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Star covering properties

E.K. van Douwen

G.M. Reed

St. Edmund Hall, Oxford OX1 4AR, UK

A.W. Roscoe

University College, Oxford OX1 4BH, UK

I.J. Tree*

Magdalen College, Oxford OX1 4AU, UK

Abstract

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In this paper, the authors investigate starcompact properties between countable compactness and the discrete finite chain condition (i.e., pseudocompactness), and star-Lindelöf properties between the Lindelöf property and the discrete countable chain condition (i.e., the pseudo-Lindelöf property). This work represents a unification and extension of concepts previously studied by several authors in the literature. Theory is developed to establish connections between the various star properties and other covering conditions, and a large collection of nontrivial examples is given to make distinctions.

Keywords: Moore space, countably compact Lindelöf, discrete finite chain conditions, discrete countable chain conditions, starcompact, star-Lindelöf, pseudocompact.

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1. Introduction

Several questions concerning chain conditions in Moore spaces were raised in [21]. Most of these questions were answered in [5]. However, van Douwen and Reed were unable to decide if there exists a 3-separable (equals DCCC) Moore space which was not 2-separable (see Section 3.2 for details). The following definitions arose in the attempt to analyse this question.

Recall that if $B \subseteq X$ and $\mathscr{H}$ is a collection of subsets of X, then $\mathrm{ST}^1(B, \mathscr{H}) = \{H \in \mathscr{H}: H \cap B \neq \emptyset\}$ and $\mathrm{st}^1(B, \mathscr{H}) = \bigcup \mathrm{ST}^1(B, \mathscr{H})$. Inductively $\mathrm{ST}^{n+1}(B, \mathscr{H}) = \{H \in \mathscr{H}: H \cap \mathrm{st}^n(B, \mathscr{H}) \neq \emptyset\}$ and $\mathrm{st}^{n+1}(B, \mathscr{H}) = \bigcup \mathrm{ST}^{n+1}(B, \mathscr{H})$. For brevity we will write $\mathrm{ST}(B, \mathscr{H})$ for $\mathrm{ST}^1(B, \mathscr{H})$, $\mathrm{st}(B, \mathscr{H})$ for $\mathrm{st}^1(B, \mathscr{H})$ and $\mathrm{st}(x, \mathscr{H})$ for $\mathrm{st}(\{x\}, \mathscr{H})$. Fix $n \in \mathbb{N}^+$, the set of strictly positive integers.

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Definition 1.1. A space X is said to be *n-starcompact* if for every open cover $\mathcal{U}$ of X, there is some finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$.

Definition 1.2. A space X is said to be *strongly n-starcompact* if for every open cover $\mathcal{U}$ of X, there is some finite subset B of X such that $\mathrm{st}^n(B, \mathcal{U}) = X$.

Definition 1.3. A space X is said to be *n-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there is some countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$.

Definition 1.4. A space X is said to be *strongly n-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there is some countable subset B of X such that $\mathrm{st}^n(B, \mathcal{U}) = X$.

It is easy to see that if X is strongly n-starcompact, then X is n-starcompact, and if X is n-starcompact, then X is strongly $n+1$-starcompact. A similar hierarchy holds for the star-Lindelöf properties. For T_3 spaces, strongly 1-starcompact equals countably compact, n-starcompact equals the DFCC (the discrete finite chain condition) for $n \geqslant 2$, and n-star-Lindelöf equals the DCCC (the discrete countable chain condition) for $n \geqslant 2$. For Moore spaces, strongly 1-starcompact equals compact and metrizable, n-starcompact implies separable for $n \geqslant 1$, and 1-star-Lindelöf equals separable. Hence for Moore spaces, the star properties form a chain of implications from compact and metrizable to the DFCC and from separable to the DCCC. Proof of these facts will be given in due course. Recall that a completely regular space is pseudocompact if and only if it has the DFCC.

The existence of a 3-separable Moore space which is not 2-separable is equivalent to the existence of a 2-star-Lindelöf Moore space which is not strongly 2-star-Lindelöf. In investigating the above properties in 1984, van Douwen and Reed were able to give examples to show that all but three of the possibly distinct classes in Moore spaces were in fact distinct. These three were (1) those in the original question (i.e., 2-star-Lindelöf and strongly 2-star-Lindelöf), (2) strongly 1-starcompact and 1-starcompact, and (3) 2-starcompact and strong 2-starcompact.

The second author finally found a Moore space to answer (1) in 1989 and presented it at the Oxford Topology Symposium. Actually, it was a space he had constructed in 1987 as an example of a DCCC Moore space with a σ-locally countable base (hence σ-para-Lindelöf) which is not separable. Now, with the aid of the last two authors, the lattice of implications for Moore spaces is complete. In fact, for Moore spaces the properties of (2) are equivalent under CH (or $\mathfrak{d} = \mathfrak{c}$), and the properties of (3) are equivalent in ZFC.

In this paper, we present not only the study of the above properties in Moore spaces, but we also explore their relationships in more general spaces (e.g. first countable, regular, Hausdorff, etc.). We show that the equivalence of (2) and (3) above for Moore spaces does not hold in the class of regular first countable spaces. In particular, we show that the Tychonoff plank is 1-starcompact but not strongly 1-starcompact, and under the assumption ($\mathfrak{d} = \omega_1$), produce such an example which is both regular and first countable. Under CH we exhibit a regular first countable 2-starcompact space which is not strongly 2-starcompact.

We consider also the extension to ω-starcompact and ω-star-Lindelöf properties.

Definition 1.5. A space X is said to be *ω-starcompact* if for every open cover $\mathcal{U}$ of X, there is some $n \in \mathbb{N}^+$ and some finite subset B of X such that $\mathrm{st}^n(B, \mathcal{U}) = X$.

Definition 1.6. A space X is said to be *ω-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there is some $n \in \mathbb{N}^+$ and some countable subset B of X such that $\mathrm{st}^n(B, \mathcal{U}) = X$.

This paper represents a unification and expansion of concepts already studied on several occasions under different terminology. Fleischman introduced the concept of strong starcompactness in [7]. Later, Sarkhel in [23] extended his work, and defined the concepts of n-starcompactness and ω-starcompactness. Matveev defined k-pseudocompactness in [15] which extended Fleischman's work to strong n-starcompactness. Furthermore, it has recently come to the authors' attention that the strong star-Lindelöf conditions have been studied by Ikenaga, in [11, 12], who named them ω-n-star spaces. It is clear that the equivalence (in our terminology) of strongly 1-starcompact and countably compact and the equivalence of strongly 3-starcompact and pseudocompact were known by some of these authors. In addition, Scott Williams has informed the authors that he had independently obtained certain of our lemmas about the weight of starcompact Moore spaces in unpublished work.

Organisation of the paper

In Section 2, we present the study of the starcompactness properties, and in Section 3, we consider the star-Lindelöf properties. Within each section, we first present the positive implications between the various properties in the context of the weakest separation. Secondly, we explore the relationships for Moore spaces. The rich structure of Moore spaces provides the equivalence of certain of these properties, and it ensures nontrivial counterexamples where the properties are not equivalent. Thirdly, we consider examples in regular spaces, Hausdorff spaces, and first countable spaces which distinguish properties that are equivalent in Moore spaces. Finally, we consider more general issues about the properties in question.

2. Starcompactness

2.1. General positive implications

The results in this section follow directly from work in [1, 7, 15, 23]. For completeness, we present proofs in our current terminology. Note that unless otherwise specified, space means simply topological space. We use the term regular to include T_1.

The following trivial lemmas instantly set up a hierarchy amongst the starcompactness properties.

Lemma 2.1.1. *If X is strongly n-starcompact, then X is n-starcompact.*

Proof. If $\mathscr{U}$ is an open cover of X, by hypothesis there is a finite subset B of X such that $\mathrm{st}^n(B, \mathscr{U}) = X$. For each $b \in B$ select some $U_b \in \mathscr{U}$ such that $b \in U_b$. Let $\mathscr{V} = \{U_b\colon b \in B\}$. So $\mathscr{V}$ is a finite subset of $\mathscr{U}$ and

$$X = \mathrm{st}^n(B, \mathscr{U}) \subseteq \mathrm{st}^n(\bigcup \mathscr{V}, \mathscr{U}) \subseteq X. \qquad \square$$

A similar proof shows:

Lemma 2.1.2. *If X is n-starcompact, then X is strongly $n+1$-starcompact.*

Also, using this style of argument, it is clear that we could take the following as an alternative, though equivalent, definition of ω-starcompact (see [23]).

Definition 2.1.3. A space X is said to be ω-starcompact if for every open cover $\mathscr{U}$ of X, there is some $n \in \mathbb{N}^+$ and some finite subset $\mathscr{V}$ of $\mathscr{U}$ such that $\mathrm{st}^n(\bigcup \mathscr{V}, \mathscr{U}) = X$.

Obviously, these covering properties are all weakenings of compactness. In fact, they all lie between countable compactness and pseudocompactness, as we will shortly see.

The following two theorems are from [7], although the proof of Theorem 2.1.5 for Hausdorff spaces was omitted. Together, they show that for Hausdorff spaces countable compactness and strong 1-starcompactness are equivalent.

Theorem 2.1.4. *Every countably compact space is strongly 1-starcompact.*

Proof. Suppose X is a countably compact space which is not strongly 1-starcompact. Let $\mathscr{U}$ be an open cover such that if $B \subseteq X$ is finite, then $\mathrm{st}(B, \mathscr{U}) \neq X$ $(*)$. Pick any $x_0 \in X$ and, inductively, pick $x_n \in X - \mathrm{st}(\{x_0, x_1, \ldots, x_{n-1}\}, \mathscr{U})$ for $n > 0$, which is possible by $(*)$. Let $A = \{x_n\colon n \in \mathbb{N}\}$ and $\mathscr{V} = \{\mathrm{st}(x_n, \mathscr{U})\colon n \in \mathbb{N}\}$. Note that by the choice of the x_n, every member of $\mathscr{V}$ contains precisely one element of A. Consequently, no finite subset of $\mathscr{V}$ will cover A.

If $y \in \bar{A}$, pick some open $U \in \mathscr{U}$ such that $y \in U$ ($\mathscr{U}$ covers X). As $U \cap \bar{A} \neq \emptyset$, $U \cap A \neq \emptyset$ and hence $y \in \mathrm{st}(x_n, \mathscr{U})$ for some n. Therefore we see that $\mathscr{V}$ is a countable covering of $\bar{A}$ by sets open in X. $\bar{A}$ is countably compact, being a closed subset of X. Therefore there must exist a finite subset of $\mathscr{V}$ which covers $\bar{A}$ and hence A. This contradicts our previous observation about $\mathscr{V}$. $\square$

Theorem 2.1.5. *Strongly* 1*-starcompact Hausdorff spaces are countably compact.*

Proof. Suppose that X is a Hausdorff space that is not countably compact. Then there exists $D=\{x_n: n\in\mathbb{N}\}\subseteq X$, an infinite closed discrete subset. As D is discrete, for each n there exists an open set U_n such that $U_n\cap D=\{x_n\}$.

For every $m\in\mathbb{N}$, define $Y_m=\{x_n\in D: 2^m\leqslant n<2^{m+1}\}$, so $|Y_m|=2^m-1$. Since Y_m is finite and X is Hausdorff, there exist disjoint open sets $V_n\ni x_n$ for $2^m\leqslant n<2^{m+1}$.

Next, setting $\mathcal{V}_m=\{U_n\cap V_n: 2^m\leqslant n<2^{m+1}\}$ gives a collection of pairwise disjoint open subsets of X such that $(U_n\cap V_n)\cap D=\{x_n\}$.

Define $\mathcal{V}=\{X-D\}\cup\bigcup_{m\in\mathbb{N}}\mathcal{V}_m$. Evidently, $\mathcal{V}$ is an open cover of X.

Let A be any finite subset of X, with $|A|=M$, say. Then $|A|<2^M-1=|\mathcal{V}_M|$. So, for some $2^M\leqslant m<2^{M+1}-1$, $(U_m\cap V_m)\cap A=\emptyset$. But $U_m\cap V_m$ is the only member of $\mathcal{V}$ which contains x_m. Thus, $x_m\notin \mathrm{st}(A,\mathcal{V})$. Specifically, $\mathrm{st}(A,\mathcal{V})\neq X$. But A was an arbitrary finite subset of X, so X is not strongly 1-starcompact. □

The next three theorems show that for Tychonoff spaces, ω-starcompact spaces are pseudocompact, and that for regular spaces, 2-starcompactness, ω-starcompactness (together with all the properties in between), and the DFCC are equivalent conditions. These results can be found under different terminology in [1, 7, 15, 23].

Theorem 2.1.6. *Every ω-starcompact space X has the property that every continuous real-valued function on X is bounded.*

Proof. Suppose that X is ω-starcompact and that $f:X\to\mathbb{R}$ is continuous. Define $\mathcal{U}=\{f^{-1}(k,k+2): k\in\mathbb{Z}\}$. Then $\mathcal{U}$ is an open cover of X and for some $n\in\mathbb{N}^+$ and for some finite $\mathcal{V}\subseteq\mathcal{U}$, $\mathrm{st}^n(\bigcup\mathcal{V},\mathcal{U})=X$. Let $M=\max\{k+2: f^{-1}(k,k+2)\in\mathcal{V}\}$ and $m=\min\{k: f^{-1}(k,k+2)\in\mathcal{V}\}$.

It is now clear that $f(X)\subseteq(m-2n, M+2n)$. For if $x\in X$, then for $1\leqslant j\leqslant n$ there are $f^{-1}(k_j,k_j+2)\in\mathcal{U}$ such that $x\in f^{-1}(k_n,k_n+2)$ with $f^{-1}(k_j,k_j+2)\cap f^{-1}(k_{j+1},k_{j+1}+2)\neq\emptyset$ and $f^{-1}(k_1,k_1+2)\cap\bigcup\mathcal{V}\neq\emptyset$. By construction, $f(\bigcup\mathcal{V})\subseteq(m,M)$. An easy induction now shows that $f(x)\in(m-2n,M+2n)$, as required. □

Theorem 2.1.7. *If X is DFCC, then X is* 2*-starcompact.*

Proof. Suppose that X is not 2-starcompact and $\mathcal{U}$ is an open cover such that if $\mathcal{V}\subseteq\mathcal{U}$ is finite, then $\mathrm{st}^2(\bigcup\mathcal{V},\mathcal{U})\neq X$ (*).

