It is straightforward to verify that the operations of $P(\Gamma)$ above are well defined, and $P(\Gamma)$ forms an FL algebra. We still have to check that P defined above is a hyperdoctrine; this is done in the following lemma.

Lemma 4.2.7. *The syntactic hyperdoctrine* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ *(resp.* $\mathbf{FL}_X$*) is an* FL *(resp.* FL_X*) hyperdoctrine. In particular, it has quantifier structures satisfying the Beck-Chevalley and Frobenius Reciprocity conditions.*

Proof. Since substitution commutes with all the logical connectives, $P([t_1, ..., t_n])$ defined above is always a homomorphism of FL algebras. Thus, P is a contravariant functor.

We have to verify that the base category $\mathbf{C}$ has finite products, or equivalently, binary products. For objects Γ, Γ' in $\mathbf{C}$, we define their product $\Gamma \times \Gamma'$ as follows. Suppose that Γ is $x_1 : \sigma_1, ..., x_n : \sigma_n$, and Γ' is $y_1 : \tau_1, ..., y_m : \tau_m$. Then, $\Gamma \times \Gamma'$ is defined as $x_1 : \sigma_1, ..., x_n : \sigma_n, y_1 : \tau_1, ..., y_m : \tau_m$. An associated projection $\pi : \Gamma \times \Gamma' \to \Gamma'$ is defined as $[y_1, ..., y_m] : \Gamma \times \Gamma' \to \Gamma'$ where the context of each y_i is taken to be $x_1 : \sigma_1, ..., x_n : \sigma_n, y_1 : \tau_1, ..., y_m : \tau_m$ (rather than $y_1 : \tau_1, ..., y_m : \tau_m$). The other projection is defined in a similar way. It is easily verified that these indeed form a categorical product in $\mathbf{C}$.

In order to show that P has quantifier structures, let $\pi : \Gamma \times \Gamma' \to \Gamma'$ denote the projection in $\mathbf{C}$ defined above, and then consider $P(\pi)$, which we have to show has right and left adjoints. The right and left adjoints of $P(\pi)$ can be constructed as follows. Recall Γ is $x : \sigma_1, ..., x_n : \sigma_n$. Let $\varphi \in P(\Gamma \times \Gamma')$; here we are identifying φ with the equivalence class to which φ belongs, since every argument below respects the equivalence. Then define $\forall_\pi : P(\Gamma \times \Gamma') \to P(\Gamma')$ by $\forall_\pi(\varphi) = \forall x_1 ... \forall x_n \varphi$ where the formula on the right-hand side actually denotes the corresponding equivalence class. Similarly, we define $\exists_\pi : P(\Gamma \times \Gamma') \to P(\Gamma')$ by $\exists_\pi(\varphi) = \exists x_1 ... \exists x_n \varphi$. Let us show that $\forall_\pi$ is the right adjoint of $P(\pi)$. We first assume $P(\pi)(\psi) \leq \varphi$ in $P(\Gamma \times \Gamma')$ for $\psi \in P(\Gamma')$ and $\varphi \in P(\Gamma \times \Gamma')$. Then it follows from the definition of P and π that $P(\pi)(\psi\ [\Gamma]) = \psi\ [\Gamma, \Gamma']$ where we are making explicit the two different contexts of ψ; the role of $P(\pi)$ just lies in changing contexts. Since the $\leq$ of $P(\Gamma \times \Gamma')$ is induced by its lattice structure, we have $\varphi \wedge \psi = \psi$. It follows from the definition of $P(\Gamma \times \Gamma')$ that $\varphi \wedge \psi \vdash \psi\ [\Gamma, \Gamma']$ and $\psi \vdash \varphi \wedge \psi\ [\Gamma, \Gamma']$ are deducible in TFL^q (resp. TFL^q_X), whence $\psi \vdash \varphi\ [\Gamma, \Gamma']$ is deducible as well. By repeated applications of rule $\forall R$, it follows that $\psi \vdash \forall x_1 ... \forall x_n \varphi\ [\Gamma']$ is deducible. This implies that both $\psi \vdash \psi \wedge \forall x_1 ... \forall x_n \varphi\ [\Gamma']$ and $\psi \wedge \forall x_1 ... \forall x_n \varphi \vdash \psi\ [\Gamma']$ are deducible, whence $\psi \leq \forall x_1 ... \forall x_n \varphi$ in $P(\Gamma')$.

We show the converse. Assume that $\psi \leq \forall x_1 ... \forall x_n \varphi$ in $P(\Gamma')$. By arguing as in the above, $\psi \vdash \forall x_1 ... \forall x_n \varphi\ [\Gamma']$ is deducible. By enriching the context, $\psi \vdash \forall x_1 ... \forall x_n \varphi\ [\Gamma, \Gamma']$ is deducible. Since $\forall x_1 ... \forall x_n \varphi \vdash \varphi\ [\Gamma, \Gamma']$ is deducible by rule $\forall L$, the cut rule tells us that $\psi \vdash \varphi\ [\Gamma, \Gamma']$ is deducible; note that the contexts of two sequents-in-context must be the same when applying the cut rule to them. It finally follows that $P(\pi)(\psi) \leq \varphi$ in $P(\Gamma \times \Gamma')$. Thus, $\forall_\pi$ is the right adjoint of $P(\pi)$. Similarly, $\exists_\pi$ can be shown to be the left adjoint of $P(\pi)$.

The Beck-Chevalley condition for $\forall$ can be verified as follows. Let $\varphi \in P(\Gamma \times \Gamma')$, $\pi : \Gamma \times \Gamma' \to \Gamma'$ a projection in $\mathbf{C}$, and $\pi' : \Gamma \times \Gamma'' \to \Gamma''$ another projection in $\mathbf{C}$ for objects $\Gamma, \Gamma', \Gamma''$ in $\mathbf{C}$. Then, we have $P([t_1, ..., t_n]) \circ \forall_\pi(\varphi) = (\forall x_1 ... \forall x_n \varphi)[t_1/y_1, ..., t_n/y_m]$ where it is supposed that Γ is $x_1 : \sigma_1, ..., x_n : \sigma_n$, Γ' is $y_1 : \tau_1, ..., y_m : \tau_m$, and $t_1 : \tau_1\ [\Gamma''], ..., t_m : \tau_m\ [\Gamma'']$. We also have the following $\forall_{\pi'} \circ P([t_1, ..., t_n])(\varphi) = \forall x_1 ... \forall x_n(\varphi[t_1/y_1, ..., t_n/y_m])$. The Beck-Chevalley condition for $\forall$ thus follows. The Beck-Chevalley condition for $\exists$ can be verified in a similar way. The two Frobenius Reciprocity conditions for $\exists$ follow immediately from Lemma 4.2.1. □

The syntactic hyperdoctrine is a free or classifying hyperdoctrine in a suitable sense. It is the combination of the classifying category $\mathbf{C}$ above and the free algebras $P(\Gamma)$ above, which has the universal property inherited from both of them.

Now, there is the obvious, canonical interpretation of TFL^q (resp. TFL^q_X) in the syntactic hyperdoctrine of TFL^q (resp. TFL^q_X); it is straightforward to see:

Lemma 4.2.8. *If $\Phi \vdash \psi\ [\Gamma]$ is satisfied in the canonical interpretation in the syntactic hyperdoctrine of* TFL^q *(resp.* TFL^q_X*), it is deducible in* TFL^q *(resp.* TFL^q_X*).*

The lemmata above give us the completeness result: If $\Phi \vdash \psi\ [\Gamma]$ is satisfied in any interpretation in any FL (resp. FL_X) hyperdoctrine, then it is deducible in TFL^q (resp. TFL^q_X). Combining soundness and completeness, we obtain:

Theorem 4.2.9. *$\Phi \vdash \psi\ [\Gamma]$ is deducible in TFL^q (resp. TFL^q_X) iff it is satisfied in any interpretation in any FL (resp. FL_X) hyperdoctrine.*

In the remainder of the section we discuss hyperdoctrines induced from dual adjunctions between **Set** and **FL**, which are, so to say, many-valued powerset hyperdoctrines, and give many-valued Tarski semantics with soundness and completeness, generalising the powerset hyperdoctrine $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \mathbf{2})$, which is equivalent to Tarski semantics.

Theorem 4.2.10. *Let $\Omega \in \mathbf{FL}$. The following dual adjunction holds between **Set** and **FL**, induced by Ω as a dualising object:*

$$\mathrm{Hom}_{\mathbf{FL}}(\text{-}, \Omega)^{\mathrm{op}} \dashv \mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{FL}.$$

Proposition 4.2.11. *Let $\Omega \in \mathbf{FL}$ with Ω complete. Then, $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{FL}$ (resp. $\mathbf{FL}_X$) is an FL (resp. FL_X) hyperdoctrine.*

Proof. Let $\pi : X \times Y \to Y$ be a projection in **Set**. We define $\forall_\pi$ and $\exists_\pi$ as follows: given $v \in \mathrm{Hom}(X \times Y, \Omega)$ and $y \in Y$, let $\forall_\pi(v)(y) := \bigwedge\{v(x, y) \mid x \in X\}$ and $\exists_\pi(v)(y) := \bigvee\{v(x, y) \mid x \in X\}$. These yield the required quantifier structures with the Beck-Chevalley and Frobenius Reciprocity conditions. $\square$

Now, we aim at obtaining completeness with respect to models of form $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$. The above proof tells us that $\forall$ and $\exists$ in $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ are actually meets and joins in Ω. This implies that if Ω is not complete, in general, $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ cannot interpret quantifiers. At the same time, however, assuming completeness prevents us from obtaining completeness for any axiomatic extension TFL^q_X of TFL^q; this is why we do not assume it. Such incompleteness phenomena have already been observed (see, e.g., Ono [219]). A standard remedy to this problem is to restrict attention to "safe" interpretations while considering general Ω. In our context, a safe interpretation $[\![\text{-}]\!]$ in $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ is such that $[\![\text{-}]\!]$ uses those joins and meets only that exist in Ω, i.e., quantifiers are always interpreted via existing joins and meets only (a "non-safe" interpretation may use "non-existing" joins or meets; safety is the least condition for ensuring the very possibility of interpretation as it were). We then have completeness with respect to the special class of set-theoretical models $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$.

Theorem 4.2.12. $\Phi \vdash \psi$ $[\Gamma]$ *is deducible in* TFL^q_X *iff it is satisfied in any safe interpretation in* $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ *for any* $\Omega \in \mathbf{FL}$.

In the special case of TFL^q, *it suffices to consider complete* Ω*'s only:* $\Phi \vdash \psi$ $[\Gamma]$ *is deducible in* TFL^q *iff it is satisfied in any interpretation in any* FL *hyperdoctrine* $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ *with* $\Omega \in \mathbf{FL}$ *complete.*

This theorem can be proven in the same way as the standard proof of the algebraic completeness of quantified Full Lambek calculus via safe interpretations, since (safely) interpreting logic in $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ boils down to the standard algebraic semantics (via safe interpretations).

Focusing on a more specific context, we can further reduce the class of models $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ into a smaller one. In the strongest case of classical logic, it suffices to consider $\{0, 1\}$ only in the place of Ω; this is exactly the Tarski completeness.

For an intermediate case, consider MTL (monoidal t-norm logic; see Hájek et al. [126]), which is FL expanded with exchange, weakening, and the pre-linearity axiom, $(\varphi \to \psi) \vee (\psi \to \varphi)$. The algebras of MTL are denoted by **MTL**. We denote by MTL^q the quantified version with the additional axiom of $\forall$-$\vee$ distributivity, i.e., $\forall x(\varphi \vee \psi) \leftrightarrow \forall x\varphi \vee \psi$ where x does not occur in ψ as a free variable, and by MTL^q_X an axiomatic extension of MTL^q.

Theorem 4.2.13. $\Phi \vdash \psi$ $[\Gamma]$ *is deducible in* MTL^q_X *iff it is satisfied in any interpretation in* $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ *for any linearly ordered* $\Omega \in \mathbf{MTL}_X$.

This theorem follows from the so-called standard completeness of MTL, which may be found in any standard textbook on fuzzy logic.

We briefly discuss the tripos-topos construction in the present context of FL hyperdoctrines; it is originally due to Hyland-Johnstone-Pitts [143]. To this end, we work in the internal logic of FL hyperdoctrines $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$: i.e., we have types X and function symbols f corresponding to objects X and arrows f in $\mathbf{C}$ respectively, and also those predicate symbols R on a type $C \in \mathbf{C}$ that correspond to elements $R \in P(C)$.

Definition 4.2.14. *Let* P *be an* FL *hyperdoctrine. We define a category* $\mathbf{T}[P]$ *as follows. An object of* $\mathbf{T}[P]$ *is a partial equivalence relation, i.e., a pair* (X, E_X) *such that* X *is an object in the base category* $\mathbf{C}$, *and* E_X *is an element of* $P(X \times X)$ *and is symmetric and transitive in the internal logic of* P*:* $E_X(x, y) \vdash E_X(y, x)$ $[x, y : X]$ *and* $E_X(x, y), E_X(y, z) \vdash E_X(x, z)$ $[x, y, z : X]$.

An arrow from (X, E_X) *to* (Y, E_Y) *is* $F \in P(X \times Y)$ *such that (i) extensionality:* $E_X(x_1, x_2), E_Y(y_1, y_2), F(x_1, y_1) \vdash F(x_2, y_2)$ $[x_1, x_2 : X, y_1, y_2 : Y]$; *(ii) strictness:* $F(x, y) \vdash E_X(x, x) \wedge E_Y(y, y)$ $[x : X, y : Y]$; *(iii) single-valuedness:* $F(x, y_1), F(x, y_2) \vdash E_Y(y_1, y_2)$ $[x : X, y_1, y_2 : Y]$; *(iv) totality:* $E_X(x, x) \vdash \exists y\ F(x, y)$ $[x : X]$. *Such an* F *is called a functional relation.*

For a complete FL algebra Ω, which is a quantale with additional operations, $\mathbf{T}[\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)]$ may be called the category of Ω-valued sets. Quantale sets in the sense of Höhle et al. [138] are objects in $\mathbf{T}[\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)]$, but not vice versa: our Ω-valued sets are slightly more general than their quantale sets.

Note that if Ω is a locale, $\mathbf{T}[\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)]$ is the Higgs topos of Ω-valued sets, which is in turn equivalent to the category of sheaves on Ω.

4.3 Hyperdoctrinal Girard and Gödel Translation

In this section, we discuss Girard's and Gödel's translation theorems on the hyperdoctrinal setting. The former embeds intuitionistic logic into linear logic via exponential !; the latter embeds classical logic into intuitionistic logic via double negation $\neg\neg$. Since logic is dual to algebraic semantics, we construct intuitionistic (resp. classical) hyperdoctrines from linear (resp. intuitionistic) hyperdoctrines. Proofs of two theorems below shall be given later in more general form.

We first consider Gödel's translation. We think of $\neg\neg$ as a functor $\mathrm{Fix}_{\neg\neg}$ from **HA**, the category of heyting algebras, to **BA**, the category of boolean algebras: i.e., define $\mathrm{Fix}_{\neg\neg}(A) = \{a \in A \mid \neg\neg a = a\}$; the arrow part is defined by restriction. Here, $\mathrm{Fix}_{\neg\neg}(A)$ forms a boolean algebra.

Let us define IL hyperdoctrines as FL hyperdoctrines with values in **HA**. Likewise, CL hyperdoctrines are defined as FL hyperdoctrines with values in **BA**. Note that both kinds of hyperdoctrines are TFL_X^q hyperdoctrines with suitable choices of axioms X. Finally, Gödel's translation theorem can be understood in terms of hyperdoctrines as follows.

Theorem 4.3.1. *Let* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ *be an* IL *hyperdoctrine. Then, the following composed functor* $\mathrm{Fix}_{\neg\neg} \circ P : \mathbf{C}^{\mathrm{op}} \to \mathbf{BA}$ *forms a* CL *hyperdoctrine.*

This is a first-order and hyperdoctrinal version of the construction of boolean toposes from given toposes via double negation topologies on them.

We can treat Girard's translation along a similar line. An exponential ! on an FL algebra A is defined as a unary operation satisfying: (i) $a \leq b$ implies $!a \leq !b$; (ii)

$!!a = !a \le a$; (iii) $!\top = 1$; (iv) $!a \otimes !b = !(a \wedge b)$ (see Coumans et al. [70]). We denote by $\mathbf{FL}_c^!$ the category of commutative FL algebras with ! and maps preserving both ! and FL algebra operations; they give the algebraic counterpart of intuitionistic linear logic with !, denoted ILL.

We regard exponential ! as a functor $\mathrm{Fix}_!$ from $\mathbf{FL}_c^!$ to $\mathbf{HA}$: define $\mathrm{Fix}_!(A) = \{a \in A \mid !a = a\}$; the arrow part is defined by restriction. $\mathrm{Fix}_!(A)$ is the set of those elements of A that admit structural rules, and forms a heyting algebra. ILL hyperdoctrines are defined as FL hyperdoctrines with values in $\mathbf{FL}_c^!$.

Theorem 4.3.2. *Let $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_c^!$ be an* ILL *hyperdoctrine. Then, the following composed functor $\mathrm{Fix}_! \circ P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ forms an* IL *hyperdoctrine.*

The theorem above is slightly more general than Girard's translation theorem, in the sense that the latter corresponds to the case of syntactic hyperdoctrines in the former.

Although in this chapter we do not explicitly discuss substructural logics enriched with modalities and their hyperdoctrinal semantics, nevertheless, our method perfectly works for them as well, yielding the corresponding soundness and completeness results in terms of hyperdoctrines with values in FL algebras with modalities; Girard's ! is just a special case.

If you replace the full Lambek calculus by Sambin's basic logic in the above developments, then you can also include quantum logic as an extension of the base logic, and the technical machinery above still works in the same way. We shall get back to this issue later.

The above two translation theorems in the language of hyperdoctrines shall be extended later to what we call the universal translation theorem in categorical universal logic.

4.4 Higher-Order Full Lambek Calculus and Full Lambek Tripos

In this section we introduce higher-order full Lambek calculus HoFL, which extends quantified FL as in Ono [221] so that HoFL equipped with all the structural rules boils down to higher-order intuitionistic logic, namely the logic of topos (see Lambek-Scott [168], Jacobs [145], or Johnstone [150]). Our presentation of HoFL, especially its type-theoretic part, follows the style of Pitts [229]; thus we write, e.g., "$t : \sigma\ [\Gamma]$"

and "φ [Γ]", rather than "$\Gamma \vdash t : \sigma$" and "$\Gamma \vdash \varphi$", respectively, where t is a term of type σ in context Γ, and φ is a formula in context Γ.

HoFL is a so-called "logic over type theory" or "logic-enriched type theory" in Aczel's terms; there is an underlying type theory, upon which logic is built (see, e.g., Jacobs [145]). To begin with, let us give a bird's-eye view of the structure of HoFL. The type theory of HoFL is given by simply typed λ-calculus extended with finite product types (i.e., 1 and $\times$; these amount to the structure of Cartesian closed categories), and moreover, with the special, distinguished type

$$\mathrm{Prop}$$

which is a "proposition" type, intended to represent a truth-value object Ω on the categorical side. The logic of HoFL is given by full Lambek calculus FL. The Prop type plays the key rôle of reflecting the logical or propositional structure into the type or term structure: every formula or proposition φ may be seen as a term of type Prop. This is essentially what the subobject classifier Ω of a topos $\mathbf{E}$ is required to satisfy, that is,

$$\mathrm{Sub}_{\mathbf{E}}(\text{-}) \simeq \mathrm{Hom}_{\mathbf{E}}(\text{-}, \Omega).$$

Spelling out the meaning of this axiom in logical terms, we have got

$$\mathrm{Pred}(\sigma) \simeq \mathrm{Term}(\sigma, \mathrm{Prop})$$

which means that the structure of predicates on each type σ (or context Γ in general) is isomorphic to the structure of terms from σ to Prop. In this sense, the logical meaning of Ω may be summarised by a sort of reflection principle, namely the reflection of the propositional structure into the type structure, which may also be called the "propositions-as-terms" or "propositions-as-functions" correspondence, arguably lying at the heart of higher-order categorical logic, for Ω would presumably be the *raison d'être* of higher-order categorical logic.

The power type $P\sigma$ of a given type σ can be defined in the present framework as $\sigma \to \mathrm{Prop}$; the comprehension term $\{x : \sigma \mid \varphi\} : P\sigma$ and the membership predicate $s \in t$: Prop are definable via λ-abstraction or currying (categorically, transpose) and λ-application (categorically, evaluation), respectively. That is, $\{x : \sigma \mid \varphi\}$ may be defined as $\lambda x : \sigma.\ \varphi$ where φ is seen as a term of type Prop, and also $s \in t$ may be defined as ts where $t : \sigma \to \mathrm{Prop}$ and $s : \sigma$. These definable operations allow us to express set-theoretical reasoning in higher-order logic. There is, of course, some freedom on the choice of primitives, just as toposes can be defined in terms of either subobject classifiers or power objects. All this is to facilitate an intuitive understanding of the essential features of higher-order logic; we give a formal account below.

The syntactic details of HoFL are as follows. HoFL is equipped with the following logical connectives of full Lambek calculus:

$$\otimes, \wedge, \vee, \backslash, /, 1, 0, \top, \bot, \forall, \exists.$$

The non-commutativity of HoFL gives rise to two kinds of implication ($\backslash$ and $/$). We have basic variables and types, denoted by letters like x and σ, respectively. And as usual $x : \sigma$ is a formal expression to say that a variable x is of type σ. Note that every variable must be typed in HoFL, unlike untyped FL. A context is a finite list of typings of variables: $x_1 : \sigma_1, ..., x_n : \sigma_n$ which is often abbreviated as Γ. Formulae and terms are then defined within specific contexts. There are relation symbols and function symbols, both in context: $R(x_1, ..., x_n)\ [x_1 : \sigma_1, ..., x_n : \sigma_n]$ is a formal expression to say that R is a relation symbol with variables $x_1, ..., x_n$ of types $\sigma_1, ..., \sigma_n$ respectively; and also $f : \tau\ [x_1 : \sigma_1, ..., x_n : \sigma_n]$ is a formal expression to say that f is a function symbol with its domain (the product of) $\sigma_1, ..., \sigma_n$ and with its codomain τ.

The type constructors of HoFL are product $\times$, function space $\rightarrow$, and the proposition type Prop, which is a nullary type constructor. The term constructors of $\times$ and $\rightarrow$ are as usual: pairing $\langle \text{-}, \text{-} \rangle$ and (first and second) projections π_1, π_2 for product $\times$, and λ-abstraction and λ-application for function space $\rightarrow$. The term constructors of Prop are all the logical connectives of full Lambek calculus as listed above, the relation symbols taken to be of type Prop and thus working as generators of the terms of type Prop. Formulae in context, $\varphi\ [\Gamma]$, and terms in context, $t : \tau\ [\Gamma]$, are then defined in the usual, inductive manner (our terminology and notation mostly follow Pitts [229]; we are extending his framework so as to encompass higher-order substructural logics). Finally, sequents in contexts are defined as: $\Phi \vdash \varphi\ [\Gamma]$ where Γ is a context, Φ is a finite list of formulae $\varphi_1, ..., \varphi_n$, and all the formulae involved are in context Γ.

So far we have not touched upon any axiom (or inferential rule) involved. In the following, we first give axioms for terms, and then for sequents. The axioms for $\times$ and $\rightarrow$ are as usual (see, e.g., Pitts [229]). The axiom for Prop is as follows:

$$\frac{\varphi \vdash \psi\ [\Gamma] \quad \psi \vdash \varphi\ [\Gamma]}{\varphi = \psi : \text{Prop}\ [\Gamma]}\ (prop)$$

This axiom relates the structure of propositions to that of terms, thus guaranteeing the aforementioned "propositions-as-functions" correspondence for higher-order categorical logic. There are several standard rules for contexts and substituition, which are the same as those in Pitts [229]. We now turn to inferential rules for sequents, which are the same as before, but we shall review for the reader's convenience. We first have the identity and cut rules as follows:

$$\frac{}{\varphi \vdash \varphi\ [\Gamma]}\ (id) \qquad \frac{\Phi_1 \vdash \varphi\ [\Gamma] \quad \Phi_2, \varphi, \Phi_3 \vdash \psi\ [\Gamma]}{\Phi_2, \Phi_1, \Phi_3 \vdash \psi\ [\Gamma]}\ (cut)$$

where ψ may be empty; this applies to the following L (Left) rules as well. Note that HoFL has no structural rule other than the cut rule. The rules of governing the use of the logical connectives are as follows.

$$\frac{\Phi, \varphi, \psi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \otimes \psi, \Psi \vdash \chi\ [\Gamma]}\ (\otimes L) \qquad \frac{\Phi \vdash \varphi\ [\Gamma] \quad \Psi \vdash \psi\ [\Gamma]}{\Phi, \Psi \vdash \varphi \otimes \psi\ [\Gamma]}\ (\otimes R)$$

$$\frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \wedge \psi, \Psi \vdash \chi\ [\Gamma]}\ (\wedge L_1) \qquad \frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \psi \wedge \varphi, \Psi \vdash \chi\ [\Gamma]}\ (\wedge L_2)$$

$$\frac{\Phi \vdash \varphi\ [\Gamma] \quad \Phi \vdash \psi\ [\Gamma]}{\Phi \vdash \varphi \wedge \psi\ [\Gamma]}\ (\wedge R)$$

$$\frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma] \quad \Phi, \psi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \vee \psi, \Psi \vdash \chi\ [\Gamma]}\ (\vee L)$$

$$\frac{\Phi \vdash \varphi\ [\Gamma]}{\Phi \vdash \varphi \vee \psi\ [\Gamma]}\ (\vee R_1) \qquad \frac{\Phi \vdash \varphi\ [\Gamma]}{\Phi \vdash \psi \vee \varphi\ [\Gamma]}\ (\vee R_2)$$

$$\frac{\Phi \vdash \varphi\ [\Gamma] \quad \Psi_1, \psi, \Psi_2 \vdash \chi\ [\Gamma]}{\Psi_1, \Phi, \varphi \backslash \psi, \Psi_2 \vdash \chi\ [\Gamma]}\ (\backslash L) \qquad \frac{\varphi, \Phi \vdash \psi\ [\Gamma]}{\Phi \vdash \varphi \backslash \psi\ [\Gamma]}\ (\backslash R)$$

$$\frac{\Phi \vdash \varphi\ [\Gamma] \quad \Psi_1, \psi, \Psi_2 \vdash \chi\ [\Gamma]}{\Psi_1, \psi / \varphi, \Phi, \Psi_2 \vdash \chi\ [\Gamma]}\ (/L) \qquad \frac{\Phi, \varphi \vdash \psi\ [\Gamma]}{\Phi \vdash \psi / \varphi\ [\Gamma]}\ (/R)$$

$$\frac{\Psi_1, \Psi_2 \vdash \varphi\ [\Gamma]}{\Psi_1, 1, \Psi_2 \vdash \varphi\ [\Gamma]}\ (1L) \qquad \frac{}{\vdash 1\ [\Gamma]}\ (1R)$$

$$\frac{}{0 \vdash\ [\Gamma]}\ (0L) \qquad \frac{\Phi \vdash\ [\Gamma]}{\Phi \vdash 0\ [\Gamma]}\ (0R)$$

$$\frac{}{\Phi \vdash \top\ [\Gamma]}\ (\top R) \qquad \frac{}{\Phi_1, \bot, \Phi_2 \vdash \varphi\ [\Gamma]}\ (\bot L)$$

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi\ [x : \sigma, \Gamma]}{\Phi_1, \forall x \varphi, \Phi_2 \vdash \psi\ [x : \sigma, \Gamma]}\ (\forall L) \qquad \frac{\Phi \vdash \varphi\ [x : \sigma, \Gamma]}{\Phi \vdash \forall x \varphi\ [\Gamma]}\ (\forall R)$$

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi\ [x : \sigma, \Gamma]}{\Phi_1, \exists x \varphi, \Phi_2 \vdash \psi\ [\Gamma]}\ (\exists L) \qquad \frac{\Phi \vdash \varphi\ [x : \sigma, \Gamma]}{\Phi \vdash \exists x \varphi\ [x : \sigma, \Gamma]}\ (\exists R)$$

There are eigenvariable conditions on the quantification rules: x must not appear as a free variable in the bottom sequent of the $\forall R$ rule; x must not appear as a free variable in the bottom sequent of the $\exists L$ rule. These are all of the rules of HoFL, and the provability of sequents in context is defined in the usual manner.

For a collection X of axiom schemata (which we often simply call axioms), let us denote by HoFL_X the axiomatic extension of HoFL via X. In particular, we can recover higher-order intuitionistic logic as HoFL_{ecw}, i.e., by adding to HoFL the exchange, weakening, and contraction rules (as axiom schemata).

Full Lambek Tripos

The algebras of propositional FL are FL algebras, the definition of which is reviewed below. The algebras of first-order FL are FL hyperdoctrines as argued above; note that complete FL algebras only give us completeness in the presence of the *ad hoc* condition of so-called safe valuations, and yet FL hyperdoctrines allow us to prove completeness without any such *ad hoc* condition, and at the same time, to recover the complete FL algebra semantics as a special, set-theoretical instance of the FL hyperdoctrine semantics (in a nutshell, the condition of safe valuations is only necessary to show completeness with respect to the restricted class of FL hyperdoctrines with the category of sets their base categories). In this section we define FL triposes, which are arguably the (fibred) algebras of higher-order FL, and prove higher-order completeness, again without any *ad hoc* condition such as safe valuations or Henkin-style restrictions on quantification (set-theoretical semantics is only complete under this condition).

Now, FL triposes are defined as FL hyperdoctrines with their base categories CCCs, and with truth-value objects Ω (i.e., representability via $\Omega \in \mathbf{C}$):

Definition 4.4.1. *An* FL *(Full Lambek) tripos, or higher-order* FL *hyperdoctrine, is an* FL *hyperdoctrine* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ *such that:*

- *The base category* $\mathbf{C}$ *is a CCC (Cartesian Closed Category);*

- *There is an object* $\Omega \in \mathbf{C}$ *such that*

$$P \simeq \mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega).$$

We then call Ω *the truth-value object of the* FL *tripos* P*. Given a set* X *of axioms, an* FL_X *tripos is defined by replacing* $\mathbf{FL}$ *above with* $\mathbf{FL}_X$*.*

For an FL tripos P, each $P(C)$ is called a fibre of the FL tripos P from a fibrational point of view; intuitively, $P(C)$ may be seen as the algebra of propositions on a type or domain of discourse C. Note that it is also possible to define FL triposes in terms of fibrations, even though the present formulation in terms of indexed categories would be categorically less demanding

FL tripos semantics for HoFL is defined as follows.

Definition 4.4.2. *Let $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ be an* FL *tripos. An interpretation $[\![\text{-}]\!]$ of* HoFL *in the* FL *tripos P is defined as follows. Types and atomic symbols are interpreted in the following way:*

- *each basic type σ is interpreted as an object $[\![\sigma]\!]$ in $\mathbf{C}$;*
- *product and function types, $\sigma \times \tau$ and $\sigma \to \tau$, are interpreted, as usual, by categorical product and exponentiation;*
- *each function symbol $f : \tau\ [\Gamma]$ is interpreted as an arrow $[\![f : \tau\ [\Gamma]]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]$ in $\mathbf{C}$; if the context Γ is $x_1 : \sigma_1, ..., x_n : \sigma_n$, then $[\![\Gamma]\!]$ denotes $[\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!]$;*
- *each relation symbol $R\ [\Gamma]$ is interpreted as an element $[\![R\ [\Gamma]]\!]$ in the corresponding fibre $P([\![\Gamma]\!])$ of the* FL *tripos P at $[\![\Gamma]\!]$.*

Terms and their equality are interpreted in the following, inductive manner:

- *$[\![x : \sigma\ [\Gamma_1, x : \sigma, \Gamma_2]]\!]$ is defined as the following projection in $\mathbf{C}$:*

$$\pi : [\![\Gamma_1]\!] \times [\![\sigma]\!] \times [\![\Gamma_2]\!] \to [\![\sigma]\!].$$

- *$[\![f(t_1, ..., t_n) : \tau\ [\Gamma]]\!]$ is defined as the following arrow in $\mathbf{C}$:*

$$[\![f]\!] \circ \langle [\![t_1 : \sigma_1\ [\Gamma]]\!], ..., [\![t_n : \sigma_n\ [\Gamma]]\!] \rangle$$

  where $f : \tau\ [x_1 : \sigma_1, ..., x_n : \sigma_n]$, and $t_1 : \sigma_1\ [\Gamma], ..., t_n : \sigma_n\ [\Gamma]$ (note also that $\langle [\![t_1 : \sigma_1\ [\Gamma]]\!], ..., [\![t_n : \sigma_n\ [\Gamma]]\!] \rangle$ denotes the product/pairing of arrows in $\mathbf{C}$).

- *λ-abstraction, λ-application, projections, and pairing are interpreted, as usual, by categorical transpose, evaluation, projections, and pairing in the base CCC $\mathbf{C}$, respectively;*

Formulae are interpreted in the following, inductive manner:

- $[\![R(t_1,...,t_n)\ [\Gamma]]\!]$ *is defined as*

$$P(\langle [\![t_1 : \sigma_1[\Gamma]]\!], ..., [\![t_n : \sigma_n[\Gamma]]\!]\rangle)([\![R\ [x : \sigma_1, ..., x_n : \sigma_n]]\!])$$

  where R is a relation symbol in context $x_1 : \sigma_1, ..., x_n : \sigma_n$.

- $[\![\varphi \otimes \psi\ [\Gamma]]\!]$ *is defined as* $[\![\varphi\ [\Gamma]]\!] \otimes [\![\psi\ [\Gamma]]\!]$. *The other binary connectives $\wedge, \vee, \backslash, /$ are interpreted in a similar way.* $[\![1\ [\Gamma]]\!]$ *is defined as the monoidal unit of $P([\![\Gamma]\!])$. The other constants $0, \top, \bot$ are interpreted in a similar way.*

- $[\![\forall x\varphi\ [\Gamma]]\!]$ *is defined as* $\forall_\pi([\![\varphi\ [x : \sigma, \Gamma]]\!])$ *where* $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ *is a projection in* $\mathbf{C}$*, and φ is a formula in context $[x : \sigma, \Gamma]$. Similarly,* $[\![\exists x\varphi\ [\Gamma]]\!]$ *is defined as* $\exists_\pi([\![\varphi\ [x : \sigma, \Gamma]]\!])$.

Prop *and its terms are then interpreted as follows:*

- Prop *is interpreted as the truth-value object Ω of the* FL *tripos P:*

$$[\![\mathrm{Prop}]\!] = \Omega;$$

- *each formula* $\varphi : \mathrm{Prop}\ [\Gamma]$*, regarded as a term of type* Prop*, is interpreted as the element of* $\mathrm{Hom}_{\mathbf{C}}([\![\Gamma]\!], \Omega)$ *which corresponds to* $[\![\varphi\ [\Gamma]]\!] \in P([\![\Gamma]\!])$ *in the defining isomorphism* $P \simeq \mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega)$ *of the* FL *tripos P; in a nutshell,* $[\![\varphi : \mathrm{Prop}\ [\Gamma]]\!]$*'s and* $[\![\varphi\ [\Gamma]]\!]$*'s are linked via the isomorphism.*

Finally, the validity of sequents in context is defined as follows:

- $\varphi_1, ..., \varphi_n \vdash \psi\ [\Gamma]$ *is valid in an interpretation* $[\![\text{-}]\!]$ *in an* FL *tripos P iff the following holds in* $P([\![\Gamma]\!])$*:*

$$[\![\varphi_1\ [\Gamma]]\!] \otimes ... \otimes [\![\varphi_n\ [\Gamma]]\!] \leq [\![\psi\ [\Gamma]]\!].$$

  In case the right-hand side of a sequent is empty, $\varphi_1, ..., \varphi_n \vdash [\Gamma]$ *is valid in* $[\![\text{-}]\!]$ *iff* $[\![\varphi_1\ [\Gamma]]\!] \otimes ... \otimes [\![\varphi_n\ [\Gamma]]\!] \leq 0$ *in* $P([\![\Gamma]\!])$*. In case the left-hand side of a sequent is empty,* $\vdash \varphi\ [\Gamma]$ *is valid in* $[\![\text{-}]\!]$ *iff* $1 \leq [\![\varphi[\Gamma]]\!]$ *in* $P([\![\Gamma]\!])$*. When Φ consists of* $\varphi_1, ..., \varphi_n$*, let* $[\![\Phi\ [\Gamma]]\!]$ *denote* $[\![\varphi_1\ [\Gamma]]\!] \otimes ... \otimes [\![\varphi_n\ [\Gamma]]\!]$.

An interpretation of HoFL_X *in an* FL_X *tripos is defined by replacing* **FL** *and* HoFL *above with* $\mathbf{FL}_X$ *and* HoFL_X*, respectively.*

The categorical conception of interpretation encompasses set-theoretical interpretations and forcing-style model constructions. First of all, interpreting logic in the $\mathbf{2}$-valued tripos $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \mathbf{2})$ (where $\mathbf{2}$ is the two-element Boolean algebra) is precisely equivalent to the standard Tarski semantics. Yet there is a vast generalisation of this: given a quantale Ω, the representable functor

$$\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{FL}$$

forms an FL tripos, which gives rise to a universe of quantale-valued sets via the generalised tripos-to-topos construction as noted above; if Ω is a locale in particular (i.e., complete Heyting algebra), it is known that $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ yields $\mathbf{Sh}(\Omega)$ (i.e., the sheaf topos on Ω). This sort of FL tripos models of set theory could hopefully be applied to solve consistency problems for substructural set theories (especially, Cantor-Łukasiewicz set theory).

Note that the base category of an FL tripos is used to interpret the type theory of HoFL, and the value category is used to interpret the logic part of HoFL. In the following, we first prove soundness and then completeness.

Proposition 4.4.3. *If* $\Phi \vdash \psi\ [\Gamma]$ *is provable in* HoFL *(resp.* HoFL_X*), then it is valid in any interpretation in any* FL *(resp.* FL_X*) tripos.*

Proof. Let P be an FL or FL_X tripos, and $[\![\text{-}]\!]$ an interpretation in P. Soundness for the first-order part can be proven in mostly the same way as above; we thus focus upon Prop, which is the most distinctive part of higher-order logic. So let us prove that the rule for the Prop type preserves validity. Suppose that $[\![\varphi\ [\Gamma]]\!] \leq [\![\psi\ [\Gamma]]\!]$ and that $[\![\psi\ [\Gamma]]\!] \leq [\![\varphi\ [\Gamma]]\!]$. It then follows that

$$[\![\varphi\ [\Gamma]]\!] = [\![\psi\ [\Gamma]]\!].$$

Note that this is a "propositional" equality, i.e., an equality in the fibre $P([\![\Gamma]\!])$ of propositions on $[\![\Gamma]\!]$. Since we have the following isomorphism

$$P([\![\Gamma]\!]) \simeq \mathrm{Hom}_{\mathbf{C}}([\![\Gamma]\!], [\![\mathrm{Prop}]\!])$$

the equality above, together with the definition of the interpretation of terms of type Prop, tells us that

$$[\![\varphi : \mathrm{Prop}\ [\Gamma]]\!] = [\![\psi : \mathrm{Prop}\ [\Gamma]]\!].$$

Note that this is a "functional" equality, i.e., an equality in $\mathrm{Hom}_{\mathbf{C}}([\![\Gamma]\!], [\![\mathrm{Prop}]\!])$. Thus, the propositional equality implies the functional equality (via the isomorphism above), and this is exactly what it is for the Prop rule to preserve validity. $\square$

For the sake of a completeness proof, let us introduce the syntactic tripos construction (for logic over type theory), which is the combination of the syntactic category construction (for type theory) and the Lindenbaum-Tarski algebra construction (for propositional logic):

Definition 4.4.4. *The syntactic tripos of* HoFL *is defined as follows. Let us first define the syntactic base category* $\mathbf{C}$*: an object is a context* Γ *(up to* α*-equivalence); an arrow from* Γ *to* Γ' *is a list of terms (up to equality on terms)*

$$[t_1, ..., t_n]$$

where $t_1 : \sigma_1 \ [\Gamma], ..., t_n : \sigma_n \ [\Gamma]$ *and* Γ' *is supposed to be* $x_1 : \sigma_1, ..., x_n : \sigma_n$*. Composition is defined via substitution. The syntactic tripos* $P_{\mathrm{HoFL}} : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ *is then defined as follows. Given an object* Γ *in* $\mathbf{C}$*, let* Form_Γ *denote the set of formulas in context* Γ*, and then define*

$$P_{\mathrm{HoFL}}(\Gamma) = \mathrm{Form}_\Gamma / \sim$$

where $\sim$ *is an equivalence relation on* Form_Γ *defined as follows: for* $\varphi, \psi \in \mathrm{Form}_\Gamma$*,* $\varphi \sim \psi$ *iff* $\varphi \vdash \psi \ [\Gamma]$ *and* $\psi \vdash \varphi \ [\Gamma]$ *are provable in* HoFL*. The arrow part of* P_{HoFL} *is defined as follows. Let* $[t_1, ..., t_n] : \Gamma \to \Gamma'$ *be an arrow in* $\mathbf{C}$ *where* Γ' *is* $x_1 : \sigma_1, ..., x_n : \sigma_n$*. Then we define* $P_{\mathrm{HoFL}}([t_1, ..., t_n]) : P_{\mathrm{HoFL}}(\Gamma') \to P_{\mathrm{HoFL}}(\Gamma)$ *by*

$$P_{\mathrm{HoFL}}([t_1, ..., t_n])(\varphi \ [\Gamma']) = \varphi[t_1/x_1, ..., t_n/x_n] \ [\Gamma]$$

where it is supposed that $t_1 : \sigma_1 \ [\Gamma], ..., t_n : \sigma_n \ [\Gamma]$*, and that* φ *is a formula in context* $x_1 : \sigma_1, ..., x_n : \sigma_n$*. The syntactic tripos* P_{HoFL_X} *of* HoFL_X *is defined just by replacing* $\mathbf{FL}$ *and* HoFL *above with* $\mathbf{FL}_X$ *and* HoFL_X*, respectively.*

The syntactic tripos of higher-order logic is the fibrational analogue of the Lindenbaum-Tarski algebra of propositional logic; each fibre $P_{\mathrm{HoFL}}(\Gamma)$ of the syntactic tripos P_{HoFL} is the Lindenbaum-Tarski algebra of formulae in context Γ. The syntactic tripos of HoFL has the universal mapping property that inherits from the syntactic base category of the underlying type theory of HoFL, and also from the fibre-wise Lindenbaum-Tarski algebras of the logic part of HoFL. We of course have to verify that the syntactic tripos P_{HoFL} indeed carries an FL tripos structure; this is the crucial part of the completeness proof.

Lemma 4.4.5. *The syntactic tripos* $P_{\mathrm{HoFL}} : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ *(resp.* $\mathbf{FL}_X$*) defined above is an* FL *(resp.* FL_X*) tripos. In particular, the base category is a CCC, and there is a truth-value object* $\Omega \in \mathbf{C}$ *such that* $P_{\mathrm{HoFL}} \simeq \mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega)$.

Proof. The first-order hyperdoctrine structure of P_{HoFL} can be verified to exist in basically the same way as above. The existence of exponentials is guaranteed by the existence of function space types in the type theory of HoFL. We thus concentrate on the existence of a truth-value object Ω. Let

$$\Omega = x : \mathrm{Prop}.$$

Note that, since the objects of the base category are contexts rather than types, we cannot take Ω to be Prop *per se*; yet x : Prop practically means the same thing as Prop, thanks to α-equivalence required. We now have to show that for each context Γ,

$$P(\Gamma) \simeq \mathrm{Hom}_{\mathbf{C}}(\Gamma, x : \mathrm{Prop})$$

and this correspondence yields a natural transformation. The required isomorphism is given by mapping

$$\varphi\ [\Gamma] \in P(\Gamma)$$

to

$$\varphi : \mathrm{Prop}\ [\Gamma] \in \mathrm{Hom}_{\mathbf{C}}(\Gamma, x : \mathrm{Prop}).$$

Note that φ above is actually an equivalence class, and yet the above mapping is well defined, and also that φ : Prop $[\Gamma]$ is actually a list consisting of a single term φ : Prop $[\Gamma]$. This mapping is an isomorphism by the definition of terms of type Prop. Let us denote the above mapping by

$$\mathrm{PaF}_{\Gamma} : P(\Gamma) \to \mathrm{Hom}_{\mathbf{C}}(\Gamma, x : \mathrm{Prop})$$

with the idea of "Propositions-as-Functions" in mind. The naturality of this correspondence then means that the following diagram commutes for any arrow $[t_1, ..., t_n] : \Gamma' \to \Gamma$ in $\mathbf{C}$:

$$\begin{tikzcd}
P(\Gamma) \arrow[r, "\mathrm{PaF}_{\Gamma}"] \arrow[d, "P([t_1,...,t_n])"'] & \mathrm{Hom}_{\mathbf{C}}(\Gamma, x : \mathrm{Prop}) \arrow[d, "\mathrm{Hom}_{\mathbf{C}}([t_1,...,t_n], x:\mathrm{Prop})"] \\
P(\Gamma') \arrow[r, "\mathrm{PaF}_{\Gamma'}"'] & \mathrm{Hom}_{\mathbf{C}}(\Gamma', x : \mathrm{Prop})
\end{tikzcd}$$

By the following calculation:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{C}}([t_1, ..., t_n], \mathrm{Prop}) \circ \mathrm{PaF}_{\Gamma}(\varphi\ [\Gamma]) &= \mathrm{Hom}_{\mathbf{C}}([t_1, ..., t_n], \mathrm{Prop})(\varphi : \mathrm{Prop}\ [\Gamma]) \\
&= \varphi[t_1/x_1, ..., t_n/x_n] : \mathrm{Prop}\ [\Gamma'] \\
&= \mathrm{PaF}_{\Gamma'}(\varphi[t_1/x_1, ..., t_n/x_n]\ [\Gamma']) \\
&= \mathrm{PaF}_{\Gamma'} \circ P([t_1, ..., t_n])(\varphi\ [\Gamma])
\end{aligned}$$

we obtain the commutativity of the diagram and hence the naturality of the "propositions-as-functions" correspondence.

□

It is straightforward to see that if $\Phi \vdash \psi$ $[\Gamma]$ is valid in the canonical interpretation in the syntactic tripos P_{HoFL} (resp. P_{HoFL_X}), then it is provable in HoFL (resp. HoFL_X). And this immediately gives us completeness via the standard counter-model argument. Hence the higher-order completeness theorem:

Theorem 4.4.6. $\Phi \vdash \psi$ $[\Gamma]$ *is provable in* HoFL *(resp.* HoFL_X*) iff it is valid in any interpretation in any* FL *(resp.* FL_X*) tripos.*

This higher-order completeness theorem can be applied, with a suitable choice of axioms X, for any of classical, intuitionistic, fuzzy, relevant, paraconsistent, and (both commutative and non-commutative) linear logics; higher-order completeness has not been known for these logics except the first two. The concept of (generalised) tripos, therefore, is so broadly applicable as to encompass most logical systems. Modal logics also can readily be incorporated into this framework by working with modal FL rather than plain FL. Coalgebraic dualities for modal logics (see Chapter 2) then yield models of modal triposes for them; these modal issues are to be addressed in subsequent papers.

4.5 Tripos-Theoretical Girard and Gödel Translation

We finally analyse Gödel's double negation translation and Girard's exponential ! translation from a tripos-theoretical point of view.

Propositional Gödel translation algebraically means that, for any Heyting algebra A, the doubly negated algebra $\neg\neg A$, defined as $\{a \in A \mid \neg\neg a = a\}$, always forms a Boolean algebra. This $\neg\neg$ construction extends to a functor from the category $\mathbf{HA}$ of Heyting algebras to the category $\mathbf{BA}$ of Boolean algebras. And then the categorical meaning of first-order Gödel translation is that, for any first-order IL hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ (where IL denote intuitionistic logic), the following composed functor

$$\neg\neg \circ P : \mathbf{C}^{\mathrm{op}} \to \mathbf{BA}$$

forms a first-order CL hyperdoctrine (where CL denotes classical logic). Yet this strategy does not extend to the higher-order case: in particular, although the base category does not change in the first-order case, in which types and propositions are separated, it must nevertheless be modified in the higher-order case, in which types and propositions interact via Prop or Ω. Technicalities involved get essentially more complicated in the higher-order case. Still, we can construct from a given IL tripos $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ a CL tripos

$$P_{\neg\neg} : \mathbf{C}_{\neg\neg}^{\mathrm{op}} \to \mathbf{BA}.$$

For the sake of the description of $\mathbf{C}_{\neg\neg}$ (and $P_{\neg\neg}$), however, we work within the internal language HoFL_P of the tripos $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$: in HoFL_P, we have types C and terms f corresponding to objects C and arrows f in $\mathbf{C}$, respectively, and also formulae R on a type $C \in \mathbf{C}$ corresponding to elements $R \in P(C)$.

Now we define the translation on the internal language HoFL_P of the tripos P which allows us to describe the double negation category $\mathbf{C}_{\neg\neg}$ mentioned above. The basic strategy of translation is this: we leave everything in HoFL_P as it is, unless it involves the proposition type Ω of HoFL_P; and if something involves Ω, we always put double negation on it. Formally:

Definition 4.5.1. *We recursively define the translation on* HoFL_P *as follows. If* $\varphi : \Omega\ [\Gamma]$ *then we put* $\neg\neg$ *on every sub-formula of* φ *(do the same for* φ *seen as formulae). If* $t : \sigma\ [\Gamma, x : \Omega, \Gamma']$ *then we replace every occurrence of* x *in* t *by* $\neg\neg x$*. If* $t : \Omega \times \sigma\ [\Gamma]$ *then* t *translates into* $\langle \neg\neg\pi_1 t, \pi_2 t\rangle$*; if* $t : \sigma \times \Omega\ [\Gamma]$ *then* t *translates into* $\langle \pi_1 t, \neg\neg\pi_2 t\rangle$*. If* $t : \sigma\ [\Gamma, x : \Omega \times \sigma, \Gamma']$ *then we replace every occurrence of* x *in* t *by* $\langle \neg\neg\pi_1 x, \pi_2 x\rangle$*; if* $t : \sigma\ [\Gamma, x : \sigma \times \Omega, \Gamma']$ *then we replace every occurrence of* x *in* t *by* $\langle \pi_1 x, \neg\neg\pi_2 x\rangle$*. If* $t : \sigma \to \Omega\ [\Gamma]$ *then* t *translates into* $\lambda x : \sigma.\neg\neg tx$*; if* $t : \Omega \to \sigma\ [\Gamma]$ *then* t *translates into* $\lambda x : \Omega.t\neg\neg x$*. If* $t : \sigma\ [\Gamma, x : \sigma \to \Omega, \Gamma']$ *then we replace every occurrence of* x *in* t *by* $\lambda y : \sigma.(\neg\neg x)y$*; if* $t : \sigma\ [\Gamma, x : \Omega \to \sigma, \Gamma']$ *then we replace every occurrence of* x *in* t *by* $\lambda y : \Omega.x\neg\neg y$*. Finally, if* $t : \sigma\ [\Gamma]$ *and no* Ω *appears in it, then* t *translates into itself.*

The double negation category $\mathbf{C}_{\neg\neg}$ is then defined as follows: the objects of $\mathbf{C}_{\neg\neg}$ are contexts in HoFL_P up to α-equivalence (which are essentially the same as objects in $\mathbf{C}$), and the arrows of $\mathbf{C}_{\neg\neg}$ are the translations of lists of terms in HoFL_P up to equality on terms, with their composition defined via substitution as usual. This intuitively means that those arrows in $\mathbf{C}$ that involve Ω are double negated in $\mathbf{C}_{\neg\neg}$ whilst the other part of $\mathbf{C}_{\neg\neg}$ remains the same as that of $\mathbf{C}$ (to give the rigorous definition of this, we have worked within the internal language). Then it is not obvious that $\mathbf{C}_{\neg\neg}$ forms a category again, let alone CCC. Thus:

Lemma 4.5.2. $\mathbf{C}_{\neg\neg}$ *defined above forms a category, in particular CCC.*

Proof. Since everything involving Ω is doubly negated, we have to verify that all of the relevant categorical structures, that is, composition, identity, projection, paring, evaluation, and transpose, preserve or respect double negation.

