Topology and its Applications 47 (1992) 53-67
North-Holland

Acyclic monotone normality

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Received 4 December 1990
Revised 18 July 1991

Abstract

Moody, P. J. and A. W. Roscoe, Acyclic monotone normality, Topology and its Applications 47 (1992) 53-67.

A space X is acyclic monotonically normal if it has a monotonically normal operator $M\langle\cdot,\cdot\rangle$ such that for distinct points $x_0,\ldots,x_{n-1}$ in X and $x_n=x_0$, $\bigcap_{i=0}^{n-1} M\langle x_i, X\backslash\{x_{i+1}\}\rangle=\emptyset$. It is a property which arises from the study of monotone normality and the condition "chain (F)". In this paper it is shown that GO, metric, stratifiable and elastic spaces are all acyclic monotonically normal. In addition it is established that this property is preserved by closed continuous maps, adjunction and domination. It is known that acyclic monotonically normal spaces are K_0-spaces, this being an open question for monotonically normal spaces. The links between acyclic monotone normality, monotone normality and K_0-spaces are further investigated. In particular it is shown that the addition of a simple condition to the definition of a K_0-space yields a property, called monotonically K_0, which is equivalent to acyclic monotone normality.

Keywords: Acyclic monotone normality, K_0-spaces, GO spaces, elastic spaces, adjunction, domination.

AMS (MOS) Subj. Class.: Primary 54C20, 54C99, 54D15; secondary 54B17, 54E20, 54F05.

1. Introduction

The property of acyclic monotone normality was introduced by the authors together with Collins and Reed in [10], where it was shown to be equivalent to the condition chain (F). This condition is a natural generalisation of conditions which are equivalent to metrisable and stratifiable. It is known that any acyclic monotonically normal space is monotonically normal while the converse is an open question.[1]

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[1] Added in proof: M.E. Rudin has recently constructed a monotonically normal space which isn't a K_0-space, and hence not acyclic.

In this paper we shall show that all the major classes of monotonically normal spaces are acyclic and that acyclic monotone normality behaves in much the same way as monotone normality with respect to closed continuous maps, adjunction and domination. We also consider the general problem of when monotonically normal spaces are acyclic monotonically normal with particular emphasis on K_0-spaces. Our final theorem proves that a simple extension of the definition of a K_0-space yields a property which is equivalent to acyclic monotone normality and thus chain (F).

We begin by recalling the definition of chain (F). It applies to topological spaces X for each element x of which a family $W(x)$ of subsets containing x is given. Let $\mathcal{W} = \{W(x): x \in X\}$. We say that $\mathcal{W}$ satisfies chain (F) when it satisfies:

(1) If $x \in U$ and U is open, then there exists an open $V = V\langle x, U\rangle$ containing x such that $x \in W \subseteq U$ for some $W \in W(y)$ whenever $y \in V$.

(2) $W(x)$ is linearly ordered with respect to inclusion.

It was stated above that chain (F) is a natural generalisation of conditions which are equivalent to metrisable and stratifiable. This is self-evident from the following theorems.

Theorem 1.1 [4]. *The space X is metrisable if and only if X has $\mathcal{W}$ satisfying chain* (F) *such that for each $x \in X$, every element of $W(x)$ is open, $W(x)$ is countable and there is an enumeration of $W(x)$ as $\{W\langle n, x\rangle: n \in \mathbb{N}\}$ such that $W\langle n+1, x\rangle \subseteq W\langle n, x\rangle$ for each n.*

Theorem 1.2 [1, 4]. *The space X is stratifiable if and only if X has countable pseudo-character and $\mathcal{W}$ satisfying chain* (F) *such that for each $x \in X$, $W(x)$ is countable and there is an enumeration of $W(x)$ as $\{W\langle n, x\rangle: n \in \mathbb{N}\}$ such that $W\langle n+1, x\rangle \subseteq W\langle n, x\rangle$ for each n.*

The conditions above have previously gone by the names of "open decreasing (G)" and "decreasing (G)" respectively.

It was observed in [5] that spaces with $\mathcal{W}$ satisfying chain (F) are monotonically normal. The property of acyclic monotone normality is closely related to monotone normality and is formally stronger. Recall that a space X is monotonically normal if there is an operator $M\langle\cdot,\cdot\rangle$ which assigns to each x and each open set U containing x, an open set $M\langle x, U\rangle$ containing x which satisfies:

(1) $M\langle x, U\rangle \subseteq M\langle x, U'\rangle$ whenever $x \in U \subseteq U'$, and

(2) $M\langle x, X\backslash\{y\}\rangle \cap M\langle y, X\backslash\{x\}\rangle = \emptyset$ if $x \neq y$.

If in addition $M\langle\cdot,\cdot\rangle$ satisfies

(3) $\bigcap_{i=0}^{n-1} M\langle x_i, X\backslash\{x_{i+1}\}\rangle = \emptyset$ whenever $n \geq 2$, $x_0, \ldots, x_{n-1}$ are distinct and $x_n = x_0$,

then $M\langle\cdot,\cdot\rangle$ is an acyclic monotonically normal operator. A space is acyclic monotonically normal if it has an acyclic monotonically normal operator. Observe that by taking $n = 2$ in (3) we obtain (2). Hence (3) can be seen to be a simple strengthening of (2). As stated above, the following theorem was proved in [10].

Theorem 1.3. *A space is acyclic monotonically normal if and only if it has $\mathscr{W}$ satisfying chain* (F).

Notation and conventions. All our spaces will satisfy the T_1 separation axiom. For a set A, A° will denote the interior, and $\bar{A}$ the closure, of the set A. If X is a space and Y a subset of X, then $\mathscr{T}_Y$ is the subspace topology on Y. $\langle a_1, \ldots, a_n\rangle$ will denote an ordered n-tuple, while (a, b), $[a, b)$, $(a, b]$ and $[a, b]$ will represent the usual intervals in an ordered space. Finally, $\mathbb{N}$ is the set of natural numbers.