Pick $U_0\in\mathcal{U}$ and define $\mathcal{V}_0=\{U_0\}$. Suppose inductively that we have defined $\mathcal{V}_k$, a discrete collection of k members of $\mathcal{U}$ such that $\mathcal{V}_{k-1}\subseteq\mathcal{V}_k$ for $1\leqslant k<n$. By (*), $\mathrm{st}^2(\bigcup\mathcal{V}_{n-1},\mathcal{U})\neq X$. Pick $x_n\in X-\mathrm{st}^2(\bigcup\mathcal{V}_{n-1},\mathcal{U})$ and a $U_n\in\mathcal{U}$ such that $x_n\in U_n$. Let $\mathcal{V}_n=\mathcal{V}_{n-1}\cup\{U_n\}$.

We claim that $\mathcal{V}_n$ is discrete. Let $y\in X$ and select any $V\in\mathcal{U}$ such that $y\in V$. If there were distinct $U, U'\in\mathcal{V}_n$ such that $V\cap U\neq\emptyset$ and $V\cap U'\neq\emptyset$ (say $U=U_{n_1}$ and $U'=U_{n_2}$ with $n_1<n_2$), then

$$x_{n_2}\in U'\subseteq\mathrm{st}^2(\bigcup\mathcal{V}_{n_1},\mathcal{U})\subseteq\mathrm{st}^2(\bigcup\mathcal{V}_{n_2-1},\mathcal{U}).$$

But this contradicts the choice of x_{n_2}. So for every $y \in X$ there is an open V containing y which meets at most one element of $\mathcal{V}_n$. Hence claim.

Defining $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$, a similar argument shows that $\mathcal{V}$ is a countably infinite discrete collection of open sets. Therefore X is not DFCC and the result follows. $\square$

Theorem 2.1.8. *If X is regular and ω-starcompact, then X is DFCC.*

Proof. Suppose X is not DFCC and $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a discrete collection of nonempty open sets; say $x_n \in U_n$ for $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. Applying the regularity condition n times yields

$$x_n \in A_1^{(n)} \subseteq \overline{A_1^{(n)}} \subseteq \cdots \subseteq A_m^{(n)} \subseteq \overline{A_m^{(n)}} \subseteq \cdots \subseteq A_n^{(n)} \subseteq \overline{A_n^{(n)}} \subseteq U_n,$$

where each $A_m^{(n)}$ is open in X. Define

$$W = X - \bigcup_{n \in \mathbb{N}} \overline{A_n^{(n)}},$$

$$\begin{aligned}
V_1^{(n)} &= A_2^{(n)}, \\
V_2^{(n)} &= A_3^{(n)} - \overline{A_1^{(n)}}, \\
&\vdots \\
V_{n-1}^{(n)} &= A_n^{(n)} - \overline{A_{n-2}^{(n)}}, \\
V_n^{(n)} &= U_n - \overline{A_{n-1}^{(n)}}.
\end{aligned}$$

Note that each $V_m^{(n)} \subseteq U_n$, $x_n \in V_m^{(n)}$ if and only if $m = 1$ and $V_l^{(n)} \cap V_m^{(n)} \neq \emptyset$ implies that $|l - m| \leq 1$. Note also that $\{\overline{A_n^{(n)}} : n \in \mathbb{N}\}$ is a discrete collection since $\mathcal{U}$ is discrete and $\overline{A_n^{(n)}} \subseteq U_n$. Hence W is open and so too is $V_m^{(n)}$ for all $m \leq n$ and $n \in \mathbb{N}$.

Let $\mathcal{V} = \{W\} \cup \{V_m^{(n)} : m \leq n,\ n \in \mathbb{N}\}$, a collection of sets open in X.

Claim 1. *$\mathcal{V}$ covers X.* Since $\overline{A_m^{(n)}} \subseteq A_{m+1}^{(n)}$, it is clear that $\bigcup_{m=1}^{n} V_m^{(n)} = U_n$. Furthermore, $X - \bigcup_{n \in \mathbb{N}} U_n \subseteq X - \bigcup_{n \in \mathbb{N}} \overline{A_n^{(n)}} = W$. So for $x \in X$, either $x \in U_n$ for some n, and hence $x \in V_m^{(n)}$ for some m and n, or $x \notin \bigcup_{n \in \mathbb{N}} U_n$, in which case $x \in W$.

Claim 2. *$\mathcal{V}$ witnesses that X is not ω-starcompact.* Let $B \subseteq X$ be finite and let $n \in \mathbb{N}^+$. Because B is finite and $\mathcal{U}$ is infinite, there exist infinitely many U_n in $\mathcal{U}$ such that $U_n \cap B = \emptyset$. Thus, we can pick $N > n$ such that $U_N \cap B = \emptyset$. By the remarks made earlier, it is easily seen that $\mathrm{st}(x_N, \mathcal{V}) = V_1^{(N)} \subseteq U_N$ and more generally, $\mathrm{st}^m(x_N, \mathcal{V}) \subseteq \bigcup_{l=1}^{m} V_l^{(N)} \subseteq U_N$ for $m \leq N$. In particular, $\mathrm{st}^n(x_N, \mathcal{V}) \subseteq U_N$, so $x_N \notin \mathrm{st}^n(B, \mathcal{V})$. B and n were arbitrary, so X cannot be ω-starcompact. $\square$

Remark 2.1.9. The above results complete Fig. 1 and give us some simple connections between our initial definitions. Furthermore, it is well known that pseudocompact spaces are DFCC and that (pseudo)normal pseudocompact spaces are countably compact (see [6, Section 3.10]). So these properties are all equivalent in pseudonormal spaces.

$$
\begin{tikzcd}
\text{countably compact} \arrow[d, shift right] & \\
\text{strongly 1-starcompact} \arrow[u, shift right, "T_2"'] \arrow[d] & \\
\text{1-starcompact} \arrow[d] & \\
\text{strongly 2-starcompact} \arrow[d] & \\
\text{2-starcompact} \arrow[d] & \text{DFCC} \arrow[l] \\
\vdots \arrow[d] & \\
\text{strongly } n\text{-starcompact} \arrow[d] & \\
n\text{-starcompact} \arrow[d] & \\
\text{strongly } n+1\text{-starcompact} \arrow[d] & \\
\vdots \arrow[d] & \\
\omega\text{-starcompact} \arrow[uuuuuur, "\text{regular}"'] \arrow[d, leftrightarrow, "T_{3\frac{1}{2}}"] & \\
\text{pseudocompact} &
\end{tikzcd}
$$

Fig. 1.

2.2. Moore spaces

As noted in the last section, for T_2 spaces, countable compactness is equivalent to strong 1-starcompactness. Hence, from [17], each strongly 1-starcompact Moore space is compact and metrizable. Also we know that for regular spaces, 2-starcompactness, ω-starcompactness, and the DFCC are equivalent, and for completely regular spaces, these are equivalent to pseudocompactness. Hence, for Moore spaces, we need only to establish the relationships between 2-starcompactness and strong 2-starcompactness, between strong 2-starcompactness and 1-starcompactness, and between 1-starcompactness and compactness.

Lemma 2.2.1 [21]. *Each* 2*-starcompact Moore space is separable.*

Lemma 2.2.2. *Every regular* 2*-starcompact separable space X is strongly* 2*-starcompact.*

Proof. In regular spaces, an equivalent of the DFCC property (and hence 2-starcompactness) is that every countable open cover has a finite subset whose union is dense [1]. Let D be a countable dense subset of X and $\mathcal{U}$ any open cover. The collection of open sets $\{\mathrm{st}(d, \mathcal{U}) \colon d \in D\}$ is a countable open cover because D is both countable and dense. Hence there is a finite $B \subseteq D$ such that $\mathrm{st}(B, \mathcal{U})$ is dense in X and so $\mathrm{st}^2(B, \mathcal{U}) = X$. □

Hence, we have the following.

Theorem 2.2.3. *Each 2-starcompact Moore space is strongly 2-starcompact.*

Lemma 2.2.4. *If X is a regular space which has a closed discrete subset D such that $|D| = w(X) \geq \omega$, where $w(X)$ is the weight of X, then X cannot be 1-starcompact.*

Proof. Let $\mathcal{B}$ be a basis for X with minimum cardinality, so $|\mathcal{B}| = |D|$. Because X is regular, for each $x \in D$ there is a basic $B_x \in \mathcal{B}$ such that $x \in B_x$ and $\overline{B_x} \cap D = \{x\}$. Furthermore, for each $y \in X - D$ there is a basic $V_y \ni y$ such that $\overline{V_y} \cap D = \emptyset$. Let $\mathcal{U} = \{B_x : x \in D\} \cup \{V_y : y \in X - D\}$. Therefore,

$$|\mathcal{B}| \geq |\mathcal{U}| \geq |\{B_x : x \in D\}| = |D| = |\mathcal{B}|,$$

so $|\mathcal{U}| = |\mathcal{B}|$. Now let $\mathcal{F} = \{F \subseteq \mathcal{U} : F \text{ is a finite subset of } \mathcal{U}\}$. Hence, $|\mathcal{F}| = |\mathcal{U}| = |D|$, because $|\mathcal{U}| \geq \omega$. Pick $F \in \mathcal{F}$. Then $F = \{U_1, \ldots, U_n\}$, say, so that $\overline{\bigcup F} = \bigcup_{i=1}^n \overline{U_i}$. Therefore $|\overline{\bigcup F} \cap D| = |\bigcup_{i=1}^n (\overline{U_i} \cap D)| \leq n$. Hence for every $F \in \mathcal{F}$, $\overline{\bigcup F}$ meets D in at most finitely many points of X.

Enumerate $\mathcal{F}$ as $\{F_\alpha : \alpha < \kappa\}$. Suppose for each $\beta < \alpha$ we have defined some $x_\beta \in D$ such that $x_\beta \notin \overline{\bigcup F_\beta}$ and $x_\beta \neq x_\gamma$ for $\beta \neq \gamma < \alpha < \kappa$.

$$D - (\{x_\beta : \beta < \alpha\} \cup \overline{\bigcup F_\alpha}) \neq \emptyset \quad \text{as } D \cap \overline{\bigcup F_\alpha} \text{ is finite,}$$

so we can pick $x_\alpha \in D - (\{x_\beta : \beta < \alpha\} \cup \overline{\bigcup F_\alpha})$ and $x_\alpha \neq x_\beta$ for all $\beta < \alpha$.

For each $x \in D$, define

$$U_x = \begin{cases} B_x \cap (X - \overline{\bigcup F_\alpha}), & \text{if } x = x_\alpha \text{ for some } \alpha < \kappa, \\ B_x, & \text{otherwise.} \end{cases}$$

In either case, U_x is open and contains x. Let $\mathcal{U}' = \{U_x : x \in D\} \cup \{V_y : y \in X - D\}$, so $\mathcal{U}'$ is an open cover of X. If $G \subseteq \mathcal{U}'$ is finite, then $G = \{U_{x_1}, \ldots, U_{x_n}\} \cup \{V_{y_1}, \ldots, V_{y_m}\}$. Let $F = \{B_{x_1}, \ldots, B_{x_n}\} \cup \{V_{y_1}, \ldots, V_{y_m}\}$, so $F \in \mathcal{F}$ and $\bigcup G \subseteq \bigcup F$. For some $\alpha < \kappa$, $F = F_\alpha$, so that $x_\alpha \notin \overline{\bigcup F_\alpha}$. We observe that

$$x_\alpha \in \mathrm{st}(\bigcup G, \mathcal{U}') \quad \text{if and only if} \quad U_{x_\alpha} \cap \bigcup G \neq \emptyset,$$

since U_{x_α} is the only element of $\mathcal{U}'$ containing x_α. But $U_{x_\alpha} = B_{x_\alpha} - \overline{\bigcup F_\alpha}$, so $U_{x_\alpha} \cap \bigcup G \subseteq U_{x_\alpha} \cap \bigcup F_\alpha = \emptyset$ and hence $x_\alpha \notin \mathrm{st}(\bigcup G, \mathcal{U}')$.

Thus we have constructed an open cover $\mathcal{U}'$ of X and shown that if $G \subseteq \mathcal{U}'$ is finite, then $\mathrm{st}(\bigcup G, \mathcal{U}') \neq X$. Hence X cannot be 1-starcompact. □

Example 2.2.5. There exists a locally compact, strongly 2-starcompact Moore space (Ψ) which is not 1-starcompact.

Proof. Let $\{N_s: s \in S\}$, where $\mathbb{N} \cap S = \emptyset$, be an infinite family of infinite subsets of $\mathbb{N}$ such that the intersection $N_s \cap N_{s'}$ is finite for every pair s, s' of distinct elements of S and that $\{N_s: s \in S\}$ is maximal with respect to this property.

Generate a topology on the set $X = \mathbb{N} \cup S$ by the neighbourhood system $\{\mathcal{B}(x): x \in X\}$, where

$$\mathcal{B}(x) = \begin{cases} \{\{x\}\}, & \text{if } x \in \mathbb{N}, \\ \{\{x\} \cup (N_x - \{0, 1, \dots, n\}): n \in \mathbb{N}\}, & \text{if } x \in S. \end{cases}$$

This topology makes X a locally compact Moore space, and is the space Ψ described in [9, 18]. The set S is a closed discrete subset of X which has the same cardinality as the weight of X. So, by Lemma 2.2.4, X is not 1-starcompact.

To show that X is strongly 2-starcompact, it is sufficient to show that if $\mathcal{U}$ is any open cover of X, there is a finite subset B of $\mathbb{N}$ such that $\mathbb{N} \subseteq \mathrm{st}(B, \mathcal{U})$. This is because $\mathbb{N}$ is a dense subset of X.

Suppose to the contrary, that there is some basic open cover $\mathcal{U}$ such that if $B \subseteq \mathbb{N}$ is finite, then it is not the case that $\mathbb{N} \subseteq \mathrm{st}(B, \mathcal{U})$ (*). Let $x_0 = 0$ and $B_0 = \{x_0\}$. Suppose we have inductively defined distinct elements $x_0, x_1, \dots, x_n$ of $\mathbb{N}$, such that $x_i \notin \mathrm{st}(B_{i-1}, \mathcal{U})$, where $B_i = \{x_0, x_2, \dots, x_i\}$ for $1 \leq i \leq n$. Then by the property of $\mathcal{U}$, $A_n = \mathbb{N} - \mathrm{st}(B_n, \mathcal{U})$ must be infinite (for if it were finite, $A_n \cup B_n$ would be a finite subset of $\mathbb{N}$ and $\mathbb{N} \subseteq \mathrm{st}(A_n \cup B_n, \mathcal{U})$ contradicting (*)). Hence we may pick $x_{n+1} \in A_n$ greater than all the elements of B_n. As $x_{n+1} \in A_n$, $x_{n+1} \notin \mathrm{st}(B_n, \mathcal{U})$. Now, $B = \{x_n: n \in \mathbb{N}\}$ is an infinite subset of $\mathbb{N}$, so by maximality of S, there exists a limit point, s, of B with the property that if V is an open set containing s, then $|V \cap B| = \aleph_0$. As $\mathcal{U}$ covers X, pick some $U \in \mathcal{U}$ containing s. Let x_k, $x_{k'}$ be distinct elements of $U \cap B$, with $k < k'$. Then $x_{k'} \in U \subseteq \mathrm{st}(x_k, \mathcal{U})$. In particular, $x_{k'} \subseteq \mathrm{st}(B_{k'-1}, \mathcal{U})$, as $k \leq k' - 1$. This contradicts the original property of $\mathcal{U}$ and the result follows. □

Now, let us show that it is consistent that each 1-starcompact Moore space is compact.