Consider the case of composition. We think of single terms for simplicity. The composition of arrows $t : \sigma\ [x : \Omega]$ and $s : \sigma'\ [y : \sigma]$ in $\mathbf{C}_{\neg\neg}$ (which may be seen as $t : \Omega \to \sigma$ and $s : \sigma \to \sigma'$ in terms of the original category $\mathbf{C}$) is defined as $s[t/y] : \sigma'\ [x : \Omega]$, where every occurrence of x in $s[t/y]$ must have been replaced by $\neg\neg x$ (for $s[t/y]$ to be in $\mathbf{C}_{\neg\neg}$); this is true because every occurrence of x in t is replaced by $\neg\neg x$ by the definition of arrows in $\mathbf{C}_{\neg\neg}$. Likewise, the composition of arrows $t : \sigma'\ [x : \sigma]$ and $s : \Omega\ [y : \sigma']$ in $\mathbf{C}_{\neg\neg}$ is defined as $s[t/y] : \Omega\ [x : \sigma]$, where every sub-formula of $s[t/y]$ is doubly negated by the assumption of $s, t \in \mathbf{C}$; and hence $s[t/y] \in \mathbf{C}$. More complex cases can be proven in a similar way.

Consider the case of identity. Think of an identity on Ω, which is given by $\neg\neg x : \Omega\ [x : \Omega]$. Given $t : \Omega\ [y : \sigma']$ in $\mathbf{C}_{\neg\neg}$, $(\neg\neg x) \circ t$ is defined as $(\neg\neg x)[t/x] : \Omega\ [y : \sigma']$, which equals $\neg\neg t : \Omega\ [y : \sigma']$. By $t \in \mathbf{C}_{\neg\neg}$, t can be written as $\neg\neg t'$, and so $\neg\neg t = \neg\neg\neg\neg t' = \neg\neg t' = t$. Hence $(\neg\neg x) \circ t = t$. Likewise, given $t : \sigma'\ [y : \Omega]$ in $\mathbf{C}_{\neg\neg}$, $t \circ \neg\neg x$ is defined as $t[\neg\neg x/y] : \sigma'\ [x : \Omega]$; since every occurrence of y in t is replaced by $\neg\neg y$ because $t \in \mathbf{C}_{\neg\neg}$ and since $\neg\neg\neg\neg$ is equivalent to $\neg\neg$, we have $t[\neg\neg x/y] = t$, whence $t \circ \neg\neg x = t$. More complex cases can be shown in a similar manner.

To show the existence of finite products and exponentials involving Ω (otherwise it is trivial), it is crucial to check that doubly negated projection, pairing, evaluation, and transpose still play their own rôles, just as doubly negated identity still plays the rôle of identity as we have shown above. $\square$

Finally we obtain the following, tripos-theoretical Gödel translation theorem for higher-order logic, which may be applied to show the relative consistency of higher-order CL ($\sim$ set theory) to higher-order IL ($\sim$ intuitionistic set theory):

Theorem 4.5.3. *Let* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ *be an* IL *tripos, and* $\mathbf{C}_{\neg\neg}$ *the double negation category as defined above. Then,* $P_{\neg\neg}$ *defined as*

$$\mathrm{Hom}_{\mathbf{C}_{\neg\neg}}(\text{-}, \Omega) : \mathbf{C}_{\neg\neg} \to \mathbf{BA}$$

forms a CL *tripos, called the double negation tripos of* P.

Proof. $\mathbf{C}_{\neg\neg}$ is a CCC by the lemma, and $P_{\neg\neg}$ is represented by Ω. This completes the higher-order part of the proof. Concerning the first-order part, the existence of quantifiers follows from this fact: if φ admits the double negation elimination, then $\neg\neg\forall x\varphi$ and $\neg\neg\exists x\varphi$ are equivalent to $\forall x\neg\neg\varphi$ and $\exists x\neg\neg\varphi$, respectively. $\square$

Note that the tripos-theoretical Gödel translation does not reduce to the construction of toposes via double negation topology because there are more triposes than toposes in the adjunction between them (all toposes come from triposes, but not *vice versa*). Moreover, our tripos-theoretical method is designed modularly enough to be applicable to Girard's translation as well as Gödel's.

An exponential ! on an FL algebra A is defined as a unary operation satisfying: (i) $a \leq b$ implies $!a \leq !b$; (ii) $!!a = !a \leq a$; (iii) $!\top = 1$; (iv) $!a \otimes !b = !(a \wedge b)$ (Coumans et al. [70]). We denote by $\mathbf{FL}_c^!$ the category of commutative FL algebras with !, which are algebras for intuitionistic linear logic. $\mathrm{FL}_c^!$ triposes give sound and complete semantics for higher-order intuitionistic linear logic. The Girard category $\mathbf{C}_!$ of an $\mathrm{FL}_c^!$ tripos $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_c^!$ is defined by replacing double negation in the above definition of $\mathbf{C}_{\neg\neg}$ with Girard's exponential !. The following is the tripos-theoretical Girard translation theorem for higher-order logic; no such higher-order translation has been known so far.

Theorem 4.5.4. *Let $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_c^!$ be an* $\mathrm{FL}_c^!$ *tripos (for intuitionistic linear logic), and $\mathbf{C}_!$ the Girard category of P. Define $P_! = \mathrm{Hom}_{\mathbf{C}_!}(\text{-}, \Omega) : \mathbf{C}_! \to \mathbf{HA}$. Then, $P_!$ forms an* IL *tripos (i.e.,* $\mathrm{FL}_{ecw}^!$ *tripos), called the Girard tripos of P.*

This completes the developments of first-order and higher-order categorical substructural logics. In the following sections we discuss duality-theoretical models of categorical universal logic in a more general setting.

4.6 Introduction to the Second Part

Different sorts of categorical logic have been developed in the last few decades, including categorical intuitionistic logic (see, e.g., Johnstone [150]) and categorical quantum logic (see, e.g., Heunen-Jacobs [132] and Jacobs [147]). However, a unifying perspective upon various categorical logics is still lacking, which is the ultimate aim of categorical universal logic, and towards which we have been taking first steps in the present chapter.

In categorical universal logic, we rely upon the monad-relativised concept of Lawvere's hyperdoctrine [171]; the reason is as follows. Let us consider how we can unify, e.g., toposes and dagger kernel categories (in the sense of Heunen-Jacobs [132]). Although they appear to be rather different as single categories, nevertheless, the logical functorical substances of them are not so different: a topos $\mathbf{E}$ induces the subobject functor

$$\mathrm{Sub}_{\mathbf{E}}(\text{-}) : \mathbf{E}^{\mathrm{op}} \to \mathbf{HA}$$

where **HA** is the category of heyting algebras (there is an adjunction between toposes and higher-order hyperdoctrines; see Frey [98]); a dagger kernel category $\mathbf{H}$ induces the kernel subobject functor

$$\mathrm{KSub}_{\mathbf{H}}(\text{-}) : \mathbf{H}^{\mathrm{op}} \to \mathbf{OML}$$

where **OML** is the category of orthomodular lattices (subtleties on morphisms do not matter here). What is essential in interpreting logical concepts (e.g., quantifiers) is this fibrational or hyperdoctrinal structure, as is well known (see, e.g., Jacobs [145]).

We thus define a monad-relativised hyperdoctrine as a functor (or algebra-valued presheaf)

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

with suitable conditions to express logical concepts where T is a monad on **Set**, which amounts to a (possibly infinitary) variety in terms of universal algebra. We call our theory of monad-relativised hyperdoctrines (or fibred universal algebras) Categorical Universal Logic. Choosing different monads or varieties, we can treat different sorts of categorical logic. For instance, we have already shown that any axiomatic extension of the non-commutative Full Lambek calculus (see, e.g., Galatos et al. [104]), which encompasses classical, intuitionistic, linear, fuzzy, and relevant logics, can be given sound and complete semantics via the corresponding class of monad-relativised hyperdoctrine.[4]

A general question is how we can construct models of monad-relativised hyperdoctrines. We consider duality does the job; a duality means a dual adjunction. Let us

[4]Using the same method as developed above we can show that this is even true in the case of Faggian-Sambin's first-order quantum sequent calculus [90], which has both universal and existential quantifiers, moreover enjoying excellent proof-theoretic properties such as cut elimination. We may consider the calculus over either of cartesian and monoidal type theory, or quantum hyperdoctrines with either of cartesian and monoidal base categories. Note that Heunen-Jacobs [132] discusses quantified quantum logic, but does not give a completeness result with respect to any proof-theoretic calculus, and does not treat universal quantifier in an adequate manner (indeed, they prove that universal quantifier only exists in boolean dagger-kernel categories, whereas existential quantifier always exists). Note also that Faggian-Sambin's calculus can be adapted so as to express features of quantum physics and information, such as entanglement (see, e.g., Zizzi [285] and Battilotti-Zizzi [19]).

think of the well-known dual adjunction between frames **Frm** and topological spaces **Top**. Frames give the propositional logic of open sets. The predicate functor (of the dual adjunction)

$$\mathcal{O} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$$

then turns out to have existential quantifier (in Lawvere's sense). Note that topological geometric logic (i.e., the quantified logic of open sets) does not have universal quantifier, since open sets are not necessarily closed under arbitrary intersections (universal quantification is not geometric because in general it is not preserved under geometric morphisms of toposes). We thus think that duality for propositional logic is a hyperdoctrinal model of predicate logic.

In order to discuss such phenomena in a systematic way, we use Johnstone-Dimov-Tholen's duality-theoretic framework (some idea is due to Johnstone's "general concrete dualities" [150, VI.4]; however, crucial technical points have only been explicated later by Dimov-Tholen [81] and its expository companion Porst-Tholen [232]). They basically think of two concrete categories $\mathbf{C}$ and $\mathbf{D}$ (concreteness means the existence of faithful functors into **Set**), and assume Ω living in both $\mathbf{C}$ and $\mathbf{D}$, finally $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega)$ and $\mathrm{Hom}_{\mathbf{D}}(\text{-}, \Omega)$ yielding a dual adjunction between $\mathbf{C}$ and $\mathbf{D}$ (to be precise, we need a harmony condition as in Chapter 2). In our case, one of $\mathbf{C}$ and $\mathbf{D}$, say $\mathbf{D}$, is $\mathbf{Alg}(T)$. Based upon this general setting, we consider when the predicate functor

$$\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

of such a dual adjunction has a hyperdoctrine structure. We give general criteria, and apply them to concrete situations including dual adjunctions for convex and quantum structures as well as the topological one mentioned above (note that $\mathcal{O}$ above may be seen as $\mathrm{Hom}_{\mathbf{Top}}(\text{-}, \mathbf{2})$). If the base category $\mathbf{C}$ of the predicate functor is **Set**, then $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega)$ always has both universal and existential quantifiers (and higher-order structures as well).

In particular, we look at the case that Ω is the lattice of projection operators (or closed subspaces) on a Hilbert space. In this case, the set-based quantum hyperdoctrine $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ gives rise to a universe of Takeuti-Ozawa's quantum-valued sets (see Takeuti [266] and Ozawa [223]) via the tripos-to-topos construction, which is originally due to Hyland-Johnstone-Pitts [143]. We can then refine the completeness result for Faggian-Sambin's first-order quantum logic into that with respect to these set-based Tarskian models only (rather than all models).

The rest of the chapter is organised as follows. We first clarify a basic set-up for categorical universal logic, especially the concept of monad-relativised hyperdoctrine, and then goes on to discuss duality-theoretical models of monad-relativised hyperdoctrines. We illustrate the meaning of general results thus obtained with cases studies on topological geometric logic, convex geometric logic, and categorical quantum logic; these are sample applications. We also discuss how to reconcile Birkhoff-von Neumann's quantum logic and Abramsky-Coecke's categorical quantum mechanics via the idea of categorical universal logic. And finally we show a universal translation theorem on the basis of Lawvere-Tierney topology in a hyperdoctrinal setting.

4.7 Categorical Universal Logic

Categorical Universal Logic studies a monad-relativized concept of Lawvere's hyperdoctrine or Hyland-Johnstone-Pitts' tripos, which is the logical essence of topos, thereby allowing us to develop categorical semantics for various logical systems in a generic manner. The ultimate aim is not to obtain mere semantics, but to reach a universal conception of logic that is independent of particular syntax or semantics, and that integrate different syntax and semantics into the one concept.

For a monad T on **Set**, the concept of T-algebras is too general for our purposes, in the sense that it includes non-logical algebras such as groups, rings, C^*-algebras, and even compact Hausdorff spaces, though these facts themselves are merits of the concept of algebras of monads. Since our aim is at developing a theory of logics, we should focus on more logical contexts.

What is, then, the logicality of monads or their algebras? What kind of assumption is appropriate to consider that a monad or an algebra of it represents a logical structure? This leads us to the fundamental question "What is a logic?". One answer would be the following.

- Logic is deductive relation or consequence relation.
- Thus, logics and theories over them must come with relations. In our context, this requires us to assume that each T-algebra

  $$h : TA \to A$$

  is endowed with a partial order

  $$\leq_h$$

on A, which represent a deductive relation. Accordingly, homomorphisms of T-algebras must preserve those deducibility orderings.

This conception of logic is widely, if not totally, accepted in the (philosophy of) logic community. Note that, in concrete contexts, h and $\leq_h$ above are (the Lindenbaum-Tarski algebra of) a theory over logic concerned and the deductive relation of the theory respectively.

Thus, our notion of T-algebras is a general concept of ordered algebras. This may be considered to be a generic framework for algebraic logic, as opposed to universal algebra, which is concerned with algebras in full generality. In most logical systems, lattice operations, or additive connectives in Girard's terms, canonically give rise to such partial orders on T-algebras.

Instead of assuming the existence of partial orders $\leq_h$ on T-algebras h, we may suppose the following.

- The finite powerset monad is a submonad of T.

This implies that T-algebras have semilattice reducts, which canonically provide partial orders on T-algebras. Then, homomorphisms of T-algebras automatically preserve the partial ordering $\leq$ (or the logical consequence $\vdash$). It is crucial that most logical systems can be treated in this way; note that monoidal logics, including Girard's linear logic and Full Lambek calclulus, usually have additive, lattice-theoretic connectives as well as multiplicative, monoidal ones.

Further Discussion on Conceptions of Logic

Here we give several remarks on different conceptions of logic, in relation to the one above, which are not relevant to mathematical developments, and may thus be entirely skipped, especially if one does not like non-mathematical discussion.

Although some claim that logics are collections of theorems, that point of view cannot distinguish between classical logic and Graham Priest's paraconsistent logic (see Priest [238]), which have the same set of theorems. Their deductive relations are different, however. This provides a reason why logics should be conceived of as deductive relations, rather than collections of theorems.

Some others claim that logics are proof systems themselves. While such a conception of logic is dependent on proof systems, different proof systems can represent the same logic; e.g., the system of natural deduction, the sequent calculus, and the Hilbert-style calculus for classical logic all represent the same logic. In other words, the concept of logic is independent of proof systems, just as manifolds exist independent of local coordinate systems. Proof systems are methods to analyze logics, just as coordinate systems are methods to analyse manifolds. Such a line of thought can also be found in Lawvere's idea of presentation independence, which he emphasized in his categorical developments of algebraic and logical systems.

One might say that logics are not proof systems themselves, but equivalence classes of proof systems, just as manifolds are equivalence classes of coordinate systems. A crucial issue in this direction is what kind of equivalence (or the concept of isomorphism for proof systems) is appropriate (category theory might be useful for explication of this equivalence, e.g., by defining it in terms of equivalence of suitable logical categories). In one understanding, equivalence classes would be determined by deductive relations. In another, more proof-theoretic understanding, equivalence would be finer or weaker than that by deductive relations. Some proof theorists do pay (due or not so) attention to the structure of proofs in proof systems, thus distinguishing between proof systems with different concepts of identity of proofs. Even so, a necessary condition for equivalence of proof systems would be that it must make different proof systems for the same logic be equivalent.

This implies, however, that the identity of proofs must not be respected by equivalence of proof systems, since the system of natural deduction and the sequent calculus for intuitionistic logic have essentially different notions of the identity of proofs (i.e., identity via normalization and identity via cut elimination are not isomorphic; see, e.g., Zucker [286]). We may therefore conclude that equivalence of proof systems must be coarser or stronger than proof-theoretic equivalence respecting the identity of proofs, and would be rather closer to equivalence determined by deductive relations.

Monad-Relativized Hyperdoctrines

Roughly, a hyperdoctrine comes with a base category $\mathbf{C}$ and a contravariantly functorial assignment P of logical algebras to objects in $\mathbf{C}$. Here, $\mathbf{C}$ represents a type theory or a structure of domains of discourse. Given $C \in \mathbf{C}$, $P(C)$ represents an algebra of predicates or proposiitons on C, and, for an arrow $f : C \to D$ in $\mathbf{C}$, $F(f)$ translates propositions on D into those on C, and amounts to substitution from a syntactical point of view.

In the concept of hyperdoctrine, thus, types and propositions are not primarily supposed to be equivalent, in contrast to the Curry-Howard-Lambek isomorphism perspective. Types are represented by one category, and propositions by another algebraic category. The Lawverian, hyperdoctrine-based approach gives us more flexibility than the Curry-Howard-Lambek one, since the type structure and predicate structure of logic can be totally different in the concept of hyperdoctrine.

Accordingly, we can freely combine type theory and logic by means of hyperdoctrines, whereas, in the Curry-Howard-Lambek approach, type theory and logic must come in harmony from the very beginning; however, there seems to be no reason for presupposing such a priori harmony between type theory and logic. Then, a logic and a type theory should, arguably, turn out to be equivalent after their independent births even if they are equivalent.

Definition 4.7.1. *A T-hyperdoctrine is defined as an $\mathbf{Alg}(T)$-valued presheaf (or indexed T-algebra)*

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

where $\mathbf{C}$ is a (possibly monoidal) category with finite products. For an arrow f in $\mathbf{C}$, $P(f)$ is called the pullback of f. For $C \in \mathbf{C}$, $P(C)$ is called the fibre of P over C in the fibrational view of indexed categories.

We then define the following notions:

- *A T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ has universal quantifier $\forall$ iff, for any projection $\pi : X \times Y \to Y$ in $\mathbf{C}$, the following functor*

  $$P(\pi) : P(Y) \to P(X \times Y)$$

  has a right adjoint, denoted

  $$\forall_\pi : P(X \times Y) \to P(Y)$$

  and the corresponding Beck-Chevalley condition holds, i.e., the following diagram commutes for any arrow $f : Z \to Y$ in $\mathbf{C}$ ($\pi' : X \times Z \to Z$ below denotes a projection):

  $$\begin{tikzcd} P(X \times Y) \arrow[r, "\forall_\pi"] \arrow[d, "P(X\times f)"'] & P(Y) \arrow[d, "P(f)"] \\ P(X \times Z) \arrow[r, "\forall_{\pi'}"'] & P(Z) \end{tikzcd}$$

  Note that $P(X)$ and the like above are seen as categories; here we are using the "logicality of monad" assumption: T-algebras come with "deducibility" relations, which yield categorical structures on T-algebras.

- *A T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ has existential quantifier $\exists$ iff, for any projection $\pi : X \times Y \to Y$ in $\mathbf{C}$, $P(\pi) : P(Y) \to P(X \times Y)$ has a left adjoint, which shall be denoted as*

$$\exists_\pi : P(X \times Y) \to P(Y)$$

and the corresponding Beck-Chevalley diagram commutes for any $f : Z \to Y$ in $\mathbf{C}$ ($\pi' : X \times Z \to Z$ below is a projection):

$$\begin{tikzcd} P(X \times Y) \arrow[r, "\exists_\pi"] \arrow[d, "P(X\times f)"'] & P(Y) \arrow[d, "P(f)"] \\ P(X \times Z) \arrow[r, "\exists_{\pi'}"'] & P(Z) \end{tikzcd}$$

- *A T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ has equality $=$ iff, for any diagonal $\delta : X \to X \times X$ in $\mathbf{C}$, the following functor*

$$P(\delta) : P(X \times X) \to P(X)$$

has a left adjoint, which shall be denoted as

$$\mathrm{Eq}_\delta : P(X) \to P(X \times X).$$

A quantified T-hyperdoctrine is defined as a T-hyperdoctrine having $\forall$ and $\exists$. A first-order T-hyperdoctrine is defined as a T-hyperdoctrine having $\forall$, $\exists$, and $=$.

In standard, categorical developments of regular, coherent, and intuitionistic logics, Frobenius Reciprocity is usually assumed as well as Beck-Chevalley conditions. Here we do not generally assume Frobenius Reciprocity.

The main reason is that Frobenius Reciprocity is not appropriate for certain logical systems, including quantum logic; recall that the Frobenius Reciprocity condition for existential quantifier involves the distributivity of $\exists$ over $\wedge$, a kind of infinitary distributivity law ($\exists$ may be seen as infinite joins), which is not generally acceptable in quantum logic.

In sequent calculi with restricted context formulae or "visibility", like Basic Logic by Sambin et al., Frobenius Reciprocity is actually harmful to obtain complete semantics. Note that we always assume Beck-Chevalley conditions, since it is logically indispensable to interpret the substitution of terms for variables. Categorical models without Beck-Chevalley properties are inadequate as semantics of logic.

We sometimes consider quantified hyperdoctrines without one of $\exists$ and $\forall$. For example, (topological) geometric logic only has existential quantifier in general, and thus it is natural to regard T-hyperdoctrines with $\exists$ only as being already quantified in the case of geometric logic.

The principle of comprehension in set theory can be understood in categorical terms of fibration as originally discovered by Lawvere and Benabou. When we talk about comprehension, we assume that each T-algebra has a greatest element $\top$ with respect to its deducibility ordering, and that greatest elements are preserved by homomorphisms of T-algebras (the assumption that the finite power set monad is a submonad of T is sufficient for justifying these).

T-hyperdoctrines can be seen as indexed categories, since T-algebras can be regarded as categories. We are therefore able to apply the Grothendieck construction to a T-hyperdoctrine

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

thus obtaining a fibred category

$$\int P$$

which can be described as follows. An object of $\int P$ is a pair (X, a) where X is an object of $\mathbf{C}$, and a is an object of a T-algebra $P(X)$ seen as a category. An arrow of $\int P$ from (X, a) to (Y, b) is a pair (f, k) where f is an arrow in $\mathbf{C}$ from X to Y, and k is an arrow in $P(X)$ from a to $P(f)(b)$ (note that, in $P(X)$, at most one arrow exists between two objects).

Definition 4.7.2. *A T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ has comprehension $\{\text{-}\}$ iff the truth functor defined below*

$$\top : \mathbf{C} \to \int P$$

has a right adjoint, which shall be denoted as

$$\{\text{-}\} : \int P \to \mathbf{C}.$$

The truth functor $\top$ is defined as follows. Concerning the object part, $\top$ maps an object $X \in \mathbf{C}$ to an object

$$(X, \top_{P(X)}) \in \int P$$

where $\top_{P(X)}$ denotes the greatest element of a T-algebra $P(X)$. Regarding the arrow part, $\top$ maps an arrow $f : X \to Y$ in $\mathbf{C}$ to an arrow $(f, !)$ in $\int P$ from $(X, \top_{P(X)})$ to $(Y, \top_{P(Y)})$ where $!$ is a unique arrow from $\top_{P(X)}$ to $P(f)(\top_{P(Y)})$, which equals $\top_{P(X)}$.

Higher-order T-hyperdoctrines are defined by requiring additional conditions for higher type structures and truth value objects as follows.

Definition 4.7.3. *A higher-order T-hyperdoctrine or a T-tripos is defined as a T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ such that*

- *the base category $\mathbf{C}$ is Cartesian closed;*
- *P has quantifiers $\forall$, $\exists$, equality $=$, and comprehension $\{\text{-}\}$;*
- *P has an truth value object in the following sense: there exists*

  $$\Omega \in \mathbf{C}$$

  such that P is naturally equivalent to

  $$\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega)$$

  where P is seen as a $\mathbf{Set}$-valued functor by composition with the forgetful functor from $\mathbf{Alg}(T)$ to $\mathbf{Set}$. A truth value object is also called a generic object or truth value object.

If T represents intuitionistic logic, then higher-order T-hyperdoctrines (aka. T-triposes) are almost equivalent to toposes (technically, there is a little bit more triposes than toposes; for a precise account of the adjunction between them, see [98]). We may thus consider that the concept of higher-order T-hyperdoctrine or T-tripos logically correspond to the concept of topos relativized to a monad T.

4.8 Duality as Categorical Semantics

Let us recall the setting of duality induced by Janusian (or schizophrenic) objects in the general style of Johnstone-Dimov-Tholen. That is, we have two categories $\mathbf{C}$ and $\mathbf{D}$ with faithful functors $U : \mathbf{C} \to \mathbf{Set}$ and $V : \mathbf{D} \to \mathbf{Set}$, and a Janusian object Ω which live in both $\mathbf{C}$ and $\mathbf{D}$. Then, two representable functors $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega)$ and $\mathrm{Hom}_{\mathbf{D}}(\text{-}, \Omega)$ give us dual adjunction between $\mathbf{C}$ and $\mathbf{D}$ (under the assumption of initial lifting properties of Ω).

Now suppose that $\mathbf{D}$ is $\mathbf{Alg}(T)$ for a monad T as in our Space-Algebra duality, and that $\mathbf{C}$ has finite products. We are thus thinking of the following dual adjunction

$$\mathrm{Hom}_{\mathbf{Alg}(T)}(\text{-}, \Omega) \dashv \mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T).$$

Our proposal is to regard $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega)$ as a T-hyperdoctrine. We call T-hyperdoctrines arising in this way duality T-hyperdoctrines or Stonean T-hyperdoctrines. Note that the domain category $\mathbf{C}$ of a duality T-hyperdoctrine always comes with a faithful functor $U : \mathbf{C} \to \mathbf{Set}$.

According to our logicality assumption, every T-algebra is endowed with a partial order to represent a deductive relation. In particular, Ω is thus endowed with a partial order $\leq_{\Omega}$, which canonically induce a partial order on $\mathrm{Hom}_{\mathbf{C}}(X, \Omega)$ for $X \in \mathbf{C}$: i.e., $u \leq v$ for $u, v \in \mathrm{Hom}_{\mathbf{C}}(X, \Omega)$ iff for any $x \in U(X)$,

$$U(u)(x) \leq U(v)(x).$$

In the following, we assume that Ω is complete with respect to the ordering $\leq_{\Omega}$.

When do duality T-hyperdoctrines have logical structures such as quantifiers? The existence of adjoints of pullbacks of projections and diagonals can be shown in quite general situations, via the adjoint functor theorem, as in the following propositions.

At the same time, however, Beck-Chevalley conditions are just assumed in them, and they do not provide how those adjoints actually operate. Soon after the following three propositions, we prove more specialized propositions in which Beck-Chevalley conditions are naturally derived, and it is quite clear how those adjoints that represent logical constants operate.

Proposition 4.8.1. *Assume that a duality T-hyperdoctrine*

$$\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

satisfies the following two conditions.

- *For any $X \in \mathbf{C}$, $\mathrm{Hom}_{\mathbf{C}}(X, \Omega)$ has colimits (i.e., arbitrary joins).*
- *The faithful functor associated with $\mathbf{C}$*

$$U : \mathbf{C} \to \mathbf{Set}$$

commutes with colimits in the following sense: for any $X \in \mathbf{C}$ and any $f_i \in \mathrm{Hom}_{\mathbf{C}}(X, \Omega)$ where $i \in I$, it holds that

$$\bigvee_{i \in I}(U(f_i)) = U(\bigvee_{i \in I} f_i)$$

where $\bigvee_{i \in I}(U(f_i))$ is the meet of $\{U(f_i) \mid i \in I\}$ in $\Omega^{U(X)}$, i.e., for any $x \in X$,

$$(\bigvee_{i \in I} U(f_i))(x) = \bigvee_{i \in I}(U(f_i)(x)).$$

Then, the duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ *has universal quantifier* $\forall$, *if the corresponding Beck-Chevalley condition holds.*

Proof. Let $\pi : X \times Y \to Y$ be a projection in $\mathbf{C}$. We have to show that $\mathrm{Hom}_{\mathbf{C}}(\pi, \Omega)$ has a right adjoint. In order to prove this, it suffices to show that $\mathrm{Hom}_{\mathbf{C}}(\pi, \Omega)$ preserves every colimit, i.e., join. Let $\{f_i \mid i \in I\} \subset \mathrm{Hom}_{\mathbf{C}}(Y, \Omega)$. Since we have

$$\mathrm{Hom}_{\mathbf{C}}(\pi, \Omega)(\bigvee_{i \in I} f_i) = (\bigvee_{i \in I} f_i) \circ \pi$$

and since U is faithful, it is enough to prove the following in order to show that $\mathrm{Hom}_{\mathbf{C}}(\pi, \Omega)$ preserves any join:

$$U((\bigvee_{i \in I} f_i) \circ \pi) = U(\bigvee_{i \in I} f_i \circ \pi).$$

This is shown as follows: for $z \in U(X \times Y)$,

$$\begin{aligned}
U((\bigvee_{i \in I} f_i) \circ \pi)(y) &= U(\bigvee_{i \in I} f_i) \circ U(\pi)(z) \\
&= (\bigvee_{i \in I} U(f_i)) \circ U(\pi)(z) \\
&= \bigvee_{i \in I} (U(f_i)(U(\pi)(z))) \\
&= \bigvee_{i \in I} (U(f_i \circ \pi)(z)) \\
&= (\bigvee_{i \in I} U(f_i \circ \pi))(z) \\
&= U(\bigvee_{i \in I} f_i \circ \pi)(z).
\end{aligned}$$

This completes the proof. $\square$

We can prove the following in a similar way.

Proposition 4.8.2. *Assume that a duality T-hyperdoctrine* $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ *satisfies the following two conditions.*

- *For any $X \in \mathbf{C}$,* $\mathrm{Hom}_{\mathbf{C}}(X, \Omega)$ *has limits (i.e., arbitrary meets).*
- *The faithful functor $U : \mathbf{C} \to \mathbf{Set}$ commutes with limits in the following sense: for any $X \in \mathbf{C}$ and any $f_i \in \mathrm{Hom}_{\mathbf{C}}(X, \Omega)$ where $i \in I$, it holds that*

$$\bigwedge_{i \in I} (U(f_i)) = U(\bigwedge_{i \in I} f_i)$$

where $\bigwedge_{i \in I} (U(f_i))$ is the meet of $\{U(f_i) \mid i \in I\}$ in $\Omega^{U(X)}$.

Then, the duality T-hyperdoctrine has existential quantifier $\exists$, if the corresponding Beck-Chevalley condition holds.

Proposition 4.8.3. *Assume that a duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ satisfies the following two conditions.*

- *For any $X \in \mathbf{C}$, $\mathrm{Hom}_{\mathbf{C}}(X,\Omega)$ has limits.*
- *The faithful functor $U : \mathbf{C} \to \mathbf{Set}$ commutes with limits.*

Then, the duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ has equality $=$.

Proof. Let $\delta : X \to X \times X$ be a diagonal in $\mathbf{C}$. We have to show that $\mathrm{Hom}_{\mathbf{C}}(\delta,\Omega)$ has a left adjoint. In order to prove this, it suffices to show that $\mathrm{Hom}_{\mathbf{C}}(\delta,\Omega)$ preserves every meet. Let $\{f_i \mid i \in I\} \subset \mathrm{Hom}_{\mathbf{C}}(X \times X,\Omega)$. Since we have $\mathrm{Hom}_{\mathbf{C}}(\delta,\Omega)(\bigwedge_{i\in I} f_i) = (\bigwedge_{i\in I} f_i) \circ \delta$ and since U is faithful, it is enough to prove $U((\bigwedge_{i\in I} f_i) \circ \pi) = U(\bigwedge_{i\in I} f_i \circ \pi)$. This is shown as follows: for $x \in U(X)$,

$$\begin{aligned}
U((\bigwedge_{i\in I} f_i) \circ \delta)(x) &= U(\bigwedge_{i\in I} f_i) \circ U(\delta)(x) \\
&= (\bigwedge_{i\in I} U(f_i)) \circ U(\delta)(x) \\
&= \bigwedge_{i\in I} (U(f_i \circ \delta)(x)) \\
&= U(\bigwedge_{i\in I} f_i \circ \delta)(x).
\end{aligned}$$

This completes the proof. □

In the following proposition, we use a lifting condition analogous to the initial lifting conditions in Johnstone-Dimov-Tholen's dual adjunction theorem [81, 232].

Proposition 4.8.4. *Consider a duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ such that the associated faithful functor $U : \mathbf{C} \to \mathbf{Set}$ preserves products. Given a projection $\pi : X \times Y \to Y$ in $\mathbf{C}$ and $v \in \mathrm{Hom}_{\mathbf{C}}(X \times Y,\Omega)$, we define*

$$A_v^\pi : U(Y) \to \Omega$$

as follows: for $y \in U(Y)$,

$$A_v^\pi(y) := \bigwedge\{U(v)(x,y) \mid x \in U(X)\}.$$

If "A^π_v lifts to $\forall_\pi$", i.e., there is $\forall_\pi : \mathrm{Hom}_{\mathbf{C}}(X \times Y, \Omega) \to \mathrm{Hom}_{\mathbf{C}}(Y, \Omega)$ *such that for any* $v \in \mathrm{Hom}_{\mathbf{C}}(X \times Y, \Omega)$,

$$\forall_\pi(v) \in \mathrm{Hom}_{\mathbf{C}}(Y, \Omega) \text{ and } U(\forall_\pi(v)) = A^\pi_v,$$

then the duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ *has universal quantifier* $\forall$.

Proof. We first show that $\forall_\pi$ defined above is a right adjoint of $\mathrm{Hom}_{\mathbf{C}}(\pi, \Omega)$. To show the adjointness, suppose $u \leq \forall_\pi(v)$ for $u \in \mathrm{Hom}_{\mathbf{C}}(Y, \Omega)$ and $v \in \mathrm{Hom}_{\mathbf{C}}(X \times Y, \Omega)$. Then we have

$$U(u)(y) \leq U(\forall_\pi(v))(y)$$

for $y \in Y$. The assumptions then imply that

$$U(u)(y) \leq A^\pi_v(y) = \bigwedge \{U(v)(x, y) \mid x \in U(X)\}.$$

Thus, for any $x \in X$,

$$U(u) \circ U(\pi)(x, y) \leq U(v)(x, y).$$

Since U is a functor, we obtain

$$\mathrm{Hom}_{\mathbf{C}}(\pi, \Omega)(u) = u \circ \pi \leq v.$$

The converse can be proven in a similar way.

It remains to show the following Beck-Chevalley condition: for $f : Z \to Y$, $\pi : X \times Y \to Y$, and $\pi' : X \times Z \to Z$ in $\mathbf{C}$,

$$\mathrm{Hom}_{\mathbf{C}}(f, \Omega) \circ \forall_\pi = \forall_{\pi'} \circ \mathrm{Hom}_{\mathbf{C}}(X \times f, \Omega).$$

It is then sufficient to show the following: for $v \in \mathrm{Hom}_{\mathbf{C}}(X \times Y, \Omega)$,

$$\forall_\pi(v) \circ f = \forall_{\pi'}(v \circ (X \times f)).$$

Since U is faithful, it is enough to show

$$U(\forall_\pi(v) \circ f) = U(\forall_{\pi'}(v \circ (X \times f))),$$

which is equivalent to the following: for $z \in Z$,

$$U(\forall_\pi(v)) \circ U(f)(z) = U(\forall_{\pi'}(v \circ (X \times f)))(z).$$

The assumptions imply that this is equivalent to:

$$A^\pi_v(U(f)(z)) = A^{\pi'}_{v \circ (X \times f)}(z).$$

This can be calculated as follows:

$$\bigwedge_{x \in U(X)} U(v)(x, U(f)(z)) = \bigwedge_{x \in U(X)} U(v \circ (X \times f))(x, z).$$

At the same time, we have:

$$U(v \circ (X \times f))(x, z) = U(v) \circ U(f)(z).$$

We have thus shown the Beck-Chevalley condition. $\square$

The case of existential quantifier $\exists$ can be proven in a similar way:

Proposition 4.8.5. *Consider a duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ such that $U : \mathbf{C} \to \mathbf{Set}$ preserves products. Given a projection $\pi : X \times Y \to Y$ in $\mathbf{C}$ and $v \in \mathrm{Hom}_{\mathbf{C}}(X \times Y, \Omega)$, we define $E^{\pi}_{v} : U(Y) \to \Omega$ as follows: for $y \in U(Y)$,*

$$E^{\pi}_{v}(y) := \bigvee \{U(v)(x, y) \mid x \in U(X)\}.$$

If "E^{π}_{v} lifts to $\exists_{\pi}$", i.e., there is $\exists_{\pi} : \mathrm{Hom}_{\mathbf{C}}(X \times Y, \Omega) \to \mathrm{Hom}_{\mathbf{C}}(Y, \Omega)$ such that for any $v \in \mathrm{Hom}_{\mathbf{C}}(X \times Y, \Omega)$,

$$\exists_{\pi}(v) \in \mathrm{Hom}_{\mathbf{C}}(Y, \Omega) \text{ and } U(\exists_{\pi}(v)) = A^{\pi}_{v},$$

then the duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ has existential quantifier $\exists$.

In the following case of equality, we explicitly use the least element of Ω.

Proposition 4.8.6. *Consider a duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ such that $U : \mathbf{C} \to \mathbf{Set}$ preserves products. Given a diagonal $\delta : X \to X \times X$ in $\mathbf{C}$ and $v \in \mathrm{Hom}_{\mathbf{C}}(X, \Omega)$, we define $I^{\delta}_{v} : U(X \times X) \to \Omega$ as follows: for $x, x' \in U(X)$,*

$$I^{\delta}_{v}(x, x') = \begin{cases} U(v)(x) & \text{if } x = x' \\ \bot & \text{otherwise} \end{cases}$$

If there is $\mathrm{Eq}_{\delta} : \mathrm{Hom}_{\mathbf{C}}(X, \Omega) \to \mathrm{Hom}_{\mathbf{C}}(X \times X, \Omega)$ such that for any $v \in \mathrm{Hom}_{\mathbf{C}}(X, \Omega)$,

$$\mathrm{Eq}_{\delta}(v) \in \mathrm{Hom}_{\mathbf{C}}(Y, \Omega) \text{ and } U(\mathrm{Eq}_{\delta}(v)) = I^{\delta}_{v},$$

then the duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ has equality $=$.

Proof. To show that Eq_δ with the properties above is a left adjoint of $\mathrm{Hom}_{\mathbf{C}}(\delta,\Omega)$, we first assume that

$$\mathrm{Eq}_\delta(v) \leq u$$

where $v \in \mathrm{Hom}_{\mathbf{C}}(X,\Omega)$ and $u \in \mathrm{Hom}_{\mathbf{C}}(X \times X,\Omega)$. Then we have

$$U(\mathrm{Eq}_\delta(v))(x,x') \leq U(u)(x,x').$$

By letting $x' = x$, it follows from the assumption $U(\mathrm{Eq}_\delta(v)) = I_v^\delta$ that

$$I_v^\delta(x,x) = U(v)(x) \leq U(u) \circ U(\delta)(x) = U(u \circ \delta)(x).$$

This implies that

$$v \leq u \circ \delta.$$

The converse can be shown in a similar way. □

In the case of comprehension, we make the following additional assumption on the lifting of restricted maps that originally come from arrows in $\mathbf{C}$: for any arrow $f : Y \to X$ in $\mathbf{C}$ and any $A \subset U(X)$, if there is $X' \in \mathbf{C}$ with $U(X') = A$, then the restriction of $U(f)$ to A lifts to an arrow in $\mathbf{C}$, i.e., there is an arrow $f' : Y \to X'$ in $\mathbf{C}$ such that $U(f')$ is the restriction of $U(f)$ to A. This actually holds in most concrete categories including the category of topological spaces and the category of algebras of a monad on **Set**.

Proposition 4.8.7. *Consider a duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$, its fibred category $\int \mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega)$ derived via the Grothendieck construction, and the truth functor (see Definition 4.7.2)*

$$\top : \mathbf{C} \to \int \mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega).$$

If $U(\top_{\mathrm{Hom}(X,\Omega)})(x) = \top_\Omega$ for every $X \in \mathbf{C}$ and $x \in U(X)$, and if there is a functor

$$Z : \int \mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega) \to \mathbf{C}$$

such that the following hold:

- *for $(X,v) \in \int \mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega)$, $U(Z(X,v)) = \{x \in U(X) \mid U(v)(x) = \top_\Omega\}$;*
- *for an arrow (f,k) in $\int \mathrm{Hom}_{\mathbf{C}}(\text{-},\Omega)$, $U(Z(f,k)) = U(f)$,*

then the duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ *has comprehension* $\{\text{-}\}$ *(the assumption intuitively means the correspondence* $(X, v) \mapsto \{x \in X \mid v(x) = \top_\Omega\}$ *with* $(f, k) \mapsto f$ *over* **Set** *lifts to that over* **C***).*

Proof. We show that Z is a right adjoint of the truth functor $\top$. Let

$$(f, k) : \top(Y) \to (X, v)$$

be an arrow in $\int \mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega)$ where $v \in \mathrm{Hom}_{\mathbf{C}}(X, \Omega)$. Since by definition

$$\top(Y) = (Y, \top_{\mathrm{Hom}_{\mathbf{C}}(Y,\Omega)}),$$

and since k above exists, the following must hold:

$$\mathrm{Hom}(f, \Omega)(v) = \top_{\mathrm{Hom}_{\mathbf{C}}(X,\Omega)}.$$

It follows from $v \circ f = \top_{\mathrm{Hom}_{\mathbf{C}}(X,\Omega)}$ that for any $x \in U(X)$,

$$U(v) \circ U(f)(x) = \top_\Omega.$$

Thus, the range of $U(f) : U(Y) \to U(X)$ is actually included in $U(Z(X, v))$. Define a map

$$\tilde{f} : U(Y) \to U(Z(X, v))$$

as the map obtained by restricting the range of $U(f)$ into $U(Z(X, v))$. Then, the assumption of the proposition tells us that there is an arrow

$$g : Y \to Z(X, v)$$

such that $U(g) = \tilde{f}$. The correspondence $f \mapsto g$ gives us a natural isomorphism to show that Z is a right adjoint of the truth functor. □

All the assumptions of the propositions above are satisfied if $\mathbf{C} = \mathbf{Set}$, i.e., we consider the dual adjunction between **Set** and $\mathbf{Alg}(T)$, and the corresponding duality T-hyperdoctrine $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Alg}(T)$, which turns out to be a model of higher-order logic over T.

Theorem 4.8.8. *Let* $\mathbf{C} = \mathbf{Set}$ *in a duality T-hyperdoctrine, i.e., consider*

$$\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Alg}(T),$$

(which is part of the **Set**-**Alg***-duality mentioned above). The* **Set**-**Alg***-duality T-hyperdoctrine is a higher-order T-hyperdoctrine.*

Proof. Every duality T-hyperdoctrine has an truth value object. **Set** is Cartesian closed. It thus follows from the propositions above that $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Alg}(T)$ is a higher-order T-hyperdoctrine. □

4.9 Geometric Logic, Convexity Logic, and Quantum Logic, Categorically

In this section we give sample applications of the theory above to topological geometric logic, convex geometric logic, and categorical quantum logic, thereby illustrating what the theory means in concrete situations. "Topological geometric logic" in our terms is usually called just "geometric logic" (it is the logic that is invariant under geometric morphisms of toposes). Convex geometric logic is, to our knowledge, a new concept.

Let us think of well-known dual adjunctions between topological spaces **Top** and frames **Frm**, and in particular its predicate functor

$$\mathrm{Hom}_{\mathbf{Top}}(\text{-}, \Omega) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$$

where it should be noted that not only the two-element frame **2** but also any frame Ω induces a dual adjunction between **Top** and **Frm**; this is a simple consequence of general duality theory (any of duality theories [149, 195, 81] works for this purpose).

The following is an immediate consequence of the propositions above.

Corollary 4.9.1. *The duality hyperdoctrine*

$$\mathrm{Hom}_{\mathbf{Top}}(\text{-}, \Omega) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$$

has existential quantifier $\exists$. *In particular, the open set hyperdoctrine* $\mathrm{Hom}_{\mathbf{Top}}(\text{-}, \mathbf{2})$: $\mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$ *has existential quantifier* $\exists$. *Thus, they give hyperdoctrine models of (topological) geometric logic.*

To exemplify the underlying idea of this, let us consider the simplest case of the open set functor. It is then crucial to notice that E^{π}_{v} in Proposition 4.8.5 gives us an open set by taking the inverse image of $1 \in \mathbf{2}$ under it. This is true because any topology is closed under arbitrary unions. Since a topology is not necessarily closed under arbitrary intersections, the predicate functors above do not necessarily have universal quantifier. Note that (topological) geometric logic does not have universal quantifier.

As discussed above, there are dual adjunctions between convex structures and Scott's continuous lattices (see Jacobs [146] and Chapter 2; the Jacobs duality for preframes can be recasted in terms of continuous lattices). In light of those dualities, we consider Scott's continuous lattices **ContLat** to represent pointfree convex structures, just as frames represent pointfree topological spaces.

There are two concepts of abstract convex structures, and accordingly two kinds of dual adjunctions. Let us denote by **Conv** the category of convexity spaces (for details, see van de Vel [270]), and by $\mathbf{Alg}(\mathcal{D})$ the category of algebras of the distribution monad $\mathcal{D}$, or equivalently barycentric algebras (for details, see Jacobs [146]). The following is an immediate consequence of the propositions above.

Corollary 4.9.2. *The* **Conv***-based duality hyperdoctrine*

$$\mathrm{Hom}_{\mathbf{Conv}}(\text{-}, \Omega) : \mathbf{Conv}^{\mathrm{op}} \to \mathbf{ContLat}$$

has universal quantifier $\forall$*. The* $\mathbf{Alg}(\mathcal{D})$*-based duality hyperdoctrine*

$$\mathrm{Hom}_{\mathbf{Alg}(\mathcal{D})}(\text{-}, \Omega) : \mathbf{Alg}(\mathcal{D})^{\mathrm{op}} \to \mathbf{ContLat}$$

has universal quantifier $\forall$*.*

Thus, they give hyperdoctrine models of "convex geometric logic", which does not have existential quantifier, since in general the set of convex subsets is not closed under arbitrary unions.

We can even apply the same idea to a dual adjunction between measurable spaces and σ-complete Boolean algebras (see Chapter 2 and Chapter 5).

There are different conceptions of quantum logic and its algebras. The lattice of projection operators on a Hilbert space is a standard algebra of quantum logic. We can think of different categories encompassing those standard algebras of quantum logic, including the category of orthomodular lattices, denoted **OML** and the category of effect algebras, denoted **EA**. The latter is more general than the former, and encompasses the algebras of so-called effects of a Hilbert space as well as the algebra of projection operators.

Both **OML** and **EA** are algebraic categories, i.e., can be described as categories of algebras of monads on **Set**. Effect algebras only have negation and partial disjunction, and thus they are logically less expressive than orthomodular lattices. In this section, we mainly work with **OML**, and variants of it.

Fix a Hilbert space H, and let $\mathrm{P}(H)$ denote the lattice of projection operators on H. We can see $\mathrm{P}(H)$ as a set and as an algebra, and hence we have a dual adjunction between **Set** and **OML** induced by $\mathrm{P}(H)$ ($\mathrm{Hom}_{\mathbf{Set}}(X, \mathrm{P}(H))$ is obviously closed under the pointwise operations induced from the operations of $\mathrm{P}(H)$; this amounts to the harmony condition). Let us look at the logic of the adjunction, i.e., see

$$\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \mathrm{P}(H)) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{OML}$$

as a Q-hyperdoctrine where Q is the monad corresponding to **OML**. For the brevity of description, we drop the subscript "**Set**" of "$\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \mathrm{P}(H))$".

Lemma 4.9.3. $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$ *has quantifiers* $\forall$ *and* $\exists$.

Proof. Let $\pi : X \times Y \to Y$ be a projection in **Set**. We define

$$\forall_\pi : \mathrm{Hom}(X \times Y, \mathrm{P}(H)) \to \mathrm{Hom}(Y, \mathrm{P}(H));$$
$$\forall_\pi : \mathrm{Hom}(X \times Y, \mathrm{P}(H)) \to \mathrm{Hom}(Y, \mathrm{P}(H))$$

as follows: given $v \in \mathrm{Hom}(X \times Y, \mathrm{P}(H))$ and $y \in Y$, let

$$\begin{aligned} \forall_\pi(v)(y) &:= \bigwedge\{v(x, y) \mid x \in X\}; \\ \exists_\pi(v)(y) &:= \bigvee\{v(x, y) \mid x \in X\}. \end{aligned}$$

We claim that $\forall_\pi$ and $\exists_\pi$ are right and left adjoints of $\mathrm{Hom}(\pi, \mathrm{P}(H))$ respectively. To show the case of $\forall_\pi$, assume

$$\mathrm{Hom}(\pi, \mathrm{P}(H))(u) \leq v$$

where $u \in \mathrm{Hom}(Y, \mathrm{P}(H))$ and $v \in \mathrm{Hom}(X \times Y, \mathrm{P}(H))$. For $(x, y) \in X \times Y$, we then have

$$\mathrm{Hom}(\pi, \mathrm{P}(H))(u)(x, y) = u(y) \leq v(x, y)$$

and therefore it follows that

$$u(y) \leq \bigwedge\{v(x, y) \mid x \in X\} = \forall_\pi(v)(y).$$

We have thus shown that

$$\text{if } \mathrm{Hom}(\pi, \mathrm{P}(H))(u) \leq v, \text{ then } u(y) \leq \forall_\pi(v)(y).$$

The converse can be shown by reversing the reasoning above, and so $\forall_\pi$ is a right adjoint of $\mathrm{Hom}(\pi, \mathrm{P}(H))$. In a similar way, we can prove that $\exists_\pi$ is a left adjoint of $\mathrm{Hom}(\pi, \mathrm{P}(H))$. □

Lemma 4.9.4. $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$ *has equality* $=$.