2. Spaces which are acyclic monotonically normal

From Theorem 1.2 we know that any stratifiable space, and hence any metric space, has an acyclic monotonically normal operator. Indeed it is an easy exercise to show that the usual monotonically normal operators on stratifiable and metric spaces are in fact acyclic. The other important class of monotonically normal spaces is the class of generalized order (GO) spaces [7]. Our first result demonstrates that GO spaces are acyclic monotonically normal.

Theorem 2.1. *Every GO space is acyclic monotonically normal.*

Proof. We shall establish that every GO space is acyclic monotonically normal by showing that the monotonically normal operator defined by Heath, Lutzer and Zenor on a linearly ordered space (LOTS) [7] is an acyclic monotonically normal operator.

Let $\langle X, <, \mathscr{T}\rangle$ be a LOTS and suppose that $\lhd$ is a well ordering of X. We shall recall the definition of the monotonically normal operator defined in [7]. Suppose that $p \in X$ and U is an open neighbourhood of p. Define $I\langle p, U\rangle$ to be the convex component of U which contains p. If $I_-\langle p, U\rangle = \{y \in I\langle p, U\rangle\colon y < p\} \neq \emptyset$ let $x_{\langle p,U\rangle}$ be the $\lhd$-first element of $I_-\langle p, U\rangle$ and let $y_{\langle p,U\rangle}$ be the $\lhd$-first element of $I_+\langle p, U\rangle = \{y \in I\langle p, U\rangle\colon y > p\}$ if $I_+\langle p, U\rangle \neq \emptyset$. Finally define

$$M\langle p, U\rangle = \begin{cases} (x_{\langle p,U\rangle}, y_{\langle p,U\rangle}), & \text{if } I_-\langle p, U\rangle \neq \emptyset \neq I_+\langle p, U\rangle, \\ [p, y_{\langle p,U\rangle}), & \text{if } I_-\langle p, U\rangle = \emptyset \neq I_+\langle p, U\rangle, \\ (x_{\langle p,U\rangle}, p], & \text{if } I_-\langle p, U\rangle \neq \emptyset = I_+\langle p, U\rangle, \\ \{p\}, & \text{if } I_-\langle p, U\rangle = \emptyset = I_+\langle p, U\rangle. \end{cases}$$

By [7], $M\langle\cdot,\cdot\rangle$ is a monotonically normal operator on X. We shall show that $M\langle\cdot,\cdot\rangle$ is an acyclic monotonically normal operator.

Suppose $x_0, \ldots, x_{n-1}$ are distinct elements of X and $x_n = x_0$. Since $M\langle\cdot,\cdot\rangle$ is known to be a monotonically normal operator it will be sufficient to consider the case when $n \geqslant 3$. Let i and j be such that

$$x_i = \min\{x_m : 0 \leqslant m \leqslant n-1\} \quad \text{and} \quad x_j = \max\{x_m : 0 \leqslant m \leqslant n-1\},$$

and define y to be the $\lhd$-first element of the interval (x_i, x_j) (observe this interval is nonempty because $n \geqslant 3$). Since $x_i < x_{i+1} \leqslant x_j$ we see that

$$M\langle x_i, X\backslash\{x_{i+1}\}\rangle \subseteq (\leftarrow, y),$$

where $(\leftarrow, y) = \{x \in X: x < y\}$. Similarly

$$M\langle x_j, X\backslash\{x_{j+1}\}\rangle \subseteq (y, \rightarrow),$$

where $(y, \rightarrow) = \{x \in X: x > y\}$. Hence

$$M\langle x_i, X\backslash\{x_{i+1}\}\rangle \cap M\langle x_j, X\backslash\{x_{j+1}\}\rangle = \emptyset$$

and thus $\bigcap_{k=0}^{n-1} M\langle x_k, X\backslash\{x_{k+1}\}\rangle = \emptyset$. Therefore $M\langle\cdot,\cdot\rangle$ is an acyclic monotonically normal operator. $\square$

Stratifiable spaces have been generalised by Vaughan to linearly stratifiable spaces [13] and further, by Tamano and Vaughan, to elastic spaces [11]. Borges established in [3] that elastic spaces are monotonically normal. It should be noted that the operator he defined will not, in general, be acyclic. However, as we shall see, his proof may be modified to yield such an operator.

Recall that a pair-base $\mathcal{P} = \{P = \langle P_1, P_2\rangle\}$ for a space X is a collection of pairs $P = \langle P_1, P_2\rangle$ of subsets of X such that P_1 is open, and for each $x \in X$ and neighbourhood U of x, there exists $\langle P_1, P_2\rangle \in \mathcal{P}$ such that $x \in P_1 \subseteq P_2 \subseteq U$. Let $\mathcal{P}_1 = \{P_1: P \in \mathcal{P}\}$, $\mathcal{P}_2 = \{P_2: P \in \mathcal{P}\}$ and define $f: \mathcal{P}_1 \to \mathcal{P}_2$ by $f(P_1) = P_2$. A space is elastic provided it has a pair-base $\mathcal{P}$ and a transitive relation $\sim$ on $\mathcal{P}_1$ such that

(1) $P_1, P_1' \in \mathcal{P}_1$, $P_1 \cap P_1' \neq \emptyset \Rightarrow P_1 \sim P_1'$ or $P_1' \sim P_1$, and

(2) if $\mathcal{P}_1' \subseteq \mathcal{P}_1$ and if there exists $P_1 \in \mathcal{P}_1$ such that $(P_1' \in \mathcal{P}_1' \Rightarrow P_1' \sim P_1)$, then $\overline{\bigcup \mathcal{P}_1'} \subseteq \bigcup\{f(P_1): P_1 \in \mathcal{P}_1'\}$.

Now, by the proof of [11, Lemma 2], there is a relation $\approx$ on $\mathcal{P}_1$ such that

(a) $P_1 \cap P_1' \neq \emptyset \Rightarrow (P_1 \approx P_1'$ or $P_1' \approx P_1)$ whenever $P_1, P_1' \in \mathcal{P}_1$.