Lemma 2.2.6. *If X is a Moore space such that $w(X)$ does not have countable cofinality, then there is a closed discrete subset D of X such that $|D| = w(X)$.*

Proof. For such a space X, there is an open covering $\mathcal{U} = \{U_\alpha: \alpha < \kappa\}$ such that $\kappa = w(X)$ and $U_\beta - \bigcup_{\alpha<\beta} U_\alpha \neq \emptyset$ for all $\beta < \kappa$.

[If this were not the case, we could arrange a development $\{\mathcal{G}_n: n \in \mathbb{N}\}$ with $|\mathcal{G}_n| \leq w(X)$ and covers $\mathcal{G}'_n = \{V_{n,\alpha}: \alpha < \lambda_n\} \subseteq \mathcal{G}_n$ such that $\lambda_n < \kappa$ and $V_{n,\beta} - \bigcup_{\alpha<\beta} V_{n,\alpha} \neq \emptyset$ for all $\beta < \lambda_n$. Now, as $w(X)$ does not have countable cofinality, $|\mathcal{H}| < w(X)$, where $\mathcal{H} = \bigcup_{n \in \mathbb{N}} \mathcal{G}'_n$, and $\mathcal{H}$ is a base for X. Contradiction.]

Let $x_\beta \in U_\beta - \bigcup_{\alpha<\beta} U_\alpha$ and $E = \{x_\beta : \beta < \kappa\}$. Then E_n is a discrete subset of X where $E = \bigcup_{n\in\mathbb{N}} E_n$ and $E_n = \{x_\alpha : \mathrm{st}(x_\alpha, \mathcal{G}_n) \subseteq U_\alpha\}$. Furthermore, we can find an m such that $|E_m| = w(X)$. But every discrete subset of a perfect space is σ-closed discrete, so there is a subset of E_m with cardinality κ that is closed discrete. $\square$

Corollary 2.2.7. *If X is a 1-starcompact Moore space, then $w(X)$ has countable cofinality.*

Proof. This follows from Lemmas 2.2.4 and 2.2.6. $\square$

Theorem 2.2.8 (CH). *1-starcompact Moore spaces are compact and metrizable.*

Proof. By Lemma 2.2.1, each 1-starcompact Moore space has $w(X) \leqslant \mathfrak{c}$. As $\mathfrak{c}$ does not have countable cofinality, under CH, Corollary 2.2.7 implies such a Moore space must be second countable. $\square$

To obtain a sharper result than Theorem 2.2.8, we make use of two of the cardinals (and the notation) defined in [4]. Define ${}^\omega\omega$ as the set of all functions from ω to itself. For all $f, g \in {}^\omega\omega$, we say $f \leqslant^* g$ if and only if $f(n) \leqslant g(n)$ for all but finitely many n. The unbounding number, $\mathfrak{b}$, is the smallest cardinality of an unbounded subset of $({}^\omega\omega, \leqslant^*)$. The dominating number, $\mathfrak{d}$, is the smallest cardinality of a cofinal subset of $({}^\omega\omega, \leqslant^*)$. It is straightforward to show that $\omega_1 \leqslant \mathfrak{b} \leqslant \mathfrak{d} \leqslant \mathfrak{c}$ and it is known that $\omega_1 < \mathfrak{b} = \mathfrak{c}$, $\omega_1 = \mathfrak{d} < \mathfrak{c}$ and $\omega_1 = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$ are all consistent with the axioms of ZFC (see [4] for details).

Let $L(X) = \min\{\beta$: each open cover of X has a subcover of cardinality $\leqslant \beta\} + \omega$, the Lindelöf degree of X.

Lemma 2.2.9. *If X is a regular first countable space with $L(X) < \mathfrak{b}$, then X is pseudonormal.*

Proof. Let H, K be disjoint closed subsets of X with $K = \{x_i : i \in \mathbb{N}\}$. Let $\mathcal{V}$ be an open cover of H such that $|\mathcal{V}| \leqslant L(X) < \mathfrak{b}$ and $\bar{V} \cap K = \emptyset$ for all $V \in \mathcal{V}$. For each $x \in K$, let $\{G_n(x) : n \in \mathbb{N}\}$ be a countable neighbourhood base at x with $\overline{G_1(x)} \cap H = \emptyset$ and $\overline{G_{n+1}(x)} \subseteq G_n(x)$ for all n.

For each $V \in \mathcal{V}$, there are n_i such that $K \subseteq \bigcup_{i\in\mathbb{N}} G_{n_i}(x_i)$ and $\overline{G_{n_i}(x_i)} \cap V = \emptyset$. Let $f_V : \omega \to \omega$ by $f_V(i) = n_i$. So $|\{f_V : V \in \mathcal{V}\}| < \mathfrak{b}$ and hence there is some function $g : \omega \to \omega$ such that, for every V, $f_V(i) \leqslant g(i)$ for all but finitely many i. It follows that, if $\mathcal{G} = \{G_{g(i)}(x_i) : i \in \mathbb{N}\}$, $\bigcup \mathcal{G}$ is an open set covering K and $\overline{\bigcup \mathcal{G}} \cap H = \emptyset$. $\square$

It follows from Remark 2.1.9, Corollary 2.2.7 and Lemma 2.2.9 that, if $\mathfrak{b} = \mathfrak{c}$, 1-starcompact Moore spaces are compact and metrizable. But we can do better still.

Lemma 2.2.10. *If X is a regular first countable 1-starcompact space with $w(X) < \mathfrak{d}$, then X is countably compact.*

Proof. Let $\mathscr{B}$ be a base with cardinality less than $\mathfrak{d}$ and suppose that X were not countably compact. Then there is an infinite subset $A = \{x_n : n \in \omega\}$ with no limit point. Let $\{B_m(x_n) : m \in \omega\}$ be an open neighbourhood base for x_n such that $\overline{B_{m+1}(x_n)} \subseteq B_m(x_n)$ for all m and $B_1(x_n) \cap B_1(x_m) = \emptyset$ unless $n = m$.

Let $\mathscr{C} = \{B_1(x_n) : n \in \omega\}$. For each $x \in X - \bigcup \mathscr{C}$, pick a $U_x \in \mathscr{B}$ such that $x \in U_x$ and $\overline{U_x} \cap A = \emptyset$. Let $\mathscr{D} = \{U_x : x \notin \bigcup \mathscr{C}\}$ and $\mathscr{U} = \mathscr{C} \cup \mathscr{D}$. From $\mathscr{U}$, we construct an open refinement witnessing that X cannot be 1-starcompact. Observe that $|\mathscr{U}| \leq |\mathscr{B}| < \mathfrak{d}$.

Let $\mathscr{F}$ be the collection of all finite subsets of $\mathscr{D}$. Then $|\mathscr{F}| < \mathfrak{d}$. For each $F \in \mathscr{F}$, define $f_F \in {}^{\omega}\omega$ as follows:

$$f_F(n) = \begin{cases} 0, & U \cap B_m(x_n) = \emptyset \text{ for all } m \in \omega \\ & \text{and for all } U \in F, \\ \max\{m : U \cap B_m(x_n) \neq \emptyset,\ U \in F\}, & \text{otherwise.} \end{cases}$$

For fixed n, each $U \in F$ can meet at most finitely many $B_m(x_n)$ because $\bar{U} \cap A = \emptyset$. Hence f_F is well defined. But now $|\{f_F \in {}^{\omega}\omega : F \in \mathscr{F}\}| < \mathfrak{d}$, so there is a $g \in {}^{\omega}\omega$ such that for each $F \in \mathscr{F}$ there are infinitely many n with $g(n) > f_F(n)$.

Finally, let $\mathscr{V} = \mathscr{D} \cup \{B_{g(n)}(x_n) : n \in \omega\} \cup \{B_1(x_n) - \overline{B_{g(n)+1}(x_n)} : n \in \omega\}$. Then $\mathscr{V}$ is an open refinement of $\mathscr{U}$. By the construction of g, any finite subset $\mathscr{V}'$ of $\mathscr{V}$ fails to meet infinitely many of the $B_{g(n)}(x_n)$. Hence $|A - \mathrm{st}(\bigcup \mathscr{V}', \mathscr{V})| = \omega$; so, in particular, $\mathrm{st}(\bigcup \mathscr{V}', \mathscr{V}) \neq X$. □

Theorem 2.2.11 ($\mathfrak{d} = \mathfrak{c}$). *Each 1-starcompact Moore space X is compact and metrizable.*

Proof. X is separable since it is a DFCC Moore space. Hence $w(X) \leq \mathfrak{c}$. However, by Corollary 2.2.7, $w(X) \neq \mathfrak{c}$, since $\mathfrak{c}$ does not have countable cofinality. Therefore $w(X) < \mathfrak{d}$. By Lemma 2.2.10, X is countably compact, and hence compact and metrizable. □

The relationships between the starcompactness properties in Moore spaces are given in Fig. 2.

2.3. Non-Moore space examples

First, let us consider examples to show that 2-starcompactness, strong 2-starcompactness, 1-starcompactness, and strong 1-starcompactness are (at least consistently) all distinct in the class of first countable regular spaces.

In [24], Scott describes a regular DFCC meta-Lindelöf space that is not compact. We carefully adjust this example to obtain a regular first countable 2-starcompact space that is not strongly 2-starcompact. The continuum hypothesis is used for enumeration purposes (as in Scott's example).

$$
\begin{array}{c}
\text{compact and metrizable} \\
\updownarrow \\
\text{strongly 1-starcompact} \\
\downarrow \;\; \uparrow_{\textit{consistent}} \\
\text{1-starcompact} \\
\downarrow \;\; \not\uparrow \\
\text{strongly 2-starcompact} \\
\updownarrow \\
\text{2-starcompact} \longleftrightarrow \text{DFCC} \\
\updownarrow \\
\omega\text{-starcompact}
\end{array}
$$

Fig. 2.

This space has been named "fat-psi" because, instead of taking a maximal family of almost disjoint sequences from $\mathbb{N}$ as in the construction of Ψ, sequences of clopen subsets from $Z \times \omega$ are constructed, for some suitable space Z. To ensure the whole space is regular and DFCC, a maximal family of almost disjoint such sequences is chosen. The difficulty arises in simultaneously arranging for an open cover to witness that the space is not strongly 2-starcompact. This accounts for the complexity of the chosen Z.

Lemma 2.3.1. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be an infinite collection of topological spaces. Let $\kappa = \min\{d(X_\lambda) : \lambda \in \Lambda\}$. If $A \subseteq X = \prod_{\lambda \in \Lambda} X_\lambda$ has cardinality less than κ, then A is nowhere dense in X.*

Proof. Suppose that $A \subseteq X$ is not nowhere dense in X, i.e., there is a nonempty basic $U = \prod_{\lambda \in \Lambda} U_\lambda \subseteq \bar{A}$. For some $\lambda_0 \in \Lambda$ (in fact, for all but finitely many λ) $U_{\lambda_0} = X_{\lambda_0}$. It is easy to show that $\pi_{\lambda_0}(A)$ is dense in X_{λ_0}. Hence

$$|A| \geqslant |\pi_{\lambda_0}(A)| \geqslant d(X_{\lambda_0}) \geqslant \kappa. \qquad \square$$

Observe that the lemma is not true for finite products: the long line X is locally separable, but $d(X) = \aleph_1$.

Example 2.3.2 (CH). A regular first countable 2-starcompact space which is not strongly 2-starcompact.[1]

Proof. Let S be the Cantor set on $[0, 1]$. Then S is a compact metric space with a countable base consisting of clopen sets. Put the lexicographic order on $S \times S$ (i.e., $(a, b) < (c, d)$ if and only if $b < d$ or ($b = d$ and $a < c$)) and give $S \times S$ the order topology. Then $S \times S$ is compact T_2, first countable and has a base of 2^ω clopen subsets. Observe that $d(S \times S) = 2^\omega$.

Let $Z = (S \times S)^\omega$, the product of ω many copies of $S \times S$. Then Z is compact T_2, first countable, $|Z| = 2^\omega$ and Z has a base of 2^ω clopen subsets. By Lemma 2.3.1, if

[1] Added in proof; the fourth author has recently constructed a space with these properties within ZFC.

$A \subseteq Z$ has cardinality less than 2^ω, A is nowhere dense in Z. Let $Y = Z \times \omega$ have the usual product topology.

Henceforth assume CH.

Claim 1. *We can write* $Z = \overline{\bigcup_{\alpha<\aleph_1} B_\alpha}$, *where the* B_α *are pairwise disjoint, nonempty and clopen.* The proof of this is temporarily postponed. Let $\mathcal{B} = \{B_\alpha : \alpha < \aleph_1\}$, $\mathcal{B}_n = \{B \times \{n\} : B \in \mathcal{B}\}$ and $J = \bigcup_{n\in\omega} \mathcal{B}_n$. Let Γ be the set of all countably infinite subsets R of J such that for each n, $|R \cap \mathcal{B}_n| \leq 1$. Observe that if $\Gamma' \subseteq \Gamma$ is countable, there is an $R \in \Gamma$ with $\bigcup R \cap \bigcup\bigcup \Gamma' = \emptyset$ (*).

Let $\mathcal{C}$ be the family of all basic clopen subsets of Z such that $\mathcal{C}$ refines $\mathcal{B}$ (i.e., if $C \in \mathcal{C}$, then there is some $B \in \mathcal{B}$ with $C \subseteq B$). Let $\mathcal{C}_n = \{C \times \{n\} : C \in \mathcal{C}\}$ and $K = \bigcup_{n\in\omega} \mathcal{C}_n$. Let Σ be the set of all countably infinite subsets S of K such that $|S \cap \mathcal{C}_n| \leq 1$ for each n.