Proof. Fix a diagonal $\delta : X \to X \times X$ in **Set**. We define

$$\mathrm{Eq}_\delta : \mathrm{Hom}(X, \mathrm{P}(H)) \to \mathrm{Hom}(X \times X, \mathrm{P}(H))$$

as follows: for $v \in \mathrm{Hom}(X, \mathrm{P}(H))$ and $x, x' \in X$,

$$\mathrm{Eq}_\delta(v)(x, x') = \begin{cases} v(x) & \text{if } x = x' \\ \perp & \text{otherwise} \end{cases}$$

where $\perp$ denote the least element of $\mathrm{P}(H)$. In the following, we show that Eq_δ is a left adjoint of $\mathrm{Hom}(\delta, \mathrm{P}(H))$. Let $u \in \mathrm{Hom}(X \times X, \mathrm{P}(H))$, and $v \in \mathrm{Hom}(X, \mathrm{P}(H))$. Assume

$$\mathrm{Eq}_\delta(v) \leq u.$$

We want to show that

$$v \leq \mathrm{Hom}(\delta, \mathrm{P}(H))(u),$$

which is equivalent to the following: $v \leq u \circ \delta$, i.e., $v(x) \leq u(x, x)$ for all $x \in X$. It then follows from the assumption that

$$v(x) = \mathrm{Eq}_\delta(v)(x, x) \leq u(x, x).$$

The converse can be shown in a similar way. $\square$

Lemma 4.9.5. $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$ *has "co-equality", i.e., there is a right adjoint of* $\mathrm{Hom}(\delta, \mathrm{P}(H))$ *for any diagonal* $\delta : X \to X \times X$ *in* **Set**.

Proof. We define

$$\mathrm{CE}_\delta : \mathrm{Hom}(X, \mathrm{P}(H)) \to \mathrm{Hom}(X \times X, \mathrm{P}(H))$$

as follows: for $v \in \mathrm{Hom}(X, \mathrm{P}(H))$ and $x, x' \in X$,

$$\mathrm{CE}_\delta(v)(x, x') = \begin{cases} P & \text{if } v(x) = v(x') = P \\ \perp & \text{otherwise} \end{cases}$$

where $\top$ and $\perp$ denote the greatest and least elements of $\mathrm{P}(H)$. In the following, we show that CE_δ is a right adjoint of $\mathrm{Hom}(\delta, \mathrm{P}(H))$. Let $v \in \mathrm{Hom}(X \times X, \mathrm{P}(H))$, and $u \in \mathrm{Hom}(X, \mathrm{P}(H))$. Assume

$$\mathrm{Hom}(\delta, \mathrm{P}(H))(v) \leq u.$$

For $x \in X$, we then have

$$v \circ \delta(x) = v(x, x) \leq u(x).$$

Consider $\mathrm{CE}_\delta(u)$. If $v(x, x) = P$, then the inequality above tells us

$$P \leq u(x) = \mathrm{CE}_\delta(u)(x, x).$$

Thus we have proven $v \leq \mathrm{CE}_\delta(u)$ assuming $\mathrm{Hom}(\delta, \mathrm{P}(H))(v) \leq u$. To show the converse, suppose $v \leq \mathrm{CE}_\delta(u)$. This implies that

$$\mathrm{Hom}(\delta, \mathrm{P}(H))(v)(x) = v(x,x) \leq \mathrm{CE}_\delta(u)(x,x) = u(x).$$

This completes the proof. $\square$

Lemma 4.9.6. $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$ *has comprehension* $\{\text{-}\}$.

Proof. Consider the truth functor (see Definition 4.7.2)

$$\top : \mathbf{Set} \to \int \mathrm{Hom}(\text{-}, \mathrm{P}(H)).$$

We have to construct a right adjoint of this. Define the comprehension functor

$$\{\text{-}\} : \int \mathrm{Hom}(\text{-}, \mathrm{P}(H)) \to \mathbf{Set}$$

as follows. Given $(X, v) \in \int \mathrm{Hom}(\text{-}, \mathrm{P}(H))$, which means $X \in \mathbf{Set}$ and $v \in \mathrm{Hom}(X, \mathrm{P}(H))$, let

$$\{(X, v)\} := \{x \in X \mid v(x) = \top_{\mathrm{P}(H)}\}$$

where $\top_{\mathrm{P}(H)}$ is the greatest element of $\mathrm{P}(H)$. The arrow part of $\{\text{-}\}$ is defined by taking the first projection, i.e., (f, k) is mapped to f. To show that this gives us a right adjoint, consider an arrow

$$(f, k) : \top(Y) \to (X, v)$$

in $\int \mathrm{Hom}(\text{-}, \mathrm{P}(H))$ where $v \in \mathrm{Hom}(X, \mathrm{P}(H))$. Since by definition

$$\top(Y) = (Y, \top_{\mathrm{Hom}(Y, \mathrm{P}(H))}),$$

the following must hold because of the existence of k:

$$\mathrm{Hom}(f, \mathrm{P}(H))(v) = \top_{\mathrm{Hom}(X, \mathrm{P}(H))}.$$

This implies that the range of $f : Y \to X$ is actually included in $\{(X, v)\}$. Define a map

$$\tilde{f} : Y \to \{(X, v)\}$$

as the map obtained by restricting the range of f into $\{(X, v)\}$. The mapping $f \mapsto \tilde{f}$ gives us the natural isomorphism required for the adjunction between the truth functor and the comprehension functor. $\square$

The following proposition tells us the projection-operator-based duality hyperdoctrine $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$ forms a model of higher-order quantum logic.

Proposition 4.9.7. $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$ *is a higher-order Q-hyperdoctrine or a Q-tripos.*

Proof. It is obvious that $\mathrm{P}(H)$ is an truth value object for the Q-hyperdoctrine $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$. The base category **Set** is Cartesian closed. These together with the lemmata above tell us that $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$ is a higher-order Q-hyperdoctrine. □

Given a frame Ω, the set-based duality hyperdoctrine $\mathrm{Hom}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Frm}$ yields via the tripos-to-topos construction the Higgs topos of Ω-valued sets, or equivalently the sheaf topos on Ω, or equivalently the topos of sets in the Ω-valued model of set theory (aka. Heyting-valued models; see, e.g., Bell [27]).

Let us think of a quantum analogue of this. The tripos-to-topos construction in the present context can be defined in the same way as above, that is, as the category $\mathbf{T}(\mathrm{Hom}(\text{-}, \mathrm{P}(H)))$ of partial equivalence relations in the internal logic of a given $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$. Note that we only need deductive relations (i.e., partial orders on fibres), conjunction, and existential quantifier when defining the tripos-to-topos construction; they indeed exist in $\mathrm{Hom}(\text{-}, \mathrm{P}(H))$.

Now our question is how $\mathbf{T}(\mathrm{Hom}(\text{-}, \mathrm{P}(H)))$ compares to the known concept of Takeuti-Ozawa's quantum set theory, to be precise the $\mathrm{P}(H)$-valued model of set theory, which is defined as follows: for each ordinal α, define via the transfinite recursrion $V_\alpha = \{u \mid u : D \to \mathrm{P}(H) \text{ and } D \subset \bigcup_{\beta \leq \alpha} V_\beta\}$ and then let $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$ where Ord is the class of all ordinals. We denote by $\mathbf{Set}^{\mathrm{P}(H)}$ the category of sets in this model of set theory. We then have the following proposition.

Proposition 4.9.8. $\mathbf{T}(\mathrm{Hom}(\text{-}, \mathrm{P}(H)))$ *embeds into* $\mathbf{Set}^{\mathrm{P}(H)}$.

We finally address the issue of completeness with respect to proof-theoretic calculus, which has so far been lacking in categorical quantum logic with quantifiers (one without quantifiers has already been developed).

Let **QL** denote the category of algebras of Faggian-Sambin's propositional quantum logic FS; algebraisation of logic is automatic via the well-known methods of Abstract Algebraic Logic. For syntactic details, we refer to Faggian-Sambin [90]. FS can be quantified in the same way as Sambin's Basic Logic. The quantified FS can then be typed in the same manner as typed intuitionistic logic in Pitts [229], i.e., just as having done above. We denote by TFS^q the resulting typed quantum sequent calculus. Using the same method as in the above proof of categorical completeness for TFL^q, we can show that the class of all **QL**-hyperdoctrines $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{QL}$ gives sound and complete semantics for TFS^q. We can refine this completeness theorem into the following by focusing upon set-based duality hyperdoctrines above: The class of all set-based duality $\mathbf{QL}$-hyperdoctrines $\mathrm{Hom}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{QL}$ where $\Omega \in \mathbf{QL}$ gives sound and complete semantics for TFS^q.

We may even replace the cartesian type theory of the logic by the monoidal one, in the same way as Ambler [15] considers logic over monoidal type theory. This is a merit of the hyperdoctrine approach, in which logic and type theory are separated, and can be chosen independently of each other. That is, we choose Faggian-Sambin's quantum calculus for the logic part, and Ambler's linear type theory for the type theory part, which amounts to SMCC (symmetric monoidal closed categories). Accordingly, the base category of a hyperdoctrine is taken to be an SMCC with finite products; note that we still keep cartesian products for the purpose of defining quantifiers. Let LFS^q denote the linearly typed quantum sequent calculus. The class of all $\mathbf{QL}$-hyperdoctrines $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{QL}$ over SMCC $\mathbf{C}$ with products gives sound and complete semantics for LFS^q. In the Hilbert hyperdoctrine $\mathrm{KSub} : \mathbf{Hilb}^{\mathrm{op}} \to \mathbf{QL}$, tensor $\otimes$ maps two projections $P \in \mathrm{KSub}(X)$ and $Q \in \mathrm{KSub}(Y)$ into $P \otimes Q \in \mathrm{KSub}(X \otimes Y)$, i.e., it functions as translation between different fibres. We could then argue that dagger-SMCC-based quantum-logic-valued hyperdoctrines enriched with a structure to express this tensor translation between fibres give a synthesis of Birkhoff-von Neumann's quantum logic and Abramsky-Coecke's categorical quantum mechanics.

This approach to the unification of Birkhoff-von Neumann's quantum logic and Abramsky-Coecke's categorical quantum mechanics via categorical universal logic shall be elaborated in subsequent papers.

4.10 Lawvere-Tierney Topology as Logical Translation

Given a topos $\mathbf{E}$ with a subobject classifier Ω, a Lawvere-Tierney topology on Ω is equivalent to a natural transformation

$$j : \mathrm{Sub}_{\mathbf{E}}(\text{-}) \to \mathrm{Sub}_{\mathbf{E}}(\text{-})$$

such that j_C is a left-exact monad on $\mathrm{Sub}_{\mathbf{C}}(C)$ for every $C \in \mathbf{C}$. Having this correspondence in mind, we define the concept of Lawvere-Tierney topology in the context of Categorical Universal Logic as follows.

Definition 4.10.1. *For a T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$, a Lawvere-Tierney topology on P is a natural transformation*

$$j : P \to P$$

such that $j_C : P(C) \to P(C)$ is a left-exact monad on $P(C)$ for each $C \in \mathbf{C}$. We also call a Lawvere-Tierney topology a Lawvere-Tierney operator.

Dually, a Lawvere-Tierney cotopology (or co-operator) for P is a natural transformation $j : P \to P$ such that $j_C : P(C) \to P(C)$ is a right-exact comonad on $P(C)$ for each $C \in \mathbf{C}$.

From a logical point of view, naturality above means that a propositional operator j commutes with substituition of terms for variables. This is precisely true in the syntactic (or classifying) **HA**-hyperdoctrine

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$$

obtained from intuitionistic logic with quantifiers. Roughly speaking, an object of $\mathbf{C}$ is a collection of typed variables, and an arrow of $\mathbf{C}$ is a term. Then, P maps a collection of variables $x_1, ..., x_n$ to the Heyting algebra of formulas $\varphi(x_1, ..., x_n)$ with those variables, and $P(t)$ is the substitution of t for relevant variables. (for details, see).

An important example of Lawvere-Tierney topology on the syntactic **HA**-hyperdoctrine P is the double negation topology $d : P \to P$ defined by

$$d_C(\varphi) = \neg\neg\varphi$$

where $C \in \mathbf{C}$ and $\varphi \in P(C)$. Note that $d_C : P(C) \to P(C)$ is a closure operator on $P(C)$, and d is indeed a natural transformation, since substitution commutes with double negation:

$$(\neg\neg\varphi)[t/x] = \neg\neg(\varphi[t/x]).$$

The double negation topology induces another functor

$$P_d : \mathbf{C}^{\mathrm{op}} \to \mathbf{BA}$$

defined as follows. For $C \in \mathbf{C}$, let

$$P_d(C) := \{\neg\neg\varphi \mid \varphi \in P(C)\}$$

which is the Boolean algebra of those $\varphi \in P(C)$ that validate double negation elimination. For an arrow $t : D \to C$ in $\mathbf{C}$, which is a term, let

$$P_d(t) : P_d(C) \to P_d(D)$$

be the restriction of $P(t)$ to $P_d(C)$, which is possible because substitution commutes with double negation. Then we have the following proposition stating that the double negation topology d transforms a quantified $\mathbf{HA}$-hyperdoctrine P into a quantified $\mathbf{BA}$-hyperdoctrine P_d.

Proposition 4.10.2. *The hyperdoctrine $P_d : \mathbf{C}^{\mathrm{op}} \to \mathbf{BA}$ defined above forms a quantified $\mathbf{BA}$-hyperdoctrine. In other words, the mapping $P \mapsto P_d$ preserves quantified hyperdoctrine structures.*

These give a categorical understanding of Gödel-Gentzen's translation from classical logic to intuitionistic logic. In the following, we abstract such such phenomena from the viewpoint of Categorical Universal Logic, finally leading to Universal Translation Theorem below. The proposition above shall be derived as a corollary of Universal Translation Theorem.

In addition to Gödel-Gentzen's translation, Universal Translation Theorem encompasses Gödel-McKinsey-Tarski's translation from intuitionistic logic to S4 modal logic, Baaz' translation from Łukasiewicz logic to classical logic, and Girard's translation from intuitionistic logic to linear logic.

Definition 4.10.3. *A Lawvere-Tierney topology $j : P \to P$ on a T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ is called idempotent iff $j_C : P(C) \to P(C)$ is an idempotent monad for every $C \in \mathbf{C}$, i.e.,*

$$j_C \circ j_C = j_C.$$

In the following we fix a T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ and an idempotent Lawvere-Tierney topology j on P. Note that double negation is idempotent in intuitionistic logic; S4 modality is idempotent; Baaz' delta operator is idempotent; and Girard's exponentials are idempotent. Thus, idempotency is a natural assumption to treat a Lawvere-Tierney topology as logical translation of such type.

Definition 4.10.4. *A Lawvere-Tierney topology $j : P \to P$ on a T-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ is said to induce algebras of a monad S on* **Set** *iff the image of $j_C : P(C) \to P(C)$ forms an algebra of S for each $C \in \mathbf{C}$.*

For example, the double negation topology induces Boolean algebras.

In the following, in order to capture the situation of logical translation which we focus on, we assume that a Lawvere-Tierney topology $j : P \to P$ induces S-algebras for some monad S.

Then, we define an $\mathbf{Alg}(S)$-valued presheaf

$$P_j : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(S)$$

as follows. For an object C in $\mathbf{C}$, let

$$P_j(C) = \mathrm{Fix}(j_C)$$

which denotes the fixpoints of $j_C : P(C) \to P(C)$, i.e., $\{j_C(\varphi) \mid \varphi \in P(C)\}$. For an arrow $f : D \to C$ in $\mathbf{C}$, let

$$P_j(f) : P_j(C) \to P_j(D)$$

be the restriction of $P(f) : P(C) \to P(D)$ to $P_j(C)$, which is possible, since the naturality of j gives us

$$j_D \circ P(f) = P(f) \circ j_C,$$

which implies that the image of $P_j(f)$ is included in $P_j(D)$; hence the consistency of the definition of $P_j(f)$.

Theorem 4.10.5. *We first assume that*

$$j_D(\forall_\pi(j_{C\times D}(\varphi))) \leq \forall_\pi(j_{C\times D}(\varphi))$$

for a projection $\pi : C \times D \to D$ in $\mathbf{C}$, $\varphi \in P_j(C \times D)$, and $\psi \in P_j(D)$. If the original T-hyperdoctrine

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$$

is a first-order T-hyperdoctrine, then the following S-hyperdoctrine induced by a Lawvere-Tierney topology j on P

$$P_j : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(S)$$

is a first-order S-hyperdoctrine as well.

Proof. We first show the existence of universal quantifier. Assume that

$$P_j(\pi)(\psi) \leq \varphi$$

for a projection $\pi : C \times D \to D$ in $\mathbf{C}$, $\varphi \in P_j(C \times D)$, and $\psi \in P_j(D)$. Let $\forall_\pi : P(C \times D) \to P(D)$ denote the universal quantifier of P with respect to π. Define

$$\forall_\pi^j : P_j(C \times D) \to P_j(D)$$

by

$$\forall_\pi^j(\xi) = j_D(\forall_\pi(\xi))$$

where $\xi \in P_j(C \times D)$. Note that $\forall_\pi^j(\xi) \in P_j(D)$. Now we have

$$P_j(\pi)(\psi) = P(\pi)(\psi) \leq \varphi,$$

since $\psi \in P_j(D)$. This implies that $\psi \leq \forall_\pi(\varphi)$. Since j_D is a monad, it follows from $\forall_\pi(\varphi) \leq j_D(\forall_\pi(\varphi))$ that

$$\psi \leq j_D(\forall_\pi(\varphi)) = \forall_\pi^j(\varphi).$$

To show the converse, assume $\psi \leq \forall_\pi^j(\varphi)$. Since φ is a fixpoint of $j_{C\times D}$, we have $j_{C\times D}(\varphi) = \varphi$. Then, it holds that

$$\psi \leq j_D(\forall_\pi(j_{C\times D}(\varphi))).$$

It then follows from the assumption $j_D(\forall_\pi(j_{C\times D}(\varphi))) \leq \forall_\pi(j_{C\times D}(\varphi))$ that

$$\psi \leq \forall_\pi(j_{C\times D}(\varphi)) = \forall_\pi(\varphi).$$

Since $\forall_\pi$ is a right adjoint of $P(\pi)$, we have $P(\pi)(\psi) \leq \varphi$. It then follows from $\psi \in P_j(D)$ that $P_j(\pi)(\psi) \leq \varphi$. Thus, $\forall_\pi^j$ is a right djoint of $P_j(\pi)$.

We next show the existence of existential quantifier.

We finally show the existence of equality. For a diagonal $\delta : C \to C \times C$ in $\mathbf{C}$, let Eq_δ denote the equality of P with respect to δ. Then, define

$$\mathrm{Eq}_\delta^j : P_j(C \times C) \to P_j(C)$$

by

$$\mathrm{Eq}_\delta^j(\xi) = j_C(\mathrm{Eq}_\delta(\xi))$$

where $\xi \in P_j(C \times C)$. Note that $\mathrm{Eq}_\delta^j(\xi) \in P_j(C)$. In order to prove that this gives a left adjoint of $P_j(\delta)$, we first assume that

$$\mathrm{Eq}_\delta^j(\varphi) \leq \psi$$

for $\varphi \in P_j(C \times C)$ and $\psi \in P_j(C)$. Since j_C is a monad, we have

$$\mathrm{Eq}_\delta(\varphi) \leq j_C(\mathrm{Eq}_\delta(\varphi)) \leq \psi.$$

Since Eq_δ is a left adjoint of $P(\delta)$, it holds that $\varphi \leq P(\delta)(\psi)$. But we have $\psi \in P_j(C)$, and thus

$$\varphi \leq P_j(\delta)(\psi).$$

To prove the converse, assume $\varphi \leq P_j(\delta)(\psi)$. Then, we have $\mathrm{Eq}_\delta(\varphi) \leq \psi$. Since j_C is a monad, it follows that

$$\mathrm{Eq}^j_\delta(\varphi) = j_C(\mathrm{Eq}_\delta(\varphi)) \leq j_C(\psi) = \psi.$$

Thus, Eq^j_δ is a left adjoint of $P(\delta)$. □

The higher-order version of this universal translation theorem is left for future work; the particular case of FL triposes is shown above.

4.11 Remarks on Duality in Logic and Algebraic Geometry

Categorical logic, in particular topos theory, has shown substantial analogies between (geometric) logic and algebraic geometry. In this section let us see duality-theoretical analogies between logic and algebraic geometry, as briefly mentioned in Section **??**.

The origin of duality, presumably, is in algebraic geometry, which in turn originates in Descartes' analytic geometry. It is concerned with the relationships between polynomials and their zeros (affine varieties). In logical terms, they are formulas and their models, respectively. A symptom of duality is this: the more polynomials, the less zeros. Elaborating this relationship, we lead to Hilbert's Nullstellensatz, which shall be detailed below.

Both completeness and Hilbert's Nullstellensatz may be seen as order-theoretical dualities, and completeness and Nullstellensatz are actually equivalent, up to a certain point. Logical completeness can be reformulated as follows:

$$Form \circ Mod(T) = T.$$

$Form(\mathcal{M})$ is defined as the collection of formulae valid in any $M \in \mathcal{M}$ where $\mathcal{M}$ is a collection of models. We thus have:

$$Form \circ Mod(T) = \text{ the formulae valid in any model of } T.$$

On the other hand, Hilbert Nullstellensatz states:

$$I \circ V(J) = \sqrt{J}$$

for an ideal $J \subset k[x_1, ..., x_n]$ with k an algebraically closed field. Recall that $V(J)$ gives the collection of zeros of polynomials in an ideal J, and that $I(V)$ gives the collection of polynomials vanishing on V. Note that $\sqrt{J} := \{p \mid \exists n \in \mathbb{N}\ p^n \in J\}$, and that J is called a radical ideal iff $\sqrt{J} = J$. The analogy between completeness and Nullstellensatz would be obvious in this formulation. Why, however, is there no $\sqrt{\ }$ in logic? It is because contraction holds; in the present formulation, contraction means that if $\varphi \wedge \varphi$ is refutable, then φ refutable as well. In the following formulation, the analogy would be even more transparent:

- Completeness: $T =$ the formulas valid on $Mod(T)$.
- Nullstellensatz: $J =$ the polynomials vanishing on $V(J)$ where J is a radical ideal (this always holds in a Boolean ring).

Theories correspond to ideals, and models to zeros.

Completeness tells us order-theoretic duality between theories and models. We have:

$$T \subset T' \text{ iff } Mod(T) \supset Mod(T').$$

The same holds for $Form$. It then holds that $Form$ and Mod give a pair of (dually) adjoint functors (or Galois connection). Adding non-inclusion arrows and equipping $Mod(T)$ with a topology, this poset-level duality is upgraded into Stone duality. In this sense, Stone duality is a sophisticated form of completeness where there are arrows other than inclusions, and $Mod(T)$ is equipped with a topology (note that Stone topology preceded Zariski topology as noted in Bourbaki's history of mathematids).

Nullstellensatz, on the other hand, tells us order-theoretic duality between the radical ideals of $k[x_1, ..., x_n]$ and the affine varieties over k. By adding more arrows and equipping varieties with Zariski topologies, it becomes categorical equivalence between finitely generated reduced k-algebras and affine varieties over k. Note that ideals correspond to polynomial rings quotiented by them, and the "radical" condition on ideals to the "reduced" condition on algebras. This even extends to the scheme-theoretical duality between commutative algebras and affine schemes, in which, however, dual structures are explicitly algebraic because schemes are equipped with sheaves of algebras. Instead, the scheme-theoretical approach to duality works beyond commutative structures.

The correspondence looks clear, and yet there are actually some complications involved. There is indeed a dimensional difference: models are in Ω^κ for a cardinal κ (where Ω is a set of truth values), and varieties are in k^n for a natural number n.

Another thing is that we think of $\mathbb{F}_2$ in classical logic and yet we have to think of its algebraic closure $\bar{\mathbb{F}}_2$ in the corresponding algebraic geometry. We can nonetheless go beyond these complications, and establish the correspondence between completeness and Nullstellensatz, and between Stone duality and Hilbert duality, both up to $\mathbb{GF}(p^n)$.

We can, for example, prove that completeness follows from Nullstellensatz, in the following manner. Assume

$$v(\varphi(x_1, ..., x_n)) = 0$$

for any $\{0, 1\}$-valuation v. We can naturally consider

$$\varphi \in \mathbb{F}_2[x_1, ..., x_n]$$

by rewriting φ using 0, 1, XOR (addition) and AND (multiplication) only. Then,

$$\varphi \in I\ (\mathbb{F}_2^n) = I \circ V(J)$$

where

$$J := \langle x_1^2 - x_1, ..., x_n^2 - x_n \rangle.$$

Nullstellensatz over $\bar{\mathbb{F}}_2$ tells us:

$$I \circ V(J) = \sqrt{J}.$$

Hence, $\varphi = 0$ in $\mathbb{F}_2[x_1, ..., x_n]/\sqrt{J}$. This implies that $\neg\varphi$ is provable in any standard calculus for classical logic. Note that

$$\mathbb{F}_2[x_1, ..., x_n]/\sqrt{J}$$

can be seen as a calculus and be shown to be equivalent, with respect to provability, to LK, NK, and so fourth. Completeness thus follows from Nullstellensatz over $\bar{\mathbb{F}}_2$.

Conversely, Nullstellensatz follows from completeness. For example, consider two infinitary geometries over $\mathbb{F}_2$. One is induced by infinite coordinates $(x_1, x_2, ..., x_n, ...)$. The other is induced by infinitary multiplication. A Nullstellensatz-type theorem in the former follows from the (strong) completeness of classical propositional logic. A Nullstellensatz-type theorem in the latter follows from the completeness of infinitary logic (with respect to infinitary calculus). We need to assume that J in each Nullstellensatz contains $x_i^2 - x_i$. Note that Nullstellensatz can fail in some ∞-dimensional geometries. Exploiting the dimensional difference, therefore, logic could contribute to ∞-dimensional algebraic geometry.

The Hilbert-Stone correspondence allows us to go back and fourth between logic and algebraic geometry, thus allowing for further interactions between the two fields. For example, we have the following correspondence:

- A variety is irreducible iff its coordinate ring is an integral domain. This is known in geometry, and the logic counterpart is the following: For a theory T, $Mod(T)$ is irreducible iff T is complete.

- A variety has a unique, minimal, irreducible decomposition. This one is known in geometry as well, and the logic counterpart is the following: If T contains finitely many atomic propositions, $Mod(T)$ has a unique, minimal, irreducible decomposition.

- Compactness in logic states that $X \subset Fml$ is satisfiable iff so is any finite subset of it (Fml denotes the set of formulae). The geometric counterpart is the following: $X \subset \mathbb{F}_2[x_1, ..., x_n, ...]$ has a common zero iff so is any finite subset. Note that this is not trivial, since the ring is not Noetherian.

In general, geometry over a ring R amounts to R-valued logic. The Hilbert-Stone correspondence, however, seemingly does not hold beyond $\mathbb{GF}(p^n)$.

Chapter 5

Duality and Quantum Physics

Duality meets symmetry in the present chapter, technically through the theory of Chu spaces and the theory of coalgebras. We first develop a duality theory on the basis of Chu space theory enriched with the logical concept of closure conditions; whereas our first duality theory via categorical topology and algebra may be viewed as a sort of vast extension of the duality between spatial frames and sober space, the duality theory in this chapter can be seen as extending the duality between coatomistic frames and T_1 spaces, in which the notion of frame morphism must be changed so as to preserve maximal filters via the inverse image operation. The duality of operational quantum mechanics is of the same form, and may thus be classified as T_1-type (as opposed to sober-type). Note that algebraic varieties with Zarisiki topologies are not sober but T_1. The duality theory thus established allows us to improve upon Abramsky's representation of quantum systems and their symmetries, thus showing that the quantum symmetry groupoid embeds into a "purely coalgebraic" category (rather than a "fibred coalgebraic" category). Duality is, so to say, an underpinning of symmetry. The central tenet of the present chapter is that the quantum symmetries are the maps preserving the dynamics of measurements according to the Born rule.

5.1 Introduction to the Chapter

It is not uncommon these days to hear of applications of (the methods of) theoretical computer science to foundations of quantum physics; broadly speaking, theoretical computer science seems to be taking steps towards a new kind of "pluralistic unified science" (not monistic one in logical positivism) via the language and methodology of category theory. Among them, Abramsky [5, 7] represents quantum systems as Chu spaces and as coalgebras, giving striking characterisations of quantum symmetries based upon the classic Wigner Theorem. Revisiting his work, in the present chapter, we develop a Chu-space-based theory of dualities encompassing a form of state-observable duality in quantum physics, and thereafter improve upon his coalgebraic characterisation of quantum symmetries from our duality-theoretical perspective, in order to exhibit the meaning of duality.

In Pratt's Stone Gamut paper [236], he analyses Stone-type dualities in the language of Chu spaces, saying boldly, but with good reasons, "the notoriously difficult notion of Stone duality reduces simply to matrix transposition." The concept of Chu spaces has played significant roles in fairly broad contexts, including concurrency and semantics of linear logic; similar concepts have been used in even more diverse disciplines, like Barwise-Seligman's classifications, Sambin's formal topology and basic pairs, Scott's information systems, and state-property systems in quantum foundations. This work is inspired by Pratt's perspective on Chu spaces, extending the realm of duality theory built upon the language of Chu spaces by enriching it with a generic concept of closure conditions.

In general, we have two types of dualities, namely sober-type and T_1-type ones, between set-theoretical concepts of space and their point-free, algebraic abstractions, which shall be called point-set spaces and point-free spaces respectively. The difference between the two types of dualities in fact lies in the difference between maximal and primal spectra. Our duality theory in this chapter focuses upon T_1-type dualities between point-set and point-free spaces. The logical concept of closure conditions is contrived to the end of treating different sorts of point-set and point-free spaces in a unified manner, allowing us to discuss at once topological spaces, measurable spaces, closure spaces, convexity spaces, and so fourth. In a nutshell, the concept of closure conditions prescribes the notion of space. Whilst a typical example of sober-type duality is the well-known duality between sober spaces and spatial frames, an example of T_1-type duality is a duality between T_1 closure spaces and atomistic meet-complete lattices, including as particular instances state-observable dualities between quantum state spaces (with double negation closures) and projection operator lattices in the style of operational quantum mechanics (see Coecke and Moore [66] or Moore [208]).

Our theory of T_1-type dualities enables us to derive a number of concrete T_1-type dualities in various contexts, which include T_1-type dualities between Scott's continuous lattices and convexity spaces, between σ-complete boolean algebras and measurable spaces, and between topological spaces and frames, to name but a few. Let us illustrate by a topological example a striking difference between sober-type and T_1-type dualities. The T_1-type duality in topology is a duality between T_1

spaces and coatomistic frames in which continuous maps correspond not to frame homomorphisms but to maximal homomorphisms, which are frame homomorphisms $f : L \to L'$ such that, given a maximal join-complete ideal $M \subset L'$, $f^{-1}(M)$ is again a maximal join-complete ideal. Although the duality for T_1 spaces is not mentioned in standard references such as Johnstone [149], nevertheless, we consider it important for the reason that some spaces of interest are not sober but T_1: e.g., affine varieties in k^n with k an ACF (i.e., algebraically closed field) are non-sober T_1 spaces (if they are not singletons). Note that Bonsangue et al. [41] shows a duality for T_1 spaces via what they call observation frames, which are frames with additional structures, yet the T_1-type duality above only relies upon plain frame structures.

Whilst sober-type dualities are based upon prime spectrum "Spec", T_1-type dualities are based upon maximal spectrum "Spm". Different choices of spectrum lead to different Chu representations of algebras A concerned: maximal spectrum gives $(A, \mathrm{Spec}(A), e)$ and prime spectrum gives $(A, \mathrm{Spm}(A), e)$ where e is two-valued and defined in both cases by: $e(a, M) = 1$ iff $a \in M$. Accordingly, the corresponding classes of Chu morphisms are distinctively different: e.g., in locale theory, the Spec-based representation characterises frame homomorphisms as Chu morphisms (as shown in Pratt [236]), and the Spm-based representation characterises maximal homomorphisms as Chu morphisms (as shown in this chapter for general point-free spaces encompassing frames as just a particular instance). In this way, the Chu space formalism yields a natural account of why different concepts of homomorphisms appear in sober-type and T_1-type dualities.

As in the case above, Chu morphisms can capture different sorts of homomorphisms by choosing different Chu representations. This is true even in quantum contexts, and in particular we can represent quantum symmetries as Chu morphisms by a suitable Chu representation. Coalgebras are Chu spaces with dynamics, and we have a coalgebraic representation of quantum symmetries as well. To be precise, in moving from Chu space to coalgebras, Abramsky [7] relies upon a fibred category $\int \mathbf{F}$ obtained by gluing categories $\mathbf{Coalg}(F^Q)$'s for every $Q \in \mathbf{Set}$ where F^Q is an endofunctor on $\mathbf{Set}$. He uses the Grothendieck construction to "accommodate contravariance" within a coalgebraic framework, fully embedding the groupoid of symmetries into the fibred category $\int \mathbf{F}$.

Looking at the $\int \mathbf{F}$ representation from a duality-theoretical perspective, we consider it odd that there is no structural relationship taken into account between quantum state spaces and projection operator lattices: both are seen as mere sets. For the very reason, Q (which is a projection lattice in a quantum context) first have to be fixed in the endofunctor F^Q on **Set** (objects of which are state spaces in a quantum context), and thereafter $\mathbf{Coalg}(F^Q)$'s are glued together to accommodate contravariance regarding $Q \in \mathbf{Set}$. This two-step construction is reduced in the present chapter into a simpler, one-step one as follows.

First of all, there is actually a dual, structural relationship between quantum state spaces and projection lattices with the latter re-emerging as the fixpoints (or algebras) of double negation closures (or monads) on the former. This means that Q above can be derived, rather than independently assumed, from a closure structure, if one works on the base category of closure spaces, rather than mere sets. The closure-based reformulation of the $\int \mathbf{F}$ representation leads us to a "Born" endofunctor $\mathbf{B}$ on closure spaces, and to its coalgebra category $\mathbf{Coalg}(\mathbf{B})$, which turns out to be strictly smaller than fairly huge $\int \mathbf{F}$, but still large enough to represent the quantum symmetry groupoid, thus yielding a purely coalgebraic representation and enabling to accommodate contravariance within the single coalgebra category $\mathbf{Coalg}(\mathbf{B})$ rather than the fibred $\int \mathbf{F}$ glueing different $\mathbf{Coalg}(F^Q)$'s for all sets Q; notice that contravariance is incorporated into the dualisation process of taking the fixpoints (or algebras) of closures.

5.2 Duality and Chu Space Representation

We first review basic concepts and notations on Chu spaces and closure spaces.

Chu Spaces Let us fix a set Ω. A Chu space over Ω is a triple (S, A, e) where S and A are sets, and e is a map from $S \times A$ to Ω. Ω is called the value set, and e the evaluation map. A Chu morphism from (S, A, e) to (S', A', e') is a tuple (f^*, f_*) of two maps $f^* : S \to S'$ and $f_* : A' \to A$ such that $e(x, f_*(a')) = e'(f^*(x), a')$. The category of Chu spaces and Chu morphisms is self-dual, and forms a $*$-autonomous category, giving a fully complete model of linear logic.

For a Chu space $(S, A, e : S \times A \to \Omega)$ and $a \in A$, $e(\text{-}, a) : S \to \Omega$ is called a column of (S, A, e). We denote the set of all columns of (S, A, e) by $\mathrm{Col}(S, A, e)$. On the other hand, $e(x, \text{-}) : A \to \Omega$ is called a row of (S, A, e). We denote the set of all rows of (S, A, e) by $\mathrm{Row}(S, A, e)$. If Ω is ordered, then we equip $\mathrm{Col}(S, A, e)$ and $\mathrm{Row}(S, A, e)$ with the pointwise orderings: e.g., in the case of $\mathrm{Col}(S, A, e)$, this means that, for $a, b \in A$,

$$e(\text{-}, a) \leq e(\text{-}, b) \text{ iff } e(x, a) \leq e(x, b) \text{ for any } x \in S.$$

A Chu space (S, A, e) is called extensional iff all the columns are mutually different, i.e., if $e(x, a) = e(x, b)$ for any $x \in S$ then $a = b$. On the other hand, a Chu space (S, A, e) is called separated iff all the rows are mutually different, i.e., if $e(x, a) = e(y, a)$ for any $a \in A$ then $x = y$.

Closure Spaces Closure spaces may be seen as either a set with a closure operator or a set with a family of subsets that is closed under arbitrary intersections. We denote by $\mathcal{C}(S)$ the set of closed subsets of a closure space S, and by cl(-) the closure operator of S. In this chapter we always assume $\emptyset \in \mathcal{C}(S)$ or equivalently $\mathrm{cl}(\emptyset) = \emptyset$. Note then that there is a unique closure structure on a singleton. A map $f : S \to S'$ is called closure-preserving iff $f^{-1}(C) \in \mathcal{C}(S)$ for any $C \in \mathcal{C}(S')$ iff $f(\mathrm{cl}(A)) \subset \mathrm{cl}(f(A))$. We denote by **Clos** the category of closure spaces and closure-preserving maps, which has products and coproducts. A closure space is called T_1 iff any singleton is closed.

5.2.1 Chu Representation of Quantum Systems

Abramsky [7] represents a quantum system as a Chu space defined via the Born rule, which provides the predictive content of quantum mechanics. Given a Hilbert space H, he constructs the following Chu space over the unit interval $[0, 1]$:

$$(\mathrm{P}(H), \mathrm{L}(H), e_H : \mathrm{P}(H) \times \mathrm{L}(H) \to [0, 1])$$

where $\mathrm{P}(H)$ denotes the set of quantum states as rays (i.e., one-dimesional subspaces) in H, $\mathrm{L}(H)$ denotes the the set of projection operators (or projectors) on H, and finally the evaluation map e_H is defined as follows (let $[\varphi] = \{\alpha\varphi \mid \alpha \in \mathbb{C}\}$):

$$e_H([\varphi], P) = \frac{\langle\varphi|P\varphi\rangle}{\langle\varphi|\varphi\rangle}.$$

We consider that Chu spaces have built-in dualities, or they are dualities without structures: whilst S and A have no structure, e still specifies duality. The category of Chu spaces has duals in terms of monoidal categories; this is internal duality. Can we externalise internal duality in Chu spaces by restoring structures on S and A through e? It is an inverse problem as it were. In the quantum context, it amounts to explicating the structures of $\mathrm{P}(H)$ and $\mathrm{L}(H)$ that give (external) duality.

The first observation is the bijective correspondences:

$$\mathrm{P}(H) \simeq \{e(\varphi, \text{-}) \mid \varphi \in \mathrm{P}(H)\} \simeq \{c \in \mathrm{Col}(\mathrm{P}(H), \mathrm{L}(H), e_H) \mid \text{the precisely one 1 appears in } c\}.$$

So, the states are the atoms of $\mathrm{L}(H)$: in this way we can recover $\mathrm{P}(H)$ from $\mathrm{L}(H)$. This means $\mathrm{L}(H)$ should be equipped with the lattice structure as in Birkhoff-von-Neumann's quantum logic. Although we have $\mathrm{L}(H) \simeq \{e(\text{-}, P) \mid P \in \mathrm{L}(H)\}$, it is not clear at this stage what intrinsic structure of $\mathrm{P}(H)$ enables to recover $\mathrm{L}(H)$ from $\mathrm{P}(H)$. Let us see that a double negation operator on $\mathrm{P}(H)$ does the job.

Define $(\text{-})^{\perp} : \mathcal{P}(\mathrm{P}(H)) \to \mathcal{P}(\mathrm{P}(H))$ as follows: for $X \subset \mathrm{P}(H)$, let

$$X^{\perp} = \{[\varphi] \in \mathrm{P}(H) \mid \forall [\psi] \in \mathrm{P}(H)\ \langle \varphi | \psi \rangle = 0\}.$$

It is straightforward to see that $(\text{-})^{\perp\perp}$ is a closure operator on $\mathrm{P}(H)$. Categorically, $(\text{-})^{\perp\perp}$ is a sort of double negation monad. Taking the closed sets or algebras of $(\text{-})^{\perp\perp}$ enables us to recover $\mathrm{L}(H)$:

Proposition 5.2.1. *The lattice of closed subsets of* $\mathrm{P}(H)$, *i.e.,*

$$\{X \subset \mathrm{P}(H) \mid X^{\perp\perp} = X\},$$

is isomorphic to $\mathrm{L}(H)$. *Schematically,*

$$\mathcal{C}(\mathrm{P}(H)) \simeq \mathrm{L}(H).$$

We thus have a duality between $\mathrm{P}(H)$ qua closure space and $\mathrm{L}(H)$ qua lattice. We can reconstruct $\mathrm{P}(H)$ from $\mathrm{L}(H)$ by taking the atoms on the one hand, and $\mathrm{L}(H)$ from $\mathrm{P}(H)$ by taking the closed sets (or algebras) of $(\text{-})^{\perp\perp}$ on the other. This dualising construction generally works for T_1 closure spaces and atomistic meet-complete lattices, in particular including $\mathrm{P}(H)$ and $\mathrm{L}(H)$ respectively; orthocomplements can be added to this duality.

Categorically, we have a dual equivalence between the category of T_1 closure spaces with closure-preserving maps and the category of atomistic meet-complete lattices with maximal homomorphisms (defined below). This duality is basically known at the object level in operational quantum mechanics (see Moore [208] or Coecke and Moore [66]); nevertheless, our dualisation of arrows, i.e., the concept of maximality, may be new. In this section we aim at developing a theory of such T_1-type dualities in full generality, thereby deriving T_1-type dualities in various concrete contexts as immediate corollaries (which include the state-projector duality). We embark upon this enterprise in the next subsection.

5.2.2 Chu Theory of T_1-Type Dualities via Closure Conditions

In the following part of this section, we consider two-valued Chu spaces $(S, A, e : S \times A \to \mathbf{2})$ only, where $\mathbf{2}$ denotes $\{0, 1\}$ (with ordering $0 < 1$). This is because in the duality between states $\mathrm{P}(H)$ and property observables $\mathrm{L}(H)$ we do not need other intermediate values in $[0, 1]$; when considering duality, it suffices to take into account whether a value equals 1 or not. On the other hand, intermediate values in $[0, 1]$ play an essential role in characterising quantum symmetries coalgebraically; we need at least three values (i.e., 1, 0, and "neither 0 nor 1"). In a nutshell, duality is possibilistic, whilst symmetry is probabilistic.

In this subsection, we think of (Chu representations of) "point-set" spaces $(S, \mathcal{F})$ where $\mathcal{F} \subset \mathcal{P}(S)$, and of their "point-free" abstractions L which do not have an underlying set S whilst keeping algebraic structures corresponding to closure properties of $\mathcal{F}$. Especially, we discuss **Top**, **Set**, **Clos**, **Conv**, and **Meas** where **Conv** denotes the category of convexity spaces, which are sets S with $\mathcal{C} \subset \mathcal{P}(S)$ closed under arbitrary intersections and directed unions (quite some convex geometry can be developed based upon such abstract structures; see, e.g., van de Vel [270]); **Meas** denotes the category of measurable spaces, which are sets with $\mathcal{B} \subset \mathcal{P}(S)$ closed under complements and countable intersections.

Morphisms in all of these categories of point-set spaces are defined in the same way as continuous maps, closure-preserving maps, and measurable maps (a.k.a. Borel functions): i.e., they are $f : (S, \mathcal{F}) \to (S', \mathcal{F}')$ such that $f^{-1}(X) \in \mathcal{F}$ for any $X \in \mathcal{F}'$. Note that **Set** may be seen as the category of $(S, \mathcal{F})$ such that $\mathcal{F}$ is maximally closed, i.e., $\mathcal{F} = \mathcal{P}(S)$, with "continuous" maps as morphisms; in such a situation, any map satisfies the condition that $f^{-1}(X) \in \mathcal{F}$ for $X \in \mathcal{F}'$.

Their point-free counterparts are respectively: **Frm** (frames), **CABA** (complete atomic boolean algebras), **MCLat** (meet-complete lattices), **ContLat** (Scott's continuous lattices), and σ**BA** (σ-complete boolean algebras). Continuous lattices may be defined as meet-complete lattices with directed joins distributing over arbitrary meets (this is equivalent to the standard definition via way-below relations; see [109, Theorem I-2.7]); in the light of this, we see continuous lattices as point-free convexity spaces; later, duality justifies this view.

We emphasise that closure conditions on each type of point-set structures correspond to (possibly infinitary) algebraic operations on each type of point-free structures. An insight from our theory is that such a relationship between point-set and point-free spaces always leads us to duality; indeed, we shall show T_1-type dualities between **Top** and **Frm**; **Set** and **CABA**; **Clos** and **MCLat**; **Conv** and **ContLat**; **Meas** and σ**BA**; and even more (e.g., dcpos).

In order to treat different sorts of point-set spaces in a unified manner, we introduce a concept of closure conditions. A closure condition on $\mathcal{F} \subset \mathcal{P}(S)$ is a formula of the following form:

$$\forall \mathcal{X} \subset \mathcal{F} \ (\varphi(\mathcal{X}) \Rightarrow \mathrm{BC}(\mathcal{X}) \in \mathcal{F})$$

where $\mathrm{BC}(\mathcal{X})$ is a (possibly infinitary) boolean combination of elements of $\mathcal{X}$ and $\varphi(\mathcal{X})$ is a closed formula in the language of propositional connectives, quantifiers, equality, a binary, inclusion predicate $\subset$, and nullary, cardinality predicates[1], $\mathrm{card}_{\leq\kappa}(\mathcal{X})$ and $\mathrm{card}_{\geq\kappa}(\mathcal{X})$, for each countable cardinal κ; you may include arbitrary cardinals, though the language becomes uncountable. The domain of the intended interpretation of this language is $\mathcal{X}$, and predicates are to be interpreted in the obvious way: $X \subset Y$ with $X, Y \in \mathcal{X}$ is interpreted as saying that X is a subset of Y, $\mathrm{card}_{\leq\kappa}(\mathcal{X})$ as saying that the cardinality of $\mathcal{X}$ is less than or equal to κ, and so fourth. Note that predicates $\mathrm{card}_{=\kappa}(\mathcal{X})$, $\mathrm{card}_{<\kappa}(\mathcal{X})$, and $\mathrm{card}_{>\kappa}(\mathcal{X})$ are definable in the above language.

In this setting, for example, measurable spaces are $(S, \mathcal{F})$ such that $\mathcal{F} \subset \mathcal{P}(S)$ satisfies the following closure conditions:

$$\forall \mathcal{X} \subset \mathcal{F} \ (\mathrm{card}_{\leq\omega}(\mathcal{X}) \Rightarrow \bigcap \mathcal{X} \in \mathcal{F})$$

and

$$\forall \mathcal{X} \subset \mathcal{F} \ (\mathrm{card}(\mathcal{X}) = 1 \Rightarrow \mathcal{X}^c \in \mathcal{F})$$

where $\mathcal{X}^c$ denotes the complement of the unique element of $\mathcal{X}$. and notice that by letting $\mathcal{X} = \emptyset$ we have $\bigcap \emptyset = S \in \mathcal{F}$. Likewise, convexity spaces are $(S, \mathcal{F})$ with $\mathcal{F}$ satisfying the following:

$$\forall \mathcal{X} \subset \mathcal{F} \ (\top \Rightarrow \bigcap \mathcal{X} \in \mathcal{F})$$

and

$$\forall \mathcal{X} \subset \mathcal{F} \ (\text{“}\mathcal{X} \text{ is directed w.r.t.} \subset\text{”} \Rightarrow \bigcup \mathcal{X} \in \mathcal{F})$$

where $\top$ is any tautology and "$\mathcal{X}$ is directed w.r.t. $\subset$" is expressed as "$\forall X \, \forall Y \, \exists Z \, (X \subset Z \wedge Y \subset Z)$". It is straightforward to find closure conditions for other sorts of point-set spaces. We denote by $\mathfrak{X}_{\mathrm{top}}$ the closure conditions for **Top**, by $\mathfrak{X}_{\mathrm{meas}}$ those for **Meas**, by $\mathfrak{X}_{\mathrm{clos}}$ those for **Clos**, and by $\mathfrak{X}_{\mathrm{conv}}$ those for **Conv**.

[1]First-order logic allows us to express "there are n many elements" for each positive integer n, but cannot express certain cardinality statements (e.g., "there are at most countably many elements"; we need this when defining measurable spaces). For the very reason, we expand the language with the afore-mentioned cardinality predicates.

Let us denote by $\mathfrak{X}$ a class of closure conditions, and $(S, \mathcal{F})$ with $\mathcal{F} \subset \mathcal{P}(S)$ satisfying $\mathfrak{X}$ is called a point-set $\mathfrak{X}$-space. We always assume that $\mathfrak{X}$ contains: $\forall \mathcal{X} \subset \mathcal{F}$ $(\mathrm{card}_{=0}(\mathcal{X}) \Rightarrow \bigcup \mathcal{X} \in \mathcal{F})$. This ensures that $\emptyset$ is in $\mathcal{F}$. We denote by $\mathbf{PtSp}_{\mathfrak{X}}$ the category of point-set $\mathfrak{X}$-spaces with $\mathfrak{X}$-preserving maps (i.e., maps $f : (S, \mathcal{F}) \to (S', \mathcal{F}')$ such that $f^{-1}(X) \in \mathcal{F}$ for any $X \in \mathcal{F}'$). If this setting looks too abstract, $\mathbf{PtSp}_{\mathfrak{X}}$ in the following discussion may be thought of as any of our primary examples: **Top**, **Clos**, **Conv**, and **Meas**.