(b) If $\mathcal{P}_1' \subseteq \mathcal{P}_1$ and is nonempty, then there is a $P_1' \in \mathcal{P}_1'$ such that $P_1 \not\approx P_1'$ whenever $P_1 \in \mathcal{P}_1'$ and $P_1 \neq P_1'$.

(c) If $\mathcal{P}_1' \subseteq \mathcal{P}_1$ and if there exists $P_1' \in \mathcal{P}_1$ such that $P_1 \approx P_1'$ for every $P_1 \in \mathcal{P}_1'$, then there exists $P_1'' \in \mathcal{P}_1$ such that $P_1 \sim P_1''$ for every $P_1 \in \mathcal{P}_1'$.

Using the $\approx$ relation, rather than the $\sim$ relation as in Borges' proof, define for each open U in X and $P = \langle P_1, P_2\rangle \in \mathcal{P}$,

$$U_P = \bigcup\{V_1: \langle V_1, V_2\rangle \in \mathcal{P},\ V_1 \subseteq V_2 \subseteq U \text{ and } V_1 \approx P_1\}.$$

Observe that, by condition (c) for the relation $\approx$ and condition (2) for the relation $\sim$,

$$\overline{U_p} \subseteq \bigcup\{V_2: \langle V_1, V_2\rangle \in \mathcal{P},\ V_1 \subseteq V_2 \subseteq U \text{ and } V_1 \approx P_1\} \subseteq U.$$

Now suppose that O is an open neighbourhood of the point x. Pick a $P\langle O, x\rangle \in \mathcal{P}$ which satisfies

$$x \in P\langle O, x\rangle_1 \subseteq P\langle O, x\rangle_2 \subseteq O.$$

Define

$$O^x = P\langle O, x\rangle_1 \backslash \overline{[(X\backslash\{x\})_{P\langle O,x\rangle}]}$$

and note that $x \in O^x$. Finally define for each x and open set U containing x

$$M\langle x, U\rangle = \bigcup\{O^x : x \in O \subseteq U \text{ and } O \text{ open}\}.$$

Clearly $M\langle x, U\rangle$ is an open neighbourhood of x. We shall show that $M\langle\cdot,\cdot\rangle$ is the required acyclic monotonically normal operator on X. First observe that if $U \subseteq U'$, then $M\langle x, U\rangle \subseteq M\langle x, U'\rangle$. Thus it will be sufficient to show that if $x_0, \ldots, x_{n-1}$ are distinct points of X and $x_n = x_0$, then

$$\bigcap_{i=0}^{n-1} M\langle x_i, X\backslash\{x_{i+1}\}\rangle = \emptyset.$$

So suppose that this were not the case. Pick a point x which lies in the intersection. Then for $i = 0, \ldots, n-1$ there is an open set O_i for which $x_i \in O_i \subseteq X\backslash\{x_{i+1}\}$ and $x \in O_i^{x_i}$. Thus

$$x \in P\langle O_i, x_i\rangle_1 \backslash \overline{[(X\backslash\{x_i\})_{P\langle O_i,x_i\rangle}]}.$$

Since $\{P\langle O_i, x_i\rangle_1 : i = 0, \ldots, n-1\}$ is a nonempty subset of $\mathcal{P}_1$ there is, by (b) above, an i such that for each $j = 0, \ldots, n-1$ either

$$P\langle O_j, x_j\rangle_1 = P\langle O_i, x_i\rangle_1$$

or

$$P\langle O_j, x_j\rangle_1 \not\approx P\langle O_i, x_i\rangle_1.$$

But $x \in P\langle O_i, x_i\rangle_1 \cap P\langle O_j, x_j\rangle_1$, thus by (a) above, $P\langle O_i, x_i\rangle_1 \approx P\langle O_j, x_j\rangle_1$. Either $i < n-1$ or $i = n-1$.

Suppose $i < n-1$. Hence $P\langle O_i, x_i\rangle_1 \approx P\langle O_{i+1}, x_{i+1}\rangle_1$, but

$$x \in P\langle O_i, x_i\rangle_1 \subseteq P\langle O_i, x_i\rangle_2 \subseteq O_i \subseteq X\backslash\{x_{i+1}\}.$$

Therefore, by definition, $x \in (X\backslash\{x_{i+1}\})_{P\langle O_{i+1},x_{i+1}\rangle}$, which is a contradiction because $x \in O_{i+1}^{x_{i+1}}$. The case when $i = n-1$ is similar. Thus $M\langle\cdot,\cdot\rangle$ is an acyclic monotonically normal operator for the space X.

Hence we have the following theorem.

Theorem 2.2. *Any elastic space is acyclic monotonically normal.*

3. The stability of acyclic monotone normality

Heath, Lutzer and Zenor established in [7] that the class of monotonically normal spaces is closed under closed continuous maps and asked whether a space X must be monotonically normal provided that X is dominated by a collection of such subspaces. Miwa answered this question in the affirmative in [9] and also showed that the adjunction of two monotonically normal spaces is a monotonically normal space.[2] In this section we shall extend these results to acyclic monotonically normal spaces. This demonstrates that acyclic monotone normality appears to be as stable a condition as monotone normality and that it is not possible to construct a nonacyclic monotonically normal space by using closed maps, adjunction and domination.

It is an easy exercise to extend Heath, Lutzer and Zenor's result to the acyclic case. Thus we shall simply note the following theorem.

Theorem 3.1. *The image of an acyclic monotonically normal space under a closed continuous map is acyclic monotonically normal.*

Extending Miwa's adjunction theorem is not so easy. Recall that if X and Y are disjoint spaces, F a closed subset of X and $f: F \to Y$ is a continuous map, then the adjunction $X \cup_f Y$ of X and Y, with respect to F and f, is the set

$$\{\{x\}: x \in X \setminus F\} \cup \{\{y\} \cup f^{-1}(y): y \in Y\}$$

with topology $\mathcal{T}$, where $U \in \mathcal{T}$ if and only if $h^{-1}(U)$ and $k^{-1}(U)$ are open in X and Y respectively, with $h: X \to X \cup_f Y$ and $k: Y \to X \cup_f Y$ being defined by

$$h(x) = \begin{cases} \{x\}, & x \notin F, \\ \{f(x)\} \cup f^{-1}(f(x)), & x \in F' \end{cases}$$

$$k(y) = \{y\} \cup f^{-1}(y), \quad y \in Y.$$

In order to prove that a space which is dominated by a collection of acyclic monotonically normal subspaces is acyclic monotonically normal we require the acyclic monotonically normal operator on the adjunction space to satisfy some addition conditions which are stated in the theorem below.