Claim 2. *We can enumerate* Σ *as* $\{S_\alpha : \alpha < \aleph_1\}$ *such that if* α *is a limit ordinal,* $\bigcup S_\alpha \cap \bigcup\bigcup_{\beta<\alpha} S_\beta = \emptyset$. As with Claim 1, the proof is postponed. Let $Z = \{z_\alpha : \alpha < \aleph_1\}$ and for $\beta < \aleph_1$, $Z_\beta = \{z_\alpha : \alpha < \beta\}$. Note that $|Z_\beta| < \aleph_1$, so Z_β is nowhere dense in Z. By induction on $\alpha < \aleph_1$, we shall construct $D_\alpha \in \Sigma \cup \{\emptyset\}$ and $\Delta_\alpha \subseteq \Sigma \cup \{\emptyset\}$ so that the following hold for each $\alpha < \aleph_1$:

(1) $\Delta_\alpha = \{D_\beta : \beta \leq \alpha\}$;
(2) elements of Δ_α are almost disjoint;
(3) there is a $D \in \Delta_\alpha$ such that D is not almost disjoint from S_α;
(4) $\bigcup D_\alpha \subseteq \bigcup S_\alpha$;
(5) $(Z_\alpha \times \omega) \cap \bigcup D_\alpha = \emptyset$.

To say that sequences D_β and $D_{\beta'}$ are almost disjoint, we mean here that $d \cap d' \neq \emptyset$ for at most finitely many $d \in D_\beta$ and $d' \in D_{\beta'}$.

Suppose that $\alpha < \aleph_1$ and that D_β and Δ_β have been defined for all $\beta < \alpha$ to satisfy (1)-(5). Let $\Delta'_\alpha = \{D_\beta : \beta < \alpha\}$; then Δ'_α is almost disjoint. If $\Delta'_\alpha \cup \{S_\alpha\}$ is not almost disjoint, let $D_\alpha = \emptyset$. Otherwise, let $M = \{n \in \omega : S_\alpha \cap \mathcal{C}_n \neq \emptyset\}$ and for each $n \in M$, let $C_n = (Z \times \{n\}) \cap \bigcup S_\alpha$. For each $n \in M$, there is a $C'_n \in \mathcal{C}_n$ such that $C'_n \subseteq C_n - Z_\alpha \times \{n\}$. Let $D_\alpha = \{C'_n : n \in M\}$. In either case, set $\Delta_\alpha = \Delta'_\alpha \cup \{D_\alpha\}$. (1)-(5) are clearly satisfied at α. Furthermore, if α is a limit ordinal, $\bigcup D_\alpha \cap \bigcup\bigcup_{\beta<\alpha} D_\beta = \emptyset$ (**).

Now let $\Delta = \bigcup\{\Delta_\alpha : \alpha < \aleph_1\} - \{\emptyset\}$. Then Δ is a maximal almost disjoint subfamily of Σ. Moreover, for any $y \in Y$, $\{D \in \Delta : y \in \bigcup D\}$ is countable (by (5)), and if $\Delta' \subseteq \Delta$ is countable there are uncountably many $D \in \Delta$ with $\bigcup D \cap \bigcup\bigcup \Delta' = \emptyset$ (by (**)).

Let $|L| = \aleph_1$ with $L \cap Y = \emptyset$ and associate each $D \in \Delta$ with a unique $l_D \in L$. Let $X = Y \cup L$, and topologize X as follows: Y is an open subspace of X and basic open neighbourhoods of points of L take the form $\{l_D\} \cup \bigcup\{D - F\}$, where F is a finite subset of D. This topology makes X Hausdorff, zero-dimensional, first countable, locally compact and meta-Lindelöf. Note that L is a closed discrete subset of X.

X is DFCC: Let $\mathcal{V} = \{V_n : n \in \omega\}$ be a disjoint family of nonempty open subsets of X. Y is open and dense in X, so we may assume $\bigcup \mathcal{V} \subseteq Y$. If there is an $m \in \omega$

such that $V_n \cap (Z\times\{m\}) \neq \emptyset$ for infinitely many $n\in\omega$, then the compactness of Z ensures some point of $Z\times\{m\}$ is a cluster point of $\mathscr{V}$. Otherwise there is an $S=\{S_n: n\in\omega\}\in\Sigma$ such that $S_n\subseteq V_n$ for each $n\in\omega$, and the maximality of Δ then ensures that some $D\in\Delta$ is not almost disjoint from S; therefore l_D is a cluster point of $\mathscr{V}$.

X is not strongly 2-starcompact: Let $\mathscr{V}$ be a basic open cover of X such that each $l\in L$ is contained in a unique $V_l\in\mathscr{V}$, $\mathscr{V}-\{V_l: l\in L\}$ covers Y and each element of this collection is contained in some $Z\times\{n\}$. Enumerate the finite subsets of Y as $\{F_\alpha: \alpha<\aleph_1\}$. Suppose that for all $\alpha<\beta$ we have defined a distinct $l(\alpha)\in L$ such that if $V_{l(\alpha)}$ is the unique element of $\mathscr{V}$ containing $l(\alpha)$, then $V_{l(\alpha)}\cap\bigcup D=\emptyset$ whenever $D\in\Delta$ with $\bigcup D\cap F_\alpha\neq\emptyset$. Observe that $\Delta'=\{D\in\Delta: \bigcup D\cap\bigcup_{\alpha<\beta}F_\alpha\neq\emptyset\}$ is countable, so there are uncountably many $D'\in\Delta$ such that $\bigcup D'\cap\bigcup\bigcup\Delta'=\emptyset$ (by (**)) [N.B. crucial use of CH here]. Thus we can pick such a D' for which $l_{D'}\notin\{l(\alpha): \alpha<\beta\}$ and let $l(\beta)=l_{D'}$. Notice that $V_{l(\beta)}\cap\bigcup D=\emptyset$ whenever $D\in\Delta$ and $\bigcup D\cap F_\beta\neq\emptyset$.

Define $V'_{l(\beta)}=V_{l(\beta)}-\bigcup\{Z\times\{n\}: F_\beta\cap Z\times\{n\}\neq\emptyset\}$. So $V'_{l(\beta)}$ is open as F_β is finite. Finally, let $\mathscr{V}'=\{V'_{l(\beta)}: \beta<\aleph_1\}\cup\{V\in\mathscr{V}: V\neq V_{l(\beta)}$ for any $\beta<\aleph_1\}$. Then $\mathscr{V}'$ is an open cover of X. Now if F is a finite subset of Y, $F=F_\alpha$ for some $\alpha<\aleph_1$. If $V\in\mathscr{V}'$ and $V\cap F\neq\emptyset$, then either $V\subseteq\bigcup D\cup\{l_D\}$ for some $D\in\Delta$, or $V\subseteq Z\times\{n\}$ where $F\cap(Z\times\{n\})\neq\emptyset$. In either case, $V'_{l(\alpha)}\cap V=\emptyset$ by construction. As $V'_{l(\alpha)}$ is the unique element of $\mathscr{V}'$ containing $l(\alpha)$, $l(\alpha)\notin \operatorname{st}^2(F,\mathscr{V}')$.

Proof of Claim 1. Let $\{q_n: n\in\omega\}$ be a countable dense subset of S. Let $C_0\subseteq(0,1)$ be a clopen subset of S containing q_0. Suppose we have defined pairwise disjoint clopen subsets of S, $C_0,\ldots,C_n$ that are also subsets of $(0,1)$. If $\bigcup_{0\leq i\leq n}C_i$ is dense in S, then stop. Otherwise pick C_{n+1} clopen such that $q_{m_{n+1}}\in C_{n+1}\subseteq (0,1)-\bigcup_{0\leq i\leq n}C_i$, where m_{n+1} is the least integer such that $q_{m_{n+1}}\notin\bigcup_{0\leq i\leq n}C_i$.

This process generates a collection $\mathscr{A}$ of pairwise disjoint clopen subsets of S that are dense in S and are all subsets of $(0,1)$. Let $\mathscr{A}^+=\{A\times\{s\}: s\in S,\ A\in\mathscr{A}\}$. Then $\mathscr{A}^+$ is a collection of pairwise disjoint clopen subsets of the lexicographically ordered Cantor square, $S\times S$, and $|\mathscr{A}^+|=\aleph_1$. If $\pi: Z=(S\times S)^w\to S\times S$ is the projection onto the first coordinate, $\mathscr{B}=\{\pi^{-1}A: A\in\mathscr{A}^+\}$ does the job.

Proof of Claim 2. Enumerate Σ as $\{T_\alpha: \alpha<\aleph_1\}$. Suppose for $\alpha<\beta$ we have defined $S_\alpha\in\Sigma$ such that, if α is a limit ordinal, $\bigcup S_\alpha\cap\bigcup\bigcup_{\gamma<\alpha}S_\gamma=\emptyset$. If β is a successor, let S_β be $T_\gamma\in\Sigma$, where γ is the least ordinal such that $T_\gamma\notin\{S_\alpha\in\Sigma: \alpha<\beta\}$. For limit β, recall that for all α, $S_\alpha\subseteq R(\alpha)\in\Gamma$, so by (*), there is an $R\in\Gamma$, and hence in Σ such that $\bigcup R\cap\bigcup\bigcup_{\alpha<\beta}S_\alpha=\emptyset$. Let $S_\beta=R$. □

Example 2.3.3. The Tychonoff plank is a 1-starcompact completely regular space which is not strongly 1-starcompact.

Proof. Let $T=[0,\omega_1]\times[0,\omega]$ have the usual product topology and $T^\star=T-(\omega_1,\omega)$ have the induced topology. The space $T^\star$ is known as the Tychonoff plank.

Let $\mathcal{U}$ be a cover of $T^\star$ consisting of basic open sets. Then for each $n<\omega$, there is an $\alpha_n<\omega_1$ such that $\{(\beta, n)\in T: \alpha_n<\beta\leqslant\omega_1\}\in\mathcal{U}$. Call this set U_n. Let $\alpha=\sup\{\alpha_n\in\omega_1: n<\omega\}$, so $\alpha<\omega_1$. Then $\{(\beta, n): \alpha<\beta\leqslant\omega_1\}\subseteq U_n$ for each $n<\omega$. For some $V\in\mathcal{U}$, there is an $m<\omega$ such that $\{(\alpha+1, n)\in T: m<n\leqslant\omega\}\subseteq V$. Define $\mathcal{V}_1=\{V\}\cup\{U_n: n\leqslant m\}$. Notice that $\mathcal{V}_1\subseteq\mathcal{U}$ is finite and that $\mathrm{st}(\bigcup\mathcal{V}_1, \mathcal{U})\supseteq(\alpha, \omega_1]\times[0, \omega)$.

Define $E=[0, \alpha+1]\times[0, \omega]\subseteq T^\star$. Clearly, E is compact as a subspace of $T^\star$. So there is some finite $\mathcal{V}_2\subseteq\mathcal{U}$ that covers E.

Finally, $[0, \omega_1)\times\{\omega\}$ is homeomorphic to $[0, \omega_1)$, which is countably compact and, in particular, 1-starcompact. Hence there is some finite $\mathcal{V}_3\subseteq\mathcal{U}$ such that $\mathrm{st}(\bigcup\mathcal{V}_3, \mathcal{U})\supseteq[0, \omega_1)\times\{\omega\}$.

Define $\mathcal{V}=\mathcal{V}_1\cup\mathcal{V}_2\cup\mathcal{V}_3$, so that $\mathcal{V}$ is a finite subset of $\mathcal{U}$. Observe that

$$\begin{aligned} T^\star &=((\alpha, \omega_1]\times[0, \omega))\cup([0, \alpha+1]\times[0, \omega])\cup([0, \omega_1)\times\{\omega\}) \\ &\subseteq \mathrm{st}(\textstyle\bigcup\mathcal{V}, \mathcal{U})\subseteq T^\star. \end{aligned}$$

Thus, we have shown that for any basic open cover $\mathcal{U}$ of $T^\star$, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{st}(\bigcup\mathcal{V}, \mathcal{U})=T^\star$, i.e., $T^\star$ is 1-starcompact.

However, $T^\star$ is not countably compact because $A=\{(\omega_1, n): n<\omega\}$ is a closed, infinite discrete subset of $T^\star$. Hence, by Theorem 2.1.5, $T^\star$ is not strongly 1-starcompact. □

Notice that A is a closed subset of $T^\star$ which has the discrete topology. Hence closed subsets of 1-starcompact spaces need not be 1-starcompact.

We now give a consistent example of a regular first countable 1-starcompact space which is not strongly 1-starcompact. It is a modification of an example due to van Douwen and Nyikos in [4].

Lemma 2.3.4. ($\mathfrak{d}=\omega_1$). *There is a dominating $D\subseteq{}^\omega\omega$ that is well ordered by $<^*$.*

Proof. Let $F=\{f_\alpha: \alpha<\omega_1\}$ be a dominating subset of $({}^\omega\omega, \leqslant^*)$. Suppose we have defined $g_\alpha\in{}^\omega\omega$ for all $\alpha<\beta$ such that $f_\alpha<^* g_\alpha$ and $g_{\alpha'}<^* g_\alpha$ for all $\alpha'<\alpha$. The collection $\{g_\alpha: \alpha<\beta\}\cup\{f_\beta\}$ is countable, so there is some $g_\beta\in{}^\omega\omega$ such that $g_\alpha<^* g_\beta$ for all $\alpha<\beta$ and $f_\beta<^* g_\beta$.

The set $D=\{g_\alpha: \alpha<\omega_1\}$ is the required dominating family. □

Example 2.3.5 ($\mathfrak{d}=\omega_1$). A regular first countable 1-starcompact space which is not strongly 1-starcompact.

Proof. Let D be a dominating subset of ${}^\omega\omega$ well ordered by $<^*$. Let $X=\omega\cup(\omega\times\omega)\cup D$ and topologize X as follows: points of $\omega\times\omega$ are isolated, basic neighbourhoods of points $k\in\omega$ take the form

$$U(k, n)=\{k\}\cup(\{k\}\times[n, \omega))\quad\text{for } n\in\omega,$$

and basic neighbourhoods of points f in D take the form

$$G(f, g, S) = \{h \in D: g <^* h \leqslant^* f\} \cup ((L_f - L_g) - S),$$

where $g \in D$ satisfies $g <^* f$, $S \in [\omega \times \omega]^{<\omega}$ and $L_f = \{(m, n): n \leqslant f(m)\}$.

With this topology, X is T_3 (in fact, zero-dimensional). Because $\mathfrak{d} = \omega_1$, D is homeomorphic to ω_1 with the usual topology, and it follows that X is first countable. X is not countably compact (and hence not strongly 1-starcompact) because ω is an infinite subset with no limit point.