It plays a crucial role in our duality theory that φ in a closure condition can be interpreted in a point-free setting: in other words, it only talks about the mutual relationships between elements of $\mathcal{X}$, and does not mention elements of elements of $\mathcal{X}$ or any point of an observable region $X \in \mathcal{X}$ (which may be an open set, convex set, measurable set, or the like), thus allowing us to interpret it in any abstract poset $(L, \leq)$ by interpreting the subset symbol $\subset$ as a partial order $\leq$, and lead to the concept of point-free $\mathfrak{X}$-spaces as opposed to point-set ones. We call this interpretation of φ in a poset $(L, \leq)$ the point-free interpretation of φ. Note that the above language for φ is actually nothing but the language of the first-order theory of posets enriched with the cardinality predicates.

A point-set $\mathfrak{X}$-space $(S, \mathcal{F})$ can be regarded as a Chu space

$$(S, \mathcal{F}, e_{(S,\mathcal{F})} : S \times \mathcal{F} \to \mathbf{2})$$

where e is defined by:

$$e_{(S,\mathcal{F})}(x, X) = 1 \text{ iff } x \in X.$$

Our special focus is on T_1 point-set spaces: a point-set $\mathfrak{X}$-space $(S, \mathcal{F})$ is T_1 iff any singleton is in $\mathcal{F}$. When applying this definition to topology, we see a topological space as a set with a family of closed sets rather than open sets. The T_1 property of a Chu space is defined as follows.

Definition 5.2.2. *A Chu space (S, A, e) is called T_1 iff for any $x \in S$, there is $a \in A$ such that*

$$e(x, a) = 1 \textit{ and } e(y, a) = 0 \textit{ for any } y \neq x.$$

Intuitively, a above may be thought of as a region in which there is only one point, namely x, or a property that x does satisfy and any other $y \in S$ does not.

Lemma 5.2.3. *A point-set $\mathfrak{X}$-space $(S, \mathcal{F})$ is T_1 iff the corresponding Chu space $(S, \mathcal{F}, e_{(S,\mathcal{F})})$ defined above is a T_1 Chu space.*

Lemma 5.2.4. *For point-set $\mathfrak{X}$-spaces $(S, \mathcal{F})$ and $(S', \mathcal{F}')$, a tuple of maps (f, g) : $(S, \mathcal{F}, e_{(S,\mathcal{F})}) \to (S', \mathcal{F}', e_{(S',\mathcal{F}')})$ is a Chu morphism iff $g = f^{-1} : \mathcal{F}' \to \mathcal{F}$ iff f : $(S, \mathcal{F}) \to (S', \mathcal{F}')$ is $\mathfrak{X}$-preserving.*

Proof. This follows immediately from the observation that (f, g) is a Chu morphism iff $f(x) \in X$ is equivalent to $x \in g(X)$ for $x \in S$ and $X \in \mathcal{F}$. $\square$

Lemma 5.2.5. *If a Chu space (S, A, e) is T_1 and extensional, then for any $x \in S$ there is a unique $a \in A$ such that*

$$e(x, a) = 1 \textit{ and } e(y, a) = 0 \textit{ for any } y \neq x.$$

Each column $e(\text{-}, a)$ of a Chu space (S, A, e) can be regarded as a subset of S, i.e., as $\{x \in S \mid e(x, a) = 1\}$. We say that $\mathrm{Col}(S, A, e)$ satisfies closure conditions iff the corresponding family of subsets of S satisfies them. The same property can be defined for $\mathrm{Row}(S, A, e)$ as well. The following proposition shows that a broad variety of point-set spaces can be represented as Chu spaces.

Proposition 5.2.6. *The category $\mathbf{PtSp}_{\mathfrak{X}}$ is equivalent to the category of extensional Chu spaces (S, A, e) such that $\mathrm{Col}(S, A, e)$ satisfies the closure conditions $\mathfrak{X}$, denoted $\mathbf{ExtChu}_{\mathfrak{X}}$. In particular, this can be instantiated for $\mathfrak{X}_{\mathrm{top}}$, $\mathfrak{X}_{\mathrm{meas}}$, $\mathfrak{X}_{\mathrm{clos}}$, and $\mathfrak{X}_{\mathrm{conv}}$.*

Proof. We define a functor $G : \mathbf{ExtChu}_{\mathfrak{X}} \to \mathbf{PtSp}_{\mathfrak{X}}$ as follows. Given a Chu space (S, A, e) with $\mathrm{Col}(S, A, e)$ satisfying $\mathfrak{X}$, we take $G(S, A, e)$ to be the following point-set $\mathfrak{X}$-space

$$(S, \mathcal{F}_S)$$

where $\mathcal{F}_S$ denotes the set of those $X \subset S$ such that there is a column $e(\text{-}, a)$ of (S, A, e) with

$$X = \{x \in S \mid e(x, a) = 1\}$$

where such a column is unique by the extensionality of (S, A, e). By the closure property of the original Chu space, $\mathcal{F}_S$ satisfies the closure conditions $\mathfrak{X}$.

Given a Chu morphism $(f, g) : (S, A, e) \to (S', A', e')$ (between two Chu spaces satisfying $\mathfrak{X}$), we define $G(f, g) = f : (S, \mathcal{F}_S) \to (S', \mathcal{F}_{S'})$. We must prove that f is $\mathfrak{X}$-preserving, i.e., $f^{-1}(Y) \in \mathcal{F}_S$ for $Y \in \mathcal{F}_{S'}$, which is equivalent to the following: there is $a \in A$ such that

$$f^{-1}(Y) = \{x \in S \mid e(x, a) = 1\}.$$

It follows from $Y \in \mathcal{F}_{S'}$ that there is $b \in A'$ such that

$$Y = \{y \in S' \mid e'(y, b) = 1\}.$$

Now we define $a = g(b)$, which is in A. Since we have $x \in f^{-1}(Y)$ iff $e'(f(x), b) = 1$, and since the Chu morphism condition tells us that $e'(f(x), b) = 1$ iff $e(x, a) = 1$, it finally follows that $f^{-1}(Y) = \{x \in S \mid e(x, a) = 1\}$.

We then define another functor $F : \mathbf{PtSp}_{\mathfrak{X}} \to \mathbf{ExtChu}_{\mathfrak{X}}$ as follows. A point-set $\mathfrak{X}$-space $(S, \mathcal{F})$ induces a Chu space

$$F(S, \mathcal{F}) := (S, \mathcal{F}, e_{(S,\mathcal{F})}),$$

and a $\mathfrak{X}$-preserving map $f : (S, \mathcal{F}) \to (S', \mathcal{F}')$ induces a Chu morphism

$$F(f) := (f : S \to S', f^{-1} : \mathcal{F}' \to \mathcal{F}).$$

$F(f)$ is indeed a Chu morphism, since

$$e_{(S',\mathcal{F}')}(f(x), X') = 1 \text{ iff } f(x) \in X' \text{ iff } x \in f^{-1}(X') \text{ iff } e_{(S,\mathcal{C})}(x, f^{-1}(X')) = 1.$$

Here note also that $\mathrm{Col}(F(S, \mathcal{C}))$ satisfies the closure conditions $\mathfrak{X}$, and that $F(S, \mathcal{F})$ is extensional.

Now it is straightforward to see that F and G defined above give a categorical equivalence between $\mathbf{PtSp}_{\mathfrak{X}}$ and $\mathbf{ExtChu}_{\mathfrak{X}}$. $\square$

In the following, we focus on a more specific class of closure conditions. A closure condition $\forall \mathcal{X} \subset \mathcal{F}\ (\varphi(\mathcal{X}) \to \mathrm{BC}(\mathcal{X}) \in \mathcal{F})$ is called pure iff $\mathrm{BC}(\mathcal{X})$ contains precisely one of unions, intersections, and complements. A pure closure condition is monolithic, and does not blend different operations; this is true in any major example mentioned above.

In order to define point-free $\mathfrak{X}$-spaces, we let $\mathfrak{X}$ be a class of pure closure conditions satisfying the following: if a closure condition in $\mathfrak{X}$ contains complementation in its boolean combination part, then the following two closure conditions are in $\mathfrak{X}$: $\forall \mathcal{X} \subset \mathcal{F}\ (\mathrm{card}_{<\omega}(\mathcal{X}) \to \bigcap \mathcal{X} \in \mathcal{F})$ and $\forall \mathcal{X} \subset \mathcal{F}\ (\mathrm{card}_{<\omega}(\mathcal{X}) \to \bigcup \mathcal{X} \in \mathcal{F})$. These additional conditions ensure that once we have complementation on the point-set side we can define boolean negation on the point-free side. Note that, although complementation on sets is, and should be, interpreted as boolean negation on posets of subsets, nevertheless, we are not excluding intuitionistic negation (or interiors of complements of opens), which does not arise from complements in closure conditions (i.e., complements without interiors are boolean), but from unions and finite intersections in them, by which we can define intuitionistic implication, and so intuitionistic negation.

We then define a point-free $\mathfrak{X}$-space as a bounded poset $(L, \leq, 0, 1)$ satisfying the following. If a closure condition in $\mathfrak{X}$ have unions (intersections, complements) in its $\mathrm{BC}(\mathcal{X})$ under the condition φ, then we require L to have joins (meets, boolean negation) under the point-free interpretation of φ (i.e., the subset symbol $\subset$ is interpreted as $\leq$). If one closure condition in $\mathfrak{X}$ contains unions and another contains intersections under the conditions $\varphi(\mathcal{X})$ and $\psi(\mathcal{X})$ respectively, then we require L to satisfy the following (possibly infinitary) distributive law: for any doubly indexed family $\{x_{i,j} \mid i \in I,\ j \in J_i\} \subset L$ with $F := \prod_{i\in I} J_i$, if $\{x_{i,j} \mid j \in J_i\}$ denoted by L_1 and $\{\bigwedge_{i\in I} x_{i,f(i)} \mid f \in F\}$ denoted by L_2 satisfy $\varphi(L_1)$ and $\varphi(L_2)$ respectively, and if $\{x_{i,f(i)} \mid i \in I\}$ denoted by L_3 and $\{\bigvee_{j\in J_i} x_{i,j} \mid i \in I\}$ denoted by L_4 satisfy $\psi(L_3)$ and $\psi(L_4)$ respectively, then $\bigwedge_{i\in I} \bigvee_{j\in J_i} x_{i,j} = \bigvee_{f\in F} \bigwedge_{i\in I} x_{i,f(i)}$. Note that this reduces to the ordinary infinite distributive law in the case of frames, and to distributivity between meets and directed joins in the case of continuous lattices.

There is a subtlety in defining maps f preserving possibly partial operations: e.g., even if $\bigwedge X$ is defined, $\bigwedge f(X)$ is not necessarily defined. In the case of directed joins of continuous lattices, however, this causes no problem, since directedness is preserved under monotone maps, i.e., if X is directed then $\bigwedge f(X)$ is directed as well. This is also true in the case of σ-complete boolean algebras, since $\mathrm{card}_{\leq\omega}(\text{-})$ is always preserved. With these in mind, we assume: φ in each closure condition in $\mathfrak{X}$ is preserved under monotone maps, i.e., for a monotone map $f : L \to L'$ between point-free $\mathfrak{X}$-spaces L and L', if $\varphi(X)$ holds for $X \subset L$ then $\varphi(f(X))$ holds as well. Homomorphisms of point-free $\mathfrak{X}$-spaces are defined as monotone maps preserving (in general partial) operations induced from the closure conditions in $\mathfrak{X}$. The category of point-free $\mathfrak{X}$-spaces and homomorphisms is denoted by $\mathbf{PfSp}_{\mathfrak{X}}$.

For a point-free $\mathfrak{X}$-space L, we denote the set of atoms in L by $\mathrm{Spm}(L)$, which is called the maximal spectrum of L for the following reason. In the cases of **Frm**, **ContLat**, and **MCLat**, $\mathrm{Spm}(L)$ is actually isomorphic to the maximal filters or ideals with suitable completeness conditions; furthermore, the maximal spectrum of the coordinate ring of an affine variety V in k^n with k an ACF is homeomorphic to $\mathrm{Spm}(L)$ by taking L to be the closed set lattice of V. To exemplify the meaning of “completeness conditions”, let us consider **MCLat**. A meet-complete filter is defined as a filter that is closed under arbitrary meets. Since the meet-complete filters of $L \in$ **MCLat** bijectively correspond to the principal filters of L, we have an isomorphism between $\mathrm{Spm}(L)$ and the maximal meet-complete filters of L, which holds even in the presence of natural closure structures on them. Alternatively, we may also define $\mathrm{Spm}(L) = \{\uparrow a \mid a \text{ is an atom}\}$ where $\uparrow a = \{x \in L \mid a \leq x\}$. This definition is sometimes more useful than the former.

The continuous maps between T_1 spaces (e.g., affine varieties in $\mathbb{C}^n$) do not correspond to the frame homomorphisms between their open set frames, but to a more restricted class of frame homomorphisms; this exhibits a sharp difference from the case of sober spaces. A maximal homomorphism of point-free $\mathfrak{X}$-spaces is a homomorphism $f : L \to L'$ of them satisfying the maximality condition: for any $b \in \mathrm{Spm}(L')$ there is $a \in \mathrm{Spm}(L)$ such that

$$\uparrow a = f^{-1}(\uparrow b),$$

where note that such an $a \in \mathrm{Spm}(L)$ is necessarily unique. If $\mathrm{Spm}(L)$ is defined as $\{\uparrow a \mid a \text{ is an atom}\}$, then we may state maximality in a more familiar manner: $f^{-1}(M) \in \mathrm{Spm}(L)$ for any $M \in \mathrm{Spm}(L')$. The category of atomistic point-free $\mathfrak{X}$-spaces and maximal homomorphisms is denoted by $\mathbf{AtmsPfSp}_{\mathfrak{X}}$ where recall that a poset with the least element is called atomistic iff any element can be described as the join of a set of atoms. Note that atomic posets and atomistic posets are different in general.

The atomisticity of a Chu space is defined in the following way.

Definition 5.2.7. *A Chu space (A, S, e) is called atomistic iff there are $A' \subset A$ and a bijection $\eta : S \to A'$ such that*

1. *any two elements of* $\mathrm{Row}(A', S, e')$ *are incomparable (with respect to its pointwise ordering) where e' is defined by $e'(a, x) = e(a, x)$;*
2. *for any $x \in S$ and $a \in A$, $e(a, x) = 1$ iff $e(\eta(x), \text{-}) \leq e(a, \text{-})$.*

The intended meaning of A' above is $\mathrm{Spm}(A)$, or the set of atoms of A. In the context of quantum mechanics, item 1 above means that any two quantum states, when seen as one-dimensional subspaces or projectors onto them, are incomparable, and item 2 means that there is a canonical correspondence between the quantum state space $\mathrm{P}(H)$ and the projection lattice $\mathrm{L}(H)$, by mapping the quantum states to the atoms of the lattice.

Proposition 5.2.8. *A Chu space (S, A, e) is T_1 and extensional iff its dual $(A, S, \hat{e})$ is atomistic and separated where we define $\hat{e}(a, x) = e(x, a)$.*

Proof. Assume that (S, A, e) is T_1 and extensional. Obviously, $(A, S, \hat{e})$ is separated. We show it is atomistic. For each $x \in S$ we can choose $a_x \in A$ such that $e(x, a_x) = 1$ and $e(y, a_x) = 0$ for any $y \neq x$. Note that, since a_x is unique by extensionality (see Lemma 5.2.5), we do not need the axiom of choice to choose a_x for $x \in S$. Let $A' = \{a_x \mid x \in S\}$, and define $\eta : S \to A'$ by $\eta(x) = a_x$. The afore-mentioned property of a_x ensures that η is a bijection and any two elements of $\mathrm{Row}(A', S, \hat{e}')$ are incomparable. Assume $\hat{e}(a, x) = 1$. Then, $e(x, \eta(x)) = 1$, and $e(y, \eta(x)) = 0$ for $y \neq x$. By assumption, $\hat{e}(\eta(x), \text{-}) \leq \hat{e}(a, \text{-})$. Conversely, assume $\hat{e}(\eta(x), \text{-}) \leq \hat{e}(a, \text{-})$. Then, we have $1 = \hat{e}(\eta(x), x) \leq \hat{e}(a, x)$, and thus $\hat{e}(a, x) = 1$.

To show the converse, assume that $(A, S, \hat{e})$ is atomistic and separated. Extensionality is obvious. We show (S, A, e) is T_1. Fix $x \in S$. We claim that

$$e(x, \eta(x)) = 1 \text{ and that } e(y, \eta(x)) = 0 \text{ for any } y \neq x.$$

By assumption, we have: $e(x, \eta(x)) = 1$ iff $e(\text{-}, \eta(x)) \leq e(\text{-}, \eta(x))$, whence it follows that $e(x, \eta(x)) = 1$. Now, suppose for contradiction that there is $y \neq x$ such that $e(y, \eta(x)) = 1$. It then follows from assumption that

$$e(\text{-}, \eta(y)) \leq e(\text{-}, \eta(x)).$$

This is a contradiction for the following reason: $e(\text{-}, \eta(y))$ and $e(\text{-}, \eta(x))$ are different because η is bijective and $(A, S, \hat{e})$ is separated, and hence must be incomparable by assumption. We thus obtain

$$e(y, \eta(x)) = 0 \text{ for any } y \neq x.$$

□

It does not necessarily hold that (S, A, e) is T_1 iff $(A, S, \hat{e})$ is atomistic. As a corollary of the above proposition, we obtain:

Corollary 5.2.9. *If a Chu space* (A, S, e) *is atomistic and separated, and* $\mathrm{Row}(A, S, e)$ *has a least element, then* $\mathrm{Row}(A, S, e)$ *is an atomistic poset with its atoms given by* $\{e(\eta(x), \text{-}) \mid x \in S\}$.

Proof. The proof of Proposition 5.2.8 tells us that (A, S, e) is T_1, and for each $x \in S$ we have $\eta(x) \in A$ such that $e(\eta(x), x) = 1$ and, for $y \neq x$, $e(\eta(x), y) = 0$. By this property of $\eta(x)$, $\{e(\text{-}, \eta(x)) \mid x \in S\}$ gives us the required set of atoms of $\mathrm{Row}(A, S, e)$. □

Given a point-free $\mathfrak{X}$-space L, we can construct a Chu space $(L, \mathrm{Spm}(L), e_L)$ where e_L is defined by: $e_L(b, a) = 1$ iff $a \leq b$. If we define $\mathrm{Spm}(L) = \{\uparrow a \mid a \text{ is an atom}\}$, the corresponding e_L is specified by: $e_L(a, M) = 1$ iff $a \in M$.

Lemma 5.2.10. *A point-free $\mathfrak{X}$-space L is atomistic iff $(L, \mathrm{Spm}(L), e_L)$ is an atomistic Chu space.*

Proof. Corollary 5.2.9 tells us that if $(L, \mathrm{Spm}(L), e_L)$ is atomistic then L is atomistic. Assume that a point-free $\mathfrak{X}$-space L is atomistic. Item 1 in the definition of an atomistic Chu space is obvious. Item 2 follows from the fact that for any $a \in L$ and $x \in \mathrm{Spm}(L)$, $x \leq a$ iff, for any $y \in \mathrm{Spm}(L)$, $y \leq x$ implies $y \leq a$ iff $e_L(x, \text{-}) \leq e_L(a, \text{-})$; this intuitively says that all the elements of the algebra L are classified by the maximal filters or ideals of it. $\square$

If we define $\mathrm{Spm}(L) = \{\uparrow a \mid a \text{ is an atom}\}$, we can take $\tilde{f}$ in the following lemma to be f^{-1}; in this case, the alternative definition of $\mathrm{Spm}(L)$ seems more transparent than the definition of it as the set of atoms themselves.

Lemma 5.2.11. *Let L and L' be atomistic point-free $\mathfrak{X}$-spaces. A pair of maps, $(f, g) : (L, \mathrm{Spm}(L), e_L) \to (L', \mathrm{Spm}(L'), e_{L'})$, is a Chu morphism iff f is a maximal homomorphism and $g = \tilde{f}$ where $\tilde{f} : \mathrm{Spm}(L') \to \mathrm{Spm}(L)$ is such that, for any $b \in \mathrm{Spm}(L')$, $f^{-1}(\uparrow b) = \uparrow \tilde{f}(b)$ (note $\tilde{f}$ is well defined because f is maximal).*

Proof. Assume that $(f, g) : (L, \mathrm{Spm}(L), e_L) \to (L', \mathrm{Spm}(L'), e_{L'})$ is a Chu morphism. Then we have $e_L(f(x), b) = e_{L'}(x, g(b))$, and so $b \leq f(x)$ iff $g(b) \leq x$; this equivalence shall freely be exploited in the following. We first show that f is monotone. Suppose $x \leq y$ in L. In order to show $f(x) \leq f(y)$, by the atomisticity of L', it suffices to prove that for any $a \in \mathrm{Spm}(L')$, if $a \leq f(x)$ then $a \leq f(y)$. If $a \leq f(x)$, then we have $g(a) \leq x \leq y$, and hence $a \leq f(y)$.

We next show that f is a homomorphism. Suppose that L has a meet operation $\bigwedge$ under a condition φ. We must prove that $f(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} f(x_i)$ for $\{x_i \mid i \in I\} \subset L$ satisfying φ. By atomisticity, it is enough to show that for any $a \in \mathrm{Spm}(L)$, $a \leq f(\bigwedge_{i \in I} x_i)$ iff $a \leq \bigwedge_{i \in I} f(x_i)$. If $a \leq f(\bigwedge_{i \in I} x_i)$, then $g(a) \leq \bigwedge_{i \in I} x_i \leq x_i$, whence $a \leq f(x_i)$, and so $a \leq \bigwedge_{i \in I} f(x_i)$. The converse follows simply by reversing this argument.

Suppose that L has a join operation $\bigvee$ under a condition φ. We must prove that $f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$ for $\{x_i \mid i \in I\} \subset L$ satisfying φ. By atomisticity, it is enough to show that for any $a \in \mathrm{Spm}(L)$, $a \leq f(\bigvee_{i \in I} x_i)$ iff $a \leq \bigvee_{i \in I} f(x_i)$. If $a \leq f(\bigvee_{i \in I} x_i)$, then $g(b) \leq \bigvee_{i \in I} x_i$. Since $g(b)$ is an atom, there is $i \in I$ such that $g(b) \leq x_i$. We then have $b \leq f(x_i)$ and hence $b \leq \bigvee_{i \in I} f_(x_i)$.

Suppose that L has a boolean negation $\neg$ under a condition φ. We must prove that $f(\neg x) = \neg f(x)$ for $\{x\}$ satisfying φ. By atomisticity, it is enough to show that for any $a \in \mathrm{Spm}(L)$, $a \leq f(\neg x)$ iff $a \leq \neg f(x)$. Assume $a \leq f(\neg x)$. Since a is an atom in a Boolean algebra, we have either $a \leq \neg f(x)$ or $a \leq f(x)$. If $a \leq f(x)$, then $a \leq f(x) \wedge f(\neg x) = f(x \wedge \neg x) = f(0) = 0$ (note that by our previous convention boolean negation $\neg$ is only defined in the presence of $\vee$ and $\wedge$; we have already shown $f(x \wedge y) = f(x) \wedge f(y)$ and $f(0) = 0$ in the above arguments), which contradicts that a is an atom. Thus we have $a \leq \neg f(x)$. The converse follows by a similar argument.

It remains to show the maximality of f, i.e., $f^{-1}(\uparrow b) = \uparrow g(b)$, which is proven as follows: $x \in f^{-1}(\uparrow b)$ iff $f(x) \in \uparrow b$ iff $b \leq f(x)$ iff $g(b) \leq x$ iff $x \in \uparrow g(b)$. □

Proposition 5.2.12. *The category* $\mathbf{AtmsPfSp}_{\mathfrak{X}}$ *is equivalent to the category of atomistic separated Chu spaces* (A, S, e) *such that* $\mathrm{Row}(A, S, e)$ *satisfies the closure conditions* $\mathfrak{X}$*, denoted by* $\mathbf{AtmsSepChu}_{\mathfrak{X}}$*.*

Proof. We define a functor

$$F : \mathbf{AtmsPfSp}_{\mathfrak{X}} \to \mathbf{AtmsSepChu}_{\mathfrak{X}}$$

as follows. Given a point-free $\mathfrak{X}$-space L, we let $F(L) = (L, \mathrm{Spm}(L), e_L)$. Given a maximal homomorphism $f : L \to L'$, we define $F(f) = (f, \tilde{f})$. It is easy to show that $F(L)$ is a Chu space in $\mathbf{AtmsSepChu}_{\mathfrak{X}}$; note that the atomisticity of L implies both the separatedness and the atomisticity of $F(L)$.

We show that $F(f)$ is a Chu morphism, i.e., $e_{L'}(f(x), b) = e_L(x, \tilde{f}(b))$. This is equivalent to: $b \leq f(x)$ iff $\tilde{f}(b) \leq x$. If $b \leq f(x)$ then $\uparrow b \supset \uparrow f(x)$, and so $f^{-1}(\uparrow b) \supset f^{-1}(\uparrow f(x))$. Since $\uparrow \tilde{f}(b) = f^{-1}(\uparrow b)$ by the definition of $\tilde{f}$, and since $f^{-1}(\uparrow f(x)) \supset \uparrow x$, we have $\uparrow \tilde{f}(b) \supset \uparrow x$, whence it follows that $\tilde{f}(b) \leq x$. Conversely, if $\tilde{f}(b) \leq x$ then $\uparrow \tilde{f}(b) \supset \uparrow x$, and so $f^{-1}(\uparrow b) \supset \uparrow x$. We then have $x \in f^{-1}(\uparrow b)$, and hence $f(x) \in \uparrow b$, which means $b \leq f(x)$.

We define a functor

$$G : \mathbf{AtmsSepChu}_{\mathfrak{X}} \to \mathbf{AtmsPfSp}_{\mathfrak{X}}$$

as follows. Given a Chu space (A, S, e) in $\mathbf{AtmsSepChu}_{\mathfrak{X}}$, we define $G(A, S, e) = \mathrm{Row}(A, S, e)$ where $\mathrm{Row}(A, S, e)$ is ordered pointwise. Then, $G(A, S, e)$ is an atomistic point-free $\mathfrak{X}$-space. Given a Chu morphism $(f, g) : (A, S, e) \to (A', S', e')$, we define $G(f,g)=f$ by identifying $\mathrm{Row}(A,S,e)$ with A (i.e., $e(a,\text{-})$ with a); this identification is allowed because (A,S,e) is separated. We must show that $G(f,g)$ is a maximal homomorphism. Once we prove (A,S,e) and (A',S',e') are isomorphic to $(A,\mathrm{Spm}(A),e_A)$ and $(A',\mathrm{Spm}(A'),e_{A'})$ respectively, Lemma 5.2.11 tells us $G(f,g)$ is indeed a maximal homomorphism.

Let us show that (A,S,e) is isomorphic to $(A,\mathrm{Spm}(A),e_A)$. Since (A,S,e) is atomistic, we have a bijection $\eta : S \to A'$ for some $A' \subset A$, and A' is in turn the set of atoms in A when A is identified with $\mathrm{Row}(A,S,e)$; this is a consequence of Corollary 5.2.9. We thus have a canonical bijection $\varepsilon : S \to \mathrm{Spm}(A)$. Since $e(a,x)=1$ iff $e(\eta(x),\text{-}) \leq e(a,\text{-})$ iff $\varepsilon(x) \leq a$ iff $e_A(a,\varepsilon(x))=1$, it follows that (A,S,e) is isomorphic to $(A,\mathrm{Spm}(A),e_A)$.

It is straightforward to see that F and G give us a categorical equivalence between $\mathbf{AtmsPfSp}_{\mathfrak{X}}$ and $\mathbf{AtmsSepChu}_{\mathfrak{X}}$. $\square$

We finally lead to the main duality theorem, exposing and unifying T_1-type dualities in diverse contexts, including sets, topology, measurable spaces, closure spaces, domain theory, and convex geometry.

Theorem 5.2.13. $\mathrm{T}_\mathbf{1}\mathbf{ExtChu}_{\mathfrak{X}}$ *is dually equivalent to* $\mathbf{AtmsSepChu}_{\mathfrak{X}}$*; therefore,* $\mathrm{T}_\mathbf{1}\mathbf{PsSp}_{\mathfrak{X}}$ *is dually equivalent to* $\mathbf{AtmsPfSp}_{\mathfrak{X}}$*. In particular, this universal duality can be instantiated for* $\mathfrak{X}_{\mathrm{top}}$*,* $\mathfrak{X}_{\mathrm{meas}}$*,* $\mathfrak{X}_{\mathrm{clos}}$*, and* $\mathfrak{X}_{\mathrm{conv}}$*.*

Proof. The first part is a corollary of Proposition 5.2.8. The second part follows immediately from Proposition 5.2.6, Proposition 5.2.12, and the first part. $\square$

Although many sorts of point-free spaces are complete, nevertheless, the case of $\mathfrak{X}_{\mathrm{meas}}$ is different, and only requires σ-completeness. In this case, the universal duality above yields a duality between atomistic σ-complete boolean algebras and T_1 measurable spaces. As noted above, **Set** may be seen as the category of $(S,\mathcal{P}(S))$'s with measurable maps (note any map is measurable on $(S,\mathcal{P}(S))$), so that the duality for measurable spaces turns out to restrict to the classic Stone duality between **Set** and **CABA** (note "atomic" and "atomistic" are equivalent in boolean algebras). It is thus a vast globalisation of the classic Stone duality.

Furthermore, we can apply the theorem above to dcpos (with 0), which is not complete in general, by considering closure under directed unions, which yields point-set spaces $(S,\mathcal{F})$ with $\mathcal{F}$ closed under directed unions; dcpos are their duals. Likewise, preframes fall into the picture as well. We are able to derive even more dualities in the same, simple way; although some general theories of dualities require much labour in deriving concrete dualities (this is a typical complaint on abstract duality theory from the practicing duality theorist), the universal duality above immediately gives us concrete dualities of T_1-type.

The duality obtained in the case of $\mathfrak{X}_{\mathrm{top}}$ is not subsumed by the orthodox duality between sober spaces and spatial frames, since "sober" does not imply "T_1"; there are important examples of non-sober T_1 spaces, including affine varieties in k^n with the Zariski topologies where k is an ACF. As discussed in the Introduction, furthermore, the morphism part of the T_1-type duality is distinctively different from that of the sober-type one.

In the case of $\mathfrak{X}_{\mathrm{conv}}$, we obtain a duality between atomistic continuous lattices and T_1 convexity spaces, exposing a new connection between domains and convex structures. Maruyama [189] also gives closely related dualities for convexity spaces. Jacobs [145] shows a dual adjunction between preframes and algebras of the distribution monad, which are abstract convex structures as well as convexity spaces. We can actually relate the two sorts of abstract convex structures, and thus dualities for them, by several adjunctions and equivalences, though here we do not have space to work out the details.

In the case of sober-type dualities, we first have dual adjunctions for general point-free spaces, which then restrict to dualities (i.e., dual equivalences). In the case of T_1-type dualities, however, we do not have dual adjunctions behind them because we use maximal spectrum Spm rather than prime spectrum Spec. This is the reason why in this chapter we have concentrated on the Chu representation of atomistic point-free spaces, rather than point-free spaces in general. We leave it for future work to work out the dual adjunction between $\mathbf{PsSp}_{\mathfrak{X}}$ and $\mathbf{PfSp}_{\mathfrak{X}}$ which restricts to the corresponding sober-type duality.

5.3 Quantum Symmetries and Closure-Based Coalgebras

We first review the Grothendieck construction for later discussion.

Grothendieck Construction The Grothendieck construction enables us to glue different categories together into a single category, or turn an indexed category into a fibration. Given a functor $\mathbf{I} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$, we define a category

$$\int \mathbf{I} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$$

as follows (**CAT** denotes the category of (small) categories and functors). The objects of $\int \mathbf{I}$ consist of tuples (C, X) where $C \in \mathbf{C}$ and $X \in \mathbf{I}(C)$. An arrow from (C, X) to (D, Y) in $\int \mathbf{I}$ is defined as a pair (f, g) where $f : D \to C$ and $g : \mathbf{I}(f)(X) \to Y$. Finally, composition of $(f : D \to C, g : \mathbf{I}(f)(X) \to Y) : (C, X) \to (D, Y)$ and $(p : E \to D, q : \mathbf{I}(p)(Y) \to Z) : (D, Y) \to (E, Z)$ is defined as:

$$(f \circ p, q \circ \mathbf{I}(p)(g)) : (C, X) \to (E, Z).$$

Note that the type of $\mathbf{I}(p)(g)$ is $\mathbf{I}(p)(\mathbf{I}(f)(X)) \to \mathbf{I}(p)(Y)$, which in turn equals $\mathbf{I}(f \circ p)(X) \to \mathbf{I}(p)(Y)$. We call $\int \mathbf{I}$ the fibred category constructed from the indexed category $\mathbf{I}$. The obvious forgetful functor from the fibred category $\int \mathbf{I}$ to the base category $\mathbf{C}$ which maps (C, X) to C gives a fibration.

5.3.1 Born Coalgebras on Closure Spaces

Now, we define an endofunctor $\mathbf{B} : \mathbf{Clos} \to \mathbf{Clos}$ on the category of closure spaces. For a closure space X, let

$$\mathbf{B}(X) := (\{0\} + (0, 1] \times X)^{\mathcal{C}(X)}$$

where $(\{0\} + (0, 1] \times X)^{\mathcal{C}(X)}$ is the product of $\mathcal{C}(X)$-many copies of $\{0\} + (0, 1] \times X$. For a closure-preserving map $f : X \to Y$, we define a map

$$\mathbf{B}(f) : (\{0\} + (0, 1] \times X)^{\mathcal{C}(X)} \to (\{0\} + (0, 1] \times Y)^{\mathcal{C}(Y)}$$

by

$$\mathbf{B}(f)(h)(C) = (id_{\{0\}} + id_{(0,1]} \times f) \circ h \circ f^{-1}(C)$$

where $h \in (\{0\} + (0, 1] \times X)^{\mathcal{C}(X)}$ and $C \in \mathcal{C}(Y)$. In the following, we verify that $\mathbf{B} : \mathbf{Clos} \to \mathbf{Clos}$ defined in this way actually forms a functor.

Lemma 5.3.1. *For a closure-preserving map $f : X \to Y$, $\mathbf{B}(f)$ is closure-preserving.*

Proof. It is sufficient to prove that $\mathbf{B}(f)(\mathrm{cl}(Z)) \subset \mathrm{cl}(\mathbf{B}(f)(Z))$ for a subset Z of $(\{0\} + (0, 1] \times X)^{\mathcal{C}(X)}$. Let $C \in \mathcal{C}(Y)$. We then have:

$$\begin{aligned}
\mathbf{B}(f)(\mathrm{cl}(Z))(C) &= (id_{\{0\}} + id_{(0,1]} \times f) \circ \mathrm{cl}(Z) \circ f^{-1}(C) \\
&= (id_{\{0\}} + id_{(0,1]} \times f) \circ \mathrm{cl}(Z \circ f^{-1}(C)) \\
&\subset \mathrm{cl}((id_{\{0\}} + id_{(0,1]} \times f) \circ Z \circ f^{-1}(C)) \\
&= \mathrm{cl}(\mathbf{B}(f)(Z)(C)) \\
&= \mathrm{cl}(\mathbf{B}(f)(Z))(C).
\end{aligned}$$

The second equality and the fifth equality hold because those closure operators are defined for the product spaces $(\{0\}+(0,1]\times X)^{\mathcal{C}(X)}$ and $(\{0\}+(0,1]\times Y)^{\mathcal{C}(Y)}$ respectively. The third inclusion follows from the assumption that f is closure-preserving. We have thus shown:

$$\prod_{C\in\mathcal{C}(Y)} \mathbf{B}(f)(\mathrm{cl}(Z))(C) \quad \subset \quad \prod_{C\in\mathcal{C}(Y)} \mathrm{cl}(\mathbf{B}(f)(Z))(C).$$

Since $\mathrm{cl}(\mathbf{B}(f)(Z))$ is a closed subset of the product space, we actually have

$$\mathrm{cl}(\mathbf{B}(f)(Z)) = \prod_{C\in\mathcal{C}(Y)} \mathrm{cl}(\mathbf{B}(f)(Z))(C).$$

These, together with the following fact that

$$\mathbf{B}(f)(\mathrm{cl}(Z)) \subset \prod_{C\in\mathcal{C}(Y)} \mathbf{B}(f)(\mathrm{cl}(Z))(C),$$

imply that $\mathbf{B}(f)(\mathrm{cl}(Z)) \subset \mathrm{cl}(\mathbf{B}(f)(Z))$. □

Lemma 5.3.2. *Let X, Y, Z be closure spaces. (i) $\mathbf{B}(id_X) = id_{\mathbf{B}(X)}$. (ii) $\mathbf{B}(g\circ f) = \mathbf{B}(g)\circ\mathbf{B}(f)$ for closure-preserving maps $f : X \to Y$ and $g : Y \to Z$.*

Proof. Since (i) is trivial, we prove (ii) in the following. For $C \in \mathcal{C}(Z)$ and $h \in (\{0\}+(0,1]\times X)^{\mathcal{C}(X)}$, the following holds:

$$\begin{aligned}
(\mathbf{B}(g)\circ\mathbf{B}(f)(h))(C) &= (id_{\{0\}} + id_{(0,1]}\times g)\circ\mathbf{B}(f)(h)\circ g^{-1}(C) \\
&= (id_{\{0\}} + id_{(0,1]}\times g)\circ(id_{\{0\}} + id_{(0,1]}\times f)\circ h\circ f^{-1}(g^{-1}(C)) \\
&= (id_{\{0\}} + id_{(0,1]}\times g\circ f)\circ h\circ(g\circ f)^{-1}(C) \\
&= \mathbf{B}(g\circ f)(h)(C)
\end{aligned}$$

This completes the proof. □

The lemmata above show that $\mathbf{B}$ is indeed a functor.

Now, we describe primary examples of $\mathbf{B}$-coalgebras, which are of central importance in our investigation.

Example 5.3.3. *Given a Hilbert space H, we define a $\mathbf{B}$-coalgebra*

$$(\mathrm{P}(H), \alpha_H : \mathrm{P}(H) \to \mathbf{B}(\mathrm{P}(H)))$$

as follows. Let us define

$$\alpha_H : \mathrm{P}(H) \to (\{0\}+(0,1]\times\mathrm{P}(H))^{\mathcal{C}(\mathrm{P}(H))}$$

by

$$\alpha_H([\varphi])(S) = \begin{cases} 0 & \textit{if } \langle\varphi|P_S\varphi\rangle = 0 \\ (\frac{\langle\varphi|P_S\varphi\rangle}{\langle\varphi|\varphi\rangle}, [P_S\varphi]) & \textit{otherwise} \end{cases}$$

where $[\varphi] \in \mathrm{P}(H)$, $S \in \mathrm{L}(H)$ $(\simeq \mathcal{C}(\mathrm{P}(H)))$, *and* P_S *is the projection operator corresponding to* S.

The coalgebra $(\mathrm{P}(H), \alpha_H)$ expresses the dynamics of repeated Born-rule-based measurements of a quantum system represented by a Hilbert space H.

As in Abramsky [7], we define the groupoid of quantum symmetries as follows.

Definition 5.3.4. **QSym** *is the category whose objects are projective spaces of Hilbert spaces of dimension greater than* 2 *and whose arrows are semi-unitary maps identified up to a phase factor* $e^{i\theta}$.

Wigner's theorem (or Wigner-Bargmann's theorem) clarifies the physical meaning of **QSym** as follows. Note that "surjections" below are actually bijections, since injectivity is a consequence of the other properties.

Theorem 5.3.5 (Wigner's theorem [278, 21, 207])**.** **QSym** *is equivalent to the category whose objects are projective spaces of Hilbert spaces (i.e., quantum state spaces) and whose arrows are symmetry transformations (i.e., those surjections between projective spaces that preserve transition probabilities* $\frac{|\langle\varphi|\psi\rangle|^2}{|\varphi|^2|\psi|^2}$ *between quantum states* $[\varphi]$ *and* $[\psi]$*).*

Our aim is to establish a purely coalgebraic understanding of **QSym**. We remark that symmetries are of central importance in physics: they are higher laws of conservation of various physical quantities (Nöther's theorem); in quantum mechanics in particular, we can even derive the Schrödinger-equation-based dynamics of quantum systems from a continuous one-parameter group of symmetries (Stone's theorem).

5.3.2 Quantum Symmetries Are Purely Coalgebraic

For an endofunctor $G : \mathbf{C} \to \mathbf{C}$ on a category $\mathbf{C}$, let $\mathbf{Coalg}(G)$ denote the category of G-coalgebras.

Let us briefly review Abramsky's fibred category $\int \mathbf{F}$ of coalgebras in the following. For a fixed set Q, we define a functor $F^Q : \mathbf{Set} \to \mathbf{Set}$. Given a set X, let $F^Q(X) = (\{0\} + (0, 1] \times X)^Q$. The arrow part is then defined canonically.

An indexed category

$$\mathbf{F} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{CAT}$$

is then defined as follows. Given $Q \in \mathbf{Set}$, let $\mathbf{F}(Q) = \mathbf{Coalg}(F^Q)$. For a map $f : Q' \to Q$, we define a functor

$$\mathbf{F}(f) : \mathbf{Coalg}(F^Q) \to \mathbf{Coalg}(F^{Q'})$$

in the following way. Given an object $(X, \alpha : X \to F^Q(X))$ in $\mathbf{Coalg}(F^Q)$, let

$$\mathbf{F}(f)(X, \alpha) = (X, t^f_X \circ \alpha)$$

where $t^f_X : F^Q(X) \to F^{Q'}(X)$ is defined by $t^f_X(g) = g \circ f$. Given an arrow $g : (X, \alpha) \to (Y, \beta)$, let $\mathbf{F}(f)(g) = g : (X, t^f_X \circ \alpha) \to (Y, t^f_Y \circ \beta)$.

As Wigner's theorem above has the assumption of surjectivity, Abramsky [7] requires surjectivity on the first components f of morphisms (f, g) in $\int \mathbf{F}$. Let us denote by $\int \mathbf{F_s}$ the resulting category with the restricted class of morphisms. On the other hand, we require injectivity on the morphisms $f : (X, \alpha) \to (Y, \beta)$ of $\mathbf{Coalg}(\mathbf{B})$, and denote by $\mathbf{Coalg_i}(\mathbf{B})$ the resulting category with the restricted class of morphisms. The surjectivity/injectivity conditions ensure that $\mathbf{QSym}$ is not only faithfully but also fully represented in $\int \mathbf{F_s}$ and in $\mathbf{Coalg_i}(\mathbf{B})$.

In the following we observe that $\mathbf{Coalg}(\mathbf{B})$ is much smaller than $\int \mathbf{F}$, but still large enough to encompass the quantum symmetry groupoid $\mathbf{QSym}$. To be precise, it shall be shown that $\mathbf{Coalg}(\mathbf{B})$ is a non-full proper subcategory of $\int \mathbf{F}$, and that $\mathbf{QSym}$ is a full subcategory of $\mathbf{Coalg_i}(\mathbf{B})$.

We then introduce a functor $\mathbf{BF}$ from $\mathbf{Coalg}(\mathbf{B})$ to $\int \mathbf{F}$, which will turn out to be a non-full embedding of categories.

Definition 5.3.6. *The object part of* $\mathbf{BF} : \mathbf{Coalg}(\mathbf{B}) \to \int \mathbf{F}$ *is defined by*

$$\mathbf{BF}(X, \alpha : X \to \mathbf{B}(X)) = (\mathcal{C}(X), (X, \alpha) \in \mathbf{Coalg}(F^{\mathcal{C}(X)})).$$

The arrow part of $\mathbf{BF} : \mathbf{Coalg}(\mathbf{B}) \to \int \mathbf{F}$ *is defined by*

$$\mathbf{BF}(f : (X, \alpha) \to (Y, \beta)) = (f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X), \tilde{f} : \mathbf{F}(f^{-1})(X, \alpha) \to (Y, \beta))$$

where $\tilde{f}$ *has the same underlying function as* f *(i.e.,* $\tilde{f}(x) = f(x)$ *for any* $x \in X$*; thus, the difference only lies in their types).*

In order to justify the definition above, we have to verify that $\tilde{f}$ is actually a morphism in $\mathbf{Colag}(F^{\mathcal{C}(Y)})$.

The commutative diagram below would be useful to understand what is going on in the definition above and the two lemmata below.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"'] & F^{\mathcal{C}(X)}(X) \arrow[r, "t_X^{f^{-1}}"] \arrow[d, "\mathbf{B}(f)"'] & F^{\mathcal{C}(Y)}(X) \arrow[dl, "F^{\mathcal{C}(Y)}(f)"] \arrow[ddr, "t_X^{g^{-1}}"] & \\
Y \arrow[r, "\beta"] \arrow[d, "g"'] & F^{\mathcal{C}(Y)}(Y) \arrow[r, "t_Y^{g^{-1}}"'] \arrow[d, "\mathbf{B}(g)"'] & F^{\mathcal{C}(Z)}(Y) \arrow[dl, "F^{\mathcal{C}(Z)}(g)"] & \\
Z \arrow[r, "\gamma"'] & F^{\mathcal{C}(Z)}(Z) & & F^{\mathcal{C}(Z)}(X) \arrow[ll, "F^{\mathcal{C}(Z)}(g\circ f)"]
\end{tikzcd}$$

where α, β, γ are $\mathbf{B}$-coalgebras, and f, g are morphisms of $\mathbf{B}$-coalgebras.

Lemma 5.3.7. *$\tilde{f} : \mathbf{F}(f^{-1})(X, \alpha) \to (Y, \beta)$ is an arrow in $\mathbf{Colag}(F^{\mathcal{C}(Y)})$.*

Proof. For $C \in \mathcal{C}(Y)$, we have:

$$\begin{aligned}
(F^{\mathcal{C}(Y)}(\tilde{f}) \circ t_X^{f^{-1}} \circ \alpha(x))(C) &= F^{\mathcal{C}(Y)}(\tilde{f})(\alpha(x) \circ f^{-1})(C) \\
&= (id_{\{0\}} + id_{(0,1]} \times \tilde{f}) \circ \alpha(x) \circ f^{-1}(C) \\
&= (id_{\{0\}} + id_{(0,1]} \times f) \circ \alpha(x) \circ f^{-1}(C) \\
&= \mathbf{B}(f)(\alpha(x))(C) \\
&= (\beta \circ f(x))(C) \\
&= (\beta \circ \tilde{f}(x))(C).
\end{aligned}$$

This completes the proof. $\square$

We then need to check that $\mathbf{BF} : \mathbf{Coalg}(\mathbf{B}) \to \int \mathbf{F}$ forms a functor.

Lemma 5.3.8. *(i) $\mathbf{BF}(id_{(X,\alpha)}) = id_{\mathbf{BF}(X,\alpha)}$. (ii) For $f : (X, \alpha) \to (Y, \beta)$ and $g : (Y, \beta) \to (Z, \gamma)$ in $\mathbf{Coalg}(\mathbf{B})$, $\mathbf{BF}(g \circ f) = \mathbf{BF}(g) \circ \mathbf{BF}(f)$ where the latter composition is that in $\int \mathbf{F}$.*

Proof. We prove only (ii), since (i) is easier to show. By definition we have:

$$\begin{aligned}
\mathbf{BF}(g \circ f) &= ((g \circ f)^{-1}, \widetilde{g \circ f}) \\
\mathbf{BF}(g) \circ \mathbf{BF}(f) &= (f^{-1} \circ g^{-1}, \tilde{g} \circ \mathbf{F}(g^{-1})(\tilde{f})).
\end{aligned}$$

Since $\mathbf{F}(g^{-1})(\tilde{f})(x) = f(x)$ for $x \in X$, it follows that $\widetilde{g \circ f}$ and $\tilde{g} \circ \mathbf{F}(g^{-1})(\tilde{f})$ have the same underlying function. Thus it only remains to show that their types are also the same. The type of $\widetilde{g \circ f}$ is:

$$(X, t_X^{(g \circ f)^{-1}} \circ \alpha) \to (Z, \gamma).$$

The type of $\tilde{g} \circ \mathbf{F}(g^{-1})(\tilde{f})$ is

$$(X, t_X^{g^{-1}} \circ t_X^{f^{-1}} \circ \alpha) \to (Y, t_Y^{g^{-1}}) \to (Z, \gamma).$$

These, together with the fact that $t_X^{g^{-1}} \circ t_X^{f^{-1}} = t_X^{(g \circ f)^{-1}}$, complete the proof. □

Proposition 5.3.9. $\mathbf{Coalg}(\mathbf{B})$ *can be embedded into* $\int \mathbf{F}$ *via the functor* $\mathbf{BF}$. *This is not a full embedding (i.e.,* $\mathbf{BF}$ *is not full).*

Proof. If f, g are different morphisms in $\mathbf{Coalg}(\mathbf{B})$, then $\mathbf{BF}(f), \mathbf{BF}(g)$ are also different, since $\tilde{f}, \tilde{g}$ have different underlying functions; hence the faithfulness of $\mathbf{BF}$, telling us that $\mathbf{Coalg}(\mathbf{B})$ can be embedded into $\int \mathbf{F}$.

$\mathbf{BF}$ is not full for the following reason. In $\mathbf{BF}(f : (X, \alpha) \to (Y, \beta))$, transformations from $\mathcal{C}(Y)$ to $\mathcal{C}(X)$ are always inverse image maps f^{-1}, whilst, in morphisms of $\int \mathbf{F}$, transformations from $\mathcal{C}(Y)$ to $\mathcal{C}(X)$ may be arbitrary functions from $\mathcal{C}(Y)$ to $\mathcal{C}(X)$. □

The non-fullness of $\mathbf{BF}$ implies that $\mathbf{Coalg}(\mathbf{B})$ is a smaller category than $\int \mathbf{F}$ with respect to arrows as well as objects.

We now introduce a functor $\mathbf{SC}$ from $\mathbf{QSym}$ to $\mathbf{Coalg_i}(\mathbf{B})$, which will turn out to be a full embedding of categories.

Definition 5.3.10. *The object part of* $\mathbf{SC} : \mathbf{QSym} \to \mathbf{Coalg_i}(\mathbf{B})$ *is defined by*

$$\mathbf{SC}(\mathrm{P}(H)) = (\mathrm{P}(H), \alpha_H).$$

The arrow part of $\mathbf{SC} : \mathbf{QSym} \to \mathbf{Coalg_i}(\mathbf{B})$ *is defined by*

$$\mathbf{SC}(U) = U : (\mathrm{P}(H), \alpha_H) \to (\mathrm{P}(H'), \alpha_{H'})$$

where $U : \mathrm{P}(H) \to \mathrm{P}(H')$ *is a semi-unitary map from* H *to* H' *(up to a phase).*

We need to check the well-definedness of $\mathbf{SC}$.