Theorem 3.2. *Suppose X and Y are acyclic monotonically normal spaces, with operators $M_X\langle\cdot,\cdot\rangle$ and $M_Y\langle\cdot,\cdot\rangle$ respectively. In addition suppose F is a closed subset of X and $f: F \to Y$ is a continuous mapping. Then the adjunction $Z = X \cup_f Y$ of X and Y, with respect to f and F, has an acyclic monotonically normal operator $M_Z\langle\cdot,\cdot\rangle$ which satisfies*

(1) *$M_Z\langle\{x\}, U\rangle \cap k(Y) = \emptyset$ whenever $x \in X \setminus F$ and U is an open neighbourhood of $\{x\}$,*

(2) *$M_Z\langle\{y\} \cup f^{-1}(y), U\rangle \cap k(Y) = k(M_Y\langle y, k^{-1}(U)\rangle)$ whenever $y \in Y$ and U is an open neighbourhood of $\{y\} \cup f^{-1}(y)$.*

Proof. X has an acyclic monotonically normal operator, hence X has $\mathcal{W}$ satisfying chain (F). Suppose that $z \in Z$ and U is an open neighbourhood of z. We shall define an open neighbourhood $M_Z\langle z, U\rangle$ of z.

[2] Added in proof: These results were originally proved by San-ou. Reference: S. San-ou, A note on monotonically normal spaces, Sci. Rep. Tokyo Kyoiku Daigaku 12 (1974) 214–217.

First suppose that $z=\{x\}$ where $x\in X\backslash F$. Define $A\langle z, U\rangle=\{x\}$ and let $H\langle z, U\rangle= h^{-1}(U)\backslash F$. Set

$$O\langle z, U\rangle=\{\{a\}: a\in X\backslash F, \exists W\in W(a), W\cap A\langle z, U\rangle\neq\emptyset \wedge W\subseteq H\langle z, U\rangle\}^{\circ}$$

and define $M_Z\langle z, U\rangle=O\langle z, U\rangle$. Clearly $M_Z\langle z, U\rangle$ is open. Observe that $h(V\langle x, h^{-1}(U)\backslash F\rangle)$ is open in Z and so

$$\{x\}\in h(V\langle x, h^{-1}(U)\backslash F\rangle)\subseteq O\langle z, U\rangle$$

($V\langle\cdot,\cdot\rangle$ is the operator associated with "chain (F)"). Hence $M_Z\langle z, U\rangle$ is an open neighbourhood of z.

Now suppose $z=\{y\}\cup f^{-1}(y)$, where $y\in Y$. Define $A\langle z, U\rangle$ to be $f^{-1}(M_Y\langle y, k^{-1}(U)\rangle)$ and set

$$H\langle z, U\rangle=h^{-1}(U)\backslash(F\backslash f^{-1}(M_Y\langle y, k^{-1}(U)\rangle)).$$

Observe that $H\langle z, U\rangle$ is open and that $A\langle z, U\rangle\subseteq H\langle z, U\rangle$. As above set

$$O\langle z, U\rangle=\{\{a\}: a\in X\backslash F, \exists W\in W(a), W\cap A\langle z, U\rangle\neq\emptyset \wedge W\subseteq H\langle z, U\rangle\}^{\circ}$$

and define $M_Z\langle z, U\rangle=O\langle z, U\rangle\cup k(M_Y\langle y, k^{-1}(U)\rangle)$. Clearly $z\in M_Z\langle z, U\rangle$. It can be shown that $h^{-1}(M_Z\langle z, U\rangle)$ and $k^{-1}(M_Z\langle z, U\rangle)$ are open in X and Y respectively, thus $M_Z\langle z, U\rangle$ is open in Z.

We shall show that $M_Z\langle\cdot,\cdot\rangle$ is the required acyclic monotonically normal operator. First, it is clear that conditions (1) and (2) of the theorem are satisfied. Also observe that if $z\in U\subseteq U'$ where U and U' are open in Z, then $M_Z\langle z, U\rangle\subseteq M_Z\langle z, U'\rangle$. Therefore it only remains to show that if $z_0,\ldots,z_{n-1}$ are distinct elements of Z and $z_n=z_0$, then

$$\bigcap_{i=0}^{n-1} M_Z\langle z_i, Z\backslash\{z_{i+1}\}\rangle=\emptyset.$$

Suppose this were not the case, then pick a $z\in Z$ which lies in the above intersection. Either $z=\{y\}\cup f^{-1}(y)$ for some $y\in Y$ or $z=\{x\}$ for some $x\in X\backslash F$.

Suppose $z=\{y\}\cup f^{-1}(y)$ where $y\in Y$. Since, for each $i=0,\ldots,n-1$, $O\langle z_i, Z\backslash\{z_{i+1}\}\rangle\subseteq h(X\backslash F)$, $z\notin O\langle z_i, Z\backslash\{z_{i+1}\}\rangle$. Thus for each i, there must exist a $y_i\in Y$ such that $z_i=\{y_i\}\cup f^{-1}(y_i)$. Observe that $y_0,\ldots,y_{n-1}$ are distinct and set $y_n=y_0$. For $i=0,\ldots,n-1$, $z\in k(M_Y\langle y_i, k^{-1}(Z\backslash\{z_{i+1}\})\rangle)$. Hence $y\in M_Y\langle y_i, Y\backslash\{y_{i+1}\}\rangle$ which is a contradiction since $M_Y\langle\cdot,\cdot\rangle$ is acyclic.