In preparation for the proof that X is 1-starcompact, we prove the following claims:

Claim 1. *If $n_k \in \omega$ for each $k \in \omega$, there is an $f \in D$ such that whenever U is open and $f \in U$, U meets $\{k\} \times [n_k, \omega)$ for all but finitely many k.*

Claim 2. *If A is an infinite subset of $\omega \times \omega$ with $|A \cap (\{k\} \times \omega)| < \omega$ for all k, then there is a limit point f of A in D.*

Proof of Claim 1. Define $h \in {}^\omega\omega$ by $h(k) = n_k$ for $k \in \omega$. As D is dominating, there is an $f \in D$ such that $h <^* f$. Consider $G(f, g, S)$ for any $g \in D$ with $g <^* f$ and $S \in [\omega \times \omega]^{<\omega}$. If $G(f, g, S) \cap \{k\} \times [n_k, \omega) = \emptyset$ for infinitely many k, then $(L_f - L_g) \cap \{k\} \times [n_k, \omega) = \emptyset$ for infinitely many k (S is finite). Therefore $g(k) \geqslant f(k)$ for infinitely many k, since $f(k) \geqslant n_k$ for all but finitely many k. But this contradicts $g <^* f$.

Proof of Claim 2. Define $h \in {}^\omega\omega$ as follows:

$$h(k) = \begin{cases} 0, & \text{if there is no } n \text{ with } (k, n) \in A, \\ \min\{n: (k, n) \in A\}, & \text{otherwise.} \end{cases}$$

As D is dominated and well ordered by $<^*$, let f be the $<^*$-least element of D with $h \leqslant^* f$. Consider $G(f, g, S)$ where $g \in D$ satisfies $g <^* f$ and $S \in [\omega \times \omega]^{<\omega}$. If $G(f, g, S) \cap A$ were finite, $G(f, g, S) \cap h$ would be finite (regarding h as a collection of ordered pairs). But $G(f, g, S) \cap h = ((L_f - L_g) - S) \cap h$, S is finite and $(k, h(k)) \in L_f$ for all but finitely many k. Hence $(k, h(k)) \in L_g$ for all but finitely many k, i.e., $h \leqslant^* g$. This contradicts the minimality of f.

That X is 1-starcompact: Let $\mathcal{U}$ be any open covering of X. For every $k \in \omega$, for some $U_k \in \mathcal{U}$ and some $n_k \in \omega$, $\{k\} \cup (\{k\} \times [n_k, \omega)) \subseteq U_k$. By Claim 1, there is some $f \in D$ such that, if $f \in V \in \mathcal{U}$, V meets all but finitely many of the sets U_k. Hence, $\mathrm{st}(V, \mathcal{U})$ contains all but finitely many points of ω. For each point $k \in \omega - \mathrm{st}(V, \mathcal{U})$ pick a $U_k \in \mathcal{U}$ with $k \in U_k$. Let $\mathcal{U}'$ be the set containing V and these U_k. Then $\mathcal{U}' \subseteq \mathcal{U}$ is finite and $\omega \subseteq \mathrm{st}(\bigcup \mathcal{U}', \mathcal{U})$. As a subspace of X, D is countably compact; so there is certainly a finite $\mathcal{U}'' \subseteq \mathcal{U}$ satisfying $\mathrm{st}(\bigcup \mathcal{U}'', \mathcal{U}) \supseteq D$. Finally, it follows from Claim 2 that any infinite subset of $\omega \times \omega$ has a limit point in $\omega \cup D$. Hence $X - \mathrm{st}(\bigcup \mathcal{U}' \cup \bigcup \mathcal{U}'', \mathcal{U})$ is a finite subset of $\omega \times \omega$. So there is certainly a finite $\mathcal{U}''' \subseteq \mathcal{U}$ covering this remaining subset. Then $\mathcal{V} = \mathcal{U}' \cup \mathcal{U}'' \cup \mathcal{U}'''$ is a finite subset of $\mathcal{U}$ and $\mathrm{st}(\bigcup \mathcal{V}, \mathcal{U}) = X$ as required. $\square$

We conclude the regular examples with the following:

Example 2.3.6. A regular space that has every real-valued function bounded (in fact, constant) but which is not DFCC.

Proof. Let Y be any space such that $|Y| \geq 2$, Y is T_3 and every continuous real-valued function on Y is constant (for example, see [6, Section 2.7]). Pick any $y_0 \in Y$. Let $Y_i = Y \times \{i\}$ and $X = \bigoplus_{i \in \mathbb{N}} Y_i$. Define an equivalence relation $\sim$ on X as follows:

$$(y_1, i_1) \sim (y_2, i_2) \quad \text{if and only if} \quad ((y_1 = y_2 \text{ and } i_1 = i_2) \text{ or } (y_1 = y_2 = y_0)),$$

and let $X^\star = X/\sim$. Then $X^\star$ is T_3 because each Y_i is. Pick any $y \in Y - \{y_0\}$ and disjoint open U, U' in Y such that $y_0 \in U$ and $y \in U'$. Then $\{\pi(U' \times \{i\})\colon i \in \mathbb{N}\}$ is a countably infinite discrete collection of open sets in $X^\star$, where $\pi : X \to X^\star$ is the quotient map. So $X^\star$ is not DFCC.

Suppose $f^\star : X^\star \to \mathbb{R}$ is continuous. Then $f_i : Y_i \to \mathbb{R}$ defined by $f_i = f^\star \circ \pi|_{Y_i}$ is continuous. Because of the property of Y, each $f_i : Y_i \to \mathbb{R}$ is constant. Therefore for any $y \in Y$,

$$f^\star(\pi(y, i)) = f_i(y, i) = f_i(y_0, i) = f^\star(\pi(y_0, i)).$$

But we know $\pi(y_0, i) = \pi(y_0, j)$ for all i, j. Hence we have that for any $y \in Y$ and any i, j,

$$f^\star(\pi(y, i)) = f^\star(\pi(y_0, i)) = f^\star(\pi(y_0, j)).$$

From this it is clear that $f^\star$ is constant on $X^\star$. $\square$

The situation for starcompactness in regular first countable spaces is summarised in Fig. 3.

Finally, we consider nonregular distinctions between the various starcompactness properties.

countably compact

$\updownarrow$

strongly 1-starcompact

$\downarrow \quad \not\uparrow_{\textit{consistent}}$

1-starcompact

$\downarrow \quad \not\uparrow$

strongly 2-starcompact

$\downarrow \quad \not\uparrow_{\textit{consistent}}$

2-starcompact $\longleftrightarrow$ DFCC

$\updownarrow$

ω-starcompact

Fig. 3.

Example 2.3.7. A strongly 1-starcompact T_1 space which is not countably compact.

Proof. Let $X=\mathbb{R}$ with the cocountable topology. Then X is a T_1 space which is not T_2. It is certainly not countably compact.

Let $\mathcal{V}$ be any open cover of X and pick any nonempty $V_0 \in \mathcal{V}$. If $V_0=X$, then let $B=\{0\}$; clearly, $\operatorname{st}(B, \mathcal{V})=X$. If $V_0 \neq X$, then $X-V_0=\{x_n : n \in \mathbb{N}^+\}$ (repetitions allowed). For each $n \in \mathbb{N}^+$, pick some $V_n \in \mathcal{V}$ with $x_n \in V_n$. Then $\mathcal{V}'=\{V_n : n \in \mathbb{N}\} \subseteq \mathcal{V}$ is an open cover of X. Now,

$$X-\bigcap_{n\in\mathbb{N}} V_n=\bigcup_{n\in\mathbb{N}}(X-V_n),$$

which is countable (V_n is open, so $X-V_n$ is countable). Therefore, $\bigcap_{n\in\mathbb{N}} V_n$ is nonempty, because X is uncountable. Select any $x \in \bigcap_{n\in\mathbb{N}} V_n$, let $B=\{x\}$ and observe that

$$X=\bigcup_{n\in\mathbb{N}} V_n=\operatorname{st}(x, \mathcal{V}')\subseteq \operatorname{st}(B, \mathcal{V})\subseteq X.$$

Hence X is strongly 1-starcompact. □

In [23], a scheme is constructed which, for any $n \in \mathbb{N}^+$, will generate a Hausdorff strongly $n+1$-starcompact space that is not n-starcompact. This scheme also enables Sarkhel to create a Hausdorff space that is ω-starcompact, but not n-starcompact for *any* $n \in \mathbb{N}^+$.

Here, we modify this technique so that, for any $n \in \mathbb{N}^+$, we also get an n-starcompact Hausdorff space that is not strongly n-starcompact. In the light of Theorem 2.1.8, this is the best "construction scheme" that we could hope for.

Example 2.3.8. An n-starcompact Hausdorff space which is not strongly n-starcompact.

Proof. Take the compact interval $I=[0,1]$ and express it as the union of pairwise disjoint sets $A_1, \ldots, A_{2n+1}$, each dense in I with $0, 1 \in A_{2n+1}$. Let $E_k=A_{k-1}\cup A_k \cup A_{k+1}$ for $k=1,3,5,\ldots,2n+1$ and $E_k=A_k$ for $k=2,4,6,\ldots,2n$, where $A_0=A_1$ and $A_{2n+2}=A_{2n+1}$. Notice that $E_i \cap E_j$ is dense in I if and only if $|i-j|\leq 1$ or i and j are consecutive odd numbers. Note also that for each $x \in I$ there is a unique index $k(x)$ such that $x \in A_{k(x)}$.

Now let X denote the set I with the topology $\mathcal{T}_n(X)$ consisting of all subsets $G \subseteq I$ such that for every $x \in G$ there is an open interval I_x satisfying $x \in I_x \cap E_{k(x)} \subseteq G$. This topology makes X a Hausdorff space that is n-starcompact, but not strongly n-starcompact.

Let $\mathcal{V}$ be any open cover of X. For each $x \in X$ we select an open $V_x \in \mathcal{V}$ and an open interval I_x satisfying $x \in I_x \cap E_{k(x)} \subseteq V_x$ (*). Because X is compact in the metric topology, $X=\bigcup\{I_b : b \in B\}$ for some finite $B \subseteq X$. Then given $x \in X$, we have $x \in I_b$ for some $b \in B$. It follows that $x \in \operatorname{st}^n(V_b, \mathcal{V})$; thus X is n-starcompact (take $\mathcal{V}'=\{V_b \in \mathcal{V} : b \in B\}$).

We now show that X cannot be strongly n-starcompact. Fix a strictly increasing sequence $\{c_k\} \subseteq A_{2n+1}$ converging to 1 with $c_1 = 0$. Select $a_k \in I_k \cap A_1$, where $I_k = (c_k, c_{k+1})$ and let $J_k = (c_k, a_k) \cup (a_k, c_{k+1})$. Then the family $\mathcal{U}$ consisting of the set E_{2n+1} together with the sets $I_k \cap E_1$ and $J_k \cap E_i$ ($k = 1, 2, \ldots$; $i = 2, 3, \ldots, 2n$) is an open cover of X. Given any finite $B \subseteq X$ there is an index m such that none of the sets $I_m \cap E_1, J_m \cap E_2, \ldots, J_m \cap E_{2n}$ meets B (this is because the sequence $\{c_k\}$ is increasing). By the construction of the J_k, $a_m \notin \mathrm{st}^n(B, \mathcal{U})$. This proves that X is not strongly n-starcompact, because $\mathrm{st}^n(B, \mathcal{U}) \neq X$ for arbitrary finite $B \subseteq X$.

One final observation is that the collection of sets $I_{2k} \cap A_1$ for $k = 1, 2, \ldots$ is discrete, so this space does not satisfy the DFCC. □

2.4. *Properties of starcompact spaces*

Here we consider further properties of starcompactness, such as the combination of starcompactness with other covering properties. First, we look at continuous images of starcompact spaces and products involving starcompactness.

Theorem 2.4.1. *The continuous image of a strongly n-starcompact (respectively n-starcompact) is strongly n-starcompact (respectively n-starcompact), for* $1 \leqslant n \leqslant \omega$.

Proof. Straightforward. □

In general, the product of two countably compact spaces need not even be pseudocompact [6, 3.10]; so no form of starcompactness is even finitely productive. Furthermore, Fleischman shows in [7] that the product of a strongly 1-starcompact space with a compact space need not be strongly 1-starcompact. However, at least we have the following result, which is also proved in [23]:

Theorem 2.4.2. *If X is n-starcompact and Y compact, then $X \times Y$ is n-starcompact* ($n \leqslant \omega$).

Proof. We give the proof for $n = \omega$. It is clear how the proof for other values of n can be obtained.

Suppose X and Y are as above and $\mathcal{U}$ is a basic open covering of $X \times Y$. For each $x \in X$, $\mathcal{U}$ is an open cover of the compact subset $\{x\} \times Y$ of $X \times Y$. Therefore, there is a finite subset of $\mathcal{U}$ covering $\{x\} \times Y$, say $U_{x1} \times V_{x1}, \ldots, U_{xn(x)} \times V_{xn(x)}$. Define $W_x = \bigcap_{i=1}^{n(x)} U_{xi}$, so that W_x is an open subset of X containing x and

$$\begin{aligned}\{x\} \times Y &\subseteq \bigcup \{W_x \times V_{xi} \colon 1 \leqslant i \leqslant n(x)\}\\ &\subseteq \bigcup \{U_{xi} \times V_{xi} \colon 1 \leqslant i \leqslant n(x)\}.\end{aligned}$$

Then $\mathcal{W} = \{W_x \colon x \in X\}$ is an open cover of X. Because X is ω-starcompact, there is some finite subset $\mathcal{W}' = \{W_{x_j} \colon 1 \leqslant j \leqslant r\} \subseteq \mathcal{W}$ and some $N \in \mathbb{N}^+$ such that $\mathrm{st}^N(\bigcup \mathcal{W}', \mathcal{W}) = X$.

Define $\mathcal{U}' = \{U_{x_j i} \times V_{x_j i} \colon 1 \leqslant i \leqslant n(x_j), 1 \leqslant j \leqslant r\}$, so that $\mathcal{U}'$ is a finite subset of $\mathcal{U}$.

Straightforward induction shows that $\mathrm{st}^m(\bigcup \mathcal{W}', \mathcal{W}) \times Y \subseteq \mathrm{st}^m(\bigcup \mathcal{U}', \mathcal{U})$ and hence $\mathrm{st}^N(\bigcup \mathcal{U}', \mathcal{U}) = X \times Y$. □

We have already seen that starcompactness does not have the same "closed hereditary" property as countable compactness—both the Tychonoff plank, Example 2.3.3, and Ψ, Example 2.2.5, have infinite closed discrete subsets. However, due to the equivalence of the DFCC and 2-starcompactness, we do have an instance where starcompactness is preserved in subspaces:

Theorem 2.4.3. *In regular spaces, 2-starcompactness is preserved in regular closed subsets.*

Proof. The proof that regular closed subsets of a regular DFCC space are DFCC can be found in [1]. $\square$

It is well known that a countably compact space that is either Lindelöf or metacompact is compact. So it is natural, therefore, to ask whether similar results hold for spaces that are both starcompact and have some other covering property.