Lemma 5.3.11. $\mathbf{SC}(U)$ *is a morphism of* $\mathbf{B}$*-coalgebras.*

Proof. By Wigner's theorem, we may conceive of a symmetry transformation U as a semi-unitary map between Hilbert spaces, up to a phase factor $e^{i\theta}$. A semi-untary map of Hilbert spaces preserves the closure operator (-)$^{\perp\perp}$ on a Hilbert space, since it is linear and preserves limits. Hence, U preserves (-)$^{\perp\perp}$ on the projective space. Moreover, we can show that

$$P_S U = U P_{U^{-1}(S)},$$

and this also implies that

$$\frac{\langle \varphi | P_{U^{-1}(S)} \varphi \rangle}{\langle \varphi | \varphi \rangle} = \frac{\langle U\varphi | P_{U^{-1}(S)} U\varphi \rangle}{\langle U\varphi | U\varphi \rangle}.$$

It thus follows that $\mathbf{SC}(U)$ is indeed a morphism of $\mathbf{B}$-coalgebras. $\square$

W finally obtain the purely coalgebraic representation of quantum symmetries **QSym** via the non-fibred, single sort of coalgebra category $\mathbf{Coalg_i}(\mathbf{B})$ based upon closure spaces.

Theorem 5.3.12. *The quantum symmetry groupoid* **QSym** *can be fully embedded into the purely coalgebraic category* $\mathbf{Coalg_i}(\mathbf{B} : \mathbf{Clos} \to \mathbf{Clos})$.

Proof. It is sufficient to show that the functor **SC** is full. Let us consider a **B**-coalgebra morphism

$$f : (\mathrm{P}(H), \alpha_H) \to (\mathrm{P}(H'), \alpha_{H'}).$$

It is enough to verify that f actually arises from a morphism in **QSym** via the functor **SC**. Now, we have the following, commutative diagram.

$$\begin{tikzcd}
\mathrm{P}(H) \arrow[r, "\alpha_H"] \arrow[d, "f"'] & F^{\mathrm{L}(H)}(\mathrm{P}(H)) \arrow[r, "t_X^{f^{-1}}"] \arrow[d, "\mathbf{B}(f)"'] & F^{\mathrm{L}(H')}(\mathrm{P}(H)) \arrow[dl, "F^{\mathrm{L}(H')}(f)"] \\
\mathrm{P}(H') \arrow[r, "\alpha_{H'}"] & F^{\mathrm{L}(H')}(\mathrm{P}(H')) &
\end{tikzcd}$$

This means that $\mathbf{BF}(f)$ is a morphism in $\int \mathbf{F}$. It then follows from the representation theorem in [5] that $\mathbf{BF}(f)$ arises from a semi-unitary map $U : H \to H'$ (identified up to a phase); more precisely, $\mathbf{BF}(f : (X, \alpha) \to (Y, \beta))$ coincides with

$$(U^{-1} : \mathrm{L}(H') \to \mathrm{L}(H), \tilde{U} : \mathbf{F}(U^{-1})(\mathrm{P}(H), \alpha_H) \to (\mathrm{P}(H'), \alpha'_H))$$

where $\tilde{U}$ has the same underlying function as U (when considered as a map from H to H'). Since **BF** is faithful, it follows that $U = f$; hence the fullness of **SC**. $\square$

Our closure-based coalgebraic approach to representation of quantum systems would allow us to develop "coalgebraic quantum logic" utilising existing work on coalgebraic logic over (duality between) general concrete categories (see, e.g., Kurz [163] or Klin [157]); this is left for future work.

5.4 Remarks on the Duality of Reproducing Kernel Hilbert Spaces

Chu space is useful for different purposes. To indicate directions of further inquiry, we finally discuss a categorical duality between kernel functions in Machine Learning and reproducing kernel Hilbert spaces (RKHS for short), which elucidates duality-theoretical structures underpinning the so-called kernel method, one of the most powerful, classic techniques in Big Data Analytics.

What are mathematical principles underpinning Artificial Intelligence or Machine Learning? Is there any fundamental structure lurking behind the scene? Our target here is the kernel method, a widely applied means of Big Data Analytics. The kernel method builds upon the notion of reproducing kernel Hilbert spaces; they were already uncovered in the early 20th century functional analysis.

To clarify the underlying context, let us briefly look back on the conceptual history of AI by classifying its developments into the following three phases:

- The birth of AI. The idea of AI goes back to Turing, and ultimately to Leibniz, especially his *characteristica universalis*. The distinction between Weak versus Strong AI was made by Searle, and the latter was argued to be impossible in his Chinese room argument. He argued the intentionality of mind cannot be realised by any computational means or behavioural simulation.[2]

- Symbolic AI. This is the second phase of AI. Its major canon was symbolic logic. AI was the logician's paradise at that time; it's now mostly lost due to what are called AI winters. It is the AI of Reasoning as opposed to Learning. In philosophical terms, Symbolic AI is Rationalist AI, whereas the following Statistical AI is Empiricist AI.

- Statistical AI. This is the third phase. Symbolic AI came to be replaced by Statistical AI, which was extremely suited for pattern recognition and classification. The kernel method comes into the play here. In Kantian terms, Symbolic AI was concerned with the faculty of Reason/Understanding, whilst Statistical AI with that of Sensibility.

[2] Searle actually had two arguments against Strong AI. The other one is what is called the observer-relativity argument in the author's philosophy paper [200], which reconsiders those fundamental issues in AI, relating it to pancomputationalism, quantum information, and a new distinction between weak vs. strong information physics. According to Searle, the observer-relativity argument shows syntax is not intrinsic to physics, whilst the Chinese room argument demonstrates that semantics is not intrinsic to syntax.

- Note that there are some attempts to combine Symbolic and Statistical AI, though it is not fully clear yet how fruitful they are. We could even think of duality between Symbolic and Statistical AI, thus elucidating the dual nature of intelligence as in the Kantian distinction between the faculty of Reason/Understanding and the faculty of Sensibility.

Chronologically, the birth was in the 50's, and the so-called golden age of AI came, thanks to Symbolic AI, in the 60's. The so-called AI winters then came in the 70's and again in the 90's. And the revival of AI was made thereafter in the course of developments of Statistical AI. (Some thinkers claim this ultimately leads to what is called the technological singularity.) The kernel method emerged in such a context, having contributed to making AI flourish again.

A kernel function $k : X \times X \to \mathbb{C}$ (or $\mathbb{R}$) on a collection X of entities represents similarity on the entities (cf. Quine's dictum "no entity without identity"; for anything to qualify as an entity in your ontology, you have to give an identity criterion for them, that is, a kernel function). Entities may be words, images, or any kind of information whatsoever, usually represented as lists of features; the process is called feature engineering. Reproducing kernel Hilbert spaes represent kernel functions in higher-dimensional spaces (typically, you reduce non-linear classification problems into linear ones through a map from a given kernel to its RKHS). We then have duality between entities (with identity) and representations, to be precise, the kernel spaces of entities and their Hilbert space representations. Technically, we are concerned with a dual adjunction between kernels and RK Hilbert spaces, and with a dual equivalence between them (which is not the restriction of the adjunction). And Chu space is useful for both.

(X, k) is called a kernel space iff X is a set and

$$k : X \times X \to \mathbb{C}$$

is symmetric and positive semidefinite (note that a kernel space is may be seen as a Chu space (X,X,k), which may in turn be regarded as a matrix; also the value set $\mathbb{C}$ may be replaced with $\mathbb{R}$).

We define morphisms of kernel spaces as a uni-directional variant of Chu morphisms: a map $f : (X, k) \to (X', k')$ is a morphism of kernel spaces iff f is a map from X to X' and there is a unique map $f' : X' \to X$ such that

$$k(x, f'(x')) = k'(f(x), x')$$

for any $x, x' \in X$. We call f' above the adjoint of f; this is by the obvious analogy with the adjoint of a linear operator.

- Define then $\Phi_x(f) = f(x)$ for $x \in X$ and $f : X \to \mathbb{C}$.
- $\{\Phi_x \ ; \ x \in X\}$ generates a vector space with the inner product induced by:

$$\langle \Phi_x | \Phi_y \rangle = k(x, y).$$

- Complete the space; denote the resulting Hilbert space by $H(X, k)$.
- $H(X, k)$ is called a reproducing kernel Hilbert space; note that a reproducing kernel Hilbert space $H(X, k)$ is always given together with the base space (X, k).

In machine learning we often reduce non-linear classification problems into linear ones through what is called a feature mapping from (X, k) to $H(X, k)$.

Let us define categories concerned:

- **RKHS** is the category of reproducing kernel Hilbert spaces and evaluation-preserving bounded linear operators, namely, bounded linear operators f preserving Φ above (i.e., for any x there is y such that $f(\Phi_x) = \Phi_y$).
- **KerSp** is the category of kernel spaces and bounded Chu morphisms, namely Chu morphisms f from (X, k) to (X', k') such that $k'(f(x), f(x)) \leq M * k(x, x)$ for some constant M.

The construction H can be extended to morphisms as well: given a morphism $f : (X, k) \to (X', k')$ in **KerSp**, define a linear operator

$$H(f) : H(X, k) \to H(X', k')$$

by

$$H(f)(\Phi_x) = \Phi_{f(x)}.$$

Note that $\{\Phi_x \ ; \ x \in X\}$ gives a basis, and so $H(f)$ is defined on the whole space by the above equation. And $H(f)$ is obviously evaluation-preserving and also bounded because of the above boundedness condition for Chu morphisms. The construction H is thus functorial.

Let us define functor K in the other direction, that is, from reproducing kernel Hilbert spaces to kernel spaces. Given a reproducing kernel Hilbert space $H(X, k)$, define

$$K(H(X, k)) = (\{\Phi_x \ ; \ x \in X\}, \langle \text{-} | \text{-} \rangle).$$

The arrow part is naturally induced; the Chu morphism condition is satisfied because any linear operator has a unique adjoint. Notice that the arrow part of K is well defined thanks to the evaluation-preservation condition, and also that the boundedness of operators ensures the boundedness of the corresponding Chu morphisms.

Theorem 5.4.1. *H and K give an equivalence between* **RKHS** *and* **KerSp**.

Proof. We can define a map

$$\varphi_X : (X, k) \to K(H(X, k))$$

by

$$\varphi(x) = \Phi_x.$$

φ_X obviously has a unique adjoint ψ_X mapping Φ_x to x. The Chu condition

$$k(x, \psi_X(\Phi_{x'})) = \langle \varphi_X(x) | \Phi_{x'} \rangle$$

holds because we have

$$k(x, x') = \langle \Phi_x | \Phi_{x'} \rangle.$$

This also guarantees the boundedness condition. φ_X is thus a morphism in **KerSp**. It is straightforward to verify that φ is a natural transformation from $\mathrm{id}_{\mathbf{KerSp}}$ to $K \circ H$. Given

$$f : (X, k) \to K(H(X', k'))$$

we have a unique $g : K(H(X, k)) \to K(H(X', k'))$ such that

$$g \circ \varphi_X = f.$$

This shows that H and K give an adjunction, which is an equivalence because φ_X is obviously an isormophism with its inverse ψ_X. □

We have thus shown the equivalence between the kernel spaces of entities with similarity, and their representing Hilbert spaces (i.e., RKHS). Since morphisms of Chu spaces are bi-directional, this can also be described as a dual equivalence.

Let us finally make several remarks on the power of Chu space representations. We can embed the entire category of Hilbert spaces into the category of Chu spaces, by mapping a Hilbert space H to

$$(H, H, \langle - | - \rangle : H \times H \to \mathbb{C}).$$

The arrow part is naturally induced; any bounded linear operator gives a Chu morphism because it has an adjoint operator. The Chu morphism condition boils down to the adjointness condition on operators. The former asymmetric Chu representation of quantum systems based on the Born rule corresponds to quantum symmetries, whilst this symmetric Chu representation based on inner product corresponds to general bounded linear operators.

The former even extends to general †-compact categories $\mathbf{C}$, in which we have states as arrows from the monoidal unit (i.e., arrows $\varphi : I^toC$) and projectors as usual (i.e., arrows $f : C \to C$ such that $f \circ f = f$ and $f^\dagger = f$). The † operation allows us to introduce inner product by

$$\langle \varphi : I \to C | \psi I \to C \rangle = \varphi^\dagger \circ \psi : I \to I.$$

It is known that the arrows from I to I form a semiring of scalars. All this machinery is sufficient to give the same type of Chu representation as above; the evaluation map based on the Born rule can be expressed in terms of states, projectors, and inner product. This Chu representation, therefore, allows us to embed any †-compact category into the category of Chu spaces.

The machinery of Chu space representation even works for categories as well. Given a category $\mathbf{C}$, we have a Chu space

$$(\mathbf{C}, \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}, e_{\mathbf{C}} : \mathbf{C} \times \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set})$$

where $e_{\mathbf{C}}(C, F) = F(C)$. A functor $F : \mathbf{C} \to \mathbf{D}$ gives rise to a Chu morphism

$$(F : \mathbf{C} \to \mathbf{D}, G : \mathbf{Set}^{\mathbf{D}^{\mathrm{op}}} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}) : (\mathbf{C}, \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}, e_{\mathbf{C}}) \to (\mathbf{D}, \mathbf{Set}^{\mathbf{D}^{\mathrm{op}}}, e_{\mathbf{D}})$$

where G is defined by

$$G(H : \mathbf{D}^{\mathrm{op}} \to \mathbf{Set})(C) = H(F(C)).$$

(F, G) thus defined is indeed a Chu morphism, since we have

$$J(F(C)) = G(J(C)) = e_{\mathbf{C}}(C, G(J)) = e_{\mathbf{D}}(F(C), J) = J(F(C)).$$

This is the Chu space representation of general categories, naturally embedding the category of (small) categories into the category of Chu spaces with the value object the category of (small) sets (i.e., $\Omega = \mathbf{Set}$).

Chapter 6

Conclusions and Prospects

Let us finally wind up the discussion with succinct answers to our very first questions, and give some prospects for further inquiry. The meaning of the title of the thesis is explicated at the end of the section.

6.1 How Duality Emerges, Changes, and Breaks

In the introductory chapter we have addressed the issues of dualism, duality, and disduality in a broader context, ranging from philosophy to different sciences. In the succeeding two chapters, we have looked into fundamental principles underpinning duality. On the one hand, duality emerges through the harmony condition in the general context of duality (or equivalences/adjunctions) between point-set space and point-free space, expressed in terms of categorical topology and algebra, respectively. And the theory thus developed has been cashed out to settle an open problem in duality theory (Chapter 2). On the other, duality comes into being via the topological dualisability condition and the Kripke condition in more logically oriented contexts of universal algebra, such as intuitionistic/non-Hausdorff dualities and coalgebraic/modal dualities (Chapter 3). Notwithstanding that duality theories in the category theory tradition and in the universal algebra tradition are quite separated today, we have attempted to reconcile and unite them in our duality theory.

In the following two chapters, we have undertaken the elucidation of the relationships between duality theory and two other fields, namely categorical logic and foundations of quantum physics. It has then turned out that there is a fairly general mechanism lurking behind (both first-order and higher-order) categorical completeness, which has been explicated by means of monad-relativised hyperdoctrines, and that some categorical models of predicate logic have their origin in duality, i.e., duality for propositional logic yields semantics of its predicate extension (this is a meeting point of Ω in categorical logic and Ω in duality theory, i.e., truth value objects and dualising objects; Chapter 4). We have finally articulated the place where duality and symmetry meet through the Born coalgebraic dynamics of measurements. Chu duality theory has also been established in order to lay foundations for operational quantum duality, the duality of classical algebraic varieties, and T_1-type dualities in general, as opposed to sober-type ones, which are subsumed under our first theory via categorical topology (Chapter 5).

In light of the duality theories elaborated throughout the present thesis, we finally give succinct answers to our first motivating questions:

- How does duality emerge? It is when the dual aspects of a single entity are in "harmony" with each other; the harmony condition explicates this harmony. Dual adjunctions emerge when algebraic structures are harmonious with topological structures, according to (the harmony condition of) the duality theory via categorical topology and algebra. In dual adjunctions between algebras and spaces, the harmony condition basically means that the algebraic operations induced on the spectra of algebras are continuous. Dual equivalences are determined by the ratio of existing term functions over all functions, according to natural duality theory.

- How does duality mutate? Dual structures get simplified as term functions increase; this is what natural duality theory tells us. As a limiting case, if existing term functions are all functions (i.e., functional completeness in logical terms), then dual spaces are Stone (aka. Boolean) spaces (this is the primal duality theorem; extra structures on space are indispensable in the quasi-primal duality theorem). If continuous functions coincide with term functions, then dual structures are coherent spaces (this could be called continuous functional completeness, which entails Stone duality with respect to coherent spaces).

- How does duality break? It is caused by either an excess of the ontic or an excess of the epistemic, as discussed in the Introduction. There are some impossibility theorems known in non-commutative algebra, which exhibits an excess of the epistemic. In the following Appendices we shall outline how to overcome this breaking of duality by virtue of generalised scheme theory. At the same time, however, quite some algebraic structures, such as sheaves of algebras, are required on dual structures in order to treat non-commutativity; the duals of algebras are sort of algebraic as well.

The idea of non-commutative duality theory is simple: we take the commutative core of a non-commutative algebra, dualise it, and equip the dual space with a sheaf structure to account for the non-commutative part. The same methods works for a broad variety of non-commutative algebras, including operator algebras, quantales, and substructural logics. In substructural logics, the method is further extended in such a way that in general we take the structural core of a substructural logic, dualise it, and endow a sheaf structure with it to take care of the substructural part. This process may be expressed by means of the general concept of Grothendieck situations; we shall discuss it in the Appendices.

6.2 A Bird's-Eye View of Stone Dualities

To elucidate how duality changes in logical contexts in particular, for example, when you weaken/strengthen your logic or extend it with operators, let us also present a bird's-eye view of different logical dualities in a rough and yet intuitive manner. Stone-type dualities basically tell us that the algebras of propositions are dual to the spaces of models in the following fashion:

- Classical logic is dual to zero-dimensional Hausdorff spaces.
  - Propositions are closed opens, for which the law of excluded middle (LEM) holds, since the union of a closed open and its complement, which is closed open again, is equal to the entire space.
- Intuitionistic logic is dual to certain non-Hausdorff spaces, that is, compact sober spaces such that its compact opens form a basis, and the interiors of their boolean combinations are compact.[1]
  - Propositions are compact opens. The topological meaning of LEM is zero-dimensionality. In general it does not hold because the complement of a compact open is not necessarily compact open.
- Modal logic is dual to Vietoris coalgebras over topological spaces.
  - Modal operators amount to Kripke relations or Vietoris hyperspaces. This is what is called Abramsky-Kupke-Kurz-Venema duality in the thesis, relating to powerdomain constructions in domain theory as well.

[1] This definition of Heyting spaces came out of my joint work with Kentaro Sato [203]; Lurie's Higher Topos Theory gives yet another definition.

Note that the existence of unit ensures that duals spaces are compact (all elements of a finitary algebra concerned yield compact subspaces, and so, if there is a unit element, the entire space is compact); otherwise they are only locally compact. The same holds for the Gelfand duality as well. There are, of course, even more logical systems you can think of:

- First-order logic may be dualised by two approaches: topological groupoids (i.e., spaces of models with automorphisms) and indexed/fibrational topological spaces (i.e., duals of Lawvere hyperdoctrines).
  - The latter approach extends to higher-order logic, thus giving duals of triposes or higher-order hyperdoctrines. It just topologically dualise the propositional value category of a hyperdoctrine or tripos.
- Infinitary logic forces us to take not even locally compact spaces into account, just like the duality for frames (aka. locales). And the resulting duality is a dual adjunction in general, rather than a dual equivalence.
  - There may not be enough models or points to separate non-equivalent propositions. There is no need for the axiom of choice thanks to infinitary operations, i.e., no need to reduce infinitaries on the topological side into finitaries on the algebraic. Note that all the other dualities require the axiom of choice to warrant the existence of enough points.
- Many-valued logics are diverse. It depends what sort of dual structure appears. It is, e.g., rational polyhedra for Łukasiewicz logic. For other logics, dual structures often include multi-ary relations on spaces as in natural duality theory.
  - Dualities for many-valued logics are mostly subsumed under the framework of dualities induced by Janusian (aka. schizophrenic) objects Ω, or Chu duality theory on value objects Ω, which may be multiple-valued.

You can combine some of these, and thereby obtain more complex dualities for more complex systems. Some compatibility conditions between different sorts of structures are usually required, and yet there is no general method to generate them so far. The structure of duality combinations and coherency conditions thus required would be worth further elucidation. Note that this is a rough picture of dualities in logic, and there are some inaccuracies and omissions. Notice also that not all of these dualities are induced by Janusian (aka. schizophrenic) objects, including those for intuitionistic and modal logics, in which implication and modality, respectively, are not pointwise operations on their spectra.

6.3 The Disclosure of Meaning

Let us close the thesis with several remarks on the meaning of *Meaning and Duality*, together with a view on the broader significance of this work and prospects for further inquiry. For one thing, it comes from Kripke's *Naming and Necessity* [161], which in turn originates from Carnap's *Meaning and Necessity* [50]. For another thing, it means meaning is dual in nature, as succinctly discussed in the Introduction as well.[2]

There are two camps in the theory of meaning: the referentialist one to account for meaning in terms of truth conditions, as advocated by Davidson [75], and the inferentialist one to account for meaning in terms of verification or use conditions, as advocated by Dummett [87] or more recent Brandom [44]. Proof-theoretic semantics, along the latter strand, is an enterprise to articulate an inferentialist account of the meaning of expressions, thus formulating the principle of meaning in terms of proof rather than truth, and by doing so, replacing Davidson's path "from truth to meaning" by another Dummettian path "from proof to meaning". In this view, meaning is autonomous in inferential structure, with no outward reference to Reality or anything outside linguistic practice. The dualism between the realist and antirealist conceptions of meaning may be called the semantic dualism.

Duality goes beyond dualism by showing that two concepts involved are two sides of the same coin. Duality in this general sense seems to witness universal features of category theory. Indeed, the classic dualism between geometry and algebra breaks down in category theory. For example, Eilenberg-Moore algebras of monads encompass topological structures as well as algebraic ones. Category theory may be algebraic at first sight, yet it is often applied to formulate geometric concepts in broad fields of geometry, ranging from algebraic and arithmetic geometry to knot theory and low-dimensional topology. It is also a vital method in representation theory and mathematical physics. In algebraic topology, algebra and space are categorically integrated into a single concept, such as ∞-categories or quasi-categories. The concept of categories captures both algebraic and geometric facets of mathematics at a deeper level. And so there is duality, rather than dualism, between algebra and geometry; they are united in the categorical endeavour of mathematics.

[2]Some of the ideas presented here have originated in the author's philosophical papers [199, 200, 201, 202].

Just as category theory enables us to transgress the dualism between algebra and geometry, categorical duality and categorical logic allow us to constructively deconstruct the generally received, orthodox distinction between syntax and semantics, or between proof theory and model theory, and presumably even the semantic dualism above, suggesting that they are merely instances of one and the same concept. As have been evidenced throughout the present thesis, there is indeed no dualism, but duality, between syntax and semantics in categorical duality theory, and there is no dualism, but duality, between model-theoretic and proof-theoretic semantics in categorical logic (i.e., it subsumes both syntactic and semantic categories/hyperdoctrines in one go; both are instances of the same sort of structures). The semantic dualism, accordingly, ought to be superseded by what could be called the semantic duality in the two senses thus articulated. Categorical unity is at work not just in the theory of meaning but also in a broader context.

Category theory, in general, allows us to make conceptual bridges between different sciences, including pure mathematics, symbolic logic, computer science, and physics (and even more), thereby establishing the unity of ideas scattered in different fields of science. Even some theoretical biologists rely on categorical methods as well. Category theory today is actually applied beyond mathematical and natural sciences, for instance, in linguistics, economics, and analytic philosophy, thus unveiling and articulating novel analogies and disanalogies across diverse sciences in a mathematically precise and rigorous fashion. We are presumably heading towards a new kind of unified science. The idea of categorical unified science, nevertheless, is in sharp contrast to the logical positivist's old-fashioned idea of unified science, which was monistic and reductionistic under the foundationalist conception of epistemology.

Categorical unified science is pluralistic unified science, emancipating logical positivism from the foundationalist doctrine of reductive physicalism. Indeed, it does not aim at grounding all sciences on one and the same absolute global foundation (cf. set theory primarily aiming at this sort of foundation of mathematics, apart from the recent multiverse view of set theory), nor revising existing sciences by reducing them into a single science of the most fundamental sort. Pluralism and the idea of relative and local foundations are arguably inherent in category theory. Think, for instance, of Grothendieck's relative point of view. Base change is a fundamental idea of category theory. There is no single category that gives the ultimate ontological foundation of everything (hence no universe in category theory, unlike set theory). There are just different categories to give local relative foundations of different fields of science (as such category theory intrinsically supports the multiverse view).

Still they are categories, so that we can apprehend what structure is shared by them, and what is not. What is here aimed at is not giving ontological or epistemological foundations, but rather giving a structural account of how each discipline's local ontology and epistemology can be interlinked with others' in the whole web of human knowledge. For example, quantum mechanics and substructural logic share quite a lot; among other things, the lack of contraction and weakening in logic corresponds to the so-called no-cloning and no-deleting properties of quantum states, in particular quantum information or qubits. This is, so to say, unity from below, originating from and sticking to the actual practice of science. In such a way, category theory yields conceptual understanding across different sciences (in the present case, logic and physics). There are numerous cases of local unity already achieved by category theory. Was there any such fruit in the logical positivist's unified science movement? Unified science must not merely be a philosophical idea; it must be practiced. Pluralistic unified science or the antifoundational naturalist unity of science is of the utmost importance in overcoming the fragmentation of science after modernisation, and in refurbishing the lost scientific image of the world as an integrated whole, in this nihilistic age of the destruction of the *cogito sum* (*á la* Heidegger [131]), the deconstruction of logocentrism (*á la* Derrida [78]), the abandonment of the Cartesian goal of a first philosophy (*á la* Quine [243]), or the end of grand narrative (*á la* Lyotard [180]). Notice that most of them, whether in the analytic or continental tradition, broadly problematises the Cartesian paradigm of philosophy or human thought.

Granted that the analytic-continental divide is still pervasive in philosophy to an unfortunately great extent, origins of analytic philosophy are, at least partly, in continental philosophy according to recent studies. In particular there was a substantial influence on logical positivism from the Marburg School of Neo-Kantianism including Cassirer [99, 100, 101, 272]; the Southwest School including Rickert had led to the formation of the Heideggerian tradition in German and then French philosophy [100]. From this perspective, the distinction would arguably be rooted in Cassirer's philosophy of *Substance and Function* [52]; his genetic conception of knowledge prioritises the functional over the substantival just as the genetic conception of space prioritises properties or observables over points. The genetic conception would amount to the process conception of the universe in Whitehead's philosophy in his *Process and Reality* [277], which explicitly supported the point-free conception of space as well. What duality tells us, as argued in the Introduction, is that we could still pave the way for uniting, or at least soothing, dualistic divides between substance and function, reality and process, or realism and antirealism. A philosophical tenet underpinning the present thesis is that the concept of duality ought to be understood in this broader context of human thought.[3]

Descartes, in his dualism, separated the realm of the objective such as matter and body from the realm of the subjective such as mind; the divorce of objective reality and subjective reality has generated a number of difficult problems in philosophy. One of them is the Benacerraf's Dilemma we discussed in the Introduction, problematising our epistemic accessibility to abstract objects, which may not exist in our tangible universe, and thus we may not have any empirical or causal access to. More generally, the fundamental questions of epistemology are why, how, and what human beings or the subject can know about the world or the object, whether it is the world of concrete entities or abstract entities. It is indeed compelling in foundations of quantum physics to elucidate the relationships between observers and observed systems, between measuring systems and measured systems, or between systems and their environments. Duality often exists between the epistemic and the ontic, which may be the subject and the object, the observer and the system, observable properties and reality, or purely mathematically, algebra and space. Duality *qua* principle of human thought exposes the unity of two realms concerned. At the same time, however, duality ought not to be confused with Hegelian dialectics. Duality does not synthesise two realms involved; rather it keeps them separated, and yet networks them via structural relationships.[4] Both ontology and epistemology are indispensable in the comprehension of the world, and there is yet another glue required to interlink the ultimate constituents of the world with those of knowledge, that is, to establish coherency between ontology and epistemology. In a nutshell, the hard problem of both science and philosophy is to give a stream-lined account of how it is possible for the epistemic and the ontic to interact with each other, which is arguably the very task of duality.

Our view could be wrapped up as follows. Since the modernist killing of Natural Philosophy seeking a universal conception of the Cosmos as an integrated whole, our system of knowledge has been optimised for the sake of each particular domain,

[3]The duality of meaning and the duality of mind have been explored in the author's philosophical papers [199, 201] and [200], respectively.

[4]Categories are networks themselves, which may further be networked in larger categories, for example, in the category of categories or in the category of dualities. Categories may represent different fields of science, just as there are logical categories, physical categories, and so fourth. And likewise they are networked in larger categories of categories via structural relationships such as dualities. The theory of categories, therefore, is a networking theory of different sciences, thus paving the way for the networking of knowledge in this age of what is called the disunity of science [105].

and has accordingly been massively fragmented and disenchanted. And we now lack a unified view of the world, living in the age of disunity surrounded by myriads of uncertainties and contingencies. (The Amane Nishi's programme "Interweaving a Hundred Sciences" [210] and the Kyoto School's ideal "Overcoming Modernity" [211] may be reconsidered in this context as a matter of the unity of science.) From this point of view, the present work is a modest attempt to "re-enchant" the world, embarking upon the enterprise of building a dually integrated image of the world as a coherent whole on the basis of category theory, a theory of everything *qua* structural networks. Broadly, our ultimate aim is at realising the Kyoto School's dream "A Construction of a Unified Worldview as the Fundamental Challenge of the Contemporary Era" (citation from Nishitani [211]; for a general account of the Kyoto School, see, e.g., Davis [76]). Granted that there is still far too long a way to go, duality could ultimately serve as such a unifying principle of human knowledge, thus leading us to a contemporary incarnation of Natural Philosophy, in which the mechanistic view, the prevailing culture of modernism, and the holistic view of the world, a sort of counterculture to mechanistic modernism, would be dually united as well.

Appendix A

Scheme-Theoretical Duality Theory

Here we outline a sheaf-theoretical duality theory within the framework of universal algebra and category theory, thereby leading to a unifying perspective on both duality for noncommutative rings and C^*-algebras and duality for logical, computational, and quantum algebras, in particular noncommutative Lambek algebras in substructural logic, Scott continuous lattices in domain theory, Birkhoff-von Neumann's quantum logics, and quantales as noncommutative frames in point-free topology. Grothendieck's concept of schemes is extended so as to account for these different structures, and accordingly, sheaves are not necessarily defined upon topological spaces, but may be based upon convex or measurable spaces, for example. The theory starts with the concept of Grothendieck situations to derive sheaf duality, then yielding what we call Representable Sheaf Duality on the basis of dualising objects, and in the end leading to Core Sheaf Duality to capture substructural algebras in terms of their structural cores; noncommutativity is understood as an instance of substructurality. We may thus elucidate a trade-off between complexity of base spaces and that of stalk algebras. An underlying conceptual view is that the geometric and algebraic conceptions of space are integrated together into the one concept of (generalised) scheme, and this particular work aims at illustrating the idea that the sheaf-thereotical conception of space as scheme makes sense in diverse disciplines far beyond algebraic geometry, giving rise to different sorts of duality between algebras and schemes.

A.1 Introduction to the Appendix

The Kyoto School of Philosophy, mainly led by Kitaro Nishida and Hajime Tanabe (see, e.g., Stanford Encyclopaedia of Philosophy's entries [76, 217]), was concerned with the concept of duality in some sense. Even though the Kyoto School rapidly declined after the WWII, and has finally disappeared, yet some present-day thinkers, not only philosophers but also scientists, are actually inspired by their philosophy, and looking into its contemporary significance in different guises. Physicist Piet Hut at Princeton IAS is one of them, shedding new light on Nishida's philosophy in relation to his philosophy of physics, with a special emphasis on duality between subjects and objects (see, e.g., Hut-van Fraassen [141]). Logician Susumu Hayashi at Kyoto University recently investigated into Tanabe's Nachlass and discovered documentary evidence for his claim that Tanabe's philosophy, in particular his so-called Logic of Speicies (his concept of spices encompass social constructs like societies), was directly influenced by the Brouwer's theory of continuums. Tanabe was concerned with a sort of duality between individuals and species, which, Tanabe conceived, was in parallel with the relationships between points and continuums in the Brouwer's theory.

This line of duality thoughts manifests in really diverse fields, mathematically as well as philosophically. One of the most obvious cases in mathematics would be duality between point-set and point-free spaces, which has beed discussed in developments of point-free geometry, such as locale theory and formal topology, while one of their roots lies at the Brouwer's theory of continuums mentioned above. It is not just mathematics, neither. We may indeed find plenty of such dualities in physics, computer science, and other sciences. Particularly in physics, the conquest of noncommutativity has been an urgent issue in consideration of mathematical foundations of quantum theory, and among other things, noncommutative duality has been one of the central topics, recently discussed extensively by a number of categorical physicists, e.g., Döring [83], Furber-Jacobs [103], von den Berg-Heunen [34], Ojima [218], and Spitters-Vickers-Wolters [263]. Yet much of the focus has remained upon a particular theory for a particular sort of noncommutative structures, such as C^*-algebras and von Neumann algebras, and no unifying perspective has been elaborated so far.

Although there are different approaches to duality, sheaf-theoretical methods have been particularly successful in pursuit of noncommutative duality. Nonetheless, sheaf-based duality theory is not just for noncommutative algebras, but actually works for a wide variety of algebraic structures, whether they are finitary or infinitary. To the best of the author's knowledge, sheaves and schemes would indeed be a most widely applicable method to construct categorical dualities for various structures, ranging from (possibly noncommutative) purely algebraic structures, such as rings, to (possibly infinitary) ordered algebraic structures like lattices, as demonstrated here.

Here, it should be noted that the category-theoretical idea of duality induced by Janusian objects (i.e., what were called schizophrenic objects; see, e.g., Dimov-Tholen [81], Johnstone [149], Porst-Tholen [232], or Chapter 2) does not really do the job for noncommutative structures. We also remark that already well-developed, universal-algebraic theories of sheaf representation (for a historical bird's-eye view, we refer to Keimel [155]; see also Knoebel [158] and references therein) do not work beyond finitary algebraic structures, since universal algebra usually limits its scope to the finitary realm of algebra (even though limiting the scope sometimes gives rise to deeper or more nuanced insights); the same applies to the so-called theory of natural dualities as well (see Clark-Davey [63]; for some more recent developments, see Chapter **??**). Putting aside the question of whether or not sheaf-theoretical duality theory is the most useful or fruitful one of these different duality theories, it would indeed be most comprehensive (to the author's eyes).

In this context of the current state of the art of duality theory, by which universal theory is meant rather than particular theory, we thus aim at elucidating the generic architecture of sheaf-theoretical duality, or at least making some efforts to approach that ultimate goal. We especially target at duality via schemes (we use this term in a generalised sense explicated later), a prominent example being the Grothendieck duality between commutative rings and Affine schemes (see Grothendieck-Dieudonne [123]). Although the original Grothendieck duality dealt with commutative rings only, nevertheless, scheme-theoretical duality theory has successfully been elaborated, along a similar line, for noncommutative algebras as well, as done in Pierce [227], for example. And our theorisation encapsulates both developments, further extending the idea of scheme-theoretical duality to different sorts of algebraic structures in logic, physics, and computer science, far beyond algebraic geometry.

Summary The fundamental concept is that of a Grothendieck situation to derive duality between algebras and schemes. Most sheaf-based dualities fall into the picture of the sheaf duality that is derived from a Grothendieck situation, which, in principle, allows for possibilities of Grothendieck topologies that are different from ordinary topologies on topological spaces. On the basis of the concept of a Grothendieck situation, we show two theorems, namely the Representable Sheaf Duality theorem, and the Core Sheaf Duality theorem. The former is a sort of one-step duality construction,

and the latter two-step one. In the former, we just take the dual space of a given algebra, and equip it with a suitable structure sheaf. In the latter, we first take the core of an algebra, and then the dual space of the core, which is in turn endowed with a structure sheaf. This is a major difference between the two duality constructions. The framework encompasses different concrete dualities ranging from those for non-commutaive rings and C^*-algebras to those for various ordered algebraic structures. We especially instantiate the theorems as sheaf dualities for different logical systems including classical, intuitionistic, fuzzy, linear, modal, geometric, and quantum logics; quite some of them are previously unknown dualities. Overall, we aim at developing a streamlined way to lead from abstract duality theory to concrete duality results.

Comparison Some of the relationships with categorical and universal-algebraic duality theories have already been touched on above; here we shed light on different issues. First of all, although there are already a vast number of works on sheaf representation, it is still not clear yet what the general architecture of sheaf representation, or rather duality, is. Universal algebra has elaborated its own deep perspective on sheaf representation, which is not totally satisfactory, however. For example, general theories of sheaf representation in universal algebra (see, e.g., Knoebel [158] and the references of Keimel [155]) are mostly focused upon sheaf representation over Stone spaces (aka. Boolean spaces) or at least some compact spaces, such as spectral spaces (aka. coherent spaces). Our theory is more flexible, allowing for more possibilities: indeed, in some of our duality results, non-Hausdorff spaces or even non-compact spaces appear as base spaces of schemes; as a matter of fact, they do not have to be topological spaces at all. Also, quite some of the existing sheaf representation results, apart from the original Grothendieck duality, are merely formulated as representation, and not as duality; for example, dualisation of arrows remains untouched. On the other hand, we formulate everything in terms of exact dual equivalences throughout the appendix. We finally emphasise that, while some general theories tend to be sort of vacuous in practice, our theory is not so: we do derive from the theory concrete dualities that are of interest on their own, apart from the theory per se.

Organisation The rest of this appendix is organised as follows. We first define the concept of a Grothendieck situation, which is a fundamental set-up to capture the architecture of sheaf-theoretical duality. In later developments, we think of the way how Grothendieck situations arise in more concrete contexts. We first show the Representable Sheaf Duality theorem via the method of so-called dualising objects, providing a number of applications to account for the meaning of the duality theorem. We then introduce the concept of the core of an algebra, and show the Core Sheaf Duality theorem, concluding with numerous consequences of the theorem, and with discussions on the significance of them.

A.2 Grothendieck Situation for Sheaf Duality

The first set-up is as follows. We think of an algebraic category **Alg**, i.e., a category monadic over **Set**, which can always be presented in terms of sets with operations, i.e., what are called (general) algebras in universal algebra (see, e.g., Manes [183] or Adámek et al. [12]). In other words, **Alg** is what is called a variety in universal algebra. We may instead think of a subcategory of **Alg** that is closed under homomorphic images (or coequalisers) and subalgebras (or equalisers), since we do not use other properties. We aim at establishing duality for **Alg** by means of sheaf structures. To this end, we first assume the following:

- **Spa** is a category with a (contravariant) functor

$$\mathcal{O} : \mathbf{Spa}^{\mathrm{op}} \to \mathbf{Pos}$$

  such that $\mathcal{O}(S)$ has a greatest element $1_{\mathcal{O}(S)}$ for any $S \in \mathbf{Spa}$, where **Pos** denotes the category of posets (typically, **Spa** is the category of topological spaces, and $\mathcal{O}$ is the functor that maps topological spaces to their open set locales).

- The following

$$\mathrm{Mod} : \mathbf{Alg}^{\mathrm{op}} \to \mathbf{Spa}$$

  is a (contravariant) functor such that, for any $A \in \mathbf{A}$, denoting by $\mathrm{Cong}(A)$ the collection of congruences of A (in the sense of universal algebra), there is an anti-monotone map

$$I_A : \mathcal{O}(\mathrm{Mod}(A)) \to \mathrm{Cong}(A)$$

  such that $I_A(1_{\mathcal{O}(\mathrm{Mod}(A))})$ is the least congruence of A, where $\mathrm{Cong}(A)$ is thought of as ordered by inclusion.

- Mod and the I operation are compatible in the sense that, given $f : A \to A'$ in **Alg** and $O \in \mathcal{O}(\mathrm{Mod}(A))$, if a equals b modulo $I_A(O)$, then $f(a)$ equals $f(b)$ modulo $I(\mathcal{O} \circ \mathrm{Mod}(f)(O))$.

The above set-up then allows us to define the following presheaf structure.

Definition A.2.1. *Given $A \in \mathbf{Alg}$, let us define a functor*

$$\mathrm{Spec}(A) : \mathcal{O}(\mathrm{Mod}(A))^{\mathrm{op}} \to \mathbf{Alg}$$

as follows. Given $O \in \mathcal{O}(\mathrm{Mod}(A))$, define $\mathrm{Spec}(A)(O) = A/I_A(O)$. Given O and O' with $O' \subset O$, define $\mathrm{Spec}(A)(f) : A/I_A(O) \to A/I_A(O')$ by mapping $[a]_{I_A(O)}$ to $[a]_{I_A(O')}$, where $[x]_R$ denotes the equivalence class including $x \in A$ under $R \in \mathrm{Cong}(A)$.

Note that $\mathrm{Spec}(A)$ is well defined because the algebraic category or variety **Alg** is closed under taking quotients. Note also that the term "spectrum" is used here for denoting a sheaf structure; yet, at the same time, it may just mean the base space somewhere else.

Now the final assumption required is the following sheaf condition:

4. There is a Grothendieck topology on $\mathcal{O}(\mathrm{Mod}(A))$ with respect to which the afore-defined presheaf $\mathrm{Spec}(A) : \mathcal{O}(\mathrm{Mod}(A))^{\mathrm{op}} \to \mathbf{Alg}$ forms a sheaf.

It is usually obvious in concrete situations how to choose a topology.

The following concept of Grothendieck situations is intended to capture the way how sheaf-theoretical duality emerges.

Definition A.2.2. *We call the following triple satisfying the above conditions*

$$(\mathbf{Alg}, \mathbf{Spa}, \mathrm{Mod} : \mathbf{Alg}^{\mathrm{op}} \to \mathbf{Spa})$$

a Grothendieck situation.

Given a Grothendieck situation $(\mathbf{Alg}, \mathbf{Spa}, \mathrm{Mod})$, we can introduce the corresponding concept of schemes in the following manner.

Definition A.2.3. *An A-scheme is defined as a pair $(\mathrm{Mod}(A), \mathrm{Spec}(A))$ for $A \in \mathbf{Alg}$, with $\mathrm{Mod}(A)$ and $\mathrm{Spec}(A)$ called the base space and the structure sheaf respectively.*

A morphism of A-schemes from $(\mathrm{Mod}(A'), \mathrm{Spec}(A'))$ to $(\mathrm{Mod}(A), \mathrm{Spec}(A))$ is a natural transformation

$$\eta : \mathrm{Spec}(A) \to \mathrm{Spec}(A') \circ \mathcal{O} \circ \mathrm{Mod}(f)$$

for $f : A \to A'$ in $\mathbf{Alg}$.

*Finally, let **Sch** denote the category of A-schemes and their morphisms (composition is well defined in a canonical way).*

We sometimes identify an A-scheme $(\mathrm{Mod}(A), \mathrm{Spec}(A))$ with the structure sheaf $\mathrm{Spec}(A)$.

We can now define a (contravariant) functor

$$\mathrm{Spec} : \mathbf{Alg}^{\mathrm{op}} \to \mathbf{Sch}$$

as follows. The object part is already defined. To define the arrow part, consider $f : A \to A'$ in $\mathbf{Alg}$. Then,

$$\mathrm{Spec}(f) : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$$

is defined as a natural transformation

$$\eta^f : \mathrm{Spec}(A) \to \mathrm{Spec}(A') \circ \mathcal{O} \circ \mathrm{Mod}(f)$$

such that, for $O \in \mathcal{O} \circ \mathrm{Mod}(A)$,

$$\eta^f_O : A/I_A(O) \to A'/I_{A'}(\mathcal{O} \circ \mathrm{Mod}(f)(O))$$

maps $[a]_{I_A(O)}$ to $[f(a)]_{I_{A'}(\mathcal{O}\circ\mathrm{Mod}(f)(O))}$. This is well defined.

$\mathrm{Spec} : \mathbf{Alg}^{\mathrm{op}} \to \mathbf{Sch}$ then induces a dual equivalence between the algebras and the schemes. The other functor $S : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Alg}$ maps an A-scheme $\mathrm{Spec}(A)$ to its value at the whole space, i.e., $\mathrm{Spec}(A)(1_{\mathcal{O}(\mathrm{Mod}(A))})$. This is is obviously isomorphic to A:

$$\mathrm{Spec}(A)(1_{\mathcal{O}(\mathrm{Mod}(A))}) = A/I_A(1_{\mathcal{O}(\mathrm{Mod}(A))}) \simeq A.$$

The arrow part of this functor then maps a morphism between A-schemes, $\eta : \mathrm{Spec}(A) \to \mathrm{Spec}(A') \circ \mathcal{O} \circ \mathrm{Mod}(f)$, to the underlying map $f : A \to A'$ under the canonical identification of $\mathrm{Spec}(A)(1_{\mathcal{O}(\mathrm{Mod}(A))})$ with A (and the same identification for A'). The double dual of an A-scheme $\mathrm{Spec}(A)$ is isomorphic to itself, since it has been shown that the dual of $\mathrm{Spec}(A)$ is isomorphic to A. We thus obtain:

Theorem A.2.4. *The algebraic category* **Alg** *and the scheme category* **Sch** *are dually equivalent.*

This encapsulates a great variety of concrete dualities; for the moment, however, we will just mention the following:

1. The Grothendieck duality between commutative rings and Affine schemes is a particular instance of the generic duality above; in this case, the Mod functor above amounts to the Zariski's prime spectrum functor (where the term "spectrum" means dual spaces, or base spaces rather than schemes themselves).

2. The Grothendieck duality for commutative rings has been extended so as to encompass noncommutative rings (see, e.g., Pierce [227]), and most such dualities fall into the general picture above as well as the original commutative one (the Pierce duality, and also the case of noncommutative C^*-algebras, shall be discussed more in a later section).

Note that the existence of such noncommutative dualities is consistent with the recently discovered No-Go theorems of von den Berg-Heunen [34].

All this functions as a sort of preliminaries for the following developments addressing the real question behind the scene, that is, when we actually have a Grothendieck situation. In the rest of the appendix, thus, we seek good sufficient conditions to derive a Grothendieck situation.

A.3 Sheaf Duality via Dualising Objects

In this section, we show that so-called dualising objects Ω naturally lead us to Grothendieck situations: that is, Ω automatically gives rise to Mod as a Hom functor into Ω; and I is necessarily derived from the structure of Ω; and the compatibility condition between Mod and the I operation turns out to be just provable.

A.3.1 Representable Sheaf Duality

The set-up here is basically the same as that of Chapter 2. **Spa** is what is called in Chapter 2 a full subcategory of a functor-costructured category $\mathbf{Spa}(Q)^{\mathrm{op}}$ that is definable by a class of classical topological coaxioms in $\mathbf{Spa}(Q)^{\mathrm{op}}$ (for the concepts of functor-(co)structured categories and topological (co)axioms in categorical topology, see Adámok et al. [12]; the naming of the concepts is due to them). For simplicity, the reader may just regard **Spa** as a full subcategory of the category of topological spaces, and what is called a generalised topology as a topology in the ordinary sense, except for the one case in which the theory is explicitly applied to convex structures later.

There are, however, other possible applications beyond topological spaces; for instance, we may think of a dual equivalence between σ-complete Boolean algebras and schemes based on measurable, rather than topological, spaces. In order to cover all such cases, the framework of Chapter 2 has to be used in its full generality.

Suppose Ω is an object in **Alg**. In the following we assume that every $A \in \mathbf{Alg}$ is not empty, equipped with a fixed element $\top_A$, and then $\top_\Omega$ shall be denoted just

$\top$. Given $A \in \mathbf{Alg}$, we are able to equip $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ with the generalised topology generated by $\{\langle a \rangle \mid a \in A\}$ where $\langle a \rangle$ is defined by

$$\langle a \rangle = \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega) \mid v(a) = \top\}$$

which, intuitively speaking, is the spatial region in which a formula a holds, or the collection of semantic models of a. This precisely gives Stone topologies used in different developments of duality theory. And the Stone topology even allows us to account for how $I_A : \mathcal{O}(\mathrm{Mod}(A)) \to \mathrm{Cong}(A)$ arises in duality:

- Mod is defined as the contravariant representable functor into Ω, i.e.,

  $$\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega) : \mathbf{Alg}^{\mathrm{op}} \to \mathbf{Spa}$$

  This can be verified to be well defined, especially on the arrow part.

- $I_A : \mathcal{O}(\mathrm{Mod}(A)) \to \mathrm{Cong}(A)$ is defined as follows. For generators $\langle a \rangle$, $I_A(\langle a \rangle)$ is defined by

  $$I_A(\langle a \rangle) = A / \langle a \rangle_\top$$

  where $\langle a \rangle_\top$ is the congruence generated by $\{(a, \top)\}$. This definition can be canonically extended to all $O \in \mathcal{O}(\mathrm{Mod}(A))$, since the generalised topology is generated by $\{\langle a \rangle \mid a \in A\}$. To ensure that $I_A(\mathrm{Mod}(A))$ is the least congruence, the truthness condition is assumed:

  $$\langle \top \rangle = \mathrm{Mod}(A).$$

  Note that this is obviously true in concrete cases where all homomorphisms preserve truth constants $\top$.

- Mod and the I operation can be proved to be compatible based on the fact that

  $$\mathrm{Mod}(f)^{-1}(\langle a \rangle) = \langle f(a) \rangle.$$

Now we only need to assume the sheaf condition, thus obtaining the following duality induced by the dualising object Ω. We call it Representable Sheaf Duality.

Theorem A.3.1. *Under the assumption of the sheaf and truthness conditions, any object Ω in* **Alg** *yields the following Grothendieck situation*

$$(\mathbf{Alg}, \mathbf{Spa}, \mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega))$$

so that the algebraic category **Alg** *and the induced scheme category* **Sch** *are dually equivalent. Such a Grothendieck situation is called a representable Grothendieck situation.*

This generic duality via the dualising object encompasses, for instance, the following concrete dualities:

1. Classical logic. The theorem gives us a sheaf duality for Boolean algebras. Since there is an isomorphism between the above Hom set into Ω and the prime filters, the contravariant Hom functor is actually the Stone spectrum functor. Of course, Ω amounts to the two-element Boolean algebra $\mathbf{2}$. Note, however, the following:

    - The above theorem tells us Ω does not have to be the two-element algebra; rather, it can be any Boolean algebra whatsoever (note that the case of the one-element algebra, if it is allowed, is trivial). We often have a canonical choice like $\mathbf{2}$, yet other choices do work as well. The same remark applies to the following cases as well.