Therefore $z=\{x\}$ where $x\in X\backslash F$. For $i=0,\ldots,n-1$, $z\in O\langle z_i, Z\backslash\{z_{i+1}\}\rangle$. Thus we can pick, for each i, an element $W_i\in W(x)$ for which $W_i\cap A\langle z_i, Z\backslash\{z_{i+1}\}\rangle\neq\emptyset$ and $W_i\subseteq H\langle z_i, Z\backslash\{z_{i+1}\}\rangle$. Since $W(x)$ is linearly ordered with respect to inclusion, there is an i such that $W_i\subseteq W_j$ for $j=0,\ldots,n-1$. Suppose that $z_i=\{x_i\}$ where $x_i\in X\backslash F$. We shall show that this leads to a contradiction when $i>0$, and a similar contradiction may be found if $i=0$.

If $z_{i-1}=\{x_{i-1}\}$ where $x_{i-1}\in X\backslash F$, then

$$\begin{aligned}W_i\subseteq W_{i-1}\subseteq H\langle z_{i-1}, Z\backslash\{z_i\}\rangle&=h^{-1}(Z\backslash\{z_i\})\backslash F\\&=(X\backslash F)\backslash\{x_i\}.\end{aligned}$$

But $W_i \cap A\langle z_i, Z\backslash\{z_{i+1}\}\rangle \neq \emptyset$. However this is a contradiction because $A\langle z_i, Z\backslash\{z_{i+1}\}\rangle = \{x_i\}$. Thus $z_{i-1} = \{y_{i-1}\} \cup f^{-1}(y_{i-1})$, where $y_{i-1} \in Y$. As above

$$\begin{aligned} W_i &\subseteq H\langle z_{i-1}, Z\backslash\{z_i\}\rangle \\ &= h^{-1}(Z\backslash\{z_i\})\backslash(F\backslash f^{-1}(M_Y\langle y_{i-1}, k^{-1}(Z\backslash\{z_i\})\rangle)) \\ &\subseteq h^{-1}(Z\backslash\{z_i\}) = X\backslash\{x_i\}. \end{aligned}$$

But as above this is a contradiction since $x_i \in W_i$.

Hence $z_i = \{y_i\} \cup f^{-1}(y_i)$ for some $y_i \in Y$. Suppose that there is a $j \leq n-1$ for which $z_j = \{x_j\}$ where $x_j \in X\backslash F$. By the definitions

$$W_j \subseteq H\langle z_j, Z\backslash\{z_{j+1}\}\rangle = h^{-1}(Z\backslash\{z_{j+1}\})\backslash F \subseteq X\backslash F,$$

and hence $W_i \subseteq X\backslash F$. However $A\langle z_i, Z\backslash\{z_{i+1}\}\rangle \cap W_i \neq \emptyset$ and $A\langle z_i, Z\backslash\{z_{i+1}\}\rangle$ is a nonempty subset of F; hence we have a contradiction. Thus for each $j = 0, \ldots, n-1$, $z_j = \{y_j\} \cup f^{-1}(y_j)$ for some $y_j \in Y$. Hence for each j,

$$\begin{aligned} W_i &\subseteq H\langle z_j, Z\backslash\{z_{j+1}\}\rangle \\ &= h^{-1}(Z\backslash\{z_{j+1}\})\backslash(F\backslash f^{-1}(M_Y\langle y_j, k^{-1}(Z\backslash\{z_{j+1}\})\rangle)) \\ &= (X\backslash f^{-1}(y_{j+1}))\backslash(F\backslash f^{-1}(M_Y\langle y_j, Y\backslash\{y_{j+1}\}\rangle)) \\ &= (X\backslash F) \cup f^{-1}(M_Y\langle y_j, Y\backslash\{y_{j+1}\}\rangle). \end{aligned}$$

Recall that $W_i \cap A\langle z_i, Z\backslash\{z_{i+1}\}\rangle \neq \emptyset$; hence $W_i \cap f^{-1}(M_Y\langle y_i, Y\backslash\{y_{i+1}\}\rangle) \neq \emptyset$. Pick a w in this intersection and observe that, by the above, for each j, $w \in f^{-1}(M_Y\langle y_j, Y\backslash\{y_{j+1}\}\rangle)$. Hence

$$f(w) \in \bigcap_{j=0}^{n-1} M_Y\langle y_j, Y\backslash\{y_{j+1}\}\rangle$$

which is a contradiction since $M_Y\langle\cdot,\cdot\rangle$ is acyclic. $\square$

We are now in a position to prove that a space which is dominated by a collection of acyclic monotonically normal spaces is itself acyclic monotonically normal. The concept of a space being dominated by a collection of subsets is due to Michael [8, Definition 8.1].

Let X be a space and $\mathcal{B}$ a collection of closed subsets of X which covers X. Then $\mathcal{B}$ *dominates* X if, whenever $A \subseteq X$ has a closed intersection with every element of some subcollection $\mathcal{B}_1$ of $\mathcal{B}$ which covers A, then A is closed.

In [8], Michael establishes that a space is paracompact if and only if it is dominated by a collection of paracompact subspaces. Analogous results have been obtained for stratifiable spaces and monotonically normal spaces by Borges [2] and Miwa [9] respectively. Having proved that the adjunction of two acyclic monotonically normal spaces is acyclic monotonically normal and that the operator satisfies conditions (1) and (2) of Theorem 3.2, the proof that a space dominated by a collection of acyclic monotonically normal spaces is acyclic monotonically normal is essentially the same as Miwa's proof in the nonacyclic case. Thus we just note the following theorem.

Theorem 3.3. *A space X is acyclic monotonically normal if and only if it is dominated by a collection of acyclic monotonically normal subspaces.*

4. Acyclic monotone normality and K_0-spaces

A space X is said to be a K_0-space if for every subspace Y of X there is a function k from $\mathcal{T}_Y$ to $\mathcal{T}_X$ such that

(1) $Y \cap k(U) = U$ for each $U \in \mathcal{T}_Y$,

(2) $k(U) \cap k(V) = k(U \cap V)$ for each U, $V \in \mathcal{T}_Y$, and

(3) $k(\emptyset) = \emptyset$.

k is called a K_0-function.