Observe that Example 2.3.7 is a strongly 1-starcompact, Lindelöf T_1 space. So we immediately see that no form of starcompactness strictly weaker than countable compactness, together with the Lindelöf condition, is sufficient to imply compactness. Furthermore, the following example is a second-countable Hausdorff space that is 1-starcompact but not compact.

Example 2.4.4. There exists a 1-starcompact, second-countable T_2 space which is not strongly 1-starcompact.

Proof. Let $Y = \bigcup\{[0,1]\times\{n\}: n\in\mathbb{N}\}$ and $X = Y\cup\{a\}$, where $a\notin Y$. Define a basis for a topology on X as follows. Basic open sets containing a take the form $\{a\}\cup\bigcup\{[0,1)\times\{n\}: n>m\}$ where $m\in\mathbb{N}$. Basic open sets about other points of X are the usual induced metric open sets. The topology which this basis generates is clearly Hausdorff and makes each of the subsets $[0,1]\times\{n\}$ compact.

This space is not countably compact since $\{(1,n): n\in\mathbb{N}\}$ is an infinite closed discrete collection of points of X; so X is not strongly 1-starcompact. It is, however, 1-starcompact. Let $\mathcal{U}$ be any basic open cover of X. Pick any $U_a\in\mathcal{U}$ that contains a. Then $U_a = \{a\}\cup\bigcup\{[0,1)\times\{n\}: n>m\}$ for some m. Each $[0,1]\times\{n\}$ for $n\leq m$ is compact so these sets are covered by some finite $\mathcal{U}'\subseteq\mathcal{U}$. If we now define $\mathcal{U}'' = \mathcal{U}'\cup\{U_a\}$, it is clear that $\operatorname{st}(\bigcup\mathcal{U}'', \mathcal{U}) = X$. $\square$

However, every regular Lindelöf space is normal. So, with Remark 2.1.9 in mind, we see that any space that is regular, Lindelöf and has some starcompactness property is compact (using the fact that countably compact Lindelöf spaces are compact).

Notice that Example 2.4.4 is also metacompact. Therefore, 1-starcompact metacompact spaces need not be compact, unlike the case with countable compactness. In particular, it shows that n-starcompact metacompact spaces need not be strongly

n-starcompact. We will now see that strong 1-starcompactness is sufficient to imply compactness, and how metacompactness does link the starcompactness properties.

Theorem 2.4.5. *Strongly* 1*-starcompact metacompact spaces are compact.*

Proof. Let $\mathcal{U}$ be any open cover of a strongly 1-starcompact metacompact space X. Let $\mathcal{V}$ be a point-finite refinement of $\mathcal{U}$. There is a finite $B \subseteq X$ such that $\mathrm{st}(B, \mathcal{V}) = X$. As $\mathcal{V}$ is point-finite, $\mathrm{ST}(B, \mathcal{V})$ is finite and covers X. As $\mathcal{V}$ refines $\mathcal{U}$, for each $V \in \mathrm{ST}(B, \mathcal{V})$ we can find $U_V \in \mathcal{U}$ with $V \subseteq U_V$. The collection $\{U_V\colon V \in \mathrm{ST}(B, \mathcal{V})\}$ is the required finite subcover. □

This provides us with an alternative, if roundabout, proof of the fact mentioned above:

Corollary 2.4.6. *Countably compact metacompact spaces are compact.*

Proof. Combine Theorems 2.1.4 and 2.4.5. □

Theorem 2.4.7. *Strongly* $n+1$*-starcompact metacompact spaces are n-starcompact.*

Proof. Adapt the proof of Theorem 2.4.5. □

In [24], Scott shows that regular DFCC metacompact spaces are compact. To conclude our investigation with metacompactness, we again use the equivalence of the DFCC and 2-starcompactness in regular spaces:

Theorem 2.4.8. *Every regular* 2*-starcompact metacompact space is compact.*

Example 2.4.4 shows that regular cannot be weakened to Hausdorff.

3. The star-Lindelöf condition

3.1. General positive results

Let us recall the definitions of Section 1.

Definition. A space X is said to be *n-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there is some countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$.

Definition. A space X is said to be *strongly n-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there is some countable subset B of X such that $\mathrm{st}^n(B, \mathcal{U}) = X$.

Definition. A space X is said to be *ω-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there is some $n \in \mathbb{N}^+$ and some countable subset B of X such that $\mathrm{st}^n(B, \mathcal{U}) = X$.

As we might expect, the theory that follows from these definitions is similar to that of Section 2. For instance, it is immediate that every (strongly) n-starcompact space is (strongly) n-star-Lindelöf for $n \leq \omega$. However, the theories are by no means completely identical; but when a corresponding theorem does hold in the star-Lindelöf case, it is often enough to replace "finite" with "countable" in the proof (perhaps with some transfinite induction). Theorem 3.1.1 summarises the results analogous to Lemmas 2.1.1 and 2.1.2 and Theorem 2.1.4.

Theorem 3.1.1. (1) *Every Lindelöf space is strongly* 1*-star-Lindelöf.*

(2) *Every strongly n-star-Lindelöf space is n-star-Lindelöf.*

(3) *Every n-star-Lindelöf space is strongly* $n+1$*-star-Lindelöf.*

(4) *Every (strongly) n-star-Lindelöf is* ω*-star-Lindelöf.*

In fact, Theorem 3.1.1(1) holds for the more general class of $\aleph_1$-compact (i.e., every uncountable subset has a limit point) T_1 spaces.

Being $\aleph_1$-compact (or by Theorem 2.1.4 and the remark above), ω_1 with the order topology is strongly 1-star-Lindelöf. Moreover, ω_1 is normal but not Lindelöf. Thus, we see that Theorem 2.1.5 does not carry over to the star-Lindelöf case.

The discrete countable chain condition is described in [25]. By modifying the proofs of Theorems 2.1.6 and 2.1.7, we obtain the following:

Theorem 3.1.2. (1) *Every DCCC space is* 2*-star-Lindelöf.*

(2) *Every regular* ω*-star-Lindelöf space is DCCC.*

Consequently, for regular spaces, the DCCC equals 2-star-Lindelöf equals ω-star-Lindelöf, and all the properties in between. Both Theorems 3.1.1 and 3.1.2 were known by Ikenaga [11, 12].

Observe that a completely regular space is pseudocompact if and only if every continuous real-valued function has compact image. We introduce a definition which is the Lindelöf analogue of pseudocompactness, i.e., keeping the "continuous function" flavour. Let H be the hedgehog of spininess ω_1 (see [6]). Then H is a non-DCCC, connected metric space, with metric d.

Definition 3.1.3. A space X is *pseudo-Lindelöf* if every continuous $f: X \to H$ has Lindelöf image.

Theorem 3.1.4. *For a completely regular space X, X is DCCC if and only if it is pseudo-Lindelöf.*

Proof. Suppose $f: X \to H$ does not have Lindelöf image, i.e., $f(X)$, as a subspace of H, is not Lindelöf. H is metric, so $f(X)$ is too. Therefore $f(X)$ is not DCCC [25]. Let $\{V_\alpha: \alpha < \omega_1\}$ be an uncountable discrete collection of nonempty open sets. It is easy to verify that $\{f^{-1}(V_\alpha): \alpha < \omega_1\}$ is an uncountable discrete collection in X.

Conversely, suppose $\{V_\alpha: \alpha < \omega_1\}$ is an uncountable discrete collection of open subsets of X and pick $x_\alpha \in V_\alpha$. Let $[0, 1]_\lambda$ be the λth spine of H. By complete regularity, there is a continuous $f_\lambda : X \to [0,1]_\lambda$ such that $f_\lambda(x_\lambda)=1$ and $f_\lambda(x)=0$ for all $x \notin V_\alpha$. Define $F: X \to H$ by

$$F(x)=\begin{cases} f_\lambda(x), & \text{if } x \in V_\lambda \text{ (and such a } \lambda \text{ is unique)},\\ 0, & \text{otherwise.}\end{cases}$$

To show that F is continuous, we show that inverse images of basic open subsets of H are open in X. First observe that if $0 < a < b \leq 1$, then $F^{-1}((a,b)_\lambda)=f_\lambda^{-1}((a,b)_\lambda)$ and $F^{-1}((a,1]_\lambda)=f_\lambda^{-1}((a,1]_\lambda)$, which are open by the continuity of f_λ. If $B_{1/n}(0)=\{h \in H: d(0,h)<1/n\}$, then it remains to show that $F^{-1}(B_{1/n}(0))$ is open in X. But

$$F^{-1}(B_{1/n}(0))=\{x \in X: F(x)=0\} \cup \bigcup_{m>n \wedge \lambda \in \omega_1} f_\lambda^{-1}\left(\frac{1}{m},\frac{1}{n}\right)$$

and the second half of this union is open in X. We must show, therefore, that for each $x \in X$ such that $F(x)=0$, there is an open U containing x with $U \subseteq F^{-1}(B_{1/n}(0))$. So suppose $F(x)=0$. If $x \notin \overline{\bigcup_{\lambda \in \omega_1} V_\lambda}$, then $U = X - \overline{\bigcup_{\lambda \in \omega_1} V_\lambda}$ works. If, on the other hand, $x \in \overline{\bigcup_{\lambda \in \omega_1} V_\lambda}$, there is an open W containing x that meets precisely one of the V_λ, say V_{λ_0}. It follows that $U=(X - f_{\lambda_0}^{-1}([1/n,1]_{\lambda_0})) \cap W$ has the required properties. Hence F is continuous. Finally, by considering the open cover $\{B_{1/2}(0)\} \cup \{(0,1]_\lambda : \lambda \in \Lambda\}$ intersected with $F(X)$, we see that $F(X)$ is not Lindelöf. $\square$

If Theorem 3.1.2 is not convincing enough, the next two easily established results are evidence that the star-Lindelöf properties are essentially chain conditions.

Theorem 3.1.5. *Every separable space is strongly 1-star-Lindelöf.*

Proof. Take B equal to some countable dense subset. $\square$

Theorem 3.1.6. *Every CCC space is 1-star-Lindelöf.*

Proof. In a CCC space X, for any open cover $\mathcal{U}$ of X there is a countable $\mathcal{U}' \subseteq \mathcal{U}$ whose union is dense in X. $\square$

We saw above that ω_1 is a normal, strongly 1-star-Lindelöf space that is not Lindelöf. In a sense this unfortunate: it would have been convenient if normal star-Lindelöf spaces were Lindelöf. A simple argument shows that a normal collectionwise Hausdorff space that is not $\aleph_1$-compact cannot be DCCC. So the search for normal spaces that distinguish the star-Lindelöf conditions may well be difficult: such spaces would have to be normal but not collectionwise Hausdorff. A further complication is that, in [8], Fleissner proved that, under $V=L$, all normal spaces with character $\leq \aleph_1$ are collectionwise Hausdorff. So any normal space defined in ZFC that distinguishes two star-Lindelöf conditions will have a large character. A normal 1-star-Lindelöf not strongly 1-star-Lindelöf space exists in ZFC and can be found in [14]. It has character $\aleph_\omega$.

The search for normal spaces distinguishing the star-Lindelöf properties is reduced considerably by a corollary of a result due to Ikenaga [12].

Theorem 3.1.7. *Let X be a normal DCCC space. If $\mathcal{U}$ is any open cover, there is a countable $B \subseteq X$ such that* $\operatorname{st}(B, \mathcal{U})$ *is dense in X.*

Proof. Suppose there were an open cover $\mathcal{U}$ that failed to have this property. Then, for $\beta < \omega_1$, we can pick $x_\beta \in X - \overline{\operatorname{st}(\{x_\alpha : \alpha < \beta\}, \mathcal{U})}$. Let

$$V_\beta = \operatorname{st}(x_\beta, \mathcal{U}) - \overline{\operatorname{st}(\{x_\alpha : \alpha < \beta\}, \mathcal{U})}.$$

Then V_β is open, $x_\beta \in V_\beta$ and $V_\beta \cap V_{\beta'} = \emptyset$ whenever $\beta \neq \beta'$. If we let $H = \{x_\alpha : \alpha < \omega_1\}$, H is closed; after all, no $U \in \mathcal{U}$ can contain more than one element of H and $\mathcal{U}$ covers X. By normality, there is an open U such that $H \subseteq U \subseteq \bar{U} \subseteq \bigcup_{\alpha < \omega_1} V_\alpha$. It is easy to verify that $\{U \cap V_\alpha : \alpha < \omega_1\}$ is an uncountable discrete collection of nonempty open sets. $\square$

Corollary 3.1.8. *Normal DCCC spaces are strongly 2-star-Lindelöf.*

The question remains whether there are normal strongly 2-star-Lindelöf spaces that are not 1-star-Lindelöf. Our final result in this section shows that there are no such spaces that are also perfect. This completes Fig. 4.

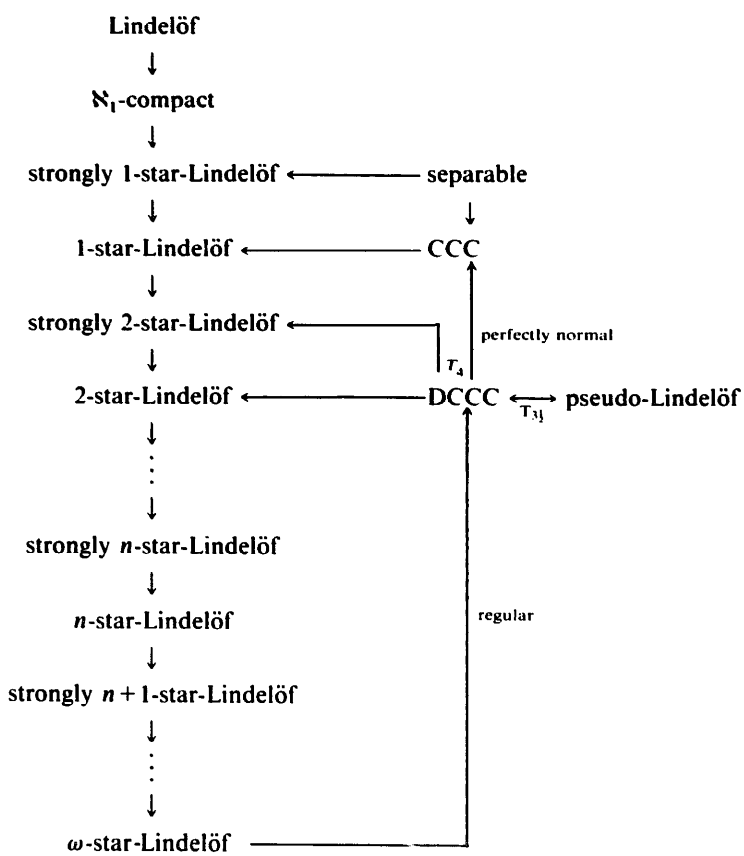

Fig. 4.