2. Intuitionistic logic. The theorem gives us sheaf dualities for distributive lattices and Heyting algebras. Note that the latter is a subclass of the former. It is immediate to get a sheaf duality for distributive lattices, just by letting Ω be the two-element distributive lattice $\mathbf{2}$ or any other larger one as remarked above. Since the Heyting algebras form a subclass of the distributive lattices, we can restrict the resulting sheaf duality for distributive lattices into the sheaf duality for Heyting algebras.

    - Note that the prime spectrum of a Heyting algebra is not $\mathrm{Hom}(A, \mathbf{2})$ in the category of Heyting algebras, but it is $\mathrm{Hom}(A, \mathbf{2})$ in the category of distributive lattices. This accounts for the reason why we first work with distributive lattices, and then restrict their duality to the one for Heyting algebras.

3. Fuzzy logic. The Representable Sheaf Duality theorem yields a sheaf duality for MV algebras, which are algebraic structures for Łukasiewiz logic. The canonical dualising object is the real unit interval $[0, 1]$, though it does not have to be as mentioned above. There is an essential difference between this case and the two cases above, as follows.

    - In the cases of Boolean and Heyting algebras, we have dualities with suitable classes of topological spaces (i.e., well-known Stone spaces and a certain class of compact sober spaces, respectively). And so it is not compelling to introduce sheaf structures. However, there is no such topological duality known in the case of (all) MV algebras, for which it is really compelling to rely on sheaves and schemes in order to construct duality.

4. Geometric logic or point-free topology. The theorem yields a sheaf duality for frames or locales. The canonical dualising object is the two-element frame **2**. Schemes in this case may be called framed spaces or localed spaces, just like ringed spaces. As far as the author knows, no such duality has appeared in the literature.

   - Compared with the well-known duality between spatial frames and sober spaces, the sheaf duality covers all frames and not just spatial ones. If a frame has no point (i.e., completely prime filter), it does not make so much sense to dualise it via sheaves; nonetheless, when it has some points, if not all, sheaf dualisation yields some insights into the structure.

5. Domain theory or convex geometric logic. The theorem can be instantiated as a sheaf duality for Scott continuous lattices, which may be seen as point-free convexity spaces (see Maruyama [189] and Chapter 2), giving the logic of convex geometry. The dualising object is the two-element continuous lattice **2**, though it does not have to be.

   - Compared with the rest of dualities given above, the sheaf duality for Scott continuous lattices is based upon convexity spaces (in the sense of van de Vel [270]). Also, compared with other known dualities for Scott continuous lattices (e.g., those in Maruyama [189] and Chapter 2), the sheaf duality works for all such structures, just as in the case of locales.

There are actually more concrete dualities that can be derived from the Representable Sheaf Duality theorem above. However, these would already be enough for the purpose of illustrating the significance of the theorem.

In the next section we head towards noncommutative structures and scheme-theoretical dualities for them.

A.4 Core Sheaf Duality via Dualising Objects

In this section we think of the situation in which the $\mathrm{Mod} : \mathbf{Alg}^{\mathrm{op}} \to \mathbf{Spa}$ functor factors through the core functor

$$\mathrm{C} : \mathbf{Alg} \to \mathbf{CA}$$

and the core dual space functor

$$\mathrm{CM} : \mathbf{CA}^{\mathrm{op}} \to \mathbf{Spa}.$$

We actually start with C and CM, and then define $\mathrm{Mod} = \mathrm{CM} \circ \mathrm{C}$, and finally $(\mathbf{Alg}, \mathbf{Spa}, \mathrm{Mod} \circ \mathrm{C})$ forms a Grothendieck situation.

The basic idea behind the construction is simple: given an algebra A of some substructural (e.g., noncommutative) sort, we find the structural (e.g., commutative) core $\mathrm{C}(A)$ of A, taking the dual space $\mathrm{CM} \circ \mathrm{C}(A)$ of the core $\mathrm{C}(A)$, on which a suitable sheaf structure $\mathrm{Spec}(A)$ is induced, allowing us to reconstruct the original algebra A as the algebra of global sections. There are, of course, other non-trivial details to be worked out; yet, this is the basic idea.

Examples of the concept of core include the centre of a ring or C^*-algebra, and much more:

1. The core of a quantale may be defined as the centre of it, which is of course commutative. At same time, however, the core of a quantale may also be defined as the frame of idempotent central elements of it. And we later use this definition of the core of a quantale.

2. The core of a Heyting algebra is the Boolean algebra of fixpoints of the double negation $\neg\neg$ operation (i.e., the elements for which the excluded middle holds). This gives an algebraic account of the Gödel-Gentzen translation.

3. The core of an S4 modal algebra is the Heyting algebra of fixpoints of the modality $\Box$ operation. This gives an algebraic account of the Gödel-McKinsey-Tarski translation.

4. The core of an MV algebra is the Boolean algebra of idempotents of it. We may also think of this in terms of the so-called Baaz delta Δ operation.

5. The core of an orhomodular lattice or more generally ortholattice is the centre of it, which forms a Boolean algebra. This is a standard idea in quantum logic.

6. The core of the full Lambek calculus with Girard's exponential ! is the logic extended with all the structural rules, which is intuitionistic. In other words, the core of a full Lambek algebra with ! is the Heyting algebra of fixpoints of the exponential ! operation. This gives an algebraic account of the Girard translation. Concerning the definition of a full Lambek algebra or FL algebra with exponential !, we refer to Galatos et al. [104] for FL algebras, and Coumans et al. [70] especially for the algebraic account of exponential !.

For a single sort of algebras, there are, in general, multiple concepts of their cores. The distinction between algebras and their cores are thus relative rather than absolute, as you can see from the list above.

A.4.1 Sheaf Duality via Core Dualisation

Here we combine the idea of core with that of Representable Sheaf Duality in Section A.3. Note that it is also possible to develop core-based duality theory without using dualising objects, in the style of Section A.2.

We first assume that

- The algebraic category **Alg** comes equipped with the following functor

$$\mathrm{C} : \mathbf{Alg} \to \mathbf{CA}$$

  where **CA** is a full subcategory of **Alg** such that $\mathrm{C}(A) \subset A$ for each $A \in \mathbf{Alg}$.

We then proceed as in the above case of the Representable Sheaf Duality construction. Suppose Ω is an object in **CA**. We assume that every $A \in \mathbf{CA}$ is not empty, equipped with a fixed element $\top_A$. Given $A \in \mathbf{CA}$, we are able to equip $\mathrm{Hom}_{\mathbf{CA}}(A, \Omega)$ with the generalised topology generated by $\{\langle a \rangle \mid a \in A\}$ where $\langle a \rangle$ is defined by $\langle a \rangle = \{v \in \mathrm{Hom}_{\mathbf{CA}}(A, \Omega) \mid v(a) = \top\}$. And finally we define $\mathrm{Mod} : \mathbf{Alg}^{\mathrm{op}} \to \mathbf{Spa}$ and $I_A : \mathcal{O}(\mathrm{Mod}(A)) \to \mathrm{Cong}(A)$ as follows.

- Mod is defined as the composed contravariant functor:

$$\mathrm{Hom}_{\mathbf{CA}}(\text{-}, \Omega) \circ \mathrm{C} : \mathbf{Alg}^{\mathrm{op}} \to \mathbf{Spa}$$

  This can be verified to be well defined, especially on the arrow part.

- Now,

$$I_A : \mathcal{O}(\mathrm{Hom}_{\mathbf{CA}}(\text{-}, \Omega) \circ \mathrm{C}(A)) \to \mathrm{Cong}(A)$$

  is defined as follows. For generators $\langle a \rangle$, $I_A(\langle a \rangle)$ is defined by $I_A(\langle a \rangle) = A/\langle a \rangle_\top$ where $\langle a \rangle_\top$ is the congruence of A that is generated by $\{(a, \top)\}$. To ensure that $I_A(\mathrm{Mod}(A))$ is the least congruence, we assume the core truthness condition: $\langle \top \rangle = \mathrm{Hom}_{\mathbf{CA}}(\text{-}, \Omega) \circ \mathrm{C}(A)$. Note that this condition is obviously satisfied in concrete situations because homomorphisms usually preserve truth constants.

We thus obtain the following Core Sheaf Duality theorem.

Theorem A.4.1. *Under the assumption of the core truthness and sheaf conditions, any object* Ω *in* **CA** *yields the following Grothendieck situation*

$$(\mathbf{Alg}, \mathbf{Spa}, \mathrm{Hom}_{\mathbf{CA}}(\text{-}, \Omega) \circ \mathrm{C}\ )$$

and therefore the algebraic category **Alg** *and the induced scheme category* **Sch** *are dually equivalent. Such a Grothendieck situation is called a core-induced Grothendieck situation.*

The significance and consequences of this theorem are discussed and analysed in the following. The Core Sheaf Duality theorem can be applied in any of the examples mentioned above, and many of the resulting dualities are previously unknown dualities, which are discussed in more detail below.

Firstly it would be instructive to recall the Pierce's sheaf-theoretical duality for noncommutative rings. In the Pierce duality, the core of a noncommutative ring is the Boolean algebra of the idempotents of the centre of the ring. To be precise, the Pierce duality is a corollary of the theory which combines the idea of core dualisation with the theory of Section A.2, and it is not a direct corollary of the above theorem, since taking Zariski spectra cannot be expressed as a representable functor. Nevertheless, it just suffices to compose the Zariski spectrum functor with the core functor to get the Mod functor that yields a Grothendieck situation; thus it anyway falls into the scope of our theory as it is, namely that of Section A.2. Note that a homomorphism of noncommutative rings is required to preserve central elements, and this allows us to define the core functor C from noncommutative to Boolean rings. The same remark applies to the other sorts of structures mentioned above; cores must be preserved.

It is also possible to define the core of a ring to be the commutative ring of central elements, and it leads us to another duality. In the former case, the base space of a scheme is a Stone space (aka. Boolean space), while in the latter case, it is a spectral space (aka. coherent space). This is a manifestation of the general fact that, the simpler the base space is, the more complex the stalk algebras are. There is thus a trade-off between complexity of base spaces and that of stalk algebras in sheaf-theoretical duality theory. We shall soon see even more cases of the trade-off.

Let us think of the relevant question of whether or not our theory encompasses any noncommutative duality for C^*-algebras. It is known that the category of unital C^*-algebras is monadic over **Set** (see, e.g., Pelletier-Rosicky [225]), and thus they fall into the scope of the present framework. The core of a unital C^*-algebra can be defined as the centre of it, which is a commutative unital C^*-algebra, and therefore the classic Gelfand duality is available for it. We can then define the Mod functor as the representable Gelfand spectrum functor Hom(-, $\mathbb{C}$) composed with the core functor, which yields a Grothendieck situation. This scheme-theoretical duality for noncommutative unital C^*-algebras is thus within the scope of our theory (for different sheaf dualities for noncommutative C^*-algebras, see, e.g., Dauns-Hofmann [74] and Hofmann [137]). In a nutshell, we may summarise all this as follows:

- The Gelfand duality for commutative C^*-algebras lifts to the scheme-theoretical duality for noncommutative C^*-algebras. In general, different Stone dualities based on the classical conception of space as topological space and the like, one of which is the Gelfand one, lift to different Grothendieck dualities based on the modern conception of space as scheme. This was the core idea of our theorisation, having triggered the developments of the present framework.

Now, we discuss previously unknown dualities resulting from the above theorem; in the following, each entry's number corresponds to that in the previous table of cores of different algebras.

1. We define the core of a quantale as the frame of idempotent central elements of it, and the dualising object Ω as the two-element frame **2**. The Core Sheaf Duality theorem gives us a duality between (all) quantales and the corresponding schemes. In this case, base spaces are not necessarily compact; they are compact in the case of rings.

2. It is obvious how to apply the theorem to Heyting algebras: the core of a Heyting algebra is the Boolean algebra of those elements that validate double negation elimination, and **2** is the dualising object. In this case, base spaces are compact. Non-compact spaces only appear if algebras are of infinitary nature, just as quantales and frames are so. Finitary algebras give rise to compact spaces. Compared with the Representable Sheaf Duality applied to Heyting algebras, in which stalk algebras enjoy the disjunction property, the present Core Sheaf Duality is enabled by allowing for more general stalk algebras.

3. It is obvious how to apply the theorem in this case. Base spaces are spectral spaces. We can make them Stone spaces by further taking the algebra of idempotents of the core of an S4 algebra. There seems to be no obvious notion of the core of a modal algebra in general; on the other hand, Representable Sheaf Duality is available for general modal algebras.

4. In this case, the dualising object Ω is the two-element Boolean algebra, different from the previous duality for MV algebras. In the previous case, stalk algebras are chains, which can be larger than $[0, 1]$, whereas in the present case they are not necessarily totally ordered. This is of course a case of the trade-off mentioned above.

5. It is obvious how to apply the theorem in this case: just take the core, and the dual space of the Boolean algebra, and equip it with the structure sheaf. Again in this case, the dualising object Ω is the two-element Boolean algebra. There seems to be no canonical choice of a dualising object for general quantum logics without taking the cores. It also seems that Representable Sheaf Duality is not directly applicable for quantum logics due to non-distributivity.

6. It is obvious how to apply the theorem in this case: we take the core, and the spectrum of the Heyting algebra with the structure sheaf. This is, so to speak, a finitary analogue of the quantale case, and a linear analogue of the S4 case. Note that quantales have been used for semantics of linear logic. Since most logical systems (classical, intuitionistic, fuzzy, linear, relevant, etc.) can be expressed as an axiomatic extension of the full Lambek calculus, dualities for them can be obtained as restrictions of the duality for full Lambek algebras.

These, as well as the previous list of representable sheaf dualities, illustrate the broad applicability of the scheme-theoretical duality theory that has been developed here. Yet they are just sample applications to explicate what the abstract sheaf duality theorems mean, and how they are substantiated, in concrete situations; hopefully, more applications of the theory will be found, and other versions of universal sheaf duality shall be elaborated, in future work.

Appendix B

Duality, Modality, and Vagueness

Here we explore the relationships between many-valued logic and fuzzy topology from the viewpoint of duality theory. We first show a fuzzy topological duality for the algebras of Łukasiewicz n-valued logic with truth constants, which generalizes Stone duality for Boolean algebras to the n-valued case via fuzzy topology. Then, based on this duality, we show a fuzzy topological duality for the algebras of modal Łukasiewicz n-valued logic with truth constants, which generalizes Jónsson-Tarski duality for modal algebras to the n-valued case via fuzzy topology. We emphasise that fuzzy topology naturally arises in the context of many-valued logic.

B.1 Introduction to the Appendix

We aim to explore relationships between many-valued logic and fuzzy topology from the viewpoint of duality theory. In particular, we consider fuzzy topological dualities for the algebras of Łukasiewicz n-valued logic $Ł_n^c$ with truth constants and for the algebras of modal Łukasiewicz n-valued logic $MŁ_n^c$ with truth constants.

Roughly speaking, a many-valued logic is a logical system in which there are more than two truth values (for a general introduction, see [120, 125, 181]). In many-valued logic, a proposition may have a truth value different from 0 (false) and 1 (true). Łukasiewicz many-valued logic introduced in [176] is one of the most prominent many-valued logics. Many-valued logics have often been studied from the algebraic point of view (see, e.g., [42, 58, 125]). MV-algebra introduced in [54] provides algebraic semantics for Łukasiewicz many-valued logic. MV_n-algebra introduced in [122] provides algebraic semantics for Łukasiewicz n-valued logic ([122] also gives an axiomatization of Łukasiewicz many-valued logic). $Ł_n^c$-algebras are considered MV_n-algebras enriched with constants.

Kripke semantics for modal logic is naturally extended to the many-valued case by allowing for more than two truth values at each possible world and so we can define modal many-valued logics by such many-valued Kripke semantics, including modal Łukasiewicz many-valued logics. Modal many-valued logics have already been studied by several authors (see [93, 94, 186, 265]).

As a major branch of fuzzy mathematics, fuzzy topology is based on the concept of fuzzy set introduced in [284, 115], which is defined by considering many-valued membership function. For example, a $[0,1]$-valued fuzzy set μ on a set X is defined as a function from X to $[0,1]$. Then, for $x \in X$ and $r \in [0,1]$, $\mu(x) = r$ intuitively means that the proposition "$x \in \mu$" has a truth value r. A fuzzy topology on a set is defined as a collection of fuzzy sets on the set which satisfies some conditions (for details, see Section B.3). Historically, Chang [55] introduced the concept of $[0,1]$-valued fuzzy topology and thereafter Goguen [116] introduced that of lattice-valued fuzzy topology. There have been many studies on fuzzy topology (see, e.g., [175, 244, 259]).

Stone duality for Boolean algebras (see [149, 262]) is one of the most important results in algebraic logic and states that there is a categorical duality between Boolean algebras (i.e., the algebras of classical propositional logic) and Boolean spaces (i.e., zero-dimensional compact Hausdorff spaces). Since both many-valued logic and fuzzy topology can be considered as based on the idea that there are more than two truth values, it is natural to expect that there is a duality between the algebras of many-valued logic and "fuzzy Boolean spaces." Stone duality for Boolean algebras was extended to Jónsson-Tarski duality (see [37, 59, 127, 248]) between modal algebras and relational spaces (or descriptive general frames), which is another classical theorem in duality theory. Thus, it is also natural to expect that there is a duality between the algebras of modal many-valued logic and "fuzzy relational spaces."

We realise the above expectations in the cases of $\mathrm{Ł}_n^c$ and $\mathrm{MŁ}_n^c$. We first develop a categorical duality between the algebras of $\mathrm{Ł}_n^c$ and $\mathbf{n}$-fuzzy Boolean spaces (see Definition B.4.5), which is a generalization of Stone duality for Boolean algebras to the $\mathbf{n}$-valued case via fuzzy topology. This duality is developed based on the following insights:

- The spectrum of an algebra of $\mathrm{Ł}_n^c$ can be naturally equipped with a certain $\mathbf{n}$-fuzzy topology (see Definition B.4.9).

- The notion of clopen subset of Boolean space in Stone duality for Boolean algebras corresponds to that of continuous function from $\mathbf{n}$-fuzzy Boolean space to

$\mathbf{n}$ ($= \{0, 1/(n-1), 2/(n-1), ..., 1\}$) equipped with the $\mathbf{n}$-fuzzy discrete topology in the duality for the algebras of $Ł_n^c$. This means that the zero-dimensionality of $\mathbf{n}$-fuzzy topological spaces is defined in terms of continuous function into $\mathbf{n}$ (see Definition B.4.4).

Moreover, based on the duality for the algebras of $Ł_n^c$, we develop a categorical duality between the algebras of $MŁ_n^c$ and $\mathbf{n}$-fuzzy relational spaces (see Definition B.6.3), which is a generalization of Jónsson-Tarski duality for modal algebras to the $\mathbf{n}$-valued case via fuzzy topology. Note that an $\mathbf{n}$-fuzzy relational space is also defined in terms of continuous functions into $\mathbf{n}$ (see the items 1 and 2 in the object part of Definition B.6.3).

There have been some studies on dualities for algebras of many-valued logics (see, e.g., [42, 57, 174, 215, 212, 265]). However, they are based on the ordinary topology and therefore do not reveal relationships between many-valued logic and fuzzy topology. By our results, we notice that fuzzy topological spaces naturally arise as spectrums of algebras of some many-valued logics and that there are categorical dualities connecting fuzzy topology and those many-valued logics which generalize Stone and Jónsson-Tarski dualities via fuzzy topology.

The rest of the appendix is organised as follows. In Section B.2, we define $Ł_n^c$ and $Ł_n^c$-algebras, and show basic properties of them. In Section B.3, we review basic concepts related to fuzzy topology. In Section B.4, we define $\mathbf{n}$-fuzzy Boolean spaces and show a fuzzy topological duality for $Ł_n^c$-algebras, which is a main theorem. In Section B.5, we define $MŁ_n^c$ and $MŁ_n^c$-algebras, and show basic properties of them, including a compactness theorem for $MŁ_n^c$. In Section B.6, we define $\mathbf{n}$-fuzzy relational spaces and show a fuzzy topological duality for $MŁ_n^c$-algebras, which is the other main theorem.

B.2 $Ł_n^c$-algebras and basic properties

Let n denote a natural number more than 1.

Definition B.2.1. $\mathbf{n}$ *denotes* $\{0, 1/(n-1), 2/(n-1), ..., 1\}$. *We equip* $\mathbf{n}$ *with all* *constants $r \in \mathbf{n}$ and the operations $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp})$ defined as follows:*

$$\begin{aligned} x \wedge y &= \min(x, y); \\ x \vee y &= \max(x, y); \\ x * y &= \max(0, x + y - 1); \\ x \wp y &= \min(1, x + y); \\ x \rightarrow y &= \min(1, 1 - (x - y)); \\ x^{\perp} &= 1 - x. \end{aligned}$$

We define Łukasiewicz n-valued logic with truth constants, which is denoted by Ł_n^c. The connectives of Ł_n^c are

$$(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1/(n-1), 2/(n-1), ..., 1),$$

where $(\wedge, \vee, *, \wp, \rightarrow)$ are binary connectives, $(\text{-})^{\perp}$ is a unary connective, and $(0, 1/(n-1), 2/(n-1), ..., 1)$ are constants. The formulas of Ł_n^c are recursively defined in the usual way. Let **PV** denote the set of propositional variables and **Form** denote the set of formulas of Ł_n^c.

$x \leftrightarrow y$ is the abbreviation of $(x \rightarrow y) \wedge (y \rightarrow x)$. For $m \in \omega$ with $m \neq 0$, $*^m x$ is the abbreviation of $x * ... * x$ (m-times). For instance, $*^3 x = x * x * x$.

Definition B.2.2. *A function $v : \mathbf{Form} \rightarrow \mathbf{n}$ is an $\mathbf{n}$-valuation iff it satisfies:*

- *$v(\varphi @ \psi) = v(\varphi) @ v(\psi)$ for $@ = \wedge, \vee, *, \wp, \rightarrow$;*
- *$v(\varphi^{\perp}) = (v(\varphi))^{\perp}$;*
- *$v(r) = r$ for $r \in \mathbf{n}$.*

Define $\text{Ł}_n^c = \{\varphi \in \mathbf{Form} \,;\, v(\varphi) = 1$ for any $\mathbf{n}$-valuation $v\}$.

Ł_n^c-algebras and homomorphisms are defined as follows.

Definition B.2.3. *$(A, \wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1/(n-1), 2/(n-1), ..., 1)$ is an Ł_n^c-algebra iff it satisfies the following set of equations: $\{\varphi = \psi \,;\, \varphi \leftrightarrow \psi \in \text{Ł}_n^c\}$.*

A homomorphism of Ł_n^c-algebras is defined as a function which preserves the operations $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1/(n-1), 2/(n-1), ..., 1)$.

We do not distinguish between formulas of Ł_n^c and terms of Ł_n^c-algebras.

Definition B.2.4. $\varphi \in \mathbf{Form}$ *is idempotent iff* $\varphi * \varphi \leftrightarrow \varphi \in \text{Ł}_n^c$.

For an Ł_n^c*-algebra* A*,* $a \in A$ *is idempotent iff* $a * a = a$.

$\mathcal{B}(A)$ *denotes the set of all idempotent elements of an* Ł_n^c*-algebra* A.

Let A be an Ł_n^c-algebra. Then, we have the following facts: (i) For $a \in A$, $*^{n-1}a$ is always idempotent. (ii) If $a \in A$ is idempotent, then either $v(a) = 1$ or $v(a) = 0$ holds for any homomorphism $v : A \to \mathbf{n}$. (iii) If $a, b \in A$ are idempotent, then $a * b = (*^{n-1}a) * (*^{n-1}b) = (*^{n-1}a) \wedge (*^{n-1}b) = a \wedge b$ and $a \wp b = (*^{n-1}a) \wp (*^{n-1}b) = (*^{n-1}a) \vee (*^{n-1}b) = a \vee b$.

It is easy to verify the following:

Proposition B.2.5. *For an* Ł_n^c*-algebra* A*,* $\mathcal{B}(A)$ *forms a Boolean algebra. In particular,* $a \vee a^{\perp} = 1$ *for any idempotent element* a *of* A.

In the following, we define a formula $\mathrm{T}_r(x)$ for $r \in \mathbf{n}$, which intuitively means that the truth value of x is exactly r.

Lemma B.2.6. *Let* A *be an* Ł_n^c*-algebra and* $r \in \mathbf{n}$*. There is an idempotent formula* $\mathrm{T}_r(x)$ *with one variable* x *such that, for any homomorphism* $v : A \to \mathbf{n}$ *and any* $a \in A$*, the following hold:*

- $v(\mathrm{T}_r(a)) = 1$ *iff* $v(a) = r$*;*
- $v(\mathrm{T}_r(a)) = 0$ *iff* $v(a) \neq r$*.*

Proof. If $r = 0$, then we can set $\mathrm{T}_r(x) = *^{n-1}(x^{\perp})$. If $r = 1$, then we can set $\mathrm{T}_r(x) = *^{n-1}x$.

Let $r = k/(n-1)$ for $k \in \{1, ..., n-2\}$. If k is a divisor of $n-1$, then we can set

$$\mathrm{T}_r(x) = *^{n-1}(x \leftrightarrow (\wp^{\frac{n-1}{k}-1}x)^{\perp}).$$

For a rational number q, let $[q]$ denote the greatest integer n such that $n \leq q$. If k is not a divisor of $n-1$, then

$$\begin{aligned} v(x) = k/(n-1) \quad &\text{iff} \quad v(\wp^{[\frac{n-1}{k}]}x) = \frac{k}{n-1}\left[\frac{n-1}{k}\right] (< 1) \\ &\text{iff} \quad v((\wp^{[\frac{n-1}{k}]}x)^{\perp}) = 1 - \frac{k}{n-1}\left[\frac{n-1}{k}\right]. \end{aligned}$$

Since

$$1 - \frac{k}{n-1}\left[\frac{n-1}{k}\right] < \frac{k}{n-1},$$

this lemma follows by induction on k. $\square$

The above lemma is more easily proved by using truth constants $r \in \mathbf{n}$. However, it must be stressed that the above proof works even if we consider Łukasiewicz n-valued logic without truth constants.

Note that any homomorphism preserves the operation $\mathrm{T}_r(\text{-})$.

Lemma B.2.7. *Let A be an Ł_n^c-algebra and $a_i \in A$ for a finite set I and $i \in I$. Then, (i) $\mathrm{T}_1(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} \mathrm{T}_1(a_i)$; (ii) $\mathrm{T}_1(\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} \mathrm{T}_1(a_i)$.*

Proof. Since $\mathbf{n}$ is totally ordered, we have (i). (ii) is immediate. $\square$

By (ii) in the above lemma, $\mathrm{T}_1(\text{-})$ is order preserving.

Lemma B.2.8. *Let A be an Ł_n^c-algebra and $r \in \mathbf{n}$. There is an idempotent formula $\mathrm{U}_r(x)$ with one variable x such that, for any homomorphism $v : A \to \mathbf{n}$ and any $a \in A$, the following two conditions hold: (i) $v(\mathrm{U}_r(a)) = 1$ iff $v(a) \geq r$; (ii) $v(\mathrm{U}_r(a)) = 0$ iff $v(a) \not\geq r$.*

Proof. It suffices to let $\mathrm{U}_r(x) = \bigvee\{\mathrm{T}_s(x) \ ; \ r \leq s\}$ by Lemma B.2.6. $\square$

Note that any homomorphism preserves the operation $\mathrm{U}_r(\text{-})$.

Lemma B.2.9. *Let A be an Ł_n^c-algebra and $r \in \mathbf{n}$. There is a formula $S_r(x)$ with one variable x such that, for any homomorphism $v : A \to \mathbf{n}$ and any $a \in A$, the following two conditions hold: (i) $v(S_r(a)) = r$ iff $v(a) = 1$; (ii) $v(S_r(a)) = 0$ iff $v(a) \neq 1$.*

Proof. Let $S_r(x) = (\mathrm{T}_1(x) \to r) \wedge ((\mathrm{T}_1(x))^{\perp} \to 0)$. $\square$

Note that any homomorphism preserves the operation $\mathrm{S}_r(\text{-})$.

Lemma B.2.10. *Let A be an Ł_n^c-algebra. Let v and u be homomorphisms from A to $\mathbf{n}$. Then, (i) $v = u$ iff (ii) $v^{-1}(\{1\}) = u^{-1}(\{1\})$.*

Proof. Clearly, (i) implies (ii). We show the converse. Assume that $v^{-1}(\{1\}) = u^{-1}(\{1\})$. Suppose for contradiction that $v(a) \neq u(a)$ for some $a \in A$. Let $r = v(a)$. Then $v(\mathrm{T}_r(a)) = 1$ and $u(\mathrm{T}_r(a)) = 0$, which contradicts $v^{-1}(\{1\}) = u^{-1}(\{1\})$. $\square$

For an Ł_n^c-algebra A and $a, b \in A$, we mean $a \vee b = b$ by $a \leq b$.

Lemma B.2.11. *Let A be an Ł_n^c-algebra. For any $a, b \in A$, the following holds:*

$$\bigwedge_{r\in\mathbf{n}} (\mathrm{T}_r(a) \leftrightarrow \mathrm{T}_r(b)) \leq a \leftrightarrow b.$$

Proof. This is proved by straightforward computation. $\square$

For a partially ordered set $(M, \leq)$, $X \subset M$ is called an upper set iff if $x \in X$ and $x \leq y$ for $y \in M$ then $y \in X$.

Definition B.2.12. *Let A be an $Ł_n^c$-algebra. A non-empty subset F of A is called an* **n***-filter of A iff F is an upper set and is closed under $*$. An* **n***-filter F of A is called proper iff $F \neq A$.*

An **n**-filter of A is closed under $\wedge$, since $a * b \leq a \wedge b$ for any $a, b \in A$.

Definition B.2.13. *Let A be an $Ł_n^c$-algebra. A proper* **n***-filter P of A is prime iff, for any $a, b \in A$, $a \vee b \in P$ implies either $a \in P$ or $b \in P$.*

Proposition B.2.14. *Let A be an $Ł_n^c$-algebra and F an* **n***-filter of A. For $b \in A$, assume $b \notin F$. Then, there is a prime* **n***-filter P of A such that $F \subset P$ and $b \notin P$.*

Proof. Let Z be the set of all those **n**-filters G of A such that $F \subset G$ and $b \notin G$. Then $F \in Z$. Clearly, every chain of Z has an upper bound in Z. Thus, by Zorn's lemma, we have a maximal element P in Z. Note that $F \subset P$ and $b \notin P$.

To complete the proof, it suffices to show that P is a prime **n**-filter of A. Assume $x \vee y \in P$. Additionally, suppose for contradiction that $x \notin P$ and $y \notin P$. Then, since P is maximal, there exists $\varphi_x \in A$ such that $\varphi_x \leq b$ and $\varphi_x = (*^{n-1}x) * p_x$ for some $p_x \in P$. Similary, there exists $\varphi_y \in A$ such that $\varphi_y \leq b$ and $\varphi_y = (*^{n-1}y) * p_y$ for some $p_y \in P$. Now, we have the following:

$$\begin{array}{rcl} b & \geq & ((*^{n-1}x) * p_x) \vee ((*^{n-1}y) * p_y) \\ & \geq & (*^{n-1}(x * p_x)) \vee (*^{n-1}(y * p_y)) \\ & = & *^{n-1}((x * p_x) \vee (y * p_y)) \\ & \geq & *^{n-1}((x \vee (y * p_y)) * (p_x \vee (y * p_y))) \\ & \geq & *^{n-1}((x \vee y) * p_y * p_x), \end{array}$$

where note that $*^{n-1}(x \vee y) = (*^{n-1}x) \vee (*^{n-1}y)$ and $x \vee (y * z) \geq (x \vee y) * (x \vee z)$ for any $x, y, z \in A$. Since $p_x, p_y, x \vee y \in P$, we have $b \in P$, which is a contradiction. Hence P is a prime **n**-filter of A. $\square$

We do not use (-)$^\perp$ or $\rightarrow$ in the above proof and therefore the above proof works even for algebras of "intuitionistic Łukasiewicz n-valued logic."

Definition B.2.15. *Let A be an $Ł_n^c$-algebra. A subset X of A has finite intersection property (f.i.p.) with respect to $*$ iff, for any $n \in \omega$ with $n \neq 0$, if $a_1, ..., a_n \in X$ then $a_1 * ... * a_n \neq 0$.*

Corollary B.2.16. *Let A be an $\mathrm{Ł}_n^c$-algebra and X a subset of A. If X has f.i.p. with respect to $*$, then there is a prime $\mathbf{n}$-filter P of A with $X \subset P$.*

Proof. By the assumption, we have a proper $\mathbf{n}$-filter F of A generated by X. By letting $b = 0$ in Proposition B.2.14, we have a prime $\mathbf{n}$-filter P of A with $X \subset P$. □

Proposition B.2.17. *Let A be an $\mathrm{Ł}_n^c$-algebra. For a prime $\mathbf{n}$-filter P of A, define $v_P : A \to \mathbf{n}$ by $v_P(a) = r \Leftrightarrow \mathrm{T}_r(a) \in P$. Then, v_P is a bijection from the set of all prime $\mathbf{n}$-filters of A to the set of all homomorphisms from A to $\mathbf{n}$ with $v_P^{-1}(\{1\}) = P$.*

Proof. Note that v_P is well-defined as a function. We prove that v_P is a homomorphism. We first show $v_P(a * b) = v_P(a) * v_P(b)$ for $a, b \in A$. Let $r = v_P(a)$ and $s = v_P(b)$. Then $\mathrm{T}_r(a) \in P$ and $\mathrm{T}_s(b) \in P$. It is easy to see that $\mathrm{T}_r(a) \wedge \mathrm{T}_s(b) \leq \mathrm{T}_{r*s}(a * b)$, which intuitively means that if the truth value of a is r and if the truth value of b is s then the truth value of $a * b$ is $r * s$. Since $\mathrm{T}_r(a) \in P$ and $\mathrm{T}_s(b) \in P$, we have $\mathrm{T}_{r*s}(a * b) \in P$, whence we have $v_P(a * b) = r * s = v_P(a) * v_P(b)$.

Next we show that $v_P(a^{\perp}) = v_P(a)^{\perp}$. Let $r = v_P(a)$. It is easy to see that $\mathrm{T}_r(a) \leq \mathrm{T}_{r^{\perp}}(a^{\perp})$. By $\mathrm{T}_r(a) \in P$, we have $\mathrm{T}_{r^{\perp}}(a^{\perp}) \in P$, whence $v_P(a^{\perp}) = r^{\perp} = v_P(a)^{\perp}$. As is well-known, $(\wedge, \vee, \wp, \to)$ can be defined by using only $(*, (\text{-})^{\perp})$ (see [58]) and so v_P preserves the operations $(\wedge, \vee, \wp, \to)$. Clearly, v_P preserves any constant $r \in \mathbf{n}$. Thus, v_P is a homomorphism. The remaining part of the proof is straightforward. □

B.3 n-valued fuzzy topology

Let us review basic concepts from fuzzy set theory and fuzzy topology.

B.3.1 n-valued fuzzy set theory

An $\mathbf{n}$-fuzzy set on a set S is defined as a function from S to $\mathbf{n}$. For $\mathbf{n}$-fuzzy sets μ, λ on S, define an $\mathbf{n}$-fuzzy set $\mu @ \lambda$ on S by $(\mu @ \lambda)(x) = \mu(x) @ \lambda(y)$ for $@ = \wedge, \vee, *, \wp, \to$, and define an $\mathbf{n}$-fuzzy set $\mu^{\perp}$ on S by $(\mu^{\perp})(x) = (\mu(x))^{\perp}$.

Let X, Y be sets and f a function from X to Y. For an $\mathbf{n}$-fuzzy set μ on X, define the direct image $f(\mu) : Y \to \mathbf{n}$ of μ under f by

$$f(\mu)(y) = \bigvee \{\mu(x) \,;\, x \in f^{-1}(\{y\})\} \text{ for } y \in Y.$$

For $f : X \to Y$ and an $\mathbf{n}$-fuzzy set λ on Y, define the inverse image $f^{-1}(\lambda) : X \to \mathbf{n}$ of λ under f by $f^{-1}(\lambda) = \lambda \circ f$. Note that f^{-1} commutes with $\bigvee$, i.e., $f^{-1}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} f^{-1}(\mu_i)$ for $\mathbf{n}$-fuzzy sets μ_i on Y.

For a relation R on a set S and an $\mathbf{n}$-fuzzy set μ on S, define an $\mathbf{n}$-fuzzy set $R^{-1}[\mu]$ on S, which is called the inverse image of μ under R, by $R^{-1}[\mu](x) = \bigvee\{\mu(y)\,;\, xRy\}$ for $x \in S$. Note that $R^{-1}[\bigvee_{i\in I}\mu_i] = \bigvee_{i\in I}(R^{-1}[\mu_i])$.

B.3.2 n-valued fuzzy topology

For sets X and Y, Y^X denotes the set of all functions from X to Y. We do not distinguish between $r \in \mathbf{n}$ and the constant function whose value is always r.

Definition B.3.1 ([284, 116, 259])**.** *For a set S and a subset $\mathcal{O}$ of $\mathbf{n}^S$, $(S, \mathcal{O})$ is an $\mathbf{n}$-fuzzy space iff the following hold:*

- *$r \in \mathcal{O}$ for any $r \in \mathbf{n}$;*
- *if $\mu_1, \mu_2 \in \mathcal{O}$ then $\mu_1 \wedge \mu_2 \in \mathcal{O}$;*
- *if $\mu_i \in \mathcal{O}$ for $i \in I$ then $\bigvee_{i\in I}\mu_i \in \mathcal{O}$,*

Then, we call $\mathcal{O}$ the $\mathbf{n}$-fuzzy topology of $(S, \mathcal{O})$, and an element of $\mathcal{O}$ an open $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$. An $\mathbf{n}$-fuzzy set λ on S is a closed $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$ iff $\lambda = \mu^\perp$ for some open $\mathbf{n}$-fuzzy set μ on $(S, \mathcal{O})$. A clopen $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$ means a closed and open $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$.

An $\mathbf{n}$-fuzzy space $(S, \mathcal{O})$ is often denoted by its underlying set S.

Definition B.3.2. *For a set S, $\mathbf{n}^S$ is called the discrete $\mathbf{n}$-fuzzy topology on S. $(S, \mathbf{n}^S)$ is called a discrete $\mathbf{n}$-fuzzy space.*

Definition B.3.3. *Let S_1 and S_2 be $\mathbf{n}$-fuzzy spaces. Then, $f : S_1 \to S_2$ is continuous iff, for any open $\mathbf{n}$-fuzzy set μ on S_2, $f^{-1}(\mu)$ (i.e., $\mu \circ f$) is an open $\mathbf{n}$-fuzzy set on S_1.*

A composition of continuous functions between $\mathbf{n}$-fuzzy spaces is also continuous (as a function between $\mathbf{n}$-fuzzy spaces).

Definition B.3.4. *Let $(S, \mathcal{O})$ be an $\mathbf{n}$-fuzzy space. Then, an open basis $\mathcal{B}$ of $(S, \mathcal{O})$ is a subset of $\mathcal{O}$ such that the following holds: (i) $\mathcal{B}$ is closed under $\wedge$; (ii) for any $\mu \in \mathcal{O}$, there are $\mu_i \in \mathcal{B}$ for $i \in I$ with $\mu = \bigvee_{i\in I}\mu_i$.*

Definition B.3.5. *An $\mathbf{n}$-fuzzy space S is Kolmogorov iff, for any $x, y \in S$ with $x \neq y$, there is an open $\mathbf{n}$-fuzzy set μ on S with $\mu(x) \neq \mu(y)$.*

Definition B.3.6. *An* $\mathbf{n}$*-fuzzy space* S *is Hausdorff iff, for any* $x, y \in S$ *with* $x \neq y$*, there are* $r \in \mathbf{n}$ *and open* $\mathbf{n}$*-fuzzy sets* μ, λ *on* S *such that* $\mu(x) \geq r$*,* $\lambda(y) \geq r$ *and* $\mu \wedge \lambda < r$*.*

Definition B.3.7 ([116])**.** *Let* S *be an* $\mathbf{n}$*-fuzzy space. An* $\mathbf{n}$*-fuzzy set* λ *on* S *is compact iff, if* $\lambda \leq \bigvee_{i \in I} \mu_i$ *for open* $\mathbf{n}$*-fuzzy sets* μ_i *on* S*, then there is a finite subset* J *of* I *such that* $\lambda \leq \bigvee_{i \in J} \mu_i$*.*

Let 1 *denote the constant function on* S *whose value is always* 1*. Then,* S *is compact iff, if* $1 = \bigvee_{i \in I} \mu_i$ *for open* $\mathbf{n}$*-fuzzy sets* μ_i *on* S*, then there is a finite subset* J *of* I *such that* $1 = \bigvee_{i \in J} \mu_i$*.*

We can construct an operation $(\text{-})^*$ which turns an $\mathbf{n}$-fuzzy space into a topological space (in the classical sense) as follows.

Definition B.3.8. *Let* $(S, \mathcal{O})$ *be an* $\mathbf{n}$*-fuzzy space. Define*

$$\mathcal{O}^* = \{\mu^{-1}(\{1\})\ ;\ \mu \in \mathcal{O}\}.$$

Then, S^* *denotes a topological space* $(S, \mathcal{O}^*)$ *(see the below proposition).*

Lemma B.3.9. *Let* $(S, \mathcal{O})$ *be an* $\mathbf{n}$*-fuzzy space. Then,* S^* *forms a topological space.*

Proof. Since $0 \in \mathcal{O}$ and $\emptyset = 0^{-1}(\{1\})$, we have $\emptyset \in \mathcal{O}^*$. Similarly, $S \in \mathcal{O}^*$. Assume $X_i \in \mathcal{O}$ for $i \in I$. Then, $X_i = {\mu_i}^{-1}(\{1\})$ for some $\mu_i \in \mathcal{O}$. Since $\mathbf{n}$ is totally ordered, $\bigcup_{i \in I} X_i = (\bigvee_{i \in I} \mu_i)^{-1}(\{1\})$. Thus, by $\bigvee_{i \in I} \mu_i \in \mathcal{O}$, we have $\bigcup_{i \in I} X_i \in \mathcal{O}^*$. It is easy to verify that $X, Y \in \mathcal{O}$ implies $X \cap Y \in \mathcal{O}^*$. $\square$

B.4 A fuzzy topological duality for $Ł_n^c$-algebras

In this section, we show a fuzzy topological duality for $Ł_n^c$-algebras, which is a generalization of Stone duality for Boolean algebras via fuzzy topology, where note that $Ł_2^c$-algebras coincide with Boolean algebras.

Definition B.4.1. $Ł_n^c$-Alg *denotes the category whose objects are* $Ł_n^c$*-algebras and whose arrows are homomorphisms of* $Ł_n^c$*-algebras.*

Our aim in this section is to show that the category $Ł_n^c$-Alg is dually equivalent to the category FBS_n, which is defined in the following subsection.

B.4.1 Category FBS_n

We equip $\mathbf{n}$ with the discrete $\mathbf{n}$-fuzzy topology.

Definition B.4.2. *Let S be an $\mathbf{n}$-fuzzy space. Then,* $\mathrm{Cont}(S)$ *is defined as the set of all continuous functions from S to $\mathbf{n}$. We endow* $\mathrm{Cont}(S)$ *with the operations* $(\wedge, \vee, *, \wp, \to, (\text{-})^{\perp}, 0, 1/(n-1), 2/(n-1), ..., 1)$ *defined pointwise: For $f, g \in \mathrm{Cont}(S)$, define $(f@g)(x) = f(x)@g(x)$, where $@ = \wedge, \vee, *, \wp, \to$. For $f \in \mathrm{Cont}(S)$, define $f^{\perp}(x) = (f(x))^{\perp}$. Finally, $r \in \mathbf{n}$ is defined as the constant function on S whose value is always r.*

We show that the operations of $\mathrm{Cont}(S)$ are well-defined:

Lemma B.4.3. *Let S be an $\mathbf{n}$-fuzzy space. Then,* $\mathrm{Cont}(S)$ *is closed under the operations* $(\wedge, \vee, *, \wp, \to, (\text{-})^{\perp}, 0, 1/(n-1), ..., (n-2)/(n-1), 1)$

Proof. For any $r \in \mathbf{n}$, a constant function $r : S \to \mathbf{n}$ is continuous, since any $s \in \mathbf{n}$ is an open $\mathbf{n}$-fuzzy set on S by Definition B.3.1. Then it suffices to show that, if $f, g \in \mathrm{Cont}(S)$, then $f^{\perp}$ and $f@g$ are continuous for $@ = \wedge, \vee, *, \wp, \to$. Throughout this proof, let $f, g \in \mathrm{Cont}(S)$ and μ an open $\mathbf{n}$-fuzzy set on $\mathbf{n}$, i.e., a function from $\mathbf{n}$ to $\mathbf{n}$. For $r \in \mathbf{n}$, define $\mu_r : \mathbf{n} \to \mathbf{n}$ by

$$\mu_r(x) = \begin{cases} \mu(r) & \text{if } x = r \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have $\mu = \bigvee_{r \in \mathbf{n}} \mu_r$.

We show that $(f^{\perp})^{-1}(\mu)$ is an open $\mathbf{n}$-fuzzy set on S. Now, we have

$$(f^{\perp})^{-1}(\mu) = (f^{\perp})^{-1}(\bigvee_{r \in \mathbf{n}} \mu_r) = \bigvee_{r \in \mathbf{n}} ((f^{\perp})^{-1}(\mu_r)).$$

Thus it suffices to show that $(f^{\perp})^{-1}(\mu_r)$ is an open $\mathbf{n}$-fuzzy set on S for any $r \in \mathbf{n}$. Define $\lambda_r : \mathbf{n} \to \mathbf{n}$ by

$$\lambda_r(x) = \begin{cases} \mu(r) & \text{if } x = 1 - r \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that $(f^{\perp})^{-1}(\mu_r) = f^{-1}(\lambda_r)$. Since f is continuous and since λ_r is an open $\mathbf{n}$-fuzzy set on $\mathbf{n}$, $f^{-1}(\lambda_r)$ is an open $\mathbf{n}$-fuzzy set on S.

Next, we show that $(f*g)^{-1}(\mu)$ is an open $\mathbf{n}$-fuzzy set on S. By the same argument as in the case of $f^{\perp}$, it suffices to show that $(f * g)^{-1}(\mu_r)$ is an open $\mathbf{n}$-fuzzy set on S for any $r \in \mathbf{n}$. For $p \in \mathbf{n}$, define $\theta_{r,p} : \mathbf{n} \to \mathbf{n}$ by

$$\theta_{r,p}(x) = \begin{cases} \mu(r) & \text{if } x = p \\ 0 & \text{otherwise.} \end{cases}$$

For $r \neq 0$, define $\kappa_{r,p} : \mathbf{n} \to \mathbf{n}$ by

$$\kappa_{r,p}(x) = \begin{cases} \mu(r) & \text{if } x = r - p + 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 0$, define $\kappa_{r,p} : \mathbf{n} \to \mathbf{n}$ by

$$\kappa_{r,p}(x) = \begin{cases} \mu(r) & \text{if } x \leq r - p + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that

$$(f * g)^{-1}(\mu_r) = \bigvee_{p \in \mathbf{n}} (f^{-1}(\theta_{r,p}) \wedge g^{-1}(\kappa_{r,p})).$$

Since $f, g \in \mathrm{Cont}(S)$, the right-hand side is an open $\mathbf{n}$-fuzzy set on S.

As is well-known, $(\wedge, \vee, \wp, \to)$ can be defined by using only $(*, (\text{-})^{\perp})$ (see [58]) and so $(f @ g)^{-1}(\mu)$ is an open $\mathbf{n}$-fuzzy set for $@ = \wedge, \vee, \wp, \to$. □

Definition B.4.4. *For an* $\mathbf{n}$*-fuzzy space* S*,* S *is zero-dimensional iff* $\mathrm{Cont}(S)$ *forms an open basis of* S*.*

Definition B.4.5. *For an* $\mathbf{n}$*-fuzzy space* S*,* S *is an* $\mathbf{n}$*-fuzzy Boolean space iff* S *is zero-dimensional, compact and Kolmogorov.*

Definition B.4.6. FBS_n *is defined as the category of* $\mathbf{n}$*-fuzzy Boolean spaces and continuous functions.*

Proposition B.4.7. *Let* S *be an* $\mathbf{n}$*-fuzzy space. Then, (i)* S *is an* $\mathbf{n}$*-fuzzy Boolean space iff (ii)* S *is zero-dimensional, compact and Hausdorff.*

Proof. Cleary, (ii) implies (i). We show the converse. Assume that S is an $\mathbf{n}$-fuzzy Boolean space. It suffices to show that S is Hausdorff. Let $x, y \in S$ with $x \neq y$. Since S is Kolmogorov and since S is zero-dimensional, there is $\mu \in \mathrm{Cont}(S)$ with $\mu(x) \neq \mu(y)$. Let $s = \mu(x)$. Then, $\mathrm{T}_s \circ \mu(x) = 1$ and $(\mathrm{T}_s \circ \mu)^{\perp}(y) = 1$. Since $\mathrm{T}_s : \mathbf{n} \to \mathbf{n}$ is continuous, $\mathrm{T}_s \circ \mu \in \mathrm{Cont}(S)$ and $(\mathrm{T}_s \circ \mu)^{\perp} \in \mathrm{Cont}(S)$ by Lemma B.4.3. Since S is zero-dimensional, $\mathrm{T}_s \circ \mu$ and $(\mathrm{T}_s \circ \mu)^{\perp}$ are open $\mathbf{n}$-fuzzy sets on S. We also have $(\mathrm{T}_s \circ \mu) \wedge (\mathrm{T}_s \circ \mu)^{\perp} = 0$. Thus, S is Hausdorff. □

Next we show that $(\text{-})^*$ turns an $\mathbf{n}$-fuzzy Boolean space into a Boolean space, i.e., a zero-dimensional compact Hausdorff space.

Proposition B.4.8. *Let* S *be an* $\mathbf{n}$*-fuzzy Boolean space. Then,* S^* *forms a Boolean space.*

Proof. By Lemma B.3.9, S^* is a topological space.