It was established in [10] that any acyclic monotonically normal space is a K_0-space, it being a question of van Douwen [12] as to whether every monotonically normal space is a K_0-space. There are K_0-spaces which are not monotonically normal, for example any retractifiable space is a K_0-space [12]. However, it is a natural question as to whether monotonically normal K_0-spaces are acyclic monotonically normal. We phrase this as a conjecture.

Conjecture 4.1. Every monotonically normal K_0-space is acyclic monotonically normal.

Our aim is to investigate the structure of any counterexample to the above conjecture. However, our first result does not require that our space be a K_0-space. It establishes that if a monotonically normal space is locally acyclic monotonically normal, then it is acyclic monotonically normal.

Theorem 4.2. *A monotonically normal space X is acyclic monotonically normal if and only if it can be covered by a collection of open acyclic monotonically normal subspaces.*

Proof. Clearly if X is acyclic monotonically normal, then it can be covered by a collection of open acyclic monotonically normal subspaces. So, suppose that $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ is an open cover of X such that each U_α has an acyclic monotonically normal operator $M_\alpha\langle \cdot, \cdot \rangle$. Let $V\langle \cdot, \cdot \rangle$ be a monotonically normal operator on X and define for each x in an open set U,

$$M\langle x, U\rangle = V\langle x, U \cap U_\alpha\rangle \cap M_\alpha\langle x, U \cap U_\alpha\rangle$$

where α is minimal such that $x \in U_\alpha$. We will show that $M\langle x, U\rangle$ is the desired acyclic monotonically normal operator on X. First observe that $M\langle x, U\rangle$ is an open neighbourhood of x and, since $V\langle \cdot, \cdot \rangle$ and $M_\alpha\langle \cdot, \cdot \rangle$ are monotonically normal operators, $M\langle x, U\rangle \subseteq M\langle x, U'\rangle$ whenever $x \in U \subseteq U'$. So suppose that $x_0, \ldots, x_{n-1}$ are distinct elements of X and $x_n = x_0$. Further suppose that

$$x \in \bigcap_{i=0}^{n-1} M\langle x_i, X\backslash\{x_{i+1}\}\rangle.$$

Let α_i be minimal such that $x_i \in U_{\alpha_i}$. We may assume that one of the following two cases occurs:

(1) There is an α such that $\alpha = \alpha_i$ for every i.

(2) $\alpha_0 > \alpha_1$.

Suppose case (1). For each $i = 0, \ldots, n-1$, $x \in M_\alpha\langle x_i, U_\alpha\backslash\{x_{i+1}\}\rangle$. But $x_0, \ldots, x_{n-1}$ are distinct elements of U_α, so this contradicts the fact that $M_\alpha\langle\cdot,\cdot\rangle$ is an acyclic monotonically normal operator.

Suppose case (2). Observe that

$$x \in M\langle x_0, X\backslash\{x_1\}\rangle \subseteq V\langle x_0, U_{\alpha_0}\backslash\{x_1\}\rangle \subseteq V\langle x_0, X\backslash\{x_1\}\rangle.$$

However, $\alpha_0 > \alpha_1$ and so $x_0 \notin U_{\alpha_1}$. Thus

$$x \in M\langle x_1, X\backslash\{x_2\}\rangle \subseteq V\langle x_1, U_{\alpha_1}\backslash\{x_2\}\rangle \subseteq V\langle x_1, X\backslash\{x_0\}\rangle.$$

But $V\langle x_0, X\backslash\{x_1\}\rangle \cap V\langle x_1, X\backslash\{x_0\}\rangle = \emptyset$ and so we have our contradiction. □

Our next lemma considers the situation where a monotonically normal space is the union of two acyclic monotonically normal subspaces, one of which is open. Its corollaries are used to derive a result concerning the structure of any counterexample to Conjecture 4.1 and a result concerning scattered monotonically normal spaces.

Lemma 4.3. *Suppose that X is a monotonically normal space such that $X = Y \cup Z$ where Y and Z are both acyclic monotonically normal spaces. In addition suppose that Y is open and that there is a function $k: \mathcal{T}_Z \to \mathcal{T}_X$ which satisfies*

(1) *$U \subseteq k(U)$ for each $U \in \mathcal{T}_Z$,*

(2) *$U \subseteq U' \Rightarrow k(U) \subseteq k(U')$ for each $U, U' \in \mathcal{T}_Z$, and*

(3) *$\bigcap_{i=0}^{n} U_i = \emptyset \Rightarrow \bigcap_{i=0}^{n} k(U_i) = \emptyset$ where $U_i \in \mathcal{T}_Z$ and $i = 0, \ldots, n$.*

Then X is an acyclic monotonically normal space.

Proof. Let $V\langle\cdot,\cdot\rangle$ be a monotonically normal operator on X and suppose $M_Y\langle\cdot,\cdot\rangle$ and $M_Z\langle\cdot,\cdot\rangle$ are acyclic monotonically normal operators on Y and Z respectively. Define, for each $x \in X$ and each open set U containing x

$$M\langle x, U\rangle = \begin{cases} M_Y\langle x, U \cap Y\rangle \cap V\langle x, Y\rangle, & \text{if } x \in Y, \\ k(M_Z\langle x, U \cap Z\rangle) \cap V\langle x, U\rangle, & \text{if } x \notin Y. \end{cases}$$

By a method similar to one employed in the proof of Theorem 4.2, it can be shown that $M\langle\cdot,\cdot\rangle$ is an acyclic monotonically normal operator on X. □

It should be noted that (1)–(3) above are the minimal conditions on the function k required for the proof of Lemma 4.3 to work. It turns out that if a space X is such that every subspace Z of X has a map $k:\mathcal{T}_Z\to\mathcal{T}_X$ satisfying conditions (1)–(3) of Lemma 4.3, then, essentially by [12, Proposition 1.5.6], X is a K_0-space. The converse of this proposition is also true and is mentioned in the proof of our first corollary of Lemma 4.3.