Theorem 3.1.9. *Every perfectly normal DCCC space is CCC.*

Proof. In a similar vein to Theorem 3.1.7. □

3.2. Moore spaces

3.2.1. n-separability in Moore spaces

The concept of n-separability in Moore spaces served as the inspiration of our study of star covering properties. The following results are from [20, 21].

Definition. The subset X of the Moore space S is *n-dense* in S with respect to the development $\mathscr{G}=\{\mathscr{G}_i\}$ for S provided for each i, $S=\mathrm{st}^n(X, \mathscr{G}_i)$.

Definition. A Moore space S is *n-separable* provided for each development $\mathscr{G}$ for S, there exists a countable subset K of S such that K is n-dense in S with respect to $\mathscr{G}$.

Definition. A Moore space is *wd-normal* provided for each open set U in S, there exists a sequence $\{U_n\}$ of open subsets of U such that for each n, $\overline{U_n}\subset U$, and $U\subseteq\bigcup\{U_n\}$. (This concept was later renamed (quite sensibly) by Blair as *countable tiling.*)

Theorem. *A Moore space has the countable chain condition if and only if it is* 2*-separable and has countable tiling.*

Theorem. *A Moore space has the discrete countable chain condition if and only if it is* 3*-separable.*

Example. There exists a Moore space with the discrete countable chain condition but without the countable chain condition.

Question. Does there exist a 3-separable Moore space which is not 2-separable?

Clearly, (by applying the star-Lindelöf property to each stage of the development) it follows that a Moore space is 2-separable if and only if it is strongly 2-star-Lindelöf, and that a Moore space is 3-separable if and only if it is 2-star-Lindelöf.

3.2.2. The Moore space machine

In [21, 22], Reed developed a construction technique which associates a Moore space $\mathscr{M}(X)$ to each regular first countable space X such that $\mathscr{M}(X)$ is separable (respectively, locally separable, CCC or DCCC) if and only if X has the corresponding property. This relationship has been extended to other chain conditions (e.g. calibers) by McIntyre [16]. Hence, one would expect similar results for the star-Lindelöf properties, whereby distinctions between these properties could be shown to be identical for the two classes of spaces. Surprisingly, this is not the case. Whereas previously, the construction has been used to raise distinctions between chain conditions in simple first countable examples to Moore space examples, it is used below to create distinction in the derived Moore spaces which are not made in the first countable spaces.

The construction. Let X denote a regular first countable space. For each $x\in X$, denote by $\{U_n(x)\}$ a sequence of open sets in X which forms a local base at x such that for each n, $\overline{U_{n+1}(x)}\subseteq U_n(x)$. Now, for each $m\in\mathbb{N}^+$, let $A_m=\{(n_1,n_2,\dots,n_m)\colon n_1=1$ and for $1\leqslant i\leqslant m$, $n_i\in\mathbb{N}^+\}$. Let $A=\bigcup_{1\leqslant m<\infty}A_m$. For each $a=(n_1,n_2,\dots,n_m)\in A$, denote by S_a a unique copy of X such that all copies are pairwise disjoint, and for each $x\in X$, denote by $(x_{n1},x_{n2},\dots,x_{nm})$ the element of S_a which is identified with x. Let $\mathcal{M}(X)=\bigcup\{S_a\colon a\in A\}$ and define a development for $\mathcal{M}(X)$ as follows: For each $j\in\mathbb{N}^+$, $a=(n_1,n_2,\dots,n_m)\in A$, and $p=(y_{n1},y_{n2},\dots,y_{nm})\in S_a$, let $G_j(p)=\{p\}\cup\{(x_{n1},x_{n2},\dots,x_{nm},x_{k1},x_{k2},\dots,x_{kc})\colon x\in X$, $c\in\mathbb{N}^+$ and $x\in U_{k1+j}(y)$, for some j such that $k_i\geqslant j$ and $1\leqslant i\leqslant c\}$.

It follows that $\mathcal{B}=\{G_j(p)\colon p\in\mathcal{M}(X)$, and $j\in\mathbb{N}^+\}$ is a basis for a topology on $\mathcal{M}(X)$ and that $\{\mathcal{G}_n\}$, where for each n, $\mathcal{G}_n=\{G_j(p)\colon\ p\in\mathcal{M}(X)$ and $j\geqslant n\}$, is a development for the Moore space $\mathcal{M}(X)$.

Theorem 3.2.2.1. *If X is Lindelöf, then $\mathcal{M}(X)$ is 1-star-Lindelöf.*

Proof. Without loss of generality, for each $p\in\mathcal{M}(X)$, let G_p denote a basic open set for p, and let $\mathcal{U}=\{G_p\colon\ p\in\mathcal{M}(X)\}$. It suffices to show that there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{st}(\bigcup\mathcal{V},\mathcal{U})$ covers $\mathcal{M}(X)$. In fact, since A is countable, it suffices to show that for each $a\in A$, there exists a countable subset $\mathcal{V}_a$ of $\mathcal{U}_a=\{G_p\colon p\in S_a\}$ such that $\mathrm{st}(\bigcup\mathcal{V}_a,\mathcal{U}_a)$ covers S_a.

Let π denote the natural projection mapping from $\mathcal{M}(X)$ onto X, i.e.,

$$\pi(x_{n1},x_{n2},\dots,x_{nm})=x.$$

Observe that for each a in A, π restricted to S_a is one-to-one. As noted in [22], π is an open countable-to-one continuous mapping from $\mathcal{M}(X)$ to X. For each $n\in\mathbb{N}^+$ and $a=(n1,n2,\dots,nm)\in A$, let an denote $(n1,n2,\dots,nm,n)\in A$.

Now, for each $G_p\in\mathcal{U}_a$, there exists $jp\in\mathbb{N}^+$ such that $G_p=G_{jp}(p)$. Observe that:

(1) $\forall p\in S_a$ and $n\geqslant jp$, $\pi(p)\in\pi(G_p\cap S_{an})=U_{n+jp}(\pi(p))$, and

(2) $\forall n\in\mathbb{N}^+$, $\mathcal{H}_n=\{\pi(G_p\cap S_{am})\colon G_p\in\mathcal{U}_a$ and $m\geqslant n\}$ is an open covering of X.

Since X is Lindelöf, for each n, let $\mathcal{K}_n$ denote a countable subcovering of $\mathcal{H}_n$ for X, and for each $U\in\mathcal{K}_n$, let $G_{U,n}\in\mathcal{U}_a$ such that there exists $p\in S_a$ and $m\geqslant n$ such that $G_{U,n}=G_p$ and $\pi(G_p\cap S_{am})=U$. Finally, let $\mathcal{V}_a=\{G_{U,n}\colon n\in\mathbb{N}^+,\ U\in\mathcal{K}_n\}$.

Claim. $\mathrm{st}(\bigcup\mathcal{V}_a,\mathcal{U}_a)$ *covers* S_a. Suppose $p\in S_a$ and consider $G_p=G_{jp}(p)\in\mathcal{U}_a$. From (2), there exists $U\in\mathcal{K}_{jp}$ such that $\pi(p)\in U$ and $U=\pi(G_q\cap S_{an})$ for some $q\in S_a$ and $n\geqslant jp$. Then $G_U\in\mathcal{V}_a$ and by (1), $\pi^{-1}(\pi(p))\cap G_{jp}(p)\cap G_U\cap S_{an}\neq\emptyset$. $\square$

3.2.3. Examples

In the last section, we saw that a regular space is DCCC if and only if it is 2-star-Lindelöf, if and only if it is ω-starcompact (and that these are all equivalent to pseudo-Lindelöfness for completely regular spaces). Furthermore, it is well known that $\aleph_1$-compact Moore spaces are Lindelöf (see [13]). Here, we show that there exist Moore spaces which distinguish those star-Lindelöf properties which have not already been eliminated by these constraints.

Examples 3.2.3.1. Strongly 1-star-Lindelöf Moore spaces which are not $\aleph_1$-compact.

Proof. Both the tangent disc space and Ψ are separable (and hence strongly 1-star-Lindelöf), but neither is $\aleph_1$-compact. □

Example 3.2.3.2. A CCC (and hence 1-star-Lindelöf) Moore space that is not strongly 1-star-Lindelöf.

Proof. Let X be the Pixley–Roy topology obtained from $\mathbb{R}$, i.e., $X=\{x\subseteq\mathbb{R}: x$ is finite$\}$ and a basic open set takes the form $[x, U]=\{y\in X: x\subseteq y\subseteq U\}$, where $x\in X$ and U is open in $\mathbb{R}$. Such a construction was first described in [19] and makes X a Moore space.

Let $\mathcal{B}$ be some countable base for $\mathbb{R}$. Defining

$$\mathcal{B}^*=\{\bigcup F: F \text{ is a finite subset of } \mathcal{B}\},$$

then $\mathcal{B}^*$ is countable. Obviously if $[x, U]$ is a basic open set in X, there is some $B\in\mathcal{B}^*$ such that $[x, B]\subseteq[x, U]$. If $\{[x_\alpha, U_\alpha]: \alpha<\omega_1\}$ were an uncountable collection of nonempty basic open sets, then, because $\mathcal{B}^*$ is countable, for some $\alpha\neq\alpha'$, $B_\alpha=B_{\alpha'}$. But then

$$x_\alpha\cup x_{\alpha'}\in[x_\alpha, B_\alpha]\cap[x_{\alpha'}, B_{\alpha'}]\subseteq[x_\alpha, U_\alpha]\cap[x_{\alpha'}, U_{\alpha'}].$$

So no uncountable collection of nonempty open sets in X can be pairwise disjoint, i.e., X is CCC and hence is 1-star-Lindelöf.

However, X is not strongly 1-star-Lindelöf. Consider the cover $\mathcal{U}=\{[\{t\},\mathbb{R}]: t\in\mathbb{R}\}$. If $A=\{x_n\in X: n\in\omega\}$ is any countable subset in X, then $\bigcup A\subseteq\mathbb{R}$ is countable. Pick some $s\in\mathbb{R}-\bigcup A$. Then $\{s\}\in U\in\mathcal{U}$ if and only if $U=[\{s\},\mathbb{R}]$. If $A\cap[\{s\},\mathbb{R}]\neq\emptyset$, then for some $x_n\in A$, $\{s\}\subseteq x_n\subseteq\mathbb{R}$, contradicting the choice of s. Hence $A\cap\operatorname{st}(\{s\},\mathcal{U})=A\cap[\{s\},\mathbb{R}]=\emptyset$ and therefore $\{s\}\notin\operatorname{st}(A,\mathcal{U})$. So if $A\subseteq X$ is countable, $\operatorname{st}(A,\mathcal{U})\neq X$, i.e., X is not strongly 1-star-Lindelöf.

So X is a Moore space that is 1-star-Lindelöf but not strongly 1-star-Lindelöf. □

Example 3.2.3.3. A 1-star-Lindelöf Moore space which is not CCC.

Proof. Let H and K denote a pair of Bernstein sets in the real line, i.e., each of H and K is uncountable, $H\cap K=\emptyset$, each uncountable subset of H has a limit point in K, and each uncountable subset of K has a limit point in H. Let $X = H \cup K$, and give X the inherited real-line topology except that points of K are isolated. Clearly X is a regular, Lindelöf, first countable space which does not have the CCC. From Theorem 3.2.2.1 and the fact that the CCC is invariant under the Moore space machine, it follows that $\mathcal{M}(X)$ is the desired Moore space. □

Example 3.2.3.4. A strongly 2-star-Lindelöf Moore space which is not 1-star-Lindelöf.

Proof. Let X denote ω_1 with the order topology, and let $\{U_n(x)\}$ denote a non-increasing local base of countable, clopen sets at each point of ω_1. As pointed out in [21], $\mathcal{M}(X)$ is 2-separable and hence strongly 2-star-Lindelöf. This follows immediately from the fact that $\mathcal{M}(X)$ is DCCC and locally separable. To see that $\mathcal{M}(X)$ is not 1-star-Lindelöf, consider a covering $\mathcal{U}$ by basic open sets. Given any countable subset $\mathcal{V}$ of $\mathcal{U}$, $\bigcup \mathcal{V}$ is countable. Hence pick a nonlimit ordinal $x \in \omega_1$ such that no element in $\bigcup \mathcal{V}$ is identified with x. It follows that no basic open set in $\mathcal{U}$ containing $(x_1) \in S_{(1)}$ meets $\bigcup \mathcal{V}$. Alternatively, observe that any locally separable 1-star-Lindelöf Moore space is separable. □

Example 3.2.3.5. A 2-star-Lindelöf Moore space which is not strongly 2-star-Lindelöf.

Remark. The existence of a 2-star-Lindelöf Moore space which is not strongly 2-star-Lindelöf has now been established by the second author. It is also an example of a DCCC Moore space with a σ-locally countable base (hence a σ-para-Lindelöf space) which is not Lindelöf. This space was obtained after the second author had seen a regular nonfirst countable space with these properties constructed by Heath; both spaces answer questions raised in [2] and will appear in [10].

Question 3.2.3.6. The construction in Example 3.2.3.5 is a complex variaton of the Moore space machine given above. It is not known in general whether $\mathcal{M}(X)$ must be a strongly 2-star-Lindelöf Moore space if X is a regular, first countable strongly 2-star-Lindelöf space. However, there is an open mapping from $\mathcal{M}(X)$ onto X. Hence, it does follow that if $\mathcal{M}(X)$ has either of the star-Lindelöf properties, then so must X. Example 3.2.3.4 shows that $\mathcal{M}(X)$ need not be 1-star-Lindelöf if X is 1-star-Lindelöf.

3.3. Non-Moore space examples

Although the examples in 3.2 are by no means trivial, we have distinguished the most important star-Lindelöf properties using Moore spaces. In this section, then, the aim is twofold. Firstly, using a technique similar to Sarkhel's scheme, we distinguish the star-Lindelöf properties for Hausdorff spaces. Secondly, several examples are presented to show how products of star-Lindelöf spaces behave.