First, we show that S^* is zero-dimensional in the classical sense. Let $\mathcal{B}^* = \{\mu^{-1}(\{1\})\ ;\ \mu \in \mathrm{Cont}(S)\}$, where, since S is zero-dimensional and so $\mu \in \mathrm{Cont}(S)$ is an open $\mathbf{n}$-fuzzy set on S, $\mu^{-1}(\{1\})$ is an open subset of S^*. We claim that $\mathcal{B}^*$ forms an open basis of S^*. It is easily verified that $\mathcal{B}^*$ is closed under $\cap$. Assume that O is an open subset of S^*, i.e., $O = \mu^{-1}(\{1\})$ for some open $\mathbf{n}$-fuzzy set μ on S. Since S is zero-dimensional, there are $\mu_i \in \mathrm{Cont}(S)$ with $\mu = \bigvee_{i\in I}\mu_i$. Since $\mathbf{n}$ is totally ordered, $O = \bigcup_{i\in I}\mu_i^{-1}(\{1\})$. It follows from $\mu_i \in \mathrm{Cont}(S)$ that $\mu_i^{-1}(\{1\}) \in \mathcal{B}^*$ for any $i \in I$. This completes the proof of the claim. If $\mu \in \mathrm{Cont}(S)$, then

$$(\mu^{-1}(\{1\}))^c = ((\mathrm{T}_1 \circ \mu)^{\perp})^{-1}(\{1\}).$$

Since $\mathrm{T}_1 : \mathbf{n} \to \mathbf{n}$ is continuous, $\mathrm{T}_1 \circ \mu \in \mathrm{Cont}(S)$, whence, by Lemma B.4.3, $(\mathrm{T}_1 \circ \mu)^{\perp} \in \mathrm{Cont}(S)$. Thus the right-hand side is open in S^* and so $\mu^{-1}(\{1\})$ is clopen in S^* for $\mu \in \mathrm{Cont}(S)$. Hence, S^* is zero-dimensional.

Second, we show that S^* is compact in the classical sense. Assume that $S^* = \bigcup_{i\in I} O_i$ for some open subsets O_i of S^*. Since $\mathcal{B}^*$ forms an open basis of S^*, we may assume that $S^* = \bigcup_{i\in I}\mu_i^{-1}(\{1\})$ for some $\mu_i \in \mathrm{Cont}(S)$. Then, $1 = \bigvee_{i\in I}\mu_i$ where 1 denotes the constant function on S $(= S^*)$ whose value is always 1. Since S is zero-dimensional, μ_i is an open $\mathbf{n}$-fuzzy set on S. Thus, since S is compact, there is a finite subset J of I such that $1 = \bigvee_{j\in J}\mu_j$, whence $S^* = \bigcup_{j\in J}\mu_j^{-1}(\{1\})$. Hence S^* is compact.

Finally, we show that S^* is Hausdorff in the classical sense. Since S^* is zero-dimensional, it suffices to show that S^* is Kolmogorov in the classical sense. Assume $x, y \in S^*$ with $x \neq y$. Since S is Kolmogorov, there is an open $\mathbf{n}$-fuzzy set μ on S with $\mu(x) \neq \mu(y)$. Since S is zero-dimensional, $\mu = \bigvee_{i\in I}\mu_i$ for some $\mu_i \in \mathrm{Cont}(S)$. There is $i \in I$ with $\mu_i(x) \neq \mu_i(y)$. Let $r = \mu_i(x)$. Then, we have $\mathrm{T}_r \circ \mu_i(x) = 1$ and $\mathrm{T}_r \circ \mu_i(y) = 0$, whence we have $x \in (\mathrm{T}_r \circ \mu_i)^{-1}(\{1\})$ and $y \notin (\mathrm{T}_r \circ \mu_i)^{-1}(\{1\})$. Since $\mathrm{T}_r : \mathbf{n} \to \mathbf{n}$ is continuous, it follows from $\mu_i \in \mathrm{Cont}(S)$ that $\mathrm{T}_r \circ \mu_i \in \mathrm{Cont}(S)$, whence $\mathrm{T}_r \circ \mu_i$ is an open $\mathbf{n}$-fuzzy set on S and so $(\mathrm{T}_r \circ \mu_i)^{-1}(\{1\})$ is an open subset of S^*. Hence S^* is Kolmogorov. $\square$

B.4.2 Functors Spec and Cont

We define the spectrum $\mathrm{Spec}(A)$ of an Ł_n^c-algebra A as follows.

Definition B.4.9. *For an* Ł_n^c*-algebra* A, $\mathrm{Spec}(A)$ *is defined as the set of all homomorphisms (of* Ł_n^c*-algebras) from* A *to* $\mathbf{n}$ *equipped with the* $\mathbf{n}$*-fuzzy topology generated by* $\{\langle a\rangle\ ;\ a\in A\}$*, where* $\langle a\rangle : \mathrm{Spec}(A)\to\mathbf{n}$ *is defined by*

$$\langle a\rangle(v) = v(a).$$

The operations $(\wedge, \vee, *, \wp, \to, (\text{-})^{\perp})$ *on* $\{\langle a\rangle\ ;\ a\in A\}$ *are defined pointwise as in Definition B.4.2.*

$\{\langle a\rangle\ ;\ a\in A\}$ forms an open basis of $\mathrm{Spec}(A)$, since $\langle a\rangle\wedge\langle b\rangle = \langle a\wedge b\rangle$.

Definition B.4.10. *We define a contravariant functor* $\mathrm{Spec} : \text{Ł}_n^c\text{-Alg}\to\mathrm{FBS}_n$*.*

For an object A *in* Ł_n^c-Alg*, define* $\mathrm{Spec}(A)$ *as in Definition B.4.9.*

For an arrow $f : A_1\to A_2$ *in* Ł_n^c-Alg*, define* $\mathrm{Spec}(f) : \mathrm{Spec}(A_2)\to\mathrm{Spec}(A_1)$ *by* $\mathrm{Spec}(f)(v) = v\circ f$ *for* $v\in\mathrm{Spec}(A_2)$*.*

The well-definedness of the functor Spec is proved by Proposition B.4.15 and Proposition B.4.16 below.

Since $\mathbf{n}$ is a totally ordered complete lattice, we have:

Lemma B.4.11. *Let* μ_i *be an* $\mathbf{n}$*-fuzzy set on a set* S *for a set* I *and* $i\in I$*. Then, (i)* $\mathrm{T}_1\circ\bigvee_{i\in I}\mu_i = \bigvee_{i\in I}(\mathrm{T}_1\circ\mu_i)$*; (ii)* $\mathrm{T}_1\circ\bigwedge_{i\in I}\mu_i = \bigwedge_{i\in I}(\mathrm{T}_1\circ\mu_i)$*.*

Lemma B.4.12. *Let* A *be an* Ł_n^c*-algebra. Then,* $\mathrm{Spec}(A)$ *is compact.*

Proof. Assume that $1 = \bigvee_{j\in J}\mu_j$ for open $\mathbf{n}$-fuzzy sets μ_j on $\mathrm{Spec}(A)$, where 1 denotes the constant function defined on $\mathrm{Spec}(A)$ whose value is always 1. Then, since $\{\langle a\rangle\ ;\ a\in A\}$ is an open basis of $\mathrm{Spec}(A)$, we may assume that $1 = \bigvee_{i\in I}\langle a_i\rangle$ for some $a_i\in A$. It follows from Lemma B.4.11 that $1 = \mathrm{T}_1\circ 1 = \mathrm{T}_1\circ\bigvee_{i\in I}\langle a_i\rangle = \bigvee_{i\in I}\mathrm{T}_1\circ\langle a_i\rangle = \bigvee_{i\in I}\langle\mathrm{T}_1(a_i)\rangle$. Thus, we have

$$0 = (\bigvee_{i\in I}\langle\mathrm{T}_1(a_i)\rangle)^{\perp} = \bigwedge_{i\in I}\langle(\mathrm{T}_1(a_i))^{\perp}\rangle.$$

Then, there is no homomorphism $v : A\to\mathbf{n}$ such that $v((\mathrm{T}_1(a_i))^{\perp}) = 1$ for any $i\in I$. Therefore, by Proposition B.2.17, there is no prime $\mathbf{n}$-filter of A which contains $\{(\mathrm{T}_1(a_i))^{\perp}\ ;\ i\in I\}$. Thus, by Corollary B.2.16, $\{(\mathrm{T}_1(a_i))^{\perp}\ ;\ i\in I\}$ does not have f.i.p. with respect to $*$ and so there is a finite subset $\{i_1, ...i_m\}$ of I such that $(\mathrm{T}_1(a_{i_1}))^{\perp} * ... * (\mathrm{T}_1(a_{i_m}))^{\perp} = 0$, whence $\mathrm{T}_1(a_{i_1})\wp ...\wp\mathrm{T}_1(a_{i_m}) = 1$. Since $\mathrm{T}_1(a_{i_k})$ is idempotent for any $k\in\{1, ..., m\}$, we have $\mathrm{T}_1(a_{i_1})\vee ...\vee\mathrm{T}_1(a_{i_m}) = 1$ and, by Lemma B.2.7, $\mathrm{T}_1(a_{i_1}\vee ...\vee a_{i_m}) = 1$. By $\mathrm{T}_1(x)\leq x$, we have $a_{i_1}\vee ...\vee a_{i_m} = 1$, whence $\langle a_{i_1}\vee ...\vee a_{i_m}\rangle = 1$. This completes the proof. $\square$

Lemma B.4.13. *Let A be an* $Ł_n^c$*-algebra. Then,* $\mathrm{Spec}(A)$ *is Kolmogorov.*

Proof. Let $v_1, v_2 \in \mathrm{Spec}(A)$ with $v_1 \neq v_2$. Then there is $a \in A$ such that $v_1(a) \neq v_2(a)$, whence we have $\langle a\rangle(v_1) \neq \langle a\rangle(v_2)$. □

Lemma B.4.14. *Let A be an* $Ł_n^c$*-algebra. Then,* $\mathrm{Spec}(A)$ *is zero-dimensional.*

Proof. Since $\{\langle a\rangle\ ;\ a \in A\}$ forms an open basis of $\mathrm{Spec}(A)$, it suffices to show that

$$\mathrm{Cont} \circ \mathrm{Spec}(A) = \{\langle a\rangle\ ;\ a \in A\}.$$

We first show that $\mathrm{Cont} \circ \mathrm{Spec}(A) \supset \{\langle a\rangle\ ;\ a \in A\}$, i.e., $\langle a\rangle$ is continuous for any $a \in A$. Let $a \in A$ and μ an $\mathbf{n}$-fuzzy set on $\mathbf{n}$. Then, by Lemma B.2.9,

$$\langle a\rangle^{-1}(\mu) = \mu \circ \langle a\rangle = \bigvee_{r\in\mathbf{n}} (\mathrm{S}_{\mu(r)} \circ \mathrm{T}_r) \circ \langle a\rangle = \langle \bigvee_{r\in\mathbf{n}} (\mathrm{S}_{\mu(r)}(\mathrm{T}_r(a)))\rangle.$$

Hence $\langle a\rangle$ is continuous.

Next we show $\mathrm{Cont} \circ \mathrm{Spec}(A) \subset \{\langle a\rangle ; a \in A\}$. Let $f \in \mathrm{Cont} \circ \mathrm{Spec}(A)$ and $r \in \mathbf{n}$. Define an $\mathbf{n}$-fuzzy set λ_r on $\mathbf{n}$ by $\lambda_r(x) = 1$ for $x = r$ and $\lambda_r(x) = 0$ for $x \neq r$. Since f is continuous, $f^{-1}(\lambda_r) = \bigvee_{i\in I}\langle a_i\rangle$ for some $a_i \in A$. Now the following holds:

$$1 = f^{-1}(\lambda_r) \vee (f^{-1}(\lambda_r))^{\perp} = (\bigvee_{i\in I}\langle a_i\rangle) \vee (f^{-1}(\lambda_r))^{\perp}.$$

Here, we have $(f^{-1}(\lambda_r))^{\perp} = (\lambda_r \circ f)^{\perp} = {\lambda_r}^{\perp} \circ f = f^{-1}({\lambda_r}^{\perp})$. Since $f^{-1}({\lambda_r}^{\perp})$ is an open $\mathbf{n}$-fuzzy set, $(f^{-1}(\lambda_r))^{\perp}$ is an open $\mathbf{n}$-fuzzy set on $\mathrm{Spec}(A)$. Since $\mathrm{Spec}(A)$ is compact by Lemma B.4.12, there is a finite subset J of I such that $1 = (\bigvee_{j\in J}\langle a_j\rangle) \vee (f^{-1}(\lambda_r))^{\perp}$. Thus, $f^{-1}(\lambda_r) \leq \bigvee_{j\in J}\langle a_j\rangle$. Since $\bigvee_{j\in J}\langle a_j\rangle \leq \bigvee_{i\in I}\langle a_i\rangle = f^{-1}(\lambda_r)$, we have $f^{-1}(\lambda_r) = \bigvee_{j\in J}\langle a_j\rangle$. Since J is finite, $f^{-1}(\lambda_r) = \bigvee_{j\in J}\langle a_j\rangle = \langle \bigvee_{j\in J} a_j\rangle$. Let $a_r = \bigvee_{j\in J} a_j$. Note that if $v \in f^{-1}(\{r\})$ then $v(a_r) = 1$ and that if $v \notin f^{-1}(\{r\})$ then $v(a_r) = 0$. We claim that $f = \langle \bigvee_{r\in\mathbf{n}}(r \wedge a_r)\rangle$. If $v \in f^{-1}(\{s\})$ for $s \in \mathbf{n}$, then

$$\langle \bigvee_{r\in\mathbf{n}}(r \wedge a_r)\rangle(v) = v(\bigvee_{r\in\mathbf{n}}(r \wedge a_r)) = \bigvee_{r\in\mathbf{n}}(r \wedge v(a_r)) = s = f(v).$$

This completes the proof. □

By the above lemmas, we obtain the following proposition.

Proposition B.4.15. *Let A be an object in* $Ł_n^c$-Alg*. Then,* $\mathrm{Spec}(A)$ *is an object in the category* FBS_n.

Proposition B.4.16. *Let A_1 and A_2 be objects in* $Ł_n^c$-Alg *and $f : A_1 \to A_2$ an arrow in* $Ł_n^c$-Alg*. Then,* $\mathrm{Spec}(f)$ *is an arrow in* FBS_n.

Proof. Since the inverse image $(\mathrm{Spec}(f))^{-1}$ commutes with $\bigvee$, it suffices to show that $(\mathrm{Spec}(f))^{-1}(\langle a\rangle)$ is an open $\mathbf{n}$-fuzzy set on $\mathrm{Spec}(A_2)$ for any $a \in A_1$. For $v \in \mathrm{Spec}(A_2)$, we have

$$(\mathrm{Spec}(f)^{-1}(\langle a\rangle))(v) = \langle a\rangle \circ \mathrm{Spec}(f)(v) = \langle a\rangle(v \circ f) = v \circ f(a) = \langle f(a)\rangle(v).$$

Hence $(\mathrm{Spec}(f))^{-1}(\langle a\rangle) = \langle f(a)\rangle$, which is an open $\mathbf{n}$-fuzzy set. □

Definition B.4.17. *We define a contravariant functor* $\mathrm{Cont} : \mathrm{FBS}_n \to Ł_n^c\text{-Alg}$.

For an object S in FBS_n, $\mathrm{Cont}(S)$ *is defined as in Definition B.4.2.*

For an arrow $f : S \to T$ in FBS_n, $\mathrm{Cont}(f) : \mathrm{Cont}(T) \to \mathrm{Cont}(S)$ *is defined by* $\mathrm{Cont}(f)(g) = g \circ f$ *for* $g \in \mathrm{Cont}(T)$.

Since the operations of $\mathrm{Cont}(S)$ are defined pointwise, $\mathrm{Cont}(S)$ is an $Ł_n^c$-algebra and the following holds, whence Cont is well-defined.

Proposition B.4.18. *Let S_1 and S_2 be objects in* FBS_n*, and $f : S_1 \to S_2$ an arrow in* FBS_n*. Then,* $\mathrm{Cont}(f)$ *is an arrow in* $Ł_n^c$-Alg.

Definition B.4.19. *Let A be an $Ł_n^c$-algebra. Then,* $\mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$ *is defined as the set of all homomorphisms of Boolean algebras from $\mathcal{B}(A)$ to $\mathbf{2}$ equipped with the (ordinary) topology generated by $\{\langle a\rangle_{\mathbf{2}} \,;\, a \in \mathcal{B}(A)\}$, where $\langle a\rangle_{\mathbf{2}} = \{v \in \mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A)) \,;\, v(a) = 1\}$.*

Proposition B.4.20. *Let A be an $Ł_n^c$-algebra. Define a function t_1 from* $\mathrm{Spec}(A)^*$ *to* $\mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$ *by $t_1(v) = \mathrm{T}_1 \circ v$. Then, t_1 is a homeomorphism.*

Proof. By Lemma B.2.10, t_1 is injective. We show that t_1 is surjective. Let $v \in \mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$. Define $u \in \mathrm{Spec}(A)$ by $u(a) = r \Leftrightarrow \mathrm{T}_r(a) \in v^{-1}(\{1\})$ for $a \in A$, where note $\mathrm{T}_r(a) \in \mathcal{B}(A)$. Then, in a similar way to Proposition B.2.17, it is verified that u is a homomorphism (i.e., $u \in \mathrm{Spec}(A)$). Moreover, we have $t_1(u) = v$ on $\mathcal{B}(A)$. Thus t_1 is bijective. It is straightforward to verify the remaining part of the proof. Note that, for $\langle a\rangle_{\mathbf{n}} = \{v \in \mathrm{Spec}(A) \,;\, v(a) = 1\}$, $\{\langle a\rangle_{\mathbf{n}} \,;\, a \in A\}$ forms an open basis of $\mathrm{Spec}(A)^*$ and that $t_1(\langle a\rangle_{\mathbf{n}}) = \langle \mathrm{T}_1(a)\rangle_{\mathbf{2}}$ for $a \in A$. □

B.4.3 A fuzzy topological duality for $Ł_n^c$-algebras

Theorem B.4.21. *Let A be an $Ł_n^c$-algebra. Then, there is an isomorphism between A and* $\mathrm{Cont} \circ \mathrm{Spec}(A)$ *in the category* $Ł_n^c$-Alg.

Proof. Define $\langle\text{-}\rangle : A \to \mathrm{Cont} \circ \mathrm{Spec}(A)$ as in Definition B.4.9. In the proof of Lemma B.4.14, it has already been proven that $\langle\text{-}\rangle$ is well-defined and surjective. Since the operations of $\mathrm{Cont} \circ \mathrm{Spec}(A)$ are defined pointwise, $\langle\text{-}\rangle$ is a homomorphism.

Thus it suffices to show that $\langle\text{-}\rangle$ is injective. Assume that $\langle a\rangle = \langle b\rangle$ for $a, b \in A$, which means that, for any $v \in \mathrm{Spec}(A)$, we have $v(a) = v(b)$. Thus, for any $v \in \mathrm{Spec}(A)$ and any $r \in \mathbf{n}$, we have $v(\mathrm{T}_r(a)) = v(\mathrm{T}_r(b))$. Thus, it follows from Proposition B.2.17 that, for any prime $\mathbf{n}$-filter P of A and any $r \in \mathbf{n}$, $\mathrm{T}_r(a) \in P$ iff $\mathrm{T}_r(b) \in P$.

We claim that $\mathrm{T}_r(a) = \mathrm{T}_r(b)$ for any $r \in \mathbf{n}$. Suppose for contradiction that $\mathrm{T}_r(a) \neq \mathrm{T}_r(b)$ for some $r \in \mathbf{n}$. We may assume without loss of generality that $\mathrm{T}_r(a) \nleq \mathrm{T}_r(b)$. Let $F = \{x \in A\, ;\, \mathrm{T}_r(a) \leq x\}$. Then, since $\mathrm{T}_r(a)$ is idempotent, F is an $\mathbf{n}$-filter of A. Cleary, $\mathrm{T}_r(b) \notin F$. Thus, by Lemma B.2.14, there is a prime $\mathbf{n}$-filter P of A such that $F \subset P$ and $\mathrm{T}_r(b) \notin P$. By $F \subset P$, we have $\mathrm{T}_r(a) \in P$, which contradicts $\mathrm{T}_r(b) \notin P$, since we have already shown that $\mathrm{T}_r(a) \in P$ iff $\mathrm{T}_r(b) \in P$. Thus, $\mathrm{T}_r(a) = \mathrm{T}_r(b)$ for any $r \in \mathbf{n}$, whence $\bigwedge_{r\in\mathbf{n}}(\mathrm{T}_r(a) \leftrightarrow \mathrm{T}_r(b)) = 1$. Hence, it follows from Lemma B.2.11 that $a = b$, and therefore $\langle\text{-}\rangle$ is injective. □

Theorem B.4.22. *Let S be an $\mathbf{n}$-fuzzy Boolean space. Then, there is an isomorphism between S and* $\mathrm{Spec} \circ \mathrm{Cont}(S)$ *in the category* FBS_n.

Proof. Define $\Psi : S \to \mathrm{Spec} \circ \mathrm{Cont}(S)$ by $\Psi(x)(f) = f(x)$ for $x \in S$ and $f \in \mathrm{Cont}(S)$. Since the operations of $\mathrm{Cont}(S)$ are defined pointwise, $\Psi(x)$ is a homomorphism and so Ψ is well-defined.

We show that Ψ is continuous. Let $f \in \mathrm{Cont}(S)$. Then $\Psi^{-1}(\langle f\rangle) = f$ by the following:

$$(\Psi^{-1}(\langle f\rangle))(x) = \langle f\rangle \circ \Psi(x) = \Psi(x)(f) = f(x).$$

Since $f \in \mathrm{Cont}(S)$ and S is zero-dimensional, f is an an open $\mathbf{n}$-fuzzy set and so $\Psi^{-1}(\langle f\rangle)$ is an open $\mathbf{n}$-fuzzy set on S. Since the inverse image Ψ^{-1} commutes with $\bigvee$, it follows that Ψ is continuous.

Next we show that Ψ is injective. Let $x, y \in S$ with $x \neq y$. Since S is Kolmogorov and zero-dimensional, there is $f \in \mathrm{Cont}(S)$ with $f(x) \neq f(y)$. Thus, $\Psi(x)(f) = f(x) \neq f(y) = \Psi(y)(f)$, whence Ψ is injective.

Next we show that Ψ is surjective. Let $v \in \mathrm{Spec}{\circ}\mathrm{Cont}(S)$. Consider $\{f^{-1}(\{1\}); v(f) = 1\}$. Define $\mu : \mathbf{n} \to \mathbf{n}$ by $\mu(1) = 0$ and $\mu(x) = 1$ for $x \neq 1$. Since $f^{-1}(\mu)$ $(= \mu \circ f)$ is an open $\mathbf{n}$-fuzzy set on S for $f \in \mathrm{Cont}(S)$, $(\mu \circ f)^{-1}(\{1\})$ is an open subset of S^*. Since $(\mu \circ f)^{-1}(\{1\}) = (f^{-1}(\{1\}))^c$, $f^{-1}(\{1\})$ is a closed subset of S^* for $f \in \mathrm{Cont}(S)$.

We claim that $\{f^{-1}(\{1\})\,;\, v(f) = 1\}$ has the finite intersection property. Since $f^{-1}(\{1\}) \cap g^{-1}(\{1\}) = (f \wedge g)^{-1}(\{1\})$ for $f, g \in \mathrm{Cont}(S)$, it suffices to show that if $v(f) = 1$ then $f^{-1}(\{1\})$ is not empty. Suppose for contradiction that $v(f) = 1$ and $f^{-1}(\{1\}) = \emptyset$. Since $f^{-1}(\{1\}) = \emptyset$, we have $\mathrm{T}_1(f) = 0$. Thus $v(\mathrm{T}_1(f)) = 0$ and so $v(f) \neq 1$, which contradicts $v(f) = 1$.

By Proposition B.4.8, S^* is compact. Thus, there is $z \in S$ such that $z \in \bigcap\{f^{-1}(\{1\})\,;\, v(f) = 1\}$. We claim that $\Psi(z) = v$. By the definition of z, if $v(f) = 1$ then $\Psi(z)(f) = 1$. We show the converse. Suppose for constradiction that $\Psi(z)(f) = 1$ and $v(f) \neq 1$. Then $v(\mathrm{T}_1(f)) = \mathrm{T}_1(v(f)) = 0$ and so $v((\mathrm{T}_1(f))^{\perp}) = 1$. By the definition of z, $(\mathrm{T}_1(f))^{\perp}(z) = 1$ and so $(\mathrm{T}_1(f))(z) = 0$. Thus $f(z) \neq 1$, which contradicts $\Psi(z)(f) = 1$. Hence, for any $f \in \mathrm{Cont}(S)$, $v(f) = 1$ iff $\Psi(z)(f) = 1$. By Lemma B.2.10, we have $\Psi(z) = v$. Hence, Ψ is surjective.

Finally we show that Ψ^{-1} is an arrow in the category FBS_n. It suffices to show that, for any open $\mathbf{n}$-fuzzy set λ on S, $\Psi(\lambda)$ is an open $\mathbf{n}$-fuzzy set on $\mathrm{Spec} \circ \mathrm{Cont}(S)$. Since S is zero-dimensional, there are $f_i \in \mathrm{Cont}(S)$ with $\lambda = \bigvee_{i \in I} f_i$. For $v \in \mathrm{Spec} \circ \mathrm{Cont}(S)$, the following holds:

$$\Psi(\lambda)(v) = \bigvee\{\lambda(x)\,;\, x \in \Psi^{-1}(\{v\})\} = \lambda(z) = v(\lambda) = v(\bigvee_{i \in I} f_i) = (\bigvee_{i \in I} \langle f_i \rangle)(v),$$

where z is defined as the unique element x such that $\Psi(x) = v$ (for the definition of the direct image of an $\mathbf{n}$-fuzzy set, see Subsection B.3.1). Hence $\Psi(\lambda) = \bigvee_{i \in I} \langle f_i \rangle$ and so $\Psi(\lambda)$ is an open $\mathbf{n}$-fuzzy set on $\mathrm{Spec} \circ \mathrm{Cont}(S)$. □

By Theorem B.4.21 and Theorem B.4.22, we obtain a fuzzy topological duality for $Ł_n^c$-algebras, which is a generalization of Stone duality for Boolean algebras to the n-valued case via fuzzy topology.

Theorem B.4.23. *The category* $Ł_n^c$-Alg *is dually equivalent to the category* FBS_n *via the functors* Spec *and* Cont.

Proof. Let Id_1 denote the identity functor on $Ł_n^c$-Alg and Id_2 denote the identity functor on FBS_n. Then, we define two natural transformations $\epsilon : \mathrm{Id}_1 \to \mathrm{Cont} \circ \mathrm{Spec}$ and $\eta : \mathrm{Id}_2 \to \mathrm{Spec} \circ \mathrm{Cont}$. For an $Ł_n^c$-algebra A, define $\epsilon_A : A \to \mathrm{Cont} \circ \mathrm{Spec}(A)$ by $\epsilon_A = \langle \text{-} \rangle$ (see Theorem B.4.21). For an $\mathbf{n}$-fuzzy Boolean space S, define $\eta_S : S \to$

$\mathrm{Spec} \circ \mathrm{Cont}(S)$ by $\eta_S = \Psi$ (see Theorem B.4.22). It is straightforward to see that η and ϵ are natural transformations. By Theorem B.4.21 and Theorem B.4.22, η and ϵ are natural isomorphisms. $\Box$

B.5 $\mathrm{MŁ}_n^c$-algebras and basic properties

We define modal Łukasiewicz n-valued logic with truth constants $\mathrm{MŁ}_n^c$ by $\mathbf{n}$-valued Kripke semantics. The connectives of $\mathrm{MŁ}_n^c$ are a unary connective $\Box$ and the connectives of $\mathrm{Ł}_n^c$. $\mathbf{Form}_\Box$ denotes the set of formulas of $\mathrm{MŁ}_n^c$.

Definition B.5.1. *Let (W, R) be a Kripke frame (i.e., R is a relation on a set W). Then, e is a Kripke $\mathbf{n}$-valuation on (W, R) iff e is a function from $W \times \mathbf{Form}_\Box$ to $\mathbf{n}$ which satisfies: For each $w \in W$ and $\varphi, \psi \in \mathbf{Form}_\Box$,*

- $e(w, \Box\varphi) = \bigwedge\{e(w', \varphi)\ ;\ wRw'\}$;
- $e(w, \varphi @ \psi) = e(w, \varphi) @ e(w, \psi)$ *for* $@ = \wedge, \vee, *, \wp, \rightarrow$;
- $e(w, \varphi^\perp) = (e(w, \varphi))^\perp$;
- $e(w, r) = r$ *for* $r \in \mathbf{n}$.

Then, (W, R, e) is called an $\mathbf{n}$-valued Kripke model. Define $\mathrm{MŁ}_n^c$ as the set of all those formulas $\varphi \in \mathbf{Form}_\Box$ such that $e(w, \varphi) = 1$ for any $\mathbf{n}$-valued Kripke model (W, R, e) and any $w \in W$.

By straightforward computation, we have the following lemma. Recall the definition of U_r (Definition B.2.8).

Lemma B.5.2. *Let $\varphi, \psi \in \mathbf{Form}_\Box$ and $r \in \mathbf{n}$. (i) $\mathrm{U}_r(\Box\varphi) \leftrightarrow \Box\mathrm{U}_r(\varphi) \in \mathrm{MŁ}_n^c$. (ii) $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi \in \mathrm{MŁ}_n^c$ and $\Box 1 \leftrightarrow 1 \in \mathrm{MŁ}_n^c$. (iii) $\Box(\varphi * \varphi) \leftrightarrow (\Box\varphi) * (\Box\varphi) \in \mathrm{MŁ}_n^c$ and $\Box(\varphi \wp \varphi) \leftrightarrow (\Box\varphi) \wp (\Box\varphi) \in \mathrm{MŁ}_n^c$.*

Definition B.5.3. *For $X \subset \mathbf{Form}_\Box$, X is satisfiable iff there are an $\mathbf{n}$-valued Kripke model (W, R, e) and $w \in W$ such that $e(w, \varphi) = 1$ for any $\varphi \in X$.*

$\mathrm{MŁ}_n^c$-algebras and homomorphisms are defined as follows.

Definition B.5.4. *Let A be an $\mathrm{Ł}_n^c$-algebra. Then, $(A, \Box)$ is an $\mathrm{MŁ}_n^c$-algebra iff it satisfies the following set of equations: $\{\varphi = \psi\ ;\ \varphi \leftrightarrow \psi \in \mathrm{MŁ}_n^c\}$.*

A homomorphism of $\mathrm{MŁ}_n^c$-algebras is defined as a homomorphism of $\mathrm{Ł}_n^c$-algebras which additionally preserves the operation $\Box$.

We do not distinguish between formulas of $\mathrm{MŁ}_n^c$ and terms of $\mathrm{MŁ}_n^c$-algebras.

Definition B.5.5. *Let A be an $\mathrm{MŁ}_n^c$-algebra. Define a relation $R_\Box$ on $\mathrm{Spec}(A)$ by*

$$vR_\Box u \Leftrightarrow \forall r \in \mathbf{n}\, \forall x \in A\, (v(\Box x) \geq r \textit{ implies } u(x) \geq r).$$

Define $e : \mathrm{Spec}(A) \times A \to \mathbf{n}$ by $e(v, x) = v(x)$ for $v \in \mathrm{Spec}(A)$ and $x \in A$. Then, $(\mathrm{Spec}(A), R_\Box, e)$ is called the $\mathbf{n}$-valued canonical model of A.

Proposition B.5.6. *Let A be an $\mathrm{MŁ}_n^c$-algebra. Then, the $\mathbf{n}$-valued canonical model $(\mathrm{Spec}(A), R_\Box, e)$ of A is an $\mathbf{n}$-valued Kripke model. In particular, $e(v, \Box x) = v(\Box x) = \bigwedge\{u(x)\ ;\ vR_\Box u\}$ for $x \in A$ and $v \in \mathrm{Spec}(A)$.*

Proof. It suffices to show that e is a Kripke $\mathbf{n}$-valuation. Since v is a homomorphism of $\mathrm{Ł}_n^c$-algebras, it remains to show $e(v, \Box x) = \bigwedge\{u(x); vR_\Box u\}$. To prove this, it is enough to show that, for any $r \in \mathbf{n}$, (i) $v(\Box x) \geq r$ iff (ii) $vR_\Box u$ implies $u(x) \geq r$. By the definition of $R_\Box$, (i) implies (ii). We show the converse. To prove the contrapositive, assume $v(\Box x) \not\geq r$, i.e., $\mathrm{U}_r(\Box x) \notin v^{-1}(\{1\})$. Let

$$F_0 = \{\mathrm{U}_s(x)\ ;\ s \in \mathbf{n} \text{ and } \mathrm{U}_s(\Box x) \in v^{-1}(\{1\})\}.$$

Let F be the $\mathbf{n}$-filter of A generated by F_0. We claim that $\mathrm{U}_r(x) \notin F$. Suppose for contradiction that $\mathrm{U}_r(x) \in F$. Then, there is $\varphi \in A$ such that $\varphi \leq \mathrm{U}_r(x)$ and φ is constructed from $*$ and elements of F_0. Since $\mathrm{U}_s(x)$ is idempotent, $\mathrm{U}_{s_1}(x_1)\mathrm{U}_{s_2}(x_2) = \mathrm{U}_{s_1}(x_1) \wedge \mathrm{U}_{s_2}(x_2)$ and so we may assume that $\varphi = \bigwedge\{\mathrm{U}_s(x)\ ;\ \mathrm{U}_s(x) \in F_1\}$ for some finite subset F_1 of F_0. By Lemma B.5.2, $\Box\varphi = \bigwedge\{\mathrm{U}_s(\Box x)\ ;\ \mathrm{U}_s(x) \in F_1\}$. By the definition of F_0, $\mathrm{U}_s(\Box x) \in v^{-1}(\{1\})$ for any $\mathrm{U}_s(x) \in F_1$ and so $\Box\varphi \in v^{-1}(\{1\})$. Since $\varphi \leq \mathrm{U}_r(x)$, we have $\Box\varphi \leq \Box\mathrm{U}_r(x) = \mathrm{U}_r(\Box x)$. Thus, $\mathrm{U}_r(\Box x) \in v^{-1}(\{1\})$, which contradicts $\mathrm{U}_r(\Box x) \notin v^{-1}(\{1\})$. Hence $\mathrm{U}_r(x) \notin F$. By Proposition B.2.14, there is a prime $\mathbf{n}$-filter P of A such that $\mathrm{U}_r(x) \notin P$ and $F \subset P$. By Proposition B.2.17, $v_P \in \mathrm{Spec}(A)$. Since $\mathrm{U}_r(x) \notin P$, we have $v_P(x) \not\geq r$. Since $F_0 \subset F \subset P$, we have $vR_\Box v_P$. Thus, (ii) does not hold. $\Box$

The following is a compactness theorem for $\mathrm{MŁ}_n^c$.

Theorem B.5.7. *Let $X \subset$ $\mathbf{Form}_\Box$. Assume that any finite subset of X is satisfiable. Then, X is satisfiable.*

Proof. Let A be the Lindenbaum algebra of $\mathrm{MŁ}_n^c$. We may consider $X \subset A$. We show that X has f.i.p. with respect to $*$. If not, then there are $n \in \omega$ with $n \neq 0$ and $x_1, ..., x_n \in X$ such that $x_1 * ... * x_n = 0$, which is a contradiction, since

$\{x_1, ..., x_n\}$ is satisfiable by assumption. Thus, by Proposition B.2.16, there is a prime $\mathbf{n}$-filter P of A with $X \subset P$. By Proposition B.2.17, v_P is a homomorphism, i.e., $v_P \in \mathrm{Spec}(A)$. Consider the $\mathbf{n}$-valued canonical model $(\mathrm{Spec}(A), R_\Box, e)$ of A. Then, $e(v_P, x) = v_P(x) = 1$ for any $x \in X$ by Proposition B.2.17. Thus, X is satisfiable. $\Box$

Proposition B.5.8. *Let A be an $\mathrm{MŁ}_n^c$-algebra. Then, $\mathcal{B}(A)$ forms a modal algebra.*

Proof. If $x \in A$ is idempotent, then $\Box x$ is also idempotent, since $\Box x * \Box x = \Box(x * x) = \Box x$ by Lemma B.5.2. Thus, $\mathcal{B}(A)$ is closed under $\Box$. By Lemma B.5.2, $\mathcal{B}(A)$ forms a modal algebra. $\Box$

Definition B.5.9. *Let A be an $\mathrm{MŁ}_n^c$-algebra. Define a relation $R_{\Box_\mathbf{2}}$ on $\mathrm{Spec}_\mathbf{2}(\mathcal{B}(A))$ by $vR_{\Box_\mathbf{2}}u \Leftrightarrow \forall x \in \mathcal{B}(A)$ ($v(\Box x) = 1$ implies $u(x) = 1$).*

Proposition B.5.10. *Let A be an $\mathrm{MŁ}_n^c$-algebra. For $v, u \in \mathrm{Spec}(A)$, $vR_\Box u$ iff $t_1(v)R_{\Box_\mathbf{2}}t_1(u)$ (for the definition of t_1, see Proposition B.4.20).*

Proof. By $\Box\mathrm{T}_1(x) = \mathrm{T}_1(\Box x)$, if $vR_\Box u$ then $t_1(v)R_{\Box_\mathbf{2}}t_1(u)$. We show the converse. Assume $t_1(v)R_{\Box_\mathbf{2}}t_1(u)$. In order to show $vR_\Box u$, it suffices to prove that, for any $r \in \mathbf{n}$ and any $x \in A$, $v(\Box\mathrm{U}_r(x)) = 1$ implies $u(\mathrm{U}_r(x)) = 1$, which follows from the assumption, since we have $\mathrm{U}_r(x) \in \mathcal{B}(A)$ and $\mathrm{T}_1(\mathrm{U}_r(x)) = \mathrm{U}_r(x)$. $\Box$

B.6 A fuzzy topological duality for $\mathrm{MŁ}_n^c$-algebras

In this section, based on the fuzzy topological duality for $\mathrm{Ł}_n^c$-algebras, we show a fuzzy topological duality for $\mathrm{MŁ}_n^c$-algebras, which is a generalization of Jónsson-Tarski duality for modal algebras via fuzzy topology, where note that $\mathrm{MŁ}_2^c$-algebras coincide with modal algebras.

Definition B.6.1. $\mathrm{MŁ}_n^c$-Alg *denotes the category of $\mathrm{MŁ}_n^c$-algebras and homomorphisms of $\mathrm{MŁ}_n^c$-algebras.*

Our aim in this section is to show that the category $\mathrm{MŁ}_n^c$-Alg is dually equivalent to the category FRS_n, which is defined in Definition B.6.3 below.

For a Kripke frame (S, R), we can define a modal operator $\Box$ on the "$\mathbf{n}$-valued powerset algebra" $\mathbf{n}^S$ of S as follows.

Definition B.6.2. *Let (S, R) be a Kripke frame and f a function from S to $\mathbf{n}$. Define $\Box_R f : S \to \mathbf{n}$ by $(\Box_R f)(x) = \bigwedge\{f(y)\ ;\ xRy\}$.*

Recall: For a Kripke frame (S, R) and an $\mathbf{n}$-fuzzy set μ on S, an $\mathbf{n}$-fuzzy set $R^{-1}[\mu]$ on S is defined by $R^{-1}[\mu](x) = \bigvee\{\mu(y)\,;\, xRy\}$ for $x \in S$.

Definition B.6.3. *We define the category* FRS_n *as follows.*

An object in FRS_n *is a tuple* (S, R) *such that* S *is an object in* FBS_n *and that a relation* R *on* S *satisfies the following conditions:*

1. *if* $\forall f \in \mathrm{Cont}(S)((\Box_R f)(x) = 1 \Rightarrow f(y) = 1)$ *then* xRy*;*
2. *if* $\mu \in \mathrm{Cont}(S)$*, then* $R^{-1}[\mu] \in \mathrm{Cont}(S)$*.*

An arrow $f : (S_1, R_1) \to (S_2, R_2)$ *in* FRS_n *is an arrow* $f : S_1 \to S_2$ *in* FBS_n *which satisfies the following conditions:*

1. *if* xR_1y *then* $f(x)R_2f(y)$*;*
2. *if* $f(x_1)R_2x_2$ *then there is* $y_1 \in S_1$ *such that* $x_1R_1y_1$ *and* $f(y_1) = x_2$*.*

An object in FRS_n is called an $\mathbf{n}$-fuzzy relational space.

The item 1 in the object part of Definition B.6.3 is an $\mathbf{n}$-fuzzy version of the tightness condition of descriptive general frames in classical modal logic (for the definition of the tightness condition in classical modal logic, see [59]).

Definition B.6.4. *We define a contravariant functor* $\mathrm{RSpec} : \mathrm{MŁ}^c_n\text{-Alg} \to \mathrm{FRS}_n$*. For an object* A *in* $\mathrm{MŁ}^c_n$-Alg*, define* $\mathrm{RSpec}(A) = (\mathrm{Spec}(A), R_\Box)$*. For an arrow* $f : A \to B$ *in* $\mathrm{MŁ}^c_n$-Alg*, define* $\mathrm{RSpec}(f) : \mathrm{RSpec}(B) \to \mathrm{RSpec}(A)$ *by* $\mathrm{RSpec}(f)(v) = v \circ f$ *for* $v \in \mathrm{Spec}(B)$*.*

We call $\mathrm{RSpec}(A)$ the relational spectrum of A. The well-definedness of RSpec is shown by Proposition B.6.6 and Proposition B.6.7 below.

Definition B.6.5. *Let* A *be an* $\mathrm{MŁ}^c_n$*-algebra. Then, we define* $\mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A))$ *as* $(\mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A)), R_{\Box_{\mathbf{2}}})$*. Let* A_1 *and* A_2 *be* $\mathrm{MŁ}^c_n$*-algebras and* $f : \mathcal{B}(A_1) \to \mathcal{B}(A_2)$*. Then, we define* $\mathrm{RSpec}_{\mathbf{2}}(f) : \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_2)) \to \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_1))$ *by* $\mathrm{RSpec}_{\mathbf{2}}(f)(v) = v \circ f$ *for* $v \in \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_2))$*.*

Proposition B.6.6. *For an* $\mathrm{MŁ}^c_n$*-algebra* A*,* $\mathrm{RSpec}(A)$ *is an object in* FRS_n*.*

Proof. It suffices to show the items 1 and 2 in the object part of Definition B.6.3. We first show the item 1 by proving the contrapositive. Assume $(v, u) \notin R_\Box$, i.e., there are $r \in \mathbf{n}$ and $x \in A$ such that $v(\Box x) \geq r$ and $u(x) \ngeq r$. By Lemma B.2.8,

$v(\mathrm{U}_r(\Box x)) = 1$ and $u(\mathrm{U}_r(x)) = 0$. Then, $\langle \mathrm{U}_r(x)\rangle(u) = 0$. By Proposition B.5.6 and Lemma B.5.2,

$$(\Box_R \langle \mathrm{U}_r(x)\rangle)(v) = \bigwedge\{\langle \mathrm{U}_r(x)\rangle(v') \,;\, vR_\Box v'\} = v(\Box \mathrm{U}_r(x)) = v(\mathrm{U}_r \Box x) = 1.$$

As is shown in the proof of Lemma B.4.14, $\langle \mathrm{U}_r(x)\rangle$ is continuous.

We show the item 2. Since $\mathrm{Cont} \circ \mathrm{Spec}(A) = \{\langle x\rangle \,;\, x \in A\}$ as is shown in the proof of Lemma B.4.14, it suffices to show that, for any $x \in A$, $R_\Box^{-1}(\langle x\rangle) \in \mathrm{Cont} \circ \mathrm{Spec}(A)$. Let $\Diamond x$ denote $(\Box(x^\perp))^\perp$. Since $(R_\Box^{-1}(\langle x\rangle))(v) = \bigvee\{u(x) \,;\, vR_\Box u\} = v(\Diamond x)$, we have $R_\Box^{-1}(\langle x\rangle) = \langle \Diamond x\rangle \in \mathrm{Cont} \circ \mathrm{Spec}(A)$. $\Box$

Proposition B.6.7. *For* $\mathrm{MŁ}_n^c$*-algebras* A_1 *and* A_2*, let* $f : A_1 \to A_2$ *be a homomorphism of* $\mathrm{MŁ}_n^c$*-algebras. Then,* $\mathrm{RSpec}(f)$ *is an arrow in* FRS_n*.*

Proof. Define $f_* : \mathcal{B}(A_1) \to \mathcal{B}(A_2)$ by $f_*(x) = f(x)$ for $x \in \mathcal{B}(A_1)$. By Proposition B.5.8, f_* is a homomorphism of modal algebras. Consider $\mathrm{RSpec}_{\mathbf{2}}(f_*) : \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_2)) \to \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_1))$. By Jónsson-Tarski duality for modal algebras (see [127, 37]), $\mathrm{RSpec}_{\mathbf{2}}(f_*)$ is an arrow in FRS_2.

We first show that $\mathrm{RSpec}(f)$ satisfies the item 2 in the arrow part of Definition B.6.3. Assume $\mathrm{RSpec}(f)(v_2)R_\Box u_1$ for $v_2 \in \mathrm{RSpec}(A_2)$ and $u_1 \in \mathrm{RSpec}(A_1)$. By Proposition B.5.10, $t_1(\mathrm{RSpec}(f)(v_2))R_{\Box_{\mathbf{2}}} t_1(u_1)$. It follows from $t_1(\mathrm{RSpec}(f)(v_2)) = \mathrm{T}_1 \circ v_2 \circ f = \mathrm{RSpec}_{\mathbf{2}}(f_*)(t_1(v_2))$ that we have $\mathrm{RSpec}_{\mathbf{2}}(f_*)(t_1(v_2))R_{\Box_{\mathbf{2}}} t_1(u_1)$. Since $\mathrm{RSpec}_{\mathbf{2}}(f_*)$ is an arrow in FRS_2, there is $u_2 \in \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_2))$ such that $t_1(v_2)R_{\Box_{\mathbf{2}}} u_2$ and $\mathrm{RSpec}_{\mathbf{2}}(f_*)(u_2) = t_1(u_1)$. Define $u_2' \in \mathrm{RSpec}(A_2)$ by $u_2'(x) = r \Leftrightarrow u_2(\mathrm{T}_r(x)) = 1$. It is verified in a similar way to Proposition B.2.17 that u_2' is a homomorphism.

We claim that $v_2 R_\Box u_2'$ and $\mathrm{RSpec}(f)(u_2') = u_1$. Let $x \in A_2$ and $r \in \mathbf{n}$. If $v_2(\Box x) \geq r$ then $(t_1(v_2))(\Box \mathrm{U}_r(x)) = 1$ and, since $t_1(v_2)R_{\Box_{\mathbf{2}}} u_2$, we have $u_2(\mathrm{U}_r(x)) = 1$, whence $u_2'(x) \geq r$. Thus, $v_2 R_\Box u_2'$. Next we show $\mathrm{RSpec}(f)(u_2') = u_1$. Let $r = (\mathrm{RSpec}(f)(u_2'))(x)$ for $x \in A_1$. Then, $u_2(\mathrm{T}_r(f(x))) = 1$ and so $(\mathrm{RSpec}_{\mathbf{2}}(f_*)(u_2))(\mathrm{T}_r(x)) = 1$. It follows from $\mathrm{RSpec}_{\mathbf{2}}(f_*)(u_2) = t_1(u_1)$ that $(t_1(u_1))(\mathrm{T}_r(x)) = 1$ and so $u_1(\mathrm{T}_r(x)) = 1$, whence $u_1(x) = r = (\mathrm{RSpec}(f)(u_2'))(x)$. Thus $\mathrm{RSpec}(f)$ satisfies the item 2.

It is easier to verify that $\mathrm{RSpec}(f)$ satisfies the item 1 in the arrow part of Definition B.6.3. $\Box$

Definition B.6.8. *A contravariant functor* $\mathrm{MCont} : \mathrm{FRS}_n \to \mathrm{MŁ}_n^c\text{-Alg}$ *is defined as follows. For an object* (S, R) *in* FRS_n*, define* $\mathrm{MCont}(S, R) = (\mathrm{Cont}(S), \Box_R)$*. For an arrow* $f : (S_1, R_1) \to (S_2, R_2)$ *in* FRS_n*, define* $\mathrm{MCont}(f) : \mathrm{MCont}(S_2, R_2) \to \mathrm{MCont}(S_1, R_1)$ *by* $\mathrm{MCont}(f)(g) = g \circ f$ *for* $g \in \mathrm{Cont}(S_2)$*.*

The well-definedness of MCont is shown by the following propositions.

Proposition B.6.9. *For an object* (S, R) *in* FRS_n, $\mathrm{MCont}(S, R)$ *is an* $\mathrm{MŁ}_n^c$*-algebra.*

Proof. We first show that if $f \in \mathrm{Cont}(S)$ then $\Box_R f \in \mathrm{Cont}(S)$. Let $f \in \mathrm{Cont}(S)$ and μ an open $\mathbf{n}$-fuzzy set on $\mathbf{n}$. Define μ_r as in the proof of Lemma B.4.3 and then it suffices to show that $(\Box_R f)^{-1}(\mu_r)$ is an open $\mathbf{n}$-fuzzy set on S for any $r \in \mathbf{n}$. By Lemma B.2.8,

$$(\Box_R f)^{-1}(\mu_r) = R^{-1}[\mu_r \circ f] \wedge (R^{-1}[(\mathrm{U}_r \circ f)^{\perp}])^{\perp}.$$

Since both $\mu_r \circ f$ and $(\mathrm{U}_r \circ f)^{\perp}$ are elements of $\mathrm{Cont}(S)$, the right-hand side is an element of $\mathrm{Cont}(S)$ by the definition of R and so is an open $\mathbf{n}$-fuzzy set on S, since S is zero-dimensional. Thus $\Box_R f \in \mathrm{Cont}(S)$.