Corollary 4.4. *If X is a monotonically normal K_0-space and is the union of two acyclic monotonically normal subspaces, one of which is open, then X is acyclic monotonically normal.*

Proof. Suppose that Y and Z are acyclic monotonically normal subspaces of X such that $X=Y\cup Z$ and assume that Y is open. Let $k:\mathcal{T}_Z\to\mathcal{T}_X$ be a K_0-function. It is easily verified that k satisfies (1)–(3) of Lemma 4.3. Thus X is acyclic monotonically normal. □

Corollary 4.5. *If the monotonically normal space $X=Y\cup\{x\}$ where Y is an acyclic monotonically normal space, then X is acyclic monotonically normal.*

Proof. Observe that Y is open since X satisfies the T_1 separation axiom. Define $k:\mathcal{T}_{\{x\}}\to\mathcal{T}_X$ by

$$k(\{x\})=X,$$

$$k(\emptyset)=\emptyset.$$

Clearly k satisfies (1)–(3) of Lemma 4.3 and $\{x\}$ is an acyclic monotonically normal space. Therefore X is an acyclic monotonically normal space. □

We may now prove our result on the structure of any counterexample to Conjecture 4.1. It establishes that any counterexample contains a subspace which is a counterexample not just globally, but locally as well.

Theorem 4.6. *If the space X is a counterexample to Conjecture* 4.1, *then there is a nonempty subspace X' of X, every nonempty open subset of which is a counterexample.*

Proof. Let

$$Y=\{x\in X: x \text{ has an open neighbourhood which is an acyclic monotonically normal space}\}.$$

Clearly Y is open and by Theorem 4.2, Y is an acyclic monotonically normal space. Define $X'=X\setminus Y$ and observe that X' is nonempty. Suppose that $U\subseteq X'$ is open in X' and has an acyclic monotonically normal operator. By Corollary 4.4, since $Y\cup U$ is a monotonically normal K_0-space, $Y\cup U$ is an acyclic monotonically normal space. However, $Y\cup U=X\setminus(X'\setminus U)$ and $X'\setminus U$ is closed in X. Thus by the definition of Y, $Y\cup U\subseteq Y$, hence $U=\emptyset$. Therefore every nonempty open subset of X' is a counterexample to Conjecture 4.1. □

Clearly Lemma 4.3 is fundamental to the proof of the above theorem. Hence, by the note following Lemma 4.3, the requirement that X be a K_0-space is essential to the proof.

A corollary of Theorem 4.6 is that every scattered, monotonically normal K_0-space is acyclic monotonically normal. One simply observes that every nonempty subspace X' of X contains a point x such that $\{x\}$ is open in X', and finite spaces are always acyclic monotonically normal. However, this result may be deduced for non-K_0-spaces by applying the method of proof as before, but using Corollary 4.5 rather than Corollary 4.4. Thus we have the following result.

Theorem 4.7. *Every scattered, monotonically normal space is acyclic monotonically normal.*

Notice that if a family of closed sets $\{C_\alpha : \alpha < \kappa\}$ dominates the space X, then $\bigcup_{\beta<\alpha} C_\beta$ is closed for each $\alpha < \kappa$. Hence our next theorem can be seen to be a generalisation of Theorem 3.3, although here we must assume that our space is monotonically normal and also satisfies a condition weaker than being a K_0-space.

Theorem 4.8. *Suppose that X is a monotonically normal space, that κ is a cardinal and $\{C_\alpha : \alpha < \kappa\}$ is a cover of X such that*

(i) *each C_α is acyclic monotonically normal,*

(ii) *there is a K_0-function from $\mathscr{T}_{C_\alpha}$ to $\mathscr{T}_X$ for each α, and*

(iii) *$\bigcup_{\beta<\alpha} C_\beta$ is closed for each α.*

Then X is acyclic monotonically normal.

Proof. Let $V\langle\cdot,\cdot\rangle$ be a monotonically normal operator on X, let $M_\alpha\langle\cdot,\cdot\rangle$ be an acyclic monotonically normal operator on C_α and let $k_\alpha : \mathscr{T}_{C_\alpha} \to \mathscr{T}_X$ be a K_0-function. Suppose $x \in X$, and let α be the least $\alpha < \kappa$ for which $x \in C_\alpha$. Whenever U is an open neighbourhood of x define

$$M\langle x, U\rangle = V\left\langle x, U \setminus \bigcup_{\beta<\alpha} C_\beta \right\rangle \cap k_\alpha(M_\alpha\langle x, U \cap C_\alpha\rangle).$$

We shall show that $M\langle\cdot,\cdot\rangle$ is the required acyclic monotonically normal operator. First observe that if $x \in U \subseteq U'$, where U and U' are open, then $M\langle x, U\rangle$ is an open neighbourhood of x and $M\langle x, U\rangle \subseteq M\langle x, U'\rangle$. So suppose that $x_0, \ldots, x_{n-1}$ are distinct elements of X and $x_n = x_0$. For each i, let α_i be the least $\alpha < \kappa$ for which $x_i \in C_\alpha$. Without loss of generality we need only consider the following two cases:

(1) There is an α such that $\alpha = \alpha_i$ for every i.

(2) $\alpha_0 < \alpha_1$.

Suppose case (1). For each i,

$$M\langle x_i, X\setminus\{x_{i+1}\}\rangle \subseteq k_\alpha(M_\alpha\langle x_i, C_\alpha\setminus\{x_{i+1}\}\rangle),$$

and thus $\bigcap_{i=0}^{n-1} M\langle x_i, X\backslash\{x_{i+1}\}\rangle = \emptyset$ since $M_\alpha\langle\cdot,\cdot\rangle$ is an acyclic monotonically normal operator and k_α is a K_0-function.