Fix a positive integer, n.

Example 3.3.1. A Hausdorff space which is n-star-Lindelöf but not strongly n-star-Lindelöf.

Proof. The long segment $(X, \mathcal{T})$ is constructed from the ordinal space $[0, \omega_1]$ by placing between each ordinal α and its successor $\alpha+1$ a copy of the unit interval $I=(0, 1)$. X is then linearly ordered and it is given the order topolgy. This makes $(X, \mathcal{T})$ compact, T_2 and connected.

Let $A_1, A_2, \ldots, A_{2n+1}$ be pairwise disjoint dense subsets covering X, with 0 and all limit ordinals in A_{2n+1}. Let $E_{2i+1}=A_{2i}\cup A_{2i+1}\cup A_{2i+2}$ for $i=0, 1, \ldots, n$ and $E_{2i}=A_{2i}$ for $i=1, 2, \ldots, n$, where $A_0=A_1$ and $A_{2n+2}=A_{2n+1}$.

Define a new topology $\mathcal{T}_n$ on X as follows: U is open in X if for every $x\in U$ there is some interval in the order topology $I_x \ni x$ such that $I_x \cap E_{n(x)} \subseteq U$, where $n(x)$ is the unique integer k satisfying $x\in A_k$.

This topology makes X Hausdorff. We will show that $(X, \mathcal{T}_n)$ is n-star-Lindelöf but not strongly n-star-Lindelöf.

Claim 1. *X is n-star-Lindelöf (in fact n-starcompact).* Identical to the proof in Example 2.3.8, since $(X, \mathcal{T})$ is compact.

Claim 2. *X is not strongly n-star-Lindelöf.* Define $x_0=0$ and choose $y_0\in A_{2n+1}$ with $x_0<y_0<\omega_1$ and y_0 not in the $(0, 1)$ segment containing or immediately following x_0. Suppose we have defined x_β, y_β in A_{2n+1} for all $\beta<\alpha$ such that $0=x_0<y_0\leqslant\cdots\leqslant x_\beta<y_\beta\leqslant\cdots$, and x_β, y_β do not lie in the same $(0, 1)$ segment and

$$\bigcup_{\gamma\leqslant\beta}(x_\gamma, y_\gamma)\supseteq\bigcup_{i=1}^{2n}A_i\cap[0, y_\beta).$$

Let $\zeta=\sup\{y_\beta\colon \beta<\alpha\}$. Then $\zeta<\omega_1$ and either ζ is a limit ordinal, or ζ is one of the y_β. In either case, $\zeta\in A_{2n+1}$. Let $x_\alpha=\zeta$ and pick $y_\alpha\in A_{2n+1}$ with $x_\alpha<y_\alpha<\omega_1$ and y_α not in the $(0, 1)$ segment containing or immediately following x_α. It remains to show that $\bigcup_{\gamma\leqslant\alpha}(x_\gamma, y_\gamma)\supseteq\bigcup_{i=1}^{2n}A_i\cap[0, y_\alpha)$. Suppose $z\in\bigcup_{i=1}^{2n}A_i\cap[0, y_\alpha)$. Then $z\neq x_\alpha$ as $x_\alpha\in A_{2n+1}$, so either $z<x_\alpha$ or $z>x_\alpha$. If $z<x_\alpha$, then $z<y_\beta$ for some $\beta<\alpha$ and so $z\in\bigcup_{\gamma\leqslant\beta}(x_\gamma, y_\gamma)$. If, on the other hand, $z>x_\alpha$, then $z\in(x_\alpha, y_\alpha)\subseteq\bigcup_{\gamma\leqslant\alpha}(x_\gamma, y_\gamma)$. Hence, by induction, $\{(x_\alpha, y_\alpha)\colon \alpha<\omega_1\}$ is a pairwise disjoint collection of nonempty intervals covering $\bigcup_{i=1}^{2n}A_i$.

If we now let $\mathcal{U}=\{E_{2n+1}\}\cup\{(x_\alpha, y_\alpha)\cap E_i\colon \alpha<\omega_1, i=1, 2, \ldots, 2n\}$, then, $\mathcal{U}$ is an open cover of $(X, \mathcal{T}_n)$. Pick $z_\alpha\in(x_\alpha, y_\alpha)\cap A_1$. Let B be any countable subset of X. Because the (x_α, y_α) are pairwise disjoint, there is some α_0 such that $(x_{\alpha_0}, y_{\alpha_0})\cap B=\emptyset$. Now, by the construction of $\mathcal{U}$,

$$\mathrm{st}^k(z_{\alpha_0}, \mathcal{U})\subseteq(x_{\alpha_0}, y_{\alpha_0})\cap(E_1\cup E_2\cup\cdots\cup E_{2n})\subseteq(x_{\alpha_0}, y_{\alpha_0})$$

for $k\leqslant n$. Therefore for any countable $B\subseteq X$, $\mathrm{st}^n(B, \mathcal{U})\neq X$, i.e., $(X, \mathcal{T}_n)$ is not strongly n-star-Lindelöf. □

A similar method (with $X = A_1 \cup \cdots \cup A_{2n}$) generates a T_2 space that is strongly n-star-Lindelöf, but not $n-1$-star-Lindelöf.

Example 3.3.2. A Hausdorff space which is ω-star-Lindelöf but not n-star-Lindelöf for any $n < \omega$.

Proof. Let $X_n = X \times \{n\}$ with topology $\mathcal{T}_n$ as above and $Y = \{0\} \cup \bigoplus_{n \in \mathbb{N}} X_n$. A basic open set containing 0 takes the form $\{0\} \cup \bigcup_{n>m} X_n$. Then Y is a Hausdorff space.

If $\mathcal{U}$ is open cover of Y, pick $U \in \mathcal{U}$ that contains 0. Then $U \supseteq \{0\} \cup \bigcup_{n>m} X_n$ for some m. Let $B_k \subseteq X_k$ be countable and satisfy $\mathrm{st}^k(B_k, \mathcal{U}) = X_k$, for $1 \leqslant k \leqslant m$. Then $B = \{0\} \cup \bigcup_{1 \leqslant k \leqslant m} B_k$ is a countable subset of Y such that $\mathrm{st}^m(B, \mathcal{U}) = Y$. This shows that Y is ω-star-Lindelöf.

Now let $\mathcal{V}_m$ be an open cover of X_m witnessing that X_m is not strongly m-star-Lindelöf. Then

$$\mathcal{U} = \mathcal{V}_m \cup \left\{ \{0\} \cup \bigcup_{n>m} X_n \right\} \cup \{X_n : n < m\}$$

is an open cover of Y that shows Y cannot be strongly m-star-Lindelöf. Hence Y is ω-star-Lindelöf, but not m-star-Lindelöf for any m. □

Example 3.3.3. A space that is the product of a Lindelöf space and a strongly 1-star-Lindelöf space but which is not strongly 1-star-Lindelöf.

Proof. Let $X = \omega_1$ with the usual topology and $Y = \omega_1 + 1$ with the following topology: if $\alpha < \omega_1$, then $\{\alpha\}$ is open. A set containing ω_1 is open if and only if its complement in Y is countable. With this topology, Y is T_3 and Lindelöf. The claim is that $X \times Y$ is not strongly 1-star-Lindelöf, despite X being strongly 1-star-Lindelöf and Y being Lindelöf.

For each $\alpha < \omega_1$ the set $U_\alpha = X \times \{\alpha\}$ is open in $X \times Y$. For each $\beta < \omega_1$, the set $V_\beta = [0, \beta] \times (\beta, \omega_1]$ is open in $X \times Y$. Certainly, the collection $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\beta : \beta < \omega_1\}$ is an open cover of $X \times Y$. Now, let $B = \{(x_n, y_n) \in X \times Y : n \in \mathbb{N}\}$ be some countable subset of $X \times Y$.

We now pick $y < \omega_1$ such that $y \neq y_n$ for all n and then pick $x \in X$ such that $x > y$ and $x > x_n$ for all n. If $(x, y) \in U_\alpha$, then $U_\alpha \cap B = \emptyset$, because $\alpha = y$. Also, $(x, y) \notin V_\beta$ for any $\beta < \omega_1$, for otherwise $(x, y) \in [0, \beta] \times (\beta, \omega_1]$ which contradicts $x > y$. Hence we see that $(x, y) \notin \mathrm{st}(B, \mathcal{U})$. As B was an arbitrary countable subset of $X \times Y$, this space cannot be strongly 1-star-Lindelöf. □

A simple modification of Theorem 2.4.2 proves that the product $X \times Y$ of an n-star-Lindelöf space X with a compact space Y is n-star-Lindelöf. A further modification shows that $X \times Y$ is strongly n-star-Lindelöf if X is strongly n-star-Lindelöf and Y is compact and separable. However, as we will now see, $X \times Y$ need not be strongly 1-star-Lindelöf if X is and Y is only compact.

Example 3.3.4. A space that is not strongly 1-star-Lindelöf, despite being the product of a strongly 1-star-Lindelöf space and a compact space.

Proof. Let X be the space Ψ in Example 2.2.5 and index S as $\{s_\alpha: \alpha<\kappa\}$. As X is separable, it is strongly 1-star-Lindelöf. Let $Y=\{y_\alpha: \alpha<\kappa\}\cup\{\infty\}$ where all elements of Y are distinct. Each $\{y_\alpha\}$ is open and open sets containing ∞ have finite complement. With this topology, Y is compact T_2.

Using the notation of Example 2.2.5, we define an open cover of $X\times Y$ as follows:

$$\mathcal{U}=\{X\times\{y_\alpha\}: \alpha<\kappa\}\cup\{N_{s_\alpha}\times Y-\{y_\alpha\}: \alpha<\kappa\}\cup\{\{n\}\times Y: n\in\mathbb{N}\}.$$

Observe that $(s_\alpha, y_\alpha)\in U\in\mathcal{U}$ if and only if $U=X\times\{y_\alpha\}$. If $B\subseteq X\times Y$ is countable, there is some $\alpha<\kappa$ such that $B\cap(X\times\{y_\alpha\})=\emptyset$ (as S is uncountable). By our observation, $(s_\alpha, y_\alpha)\notin \mathrm{st}(B, \mathcal{U})$. So $X\times Y$ is not strongly 1-star-Lindelöf, despite X being strongly 1-star-Lindelöf and Y compact.

Although $X\times Y$ is not 1-starcompact, a similar argument to that used in Example 2.2.5 shows that for any basic open cover $\mathcal{U}$ there is a finite $B\subseteq\mathbb{N}\times Y$ such that $\mathrm{st}(B, \mathcal{U})\supseteq\mathbb{N}\times Y$. As $\mathbb{N}\times Y$ is dense in $X\times Y$, $\mathrm{st}^2(B, \mathcal{U})=X\times Y$ and hence $X\times Y$ is strongly 2-starcompact. □

This example also shows that star-Lindelöf is not an inverse invariant of proper mappings. After all, the projection $\pi: X\times Y\to X$ is proper if Y is compact. Fleischman's example verifies that the same is true for starcompactness.

As the Sorgenfrey line shows, the product of Lindelöf spaces need not be Lindelöf. With this result in mind, it would be nice if we could show that the product of Lindelöf spaces had to be strongly n-star-Lindelöf for some fixed n. Sadly this is not the case for $n=1$.

The space described here is an example of Przymusiński's cited in Burke's handbook article [3] and uses the following result of Kuratowski:

Theorem 3.3.5. *There exists a partition $\{A_k: k\in\mathbb{N}\}$ of $\mathbb{R}$ such that $|A_k\cap F|=\mathfrak{c}$ for any $k\in\mathbb{N}$ and any uncountable closed subset F of $\mathbb{R}$.*

Example 3.3.6. Pick such a set A. Let $X=\mathbb{R}-A$ have the induced metric topology. Let Y be the set $\mathbb{R}$ with each point of $\mathbb{R}-A$ isolated and points of A having metric neighbourhoods. Both X and Y are Lindelöf (the property of A ensures that Y is). Furthermore, both spaces are T_3 and first countable, so their product is T_3 first countable also.

Let $D=\{(x,x)\in X\times Y: x\in X\}$. Then D is uncountable as A has uncountable complement. We claim that D is closed and discrete. For $(x,x)\in D$, $X\times\{x\}$ is open in $X\times Y$ and has intersection $\{(x,x)\}$ with D. For $(x,y)\notin D$, let $\varepsilon=\frac{1}{2}|x-y|>0$. Then $B_\varepsilon(x)\times B_\varepsilon(y)$ is open in the product space (as Y's topology is finer than the metric one), contains (x,y) and does not meet D. This verifies the claim. It is now easy to show that $\{(X\times Y)-D\}\cup\{X\times\{y\}: y\in\mathbb{R}-A\}$ is an open cover that witnesses that $X\times Y$ is neither strongly 1-star-Lindelöf nor CCC.

In fact, this space is 1-star-Lindelöf, despite not being CCC. Let $\mathscr{U}$ be any basic open cover of $X\times Y$. $X\times A$ is a separable metric space and hence Lindelöf. So there is a countable $\mathscr{V}\subseteq\mathscr{U}$ covering $X\times A$. It is sufficient to show that for at most countably many $y\in\mathbb{R}-A$, $X\times\{y\}-\mathrm{st}(\bigcup\mathscr{V},\mathscr{U})\neq\emptyset$, because then for each such y we can get a countable subset of $\mathscr{U}$, covering $X\times\{y\}$ (which is Lindelöf). The collection $\mathscr{V}'$ consisting of all the sets in these countable subsets together with all those sets in $\mathscr{V}$ is countable and satisfies $\mathrm{st}(\bigcup\mathscr{V}',\mathscr{U})=X\times Y$, as required.

To this end, suppose $X\times\{y\}-\mathrm{st}(\bigcup\mathscr{V},\mathscr{U})\neq\emptyset$ for uncountably many $y\in\mathbb{R}-A$. For each $\alpha<\omega_1$, select some $U_\alpha=V_\alpha\times\{y_\alpha\}\in\mathscr{U}$ such that U_α is not a subset of $\mathrm{st}(\bigcup\mathscr{V},\mathscr{U})$ and so that distinct α correspond to distinct y_α. Let P be a countable dense subset of X. For some $p\in P$, $p\in V_\alpha$ for uncountably many α. Let $E=\{y_\alpha: p\in V_\alpha\}$, so E is uncountable. By the property of A, there exists some $y\in A\cap\bar{E}$. Then $(p,y)\in V$ for some $V\in\mathscr{V}$ (as $\mathscr{V}$ covers $X\times A$) and must meet some U_α.

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