Next we show that $\mathrm{MCont}(S, R)$ satisfies $\{\varphi = \psi\ ;\ \varphi \leftrightarrow \psi \in \mathrm{MŁ}_n^c\}$. Consider $\mathrm{Cont}(S)$ as the set of propositional variables. Since $\mathrm{Cont}(S)$ is closed under the operations of $\mathrm{Cont}(S)$, an element of $\mathbf{Form}_\Box$ may be seen as an element of $\mathrm{Cont}(S)$. Define $e : S \times \mathbf{Form}_\Box \to \mathbf{n}$ by $e(w, f) = f(w)$ for $w \in S$ and $f \in \mathrm{Cont}(S)$. Then, (S, R, e) is an $\mathbf{n}$-valued Kripke model by the definition of the operations of $\mathrm{Cont}(S)$. Since $e(w, f) = 1$ for any $w \in S$ iff $f = 1$, it follows from the definition of $\mathrm{MŁ}_n^c$ that $\mathrm{MCont}(S, R)$ satisfies $\{\varphi = \psi\ ;\ \varphi \leftrightarrow \psi \in \mathrm{MŁ}_n^c\}$. $\Box$

Proposition B.6.10. *Let* $f : (S_1, R_1) \to (S_2, R_2)$ *be an arrow in* FRS_n*. Then,* $\mathrm{MCont}(f)$ *is a homomorphism of* $\mathrm{MŁ}_n^c$*-algebras.*

Proof. It remains to show that $\mathrm{MCont}(f)(\Box g_2) = \Box(\mathrm{MCont}(f)(g_2))$ for $g_2 \in \mathrm{Cont}(S_2)$. For $x_1 \in S_1$, $(\mathrm{MCont}(f)(\Box g_2))(x_1) = \bigwedge\{g_2(y_2)\ ;\ f(x_1)R_2y_2\}$. Let a denote the right-hand side. We also have $(\Box(\mathrm{MCont}(f)(g_2)))(x_1) = \bigwedge\{g_2(f(y_1))\ ;\ x_1R_1y_1\}$. Let b denote the right-hand side. Since $x_1R_1y_1$ implies $f(x_1)R_1f(y_1)$, we have $a \leq b$. By the item 2 in the arrow part of Definition B.6.3, we have $a \geq b$. Hence $a = b$. $\Box$

Theorem B.6.11. *Let* A *be an object in* $\mathrm{MŁ}_n^c$-Alg*. Then,* A *is isomorphic to* $\mathrm{MCont} \circ \mathrm{RSpec}(A)$ *in the category* $\mathrm{MŁ}_n^c$-Alg.

Proof. We claim that $\langle\text{-}\rangle : A \to \mathrm{MCont} \circ \mathrm{RSpec}(A)$ is an isomorphism of $\mathrm{MŁ}_n^c$-algebras. By Theorem B.4.21, it remains to show that $\langle \Box x\rangle = \Box_{R_\Box}\langle x\rangle$ for $x \in A$. By Proposition B.5.6, we have the following for $v \in \mathrm{Spec}(A)$: $(\Box_{R_\Box}\langle x\rangle)(v) = \bigwedge\{u(x)\ ;\ vR_\Box u\} = v(\Box x) = \langle \Box x\rangle(v)$. $\Box$

Theorem B.6.12. *Let* (S, R) *be an object in* FRS_n*. Then,* (S, R) *is isomorphic to* $\mathrm{RSpec} \circ \mathrm{MCont}(S, R)$ *in the category* FRS_n*.*

Proof. Define $\Phi : (S, R) \to \mathrm{RSpec} \circ \mathrm{MCont}(S, R)$ by $\Phi(x)(f) = f(x)$ for $x \in S$ and $f \in \mathrm{Cont}(S)$. We show: For any $x, y \in S$, xRy iff $\Phi(x) R_{\Box_R} \Phi(y)$. Assume xRy. Let $r \in \mathbf{n}$ and $f \in \mathrm{Cont}(S)$ with $\Phi(x)(\Box_R f) \geq r$. Since $\Phi(x)(\Box_R f) = \bigwedge\{f(z)\,;\, xRz\}$, we have $\Phi(y)(f) = f(y) \geq r$. Next we show the converse. To prove the contrapositive, assume $(x, y) \notin R$. By Definition B.6.3, there is $f \in \mathrm{Cont}(S)$ such that $(\Box_R f)(x) = 1$ and $f(y) \neq 1$. Then, $\Phi(x)(\Box_R f) = 1$ and $\Phi(y)(f) \neq 1$. Thus, we have $(\Phi(x), \Phi(y)) \notin R_{\Box_R}$.

By Theorem B.4.22, it remains to prove that Φ and Φ^{-1} satisfy the item 2 in the arrow part of Definition B.6.3, which follows from the above fact that xRy iff $\Phi(x) R_{\Box_R} \Phi(y)$, since Φ is bijective. $\Box$

By Theorem B.6.11 and Theorem B.6.12, we obtain a fuzzy topological duality for $\mathrm{MŁ}^c_n$-algebras, which is a generalization of Jónsson-Tarski duality for modal algebras to the n-valued case via fuzzy topology.

Theorem B.6.13. *The category* $\mathrm{MŁ}^c_n$-Alg *is dually equivalent to the category* FRS_n *via the functors* RSpec(-) *and* MCont(-).

Proof. By arguing as in the proof of Theorem B.4.23, this theorem follows immediately from Theorem B.6.11 and Theorem B.6.12. $\Box$

Appendix C

Artificial Intelligence Applications

Here we formalise reasoning about fuzzy belief and fuzzy common belief, especially incomparable beliefs, in multi-agent systems by using a logical system based on Fitting's many-valued modal logic, where incomparable beliefs mean beliefs whose degrees are not totally ordered. Completeness and decidability results for the logic of fuzzy belief and common belief are established while implicitly exploiting our duality-theoretical perspective on Fitting's logic. A conceptually novel feature is that incomparable beliefs and qualitative fuzziness can be formalised in the developed system, whereas they cannot be formalised in previously proposed systems for reasoning about fuzzy belief. We believe that belief degrees can ultimately be reduced to truth degrees, and we call this "the reduction thesis about belief degrees", which is assumed here and motivates an axiom of our system. We finally argue that fuzzy reasoning sheds new light on old epistemic issues such as the the coordinated attack problem.

C.1 Introduction to the Appendix

Epistemic logic has been studied in order to formalise reasoning about knowledge and belief (see [134, 92]) with widespread applications to many research areas, including computer science and artificial intelligence ([92, 254]), economics and game theory ([20]), and philosophy ([32, 134]). The logic of common knowledge and belief is one of the central concerns of epistemic logic (see [92, 254]).

We formalise reasoning about fuzzy belief and fuzzy common belief, especially incomparable beliefs, in multi-agent systems by using a logical system based on Fitting's many-valued modal logic (for this logic, see [93, 95, 96]), where incomparable beliefs are defined as beliefs whose degrees are not totally ordered. We remark that many-valued modal logics have already been studied from various perspectives (see

[43, 186, 191]). Results here are established while implicitly exploiting the duality-theoretic perspective on Fitting's logic that builds upon the author's previous study (see [186, 191]).

Let us explain our motivations for studying the logic of fuzzy belief and common belief. It is not so unusual that one believes something to some degree, or the degree of one's belief may be neither 0 nor 1. The notion of fuzzy belief is appropriate in such a case. Moreover, the notion of fuzzy common belief can be appropriate even in a case where any agent of a group does not have a fuzzy belief. To see this, consider the following question. Is there anything that all the people in the world believe? Strictly speaking, there may be no such thing as a common belief among all the people in the world. Even if so, there may be something that most of the people in the world believe. For instance, most but not all of the people in the world probably believe that any human being is mortal or that the law of identity (i.e., $\varphi \to \varphi$) is valid (note that some logicians do not believe it). The notion of fuzzy common belief is appropriate in such a case as well as in a case where an agent of a group has a fuzzy belief.

Here, we would like to clarify our philosophical standpoint. We consider that the degree of a belief φ by an agent i is equivalent to the truth degree of the proposition that i believes φ (in fact, this is imprecise; to be precise, see **B2** in Definition C.3.2; not T_a but U_a is appropriate also here), that is, degrees of belief can ultimately be reduced to degrees of truth in this way (in the sense of U_a as in **B2**), which we call "the reduction thesis about belief degrees" (this has no relation with Peirce's reduction thesis). We may identify the reduction thesis with the axiom **B2** in Definition C.3.2. Although the thesis may be contested, we work under the assumption of it. We believe that the reduction thesis is philosophically justifiable to some degrees, but anyway it is certainly beneficial (and would thus be justifiable) from a technical point of view as shown by our results.

Epistemic logic based on classical logic is inadequate to formalise reasoning about fuzzy (common) belief, which is due to the fact that either 0 or 1 is assigned to every formula in classical epistemic logic. We are thus led to consider epistemic logic based on many-valued logic, since the truth value of a proposition may be neither 0 nor 1 in many-valued logic. Among many existing many-valued logics, we employ a modified version of Fitting's lattice-valued logic, the reasons of which are explained later, and we add to the lattice-valued logic epistemic operators including a common belief operator, thus developing a logical system for reasoning about fuzzy belief and common belief.

Several authors have already developed logical systems to formalise reasoning about fuzzy belief (see, e.g., [33, 91, 114]), for example, by combining probabilistic logic and epistemic logic. However, there seems to have been no study of reasoning about fuzzy common belief via the combination of many-valued logic and epistemic logic. Moreover, the degrees of beliefs are supposed to be totally ordered in the previously proposed systems. For this reason, the notion of incomparable beliefs cannot be formalised in them.

There are indeed many incomparable beliefs in ordinary life. For instance, consider the following situations: (1) Suppose that there are two collections X, Y of grains here, that X is taller and thinner than Y and that it is not obvious whether or not each collection of grains makes a heap, as shown in the following figure (of course, this is based on the well-known sorites paradox, which motivates many-valued logic).
Let a (resp. b) be the degree of one's belief that X (resp. Y) makes a heap. Then, a and b may be incomparable, since their magnitudes are incomparable. (2) Suppose that a child believes that she loves her mother and that she loves her father. Then, the degrees of her two beliefs can be incomparable. Thus, one's beliefs are sometimes incomparable.

Hence, it would be significant to be able to formalise the notion of incomparable beliefs in a logical system. In our system, a degree of a belief is expressed as an element of a lattice which is not necessarily totally ordered. Therefore, the notion of incomparable beliefs can be formalised in our system, which is impossible in previously developed systems for reasoning about fuzzy belief such as those in [33, 91, 114]. This is one of the reasons why we employ a version of Fitting's lattice-valued logic as the underlying logic of our system for reasoning about fuzzy belief and common belief.

We remark that Fitting's lattice-valued logic (for different lattice-valued logics, see [283]) may be considered as a kind of fuzzy logic, but the prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ is not necessarily valid in Fitting's logic, while it is valid in fuzzy logics such as Łukasiewicz logic and Gödel logic (for these logics, see [125]). We also note that the lattice of truth values is finite in Fitting's logic (see also [43]), since L-valued modal logic may not be recursively axiomatizable for an infinite lattice L. In practice or in the real world, a sufficiently large finite lattice and an infinite lattice would not make a significant difference (we could not distinguish between them).

The rest of the appendix is organised as follows. In Section C.2, lattice-valued logic L-**VL** is discussed. In Section C.3, a logic of fuzzy belief in an n-agent system, L-$\mathbf{K}_n$, is discussed. In the two sections, we mainly aim to reformulate algebraic axiomatizations in [186] in terms of Hilbert-style deductive systems. In Section C.4, the usual Kripke semantics for a common belief operator is naturally extended to the L-valued case and then a logic of fuzzy belief and common belief, L-$\mathbf{K}_n^C$, is discussed. Especially, we develop a Hilbert-style deductive system for L-$\mathbf{K}_n^C$ and show that it is sound and complete with respect to the extended Kripke semantics and that L-$\mathbf{K}_n^C$ is decidable and enjoys the finite model property. We remark that we can also obtain other versions of these results such as **KD45**-style, **S5**-style, **K45**-style, and **S4**-style ones.

C.2 Lattice-Valued Logic: L-VL

Let L denote a finite distributive lattice with the top element 1 and the bottom element 0. Then, as is well known, L forms a finite Heyting algebra. For $a, b \in L$, let $a \to b$ denote the pseudo complement of a relative to b. Let $\mathbf{2}$ denote the two-element Boolean algebra.

Definition C.2.1. *We augment L with unary operations $\mathrm{T}_a(\text{-})$'s for all $a \in L$ defined as follows: $\mathrm{T}_a(x) = 1$ if $x = a$ and $\mathrm{T}_a(x) = 0$ if $x \neq a$. We also augment L with unary operations $\mathrm{U}_a(\text{-})$'s for all $a \in L$ defined by: $\mathrm{U}_a(x) = 1$ if $x \geq a$ and $\mathrm{U}_a(x) = 0$ if $x \not\geq a$*

We define L-valued logic L-**VL** as follows. The connectives of L-**VL** are $\wedge$, $\vee$, $\to$, 0, 1, T_a and U_a for each $a \in L$, where T_a and U_a are unary connectives, 0 and 1 are nullary connectives, and the others are binary connectives. Let **PV** denote the set of propositional variables. Then, the set of formulas of L-**VL**, which is denoted by **Form**, are recursively defined in the usual way. Let $\varphi \leftrightarrow \psi$ be the abbreviation of $(\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\neg\varphi$ the abbreviation of $\varphi \to 0$. The intended meaning of $\mathrm{T}_a(\varphi)$ is that the truth value of a proposition φ is an element a of L. The intended meaning of $\mathrm{U}_a(\varphi)$ is that the truth value of φ is more than or equal to a. L-valued semantics is then introduced as follows.

Definition C.2.2. *A function v : **Form** $\to L$ is an L-valuation on **Form** iff it satisfies the following:*

1. $v(\mathrm{T}_a(\varphi)) = \mathrm{T}_a(v(\varphi))$ *for each* $a \in L$*;*
2. $v(\mathrm{U}_a(\varphi)) = \mathrm{U}_a(v(\varphi))$ *for each* $a \in L$*;*
3. $v(\varphi \wedge \psi) = \inf(v(\varphi), v(\psi))$*;*
4. $v(\varphi \vee \psi) = \sup(v(\varphi), v(\psi))$*;*

5. $v(\varphi \to \psi) = v(\varphi) \to v(\psi)$;

6. $v(a) = a$ *for* $a = 0, 1$.

Then, $\varphi \in$ **Form** *is called valid in* L-**VL** *iff* $v(\varphi) = 1$ *for any* L*-valuation* v *on* **Form**.

If L is the two-element Boolean algebra **2**, then the above semantics coincides with the ordinary two-valued semantics for classical logic, where note that $\mathrm{T}_1(\varphi) \leftrightarrow \varphi$ and $\mathrm{T}_0(\varphi) \leftrightarrow \neg\varphi$ are valid in **2-VL**, whence all T_a's and U_a's are actually redundant in **2-VL**. We then give a Hilbert-style axiomatization of L-**VL**.

Definition C.2.3. $\varphi \in$ **Form** *is provable in* L-**VL** *iff it is either an instance of one of the following axioms or deduced from provable formulas by one of the following rules of inference: The axioms are*

A1. *all instances of tautologies of intuitionistic logic;*

A2. $(\mathrm{T}_a(\varphi) \wedge \mathrm{T}_b(\psi)) \to \mathrm{T}_{a@b}(\varphi@\psi)$ *for* $@ = \to, \wedge, \vee$; $\mathrm{T}_b(\varphi) \to (\mathrm{T}_{@(b)}(@(\varphi))$ *for* $@ = \mathrm{T}_a, \mathrm{U}_a$;

A3. $\mathrm{T}_0(0)$; $\mathrm{T}_a(0) \leftrightarrow 0$ *for* $a \neq 0$; $\mathrm{T}_1(1)$;
$\mathrm{T}_a(1) \leftrightarrow 0$ *for* $a \neq 1$;

A4. $\bigvee_{a \in L} \mathrm{T}_a(\varphi)$; $(\mathrm{T}_a(\varphi) \wedge \mathrm{T}_b(\varphi)) \leftrightarrow 0$ *for* $a \neq b$;
$\mathrm{T}_a(\varphi) \vee \neg\mathrm{T}_a(\varphi)$;

A5. $\mathrm{T}_1(\mathrm{T}_a(\varphi)) \leftrightarrow \mathrm{T}_a(\varphi)$; $\mathrm{T}_0(\mathrm{T}_a(\varphi)) \leftrightarrow (\mathrm{T}_a(\varphi) \to 0)$;
$\mathrm{T}_b(\mathrm{T}_a(\varphi)) \leftrightarrow 0$ *for* $b \neq 0, 1$;

A6. $\mathrm{T}_1(\varphi) \to \varphi$; $\mathrm{T}_1(\varphi \wedge \psi) \leftrightarrow \mathrm{T}_1(\varphi) \wedge \mathrm{T}_1(\psi)$;

A7. $\mathrm{U}_a(\varphi) \leftrightarrow \bigvee\{\mathrm{T}_x(\varphi)\ ;\ a \leq x \text{ and } x \in L\}$;

A8. $(\bigwedge_{a \in L}(\mathrm{T}_a(\varphi) \leftrightarrow \mathrm{T}_a(\psi))) \to (\varphi \leftrightarrow \psi)$,

where $a, b \in L$ *and* $\varphi, \psi \in$ **Form**. *The rules of inference are*

R1. *From* φ *and* $\varphi \to \psi$ *infer* ψ;

R2. *From* $\varphi \leftrightarrow \psi$ *infer* $\chi \leftrightarrow \chi'$, *where* χ' *is the formula obtained from* χ *by replacing an occurrence of* φ *with* ψ;

R3. *From* $\varphi \to \psi$ *infer* $\mathrm{T}_1(\varphi) \to \mathrm{T}_1(\psi)$, *where* $\varphi, \psi, \chi \in \mathbf{Form}$.

$(\mathrm{T}_a(\varphi) \wedge \mathrm{T}_b(\psi)) \rightarrow \mathrm{T}_{a\rightarrow b}(\varphi \rightarrow \psi)$ intuitively means that if the truth value of φ is a and the truth value of ψ is b then the truth value of $\varphi \rightarrow \psi$ is $a \rightarrow b$. The intuitive meanings of the axioms in **A2** can be explained in similar ways. Note that $\mathrm{T}_a(b)$ and $\mathrm{U}_a(b)$ in **A2** are either 0 or 1. An axiom $\bigvee_{a\in L} \mathrm{T}_a(\varphi)$ in **A4** is called the L-valued excluded middle, since the **2**-valued excluded middle coincides with the ordinary excluded middle.

The notion of deducibility for L-**VL** is defined in the usual way: For $\varphi \in \mathbf{Form}$ and $X \subset \mathbf{Form}$, φ is deducible from X in L-**VL** iff φ can be deduced from X and the axioms of L-**VL** by the inference rules of L-**VL**. We then have the following deduction theorem for L-**VL**, which can be shown in almost the same way as delta deduction theorems for fuzzy logics with Baaz delta (see [61, Theorem 6]).

Proposition C.2.4. *Let* $\varphi, \psi_1, ..., \psi_k \in \mathbf{Form}$ *where* $k \in \omega \setminus \{0\}$. *If* φ *is deducible from* $\{\psi_1, ..., \psi_k\}$ *in* L-**VL**, *then* $\mathrm{T}_1(\psi_1 \wedge ... \wedge \psi_k) \rightarrow \varphi$ *is provable in* L-**VL**.

In the following, we show that the above axiomatization of L-**VL** is sound and complete with respect to the L-valued semantics. We first define the notion of L-**VL** consistency as follows: For $X \subset \mathbf{Form}$, X is L-**VL** consistent iff 0 is not deducible from X in L-**VL**. Note that a maximal L-**VL** consistent subset of **Form** is closed under the inference rules of L-**VL** by the maximality of it.

Lemma C.2.5. *Let* X *be a maximal* L-**VL** *consistent subset of* **Form**. *Then, for any* $\varphi \in \mathbf{Form}$, *there is a unique* $a \in L$ *such that* $\mathrm{T}_a(\varphi) \in X$.

Proof. Let $\varphi \in \mathbf{Form}$. Assume that there is no $a \in L$ such that $\mathrm{T}_a(\varphi) \in X$. Since X is closed under $\wedge$ and T_1, it follows from the maximality of X and Proposition C.2.4 that for each $a \in L$, there is $\psi_a \in X$ such that $(\psi_a \wedge \mathrm{T}_a(\varphi)) \leftrightarrow 0$ is provable in L-**VL**. Let $\psi = \bigwedge_{a\in L} \psi_a$. Note that $\psi \in X$. Now, $(\psi \wedge \mathrm{T}_a(\varphi)) \leftrightarrow 0$ is provable in L-**VL**. By **A1**, $(\psi \wedge \bigvee_{a\in L} \mathrm{T}_a(\varphi)) \leftrightarrow 0$ is also provable in L-**VL**. Thus it follows from **A1**, **A4**, and **R2** that $\psi \leftrightarrow 0$ is provable in L-**VL**. Then, since X is closed under modus ponens, we have $0 \in X$ by $\psi \in X$, which is a contradiction. Hence there is $a \in L$ such that $\mathrm{T}_a(\varphi) \in X$. The uniqueness of such $a \in L$ is shown by using the L-**VL** consistency of X and the following axiom in **A4**: $(\mathrm{T}_a(\varphi) \wedge \mathrm{T}_b(\varphi)) \leftrightarrow 0$ for $a \neq b$. $\square$

Theorem C.2.6. *For* $\varphi \in \mathbf{Form}$, φ *is provable in* L-**VL** *iff* φ *is valid in* L-**VL**.

Proof. It is straightforward to show the soundness. We show the completeness by proving the contrapositive. Assume that φ is not provable in L-**VL**. Then, $\mathrm{T}_1(\varphi)$ is not provable in L-**VL** by **A6**. If $\{\neg\mathrm{T}_1(\varphi)\}$ is not L-**VL** consistent, then it follows from **A1** and an axiom $\mathrm{T}_a(\varphi)\vee\neg\mathrm{T}_a(\varphi)$ in **A4** that $\mathrm{T}_1(\varphi)$ is provable in L-**VL**, which is a contradiction. Thus, $\{\neg\mathrm{T}_1(\varphi)\}$ is L-**VL** consistent. By a standard argument using Zorn's lemma we have a maximal L-**VL** consistent subset X of **Form** containing $\neg\mathrm{T}_1(\varphi)$. Then we define a function v_X from **Form** to L as follows: For $\psi \in$ **Form**, $v_X(\psi) = a \Leftrightarrow \mathrm{T}_a(\psi) \in X$. Then it follows from Lemma C.2.5 that v_X is well defined. Since $\neg\mathrm{T}_1(\varphi) \in X$, we have $\mathrm{T}_1(\varphi) \notin X$ by **A1**, which implies that $v_X(\varphi) \neq 1$. Now it remains to show that v_X is an L-valuation. We first verify that $v_X(\psi \to \chi) = v_X(\psi) \to v_X(\chi)$. Let $a = v_X(\psi)$ and $b = v_X(\chi)$. Then we have $\mathrm{T}_a(\psi), \mathrm{T}_b(\chi) \in X$. Thus, since X is closed under modus ponens, it follows from **A2** that $\mathrm{T}_{a\to b}(\psi \to \chi) \in X$. Therefore, by the definition of v_X, we have $v_X(\psi \to \chi) = a \to b = v_X(\psi) \to v_X(\chi)$. The other cases are similarly verified. □

By using the above theorem, it is straightforward to show the following three propositions.

Proposition C.2.7. *Let $a \in L$ and $\varphi, \psi \in$ **Form**. (i) $\mathrm{U}_a(\varphi \wedge \psi) \leftrightarrow \mathrm{U}_a(\varphi) \wedge \mathrm{U}_a(\psi)$ is provable in L-**VL**. (ii) $(\varphi \to \psi) \to (\mathrm{U}_a(\varphi) \to \mathrm{U}_a(\psi))$ is provable in L-**VL**.*

Proposition C.2.8. *Let $a, b \in L$ with $a \neq 0$ and $\varphi \in$ **Form**. (i) $\mathrm{U}_a(\mathrm{U}_b(\varphi)) \leftrightarrow \mathrm{U}_b(\varphi)$ is provable in L-**VL**. (ii) $\mathrm{U}_a(\mathrm{T}_b(\varphi)) \leftrightarrow \mathrm{T}_b(\varphi)$ is provable in L-**VL**.*

Although the law of excluded middle does not necessarily hold in L-**VL**, it holds for a special kind of formulas. The same thing holds also for De Morgan's law and for the commutativity of U_a and $\vee$.

Proposition C.2.9. *Let $a, b \in L$ with $a \neq 0$ and $\varphi, \psi \in$ **Form**. Assume that $\mathrm{U}_a(\varphi) \leftrightarrow \varphi$ and $\mathrm{U}_a(\psi) \leftrightarrow \psi$ are provable in L-**VL**. (i) $\varphi \vee \neg\varphi$ is provable in L-**VL**. (ii) $\neg(\varphi \wedge \chi) \leftrightarrow (\neg\varphi \vee \neg\chi)$ and $\neg(\varphi \vee \chi) \leftrightarrow (\neg\varphi \wedge \neg\chi)$ are provable in L-**VL**. (iii) $\mathrm{U}_b(\varphi \vee \psi) \leftrightarrow \mathrm{U}_b(\varphi) \vee \mathrm{U}_b(\psi)$ is provable in L-**VL**.*

C.3 Logic of Fuzzy Belief: L-$\mathbf{K}_n$

In this section, we introduce a logical system for reasoning about fuzzy belief in an n-agent system for a non-negative integer n, which is denoted by L-$\mathbf{K}_n$. The connectives of L-$\mathbf{K}_n$ are unary connectives B_i for $i = 1, ..., n$ and the connectives of L-**VL**. Then, let $\mathbf{Form}_n$ denote the set of formulas of L-$\mathbf{K}_n$. The intended meaning of $B_i(\varphi)$ is that the i-th agent believes that φ. Thus, the intended meaning of $B_i\mathrm{U}_a(\varphi)$ is that the i-th agent believes φ at least to the degree of a or the degree of the i-th agent's belief φ is more than or equal to a.

An example of reasoning about fuzzy belief is: If the degree of the i-th agent's belief φ is a and if the degree of the i-th agent's belief ψ is more than or equal to a, then the degree of the i-th agent's belief $\varphi \to \psi$ is 1. This reasoning is expressed in L-$\mathbf{K}_n$ as $(B_i\mathrm{T}_a(\varphi) \wedge B_i\mathrm{U}_a(\psi)) \to B_i\mathrm{T}_1(\varphi \to \psi)$, which is both valid and provable in the following semantics and proof system for L-$\mathbf{K}_n$.

An example of reasoning about incomparable beliefs is: If the degree of the i-th agent's belief φ and the degree of the i-th agent's belief ψ are incomparable (examples of incomparable beliefs φ, ψ are in Section C.1), then it does not hold that either if the i-th agent believes φ then the i-th agent believes ψ or if the i-th agent believes ψ then the i-th agent believes φ. This is expressed in L-$\mathbf{K}_n$ as follows (let a and b be incomparable in L with $a \vee b \neq 1$): $(B_i\mathrm{T}_a(\varphi) \wedge B_i\mathrm{T}_b(\psi)) \to \neg\mathrm{T}_1((B_i\varphi \to B_i\psi) \vee (B_i\psi \to B_i\varphi))$, which is both valid and provable in the following semantics and proof system for a lattice L in which there are such a and b (there are indeed many such lattices L). Recall that $(B_i\varphi \to B_i\psi) \vee (B_i\psi \to B_i\varphi)$ is valid in classical epistemic logic, which is a so-called paradox of material implication. The paradox is avoided in our logical system.

L-valued Kripke semantics for L-$\mathbf{K}_n$ is defined as follows.

Definition C.3.1. *Let $(M, R_1, ..., R_n)$ be a Kripke n-frame, i.e., R_i is a binary relation on a set M for each $i = 1, ..., n$. Then, a function $e : M \times \mathbf{Form}_n \to L$ is an L-$\mathbf{K}_n$ valuation on $(M, R_1, ..., R_n)$ iff it satisfies the following for each $w \in M$:*

1. $e(w, B_i(\varphi)) = \bigwedge\{e(w', \varphi)\ ;\ wR_iw'\}$ *for* $i = 1, ..., n$;
2. $e(w, \mathrm{T}_a(\varphi)) = \mathrm{T}_a(e(w, \varphi))$ *for each* $a \in L$;
3. $e(w, \mathrm{U}_a(\varphi)) = \mathrm{U}_a(e(w, \varphi))$ *for each* $a \in L$;
4. $e(w, \varphi @ \psi) = e(w, \varphi) @ e(w, \psi)$ *for* $@ = \wedge, \vee, \to$;
5. $e(w, a) = a$ *for* $a = 0, 1$.

We call $(M, R_1, ..., R_n, e)$ an L-$\mathbf{K}_n$ Kripke model. Then, $\varphi \in \mathbf{Form}_n$ is said to be valid in L-$\mathbf{K}_n$ iff $e(w, \varphi) = 1$ for any L-$\mathbf{K}_n$ Kripke model $(M, R_1, ..., R_n, e)$ and any $w \in M$.

If L is the two-element Boolean algebra, then the above Kripke semantics coincides with the usual Kripke semantics for the **K**-style logic of belief in an n-agent system. A Hilbert-style axiomatization of L-$\mathbf{K}_n$ is given as follows.

Definition C.3.2. *$\varphi \in \mathbf{Form}_n$ is provable in L-$\mathbf{K}_n$ iff it is either an instance of one of the following axioms or deduced from provable formulas by one of the following rules of inference: The axioms are* **A1**, ..., **A8** *in Definition C.2.3 and*

B1. *$B_i(\varphi \wedge \psi) \leftrightarrow B_i(\varphi) \wedge B_i(\psi)$ for each $i = 1, ..., n$;*

B2. *$B_i \mathrm{U}_a(\varphi) \leftrightarrow \mathrm{U}_a B_i(\varphi)$ for each $i = 1, ..., n$,*

where $a \in L$ and $\varphi, \psi \in \mathbf{Form}_n$. The rules of inference are **R1**, **R2**, **R3** *in Definition C.2.3 and*

R4. *From $\varphi \to \psi$ infer $B_i(\varphi) \to B_i(\psi)$ for $i = 1, ..., n$.*

We may call the axiom **B2** the reduction thesis about belief degrees (see Section C.1). If **B2** is contested, it is possible to develop another deductive system without **B2** that corresponds to Kripke semantics with L-valued accessibility relations. L-$\mathbf{K}_n$ consistency (and L-$\mathbf{K}_n^C$ consistency in the next section) are defined in the same way as L-**VL** consistency. The axiomatic system above is sound and complete.

Theorem C.3.3. *For $\varphi \in \mathbf{Form}_n$, φ is provable in L-$\mathbf{K}_n$ iff φ is valid in L-$\mathbf{K}_n$.*

Proof. It is straightforward to show the soundness. We show the completeness by proving the contrapositive. Assume that φ is not provable in L-$\mathbf{K}_n$. Let Con be the set of all maximal L-$\mathbf{K}_n$ consistent subsets of $\mathbf{Form}_n$. We can consider the L-$\mathbf{K}_n$ Kripke model $(\mathrm{Con}, R_1, ..., R_n, e)$ such that for each $i = 1, ..., n$ and $V, W \in \mathrm{Con}$, $V R_i W$ iff, for any $a \in L$ and $\psi \in \mathbf{Form}_n$, $\mathrm{U}_a(B_i\psi) \in V$ implies $\mathrm{U}_a(\psi) \in W$ and that for each propositional variable p, $e(W, p) = a$ iff $\mathrm{T}_a(p) \in W$.

We claim that, for any $\psi \in \mathbf{Form}_n$ and $W \in \mathrm{Con}$,

$$e(W, \psi) = a \text{ iff } \mathrm{T}_a(\psi) \in W.$$

We show the claim by induction on the structure of formulas. We consider only the case that ψ is of the form $B_i(\chi)$ for $i \in \{1, ..., n\}$, since arguments in the other cases are similar to those in the proof of Theorem C.2.6. In order to show that $e(W, B_i\chi) = a$ iff $\mathrm{T}_a(B_i\chi) \in W$, it suffices to show that $e(W, B_i\chi) \geq a$ iff $\mathrm{U}_a(B_i\chi) \in W$, since by **A7** we have: $\mathrm{T}_a(B_i\chi) \in W$ iff $\mathrm{U}_x(B_i\chi) \in W$ for any $x \in L$ with $x \leq a$ and $\mathrm{U}_x(B_i\chi) \notin W$ for any $x \in L$ with $x \not\leq a$. If $\mathrm{U}_a(B_i\chi) \in W$, then

$\mathrm{U}_a(\chi) \in V$ for any $V \in \mathrm{Con}$ with WR_iV, whence by the induction hypothesis, we have $e(W, B_i\chi) = \bigwedge\{e(V, \chi) \,;\, WR_iV\} \geq a$. We next show the converse by proving the contrapositive. Assume $\mathrm{U}_a(B_i\chi) \notin W$. Let

$$G = \{\mathrm{U}_b(\eta) \,;\, \eta \in \mathbf{Form}_n,\ b \in L \text{ and } B_i\mathrm{U}_b(\eta) \in W\}.$$

We first verify that $G \cup \{\neg\mathrm{U}_a(\chi)\}$ is L-$\mathbf{K}_n$ consistent. Suppose for contradiction that $G \cup \{\neg\mathrm{U}_a(\chi)\}$ is not L-$\mathbf{K}_n$ consistent. Then, there is $\zeta \in G$ such that $\zeta \to \mathrm{U}_a(\chi)$ is provable in L-$\mathbf{K}_n$. Thus, $B_i\zeta \to B_i\mathrm{U}_a(\chi)$ is provable in L-$\mathbf{K}_n$. Since $B_i\zeta \in W$ by the definition of G, we have $B_i\mathrm{U}_a(\chi) \in W$ and so $\mathrm{U}_a(B_i\chi) \in W$ by **B2**, which contradicts $\mathrm{U}_a(B_i\chi) \notin W$. Thus, $G \cup \{\neg\mathrm{U}_a(\chi)\}$ is L-$\mathbf{K}_n$ consistent. By a standard argument using Zorn's lemma, we have a maximal L-$\mathbf{K}_n$ consistent subset H of $\mathbf{Form}_n$ containing $G \cup \{\neg\mathrm{U}_a(\chi)\}$. Since H contains $\neg\mathrm{U}_a(\chi)$, it follows from the induction hypothesis that $e(H, \chi) \ngeq a$. Since H contains G, it follows from **B2** that WR_iH. Thus we have $e(W, B_i\chi) \ngeq a$. This completes the proof of the above claim. Now it is straightforward to verify that $(\mathrm{Con}, R_1, ..., R_n, e)$ is a counter-model for φ. $\square$

C.4 Logic of Fuzzy Common Belief: L-$\mathbf{K}_n^C$

In this section, we introduce a logical system for reasoning about fuzzy belief and common belief in an n-agent system, which is denoted by L-$\mathbf{K}_n^C$. The connectives of L-$\mathbf{K}_n^C$ are unary connectives E and C, and the connectives of L-$\mathbf{K}_n$. Let $\mathbf{Form}_n^C$ denote the set of formulas of L-$\mathbf{K}_n^C$. The intended meaning of $E(\varphi)$ is that every agent in the system believes that φ. The intended meaning of $C(\varphi)$ is that it is a common belief among all the agents in the system that φ (for the difference between $E(\varphi)$ and $C(\varphi)$, see, e.g., [92]). Thus, the intended meaning of $\mathrm{U}_aC(\varphi)$ is that φ is a common belief at least to the degree of a or the degree of a common belief φ is more than or equal to a.

An example of reasoning about fuzzy common belief is: If the degree of a common belief φ in the n-agent system is a, then the degree of any agent's belief φ is more than or equal to a. This reasoning is expressed in L-$\mathbf{K}_n^C$ as $\mathrm{T}_aC(\varphi) \to (\mathrm{U}_aB_1(\varphi) \wedge ... \wedge \mathrm{U}_aB_n(\varphi))$, which is both valid and provable in the following semantics and proof system for L-$\mathbf{K}_n^C$.

L-valued Kripke semantics for L-$\mathbf{K}_n^C$ is defined as follows. For a non-negative integer k, $E^k(\varphi)$ is defined by $E^1(\varphi) = E(\varphi)$ and $E^{k+1}(\varphi) = E(E^k(\varphi))$.

Definition C.4.1. *Let* $(M, R_1, ..., R_n)$ *be a Kripke* n*-frame. Then, a function* $e : M \times \mathbf{Form}_n^C \to L$ *is an* L-$\mathbf{K}_n^C$ *valuation on* $(M, R_1, ..., R_n)$ *iff it satisfies the following for each* $w \in M$*:*

1. $e(w, E(\varphi)) = \bigwedge\{e(w, B_i(\varphi))\ ;\ i = 1, ..., n\ \}$*;*
2. $e(w, C(\varphi)) = \bigwedge\{e(w, E^k(\varphi))\ ;\ k \in \omega \setminus \{0\}\ \}$*;*
3. *The other conditions are as in Definition C.3.1.*

We call $(M, R_1, ..., R_n, e)$ *an* L-$\mathbf{K}_n^C$ *Kripke model. Then,* $\varphi \in \mathbf{Form}_n^C$ *is said to be valid in* L-$\mathbf{K}_n^C$ *iff* $e(w, \varphi) = 1$ *for any* L-$\mathbf{K}_n^C$ *Kripke model* $(M, R_1, ..., R_n, e)$ *and any* $w \in M$*.*

If L is the two-element Boolean algebra, then the above Kripke semantics coincides with the usual Kripke semantics for the (**K**-style) logic of common belief in an n-agent system (for logics of common knowledge and belief, see [92]). The following notion of reachability is useful for understanding the common belief operator C.

Definition C.4.2. *Let* $(M, R_1, ..., R_n)$ *be a Kripke* n*-frame and* $w, w' \in M$*. For a non-negative integer* k*,* w *is reachable from* w' *in* k *steps iff there are* $w_0, ..., w_k \in M$ *such that* $w_0 = w'$*,* $w_k = w$*, and for any* $l \in \{0, ..., k-1\}$*, there is* $i \in \{1, ..., n\}$ *with* $w_l R_i w_{l+1}$*.*

Proposition C.4.3. *Let* $(M, R_1, ..., R_n, e)$ *be an* L-$\mathbf{K}_n^C$ *Kripke model. For* $w \in M$*,* $a \in L$ *and* $\varphi \in \mathbf{Form}_n^C$*,* $e(w, C(\varphi)) \geq a$ *iff, for any* $k \in \omega \setminus \{0\}$*, if* w' *is reachable from* w *in* k *steps, then* $e(w', \varphi) \geq a$*.*

We now give a Hilbert-style axiomatization of L-$\mathbf{K}_n^C$.

Definition C.4.4. $\varphi \in \mathbf{Form}_n^C$ *is provable in* L-$\mathbf{K}_n^C$ *iff it is either an instance of one of the following axioms or deduced from provable formulas by one of the following rules of inference: The axioms are* $\mathbf{A1}, ..., \mathbf{A8}, \mathbf{B1}, \mathbf{B2}$ *in Definition C.2.3 and Definition C.3.2, and*

E1. $E(\varphi) \leftrightarrow (B_1(\varphi) \wedge ... \wedge B_n(\varphi))$*;*

C1. $C\mathrm{U}_a(\varphi) \leftrightarrow \mathrm{U}_a C(\varphi)$*;*

C2. $\mathrm{U}_a C(\varphi) \to \mathrm{U}_a E(\varphi \wedge C(\varphi))$*,*

where $a \in L$ *and* $\varphi \in \mathbf{Form}_n^C$*. The rules of inference are* $\mathbf{R1}, \mathbf{R2}, \mathbf{R3}, \mathbf{R4}$ *in Definition C.2.3 and Definition C.3.2, and*

R5. *From $\varphi \to E(\varphi \wedge \psi)$ infer $\varphi \to C(\psi)$.*

Using the axioms **B1** and **B2**, we can show the following.

Proposition C.4.5. *Let $a \in L$ and $\varphi \in \mathbf{Form}_n^C$. (i) $E\mathrm{U}_a(\varphi) \leftrightarrow \mathrm{U}_a E(\varphi)$ is provable in L-$\mathbf{K}_n^C$. (ii) $E(\varphi \wedge \psi) \leftrightarrow E(\varphi) \wedge E(\psi)$ is provable in L-$\mathbf{K}_n^C$.*

The following theorem states that the above axiomatization of L-$\mathbf{K}_n^C$ is sound and complete with respect to L-valued Kripke semantics for L-$\mathbf{K}_n^C$.

Theorem C.4.6. *For $\varphi \in \mathbf{Form}_n^C$, φ is provable in L-$\mathbf{K}_n^C$ iff φ is valid in L-$\mathbf{K}_n^C$.*

Our proof of the above theorem is rather long and so we put it in the last part of this section. L-$\mathbf{K}_n^C$ enjoys the finite model property:

Theorem C.4.7. *For $\varphi \in \mathbf{Form}_n^C$, φ is provable in L-$\mathbf{K}_n^C$ iff $e(w, \varphi) = 1$ for any finite L-$\mathbf{K}_n^C$ Kripke model $(M, R_1, ..., R_n, e)$ and any $w \in M$.*

Proof. The proof of Theorem C.4.6 given below contains the proof of this theorem, since a counter-model $(\mathrm{Con}_C(\varphi), R_1, ..., R_n, e)$ in the proof of Theorem C.4.6 is actually a finite L-$\mathbf{K}_n^C$ Kripke model. $\square$

By Theorem C.4.6 and Theorem C.4.7, we obtain the decidability of L-$\mathbf{K}_n^C$.

Theorem C.4.8. *For $\varphi \in \mathbf{Form}_n^C$, it is effectively decidable whether or not φ is valid in L-$\mathbf{K}_n^C$.*

Finally, we give the proof of Theorem C.4.6 by generalizing the proof of [92, Theorem 4.3]

Proof of Theorem C.4.6. It is straightforward to verify the soundness. The completeness is proved as follows. To show the contrapositive, assume that φ is not provable in L-$\mathbf{K}_n^C$. Let $\mathrm{Sub}_C(\varphi)$ be the set of the following formulas: (i) all subformulas of φ; (ii) $B_1\psi, ..., B_n\psi$ for each subformula $E\psi$ of φ; (iii) $\psi \wedge C\psi, B_1(\psi \wedge C\psi), ..., B_n(\psi \wedge C\psi), E(\psi \wedge C\psi)$ for each subformula $C\psi$ of φ. Let

$$\mathrm{Sub}_C^+(\varphi) = \{\mathrm{T}_a(\psi), \mathrm{U}_a(\psi)\ ;\ \psi \in \mathrm{Sub}_C(\varphi) \text{ and } a \in L\}.$$

Define $\mathrm{Con}_C(\varphi)$ as the set of all maximal L-$\mathbf{K}_n^C$ consistent subsets of $\mathrm{Sub}_C^+(\varphi)$. Then, we can consider the L-$\mathbf{K}_n^C$ Kripke model $(\mathrm{Con}_C(\varphi), R_1, ..., R_n, e)$ such that for each $i = 1, ..., n$ and $V, W \in \mathrm{Con}_C(\varphi)$, VR_iW iff, for any $a \in L$ and $\psi \in \mathrm{Sub}_C^+(\varphi)$, $\mathrm{U}_a(B_i\psi) \in V$

implies $\mathrm{U}_a(\psi) \in W$ and that for each propositional variable $p \in \mathrm{Sub}_C(\varphi)$, $e(W, p) = a$ iff $\mathrm{T}_a(p) \in W$. Then we claim that, for $a \in L$, $\psi \in \mathrm{Sub}_C(\varphi)$ and $W \in \mathrm{Con}_C(\varphi)$,

$$e(W, \psi) = a \text{ iff } \mathrm{T}_a(\psi) \in W.$$

If the claim holds, then $(\mathrm{Con}_C(\varphi), R_1, ..., R_n, e)$ is a counter-model for φ, since there is $W \in \mathrm{Con}_C(\varphi)$ with $\mathrm{T}_1(\varphi) \notin W$ (i.e., $e(W, \varphi) \neq 1$) by the assumption that φ is not provable in L-$\mathbf{K}_n^C$ (such W is obtained as follows: Construct a maximal L-$\mathbf{K}_n^C$ consistent subset X of $\mathbf{Form}_n^C$ containing $\neg\mathrm{T}_1(\varphi)$ and then let $W = X \cap \mathrm{Sub}_C^+(\varphi)$). Thus, to show the completeness, it suffices to verify the claim. Note that the above claim is equivalent to the following: For $a \in L$ with $a \neq 0$, $\psi \in \mathrm{Sub}_C(\varphi)$ and $W \in \mathrm{Con}_C(\varphi)$, $e(W, \psi) \geq a$ iff $\mathrm{U}_a(\psi) \in W$, since we have: $\mathrm{T}_a(\psi) \in W$ iff $\mathrm{U}_x(\psi) \in W$ for any $x \in L$ with $x \leq a$ and $\mathrm{U}_x(\psi) \notin W$ for any $x \in L$ with $x \nleq a$. We show the claim by induction on the structure of formulas. We consider only the case that ψ is of the form $C(\chi)$, since arguments in the other cases are similar to those in the proofs of Theorem C.2.6 and Theorem C.3.3.

Suppose that ψ is of the form $C(\chi)$. It suffices to show that, for any $W \in \mathrm{Con}_C(\varphi)$ and $a \in L$ with $a \neq 0$, $e(W, C\chi) \geq a$ iff $\mathrm{U}_a C(\chi) \in W$. We first show that $\mathrm{U}_a C(\chi) \in W$ implies $e(W, C\chi) \geq a$. Assume $\mathrm{U}_a C(\chi) \in W$. We claim that, for any $k \in \omega \setminus \{0\}$, if $V \in \mathrm{Con}_C(\varphi)$ is reachable from W in k steps, then both $\mathrm{U}_a(\chi)$ and $\mathrm{U}_a C(\chi)$ are in V. This is shown by induction on $k \in \omega \setminus \{0\}$. We first consider the case $k = 1$. By **C2**, $\mathrm{U}_a C(\chi) \to \mathrm{U}_a E(\chi \wedge C(\chi))$ is provable in L-$\mathbf{K}_n^C$. Then, by $W \in \mathrm{Con}_C(\varphi)$, we have $\mathrm{U}_a E(\chi \wedge C(\chi)) \in W$, whence $\mathrm{U}_a E\chi \in W$ and $\mathrm{U}_a E(C\chi) \in W$ by (ii) in Proposition C.4.5 and (i) in Proposition C.2.7. If V is reachable from W in 1 step, then it follows from **E1** and the definition of R_i that $\mathrm{U}_a(\chi)$ and $\mathrm{U}_a C(\chi)$ are in V. We next consider the case $k = k' + 1$ for $k' \in \omega \setminus \{0\}$. If V is reachable from W in $k'+1$ steps, then there is $V' \in \mathrm{Con}_C(\varphi)$ such that V is reachable from V' in 1 step and that V' is reachable from W in k' steps. By the induction hypothesis, both $\mathrm{U}_a(\chi)$ and $\mathrm{U}_a C(\chi)$ are in V'. By arguing as in the case $k = 1$, it is verified that both $\mathrm{U}_a(\chi)$ and $\mathrm{U}_a C(\chi)$ are in V. Thus, the claim has been proved. Hence, for any $k \in \omega \setminus \{0\}$, if V is reachable from W in k steps, then $\mathrm{U}_a(\chi) \in V$ and so $e(V, \chi) \geq a$ by the induction hypothesis for the first claim. By Proposition C.4.3, we have $e(W, C\chi) \geq a$.

In order to complete the proof, we show that $e(W, C\chi) \geq a$ implies $\mathrm{U}_a C(\chi) \in W$. Assume $e(W, C\chi) \geq a$. Define

$$\begin{aligned} \Gamma &= \{V \in \mathrm{Con}_C(\varphi)\, ;\, e(V, C\chi) \geq a\} \\ \zeta &= \bigvee\{\bigwedge V\, ;\, V \in \Gamma\}. \end{aligned}$$

where note that both V and $\{\bigwedge V\ ;\ V \in \Gamma\}$ are finite sets. We first show that if $\mathrm{U}_a(\zeta) \to \mathrm{U}_a E(\chi \wedge \zeta)$ is provable in L-$\mathbf{K}_n^C$, then $\mathrm{U}_a C(\chi) \in W$. Assume that $\mathrm{U}_a(\zeta) \to \mathrm{U}_a E(\chi \wedge \zeta)$ is provable in L-$\mathbf{K}_n^C$. Then, by (i) in Proposition C.4.5 and (i) in Proposition C.2.7, $\mathrm{U}_a(\zeta) \to E(\mathrm{U}_a(\chi) \wedge \mathrm{U}_a(\zeta))$ is provable in L-$\mathbf{K}_n^C$. Then it follows from **R5** and **C1** that $\mathrm{U}_a(\zeta) \to \mathrm{U}_a C(\chi)$ is provable in L-$\mathbf{K}_n^C$. Since $(\bigwedge W) \to \zeta$ is provable in L-$\mathbf{K}_n^C$ by $W \in \Gamma$, $\mathrm{U}_a(\bigwedge W) \to \mathrm{U}_a(\zeta)$ is provable in L-$\mathbf{K}_n^C$ by (ii) in Proposition C.2.7. By these facts, $\mathrm{U}_a(\bigwedge W) \to \mathrm{U}_a C(\chi)$ is provable in L-$\mathbf{K}_n^C$. Since $\mathrm{U}_a(\bigwedge W) \leftrightarrow (\bigwedge W)$ is provable in L-$\mathbf{K}_n^C$ by Proposition C.2.8, $(\bigwedge W) \to \mathrm{U}_a C(\chi)$ is provable in L-$\mathbf{K}_n^C$. Hence, we have $\mathrm{U}_a C(\chi) \in W$. Now it only remains to show that the assumption of this argument holds, i.e., $\mathrm{U}_a(\zeta) \to \mathrm{U}_a E(\chi \wedge \zeta)$ is provable in L-$\mathbf{K}_n^C$. □

We have studied the logic of fuzzy belief and common belief with emphasis on incomparable beliefs (which cannot be formalised in previously known systems in [33, 91, 114]), establishing completeness and decidability results (implicitly based on duality-theoretic intuitions). We can also obtain many other versions of the results such as **KD45**-style and **S5**-style ones (and many more). We will study their complexity issues in future work.

We emphasize that our system based on L-valued logic can treat both qualitative fuzziness and quantitative fuzziness in the context of epistemic reasoning, while systems based on $[0,1]$-valued logic or probabilistic logic (such as those in [33, 91, 114]) can only encompass the latter.

In the previous work such as [186, 191], we studied the mathematically profound aspects of Fitting's many-valued modal logic. The results suggest that Fitting's logic be beneficial also in the context of artificial intelligence.

We consider that fuzzy reasoning is useful to understand some epistemic problems such as two generals' problem or the coordinated attack problem (for this problem, see, e.g., [92]). In theory, the two generals cannot attack at any time, while, in practice, they will attack after they have sent messages a few times. We can understand this as follows. The more times they send messages, the higher the degrees of their beliefs become. Even if the degrees cannot reach 1, the generals will attack when they become sufficiently high. This seems to be a very natural understanding of the coordinated attack problem, which is impossible if we stick to classical logic and becomes possible only if we allow fuzzy reasoning.

We have considered the reduction thesis about belief degrees as being provisionally true. Although the reduction thesis would be justified to some degrees by its theoretical merit, in future work, we will discuss in more detail to what extent the reduction thesis is justifiable, since it would be of philosophical interest as well.

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