Suppose case (2). Notice that

$$M\langle x_0, X\backslash\{x_1\}\rangle \subseteq V\langle x_0, X\backslash\{x_1\}\rangle,$$

and that

$$M\langle x_1, X\backslash\{x_2\}\rangle \subseteq V\left\langle x_1, X\Bigg\backslash\bigcup_{\beta<\alpha_1} C_\beta\right\rangle \subseteq V\langle x_1, X\backslash\{x_0\}\rangle,$$

since $\alpha_0 < \alpha_1$ and $x_0 \in C_{\alpha_0}$. Hence

$$M\langle x_0, X\backslash\{x_1\}\rangle \cap M\langle x_1, X\backslash\{x_2\}\rangle = \emptyset,$$

and thus $\bigcap_{i=0}^{n-1} M\langle x_i, X\backslash\{x_{i+1}\}\rangle = \emptyset$.

Therefore $M\langle\cdot,\cdot\rangle$ is an acyclic monotonically normal operator on X. □

A corollary of Theorem 4.8 is that every monotonically normal space which is countable is acyclic monotonically normal. Simply enumerate X as $\{x_n : n \in \mathbb{N}\}$ and define $C_n = \{x_n\}$ for each n. Alternatively, observe that every countable space is semistratifiable [6]. Thus every monotonically normal space which is countable is stratifiable [7] and hence acyclic monotonically normal by Theorem 1.2.

We finish by demonstrating that the addition of a simple condition to the definition of a K_0-space yields a property which is equivalent to acyclic monotone normality. The additional condition is given below.

A space X is monotonically K_0 if for every subspace Y of X there is a function $k_Y : \mathcal{T}_Y \to \mathcal{T}_X$ such that k_Y is a K_0-function and

(4) If $U \subseteq U'$ and $(Y\backslash U) \supseteq (Y'\backslash U')$, then $k_Y(U) \subseteq k_{Y'}(U')$, where U, U' are open in Y, Y' respectively.

Observe that by setting $A = U$, $A' = U'$, $B = Y\backslash U$, $B' = Y'\backslash U'$, $H(A, B) = k_Y(U)$ and $H(A', B') = k_{Y'}(U')$ condition (4) can be written as

$$\text{if } A \subseteq A' \text{ and } B \supseteq B', \quad \text{then } H(A, B) \subseteq H(A', B'),$$

which is precisely the monotonic part of the version of monotone normality for pairs of disjoint closed sets [7].

Theorem 4.9. *The following are equivalent for a space X.*

(i) *X has $\mathcal{W}$ satisfying chain* (F).

(ii) *X is acyclic monotonically normal.*

(iii) *X is monotonically* K_0.

Proof. The equivalence of (i) and (ii) was established in [10]. We show that (i) implies (iii). Let $V\langle\cdot,\cdot\rangle$ be the operator associated with chain (F) and define for each open subset U of X and each subset A of U

$$H\langle A, U\rangle = \{x \in X : \exists\, W \in W(x), (W \subseteq U) \wedge (W \cap A \neq \emptyset)\}.$$

Now suppose that Y is a subspace of X and define k_Y from $\mathcal{T}_Y$ to $\mathcal{T}_X$ by

$$k_Y(U) = H\langle U, X\backslash(\overline{Y\backslash U})\rangle^\circ.$$

It was shown in [10] that k_Y is a K_0-function. Suppose that $U \subseteq U'$ and $(Y\backslash U) \supseteq (Y'\backslash U')$, where U, U' are open in Y, Y' respectively. Suppose that $x \in k_Y(U)$, so there is a $W \in W(x)$ for which

$$W \subseteq X\backslash(\overline{Y\backslash U}) \quad \text{and} \quad W \cap U \neq \emptyset.$$

Hence $W \subseteq X\backslash(\overline{Y'\backslash U'})$, since $(Y\backslash U) \supseteq (Y'\backslash U')$. Also $W \cap U' \neq \emptyset$, since $U \subseteq U'$. Thus $x \in H\langle U', X\backslash(\overline{Y'\backslash U'})\rangle$ and so $k_Y(U) \subseteq k_{Y'}(U')$, because $k_Y(U)$ is open. Therefore X is monotonically K_0.

We shall now show that (iii) implies (ii). Define for each x in an open set U,

$$M\langle x, U\rangle = k_{\{x\}\cup(X\backslash U)}(\{x\}).$$

Observe that $M\langle x,U\rangle$ is an open neighbourhood of x. Furthermore, if $x \in U \subseteq U'$, then setting $Y = \{x\} \cup (X\backslash U)$ and $Y' = \{x\} \cup (X\backslash U')$, we have that $Y\backslash\{x\} \supseteq Y'\backslash\{x\}$ and thus

$$M\langle x, U\rangle = k_Y(\{x\}) \subseteq k_{Y'}(\{x\}) = M\langle x, U'\rangle.$$

It remains to show that $M\langle\cdot,\cdot\rangle$ is acyclic. So let $x_0, \ldots, x_{n-1}$ be distinct elements of X and $x_n = x_0$. For each i, $M\langle x_i, X\backslash\{x_{i+1}\}\rangle = k_{\{x_i,x_{i+1}\}}(\{x_i\})$. If we let $D = \{x_0, \ldots, x_{n-1}\}$, then

$$k_{\{x_i,x_{i+1}\}}(\{x_i\}) \subseteq k_D(D\backslash\{x_{i+1}\}).$$

Thus

$$\begin{aligned}\bigcap_{i=0}^{n-1} M\langle x_i, X\backslash\{x_{i+1}\}\rangle &\subseteq \bigcap_{i=0}^{n-1} k_D(D\backslash\{x_{i+1}\}) \\ &= k_D\left(\bigcap_{i=0}^{n-1}(D\backslash\{x_{i+1}\})\right) \\ &= k_D(\emptyset) = \emptyset.\end{aligned}$$

Hence $M\langle\cdot,\cdot\rangle$ is an acyclic monotonically normal operator on X. $\square$

Observe that the above proof of "(iii) implies (ii)" only required the k_D had all the monotonically K_0 properties for finite D. In fact it can be shown that a space X is acyclic monotonically normal if and only if there is a monotonically normal operator $H\langle\cdot,\cdot\rangle$ on disjoint closed sets [7], such that if D is a finite subset of X, then

$$k_D(Y) = H\langle Y, D\backslash Y\rangle$$

is a K_0-function from $\mathcal{T}_D$ to $\mathcal{T}_X$.

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