Resumé of research achievements
Lloyd N. Trefethen

Lloyd N. ("Nick") Trefethen is a numerical analyst with wide interests in applied mathematics. His research activities have involved theory, algorithms, software and physical applications. He has made well-known contributions in each of the areas listed below (the numbering follows his list of publications) and is generally regarded as a leading figure in areas II, III, IV and VI.

Roughly half of Trefethen's work has been associated with the topic of eigenvalues, pseudospectra and dynamics for nonhermitian or more generally nonnormal systems. At issue here are phenomena that arise in problems where the eigenvectors of a matrix or linear operator, if they exist, form an ill-conditioned basis in which to represent the behaviour of interest.

II. Finite difference and spectral methods for partial differential equations

Trefethen's early work in this area concerned dispersive wave propagation in finite difference grids, a topic which is now standard but had received little attention by 1980 [II.1]. He showed that the "GKS" criterion for stability of boundary conditions can be interpreted as a group velocity condition and that this interpretation leads to new quantitative estimates of instability [II.4]. Among other applications of wave ideas he and Halpern solved the problem of which Engquist-Majda absorbing boundary conditions are well-posed [III.8].

Later Trefethen became involved with spectral methods for high-accuracy solution of ordinary and partial differential equations, one of his main ongoing activites and the subject of one of his textbooks. In the course of this work he recognized contexts in which a study of nonnormality was crucial to the understanding of behaviour of ODE and PDE discretizations. He and Reddy proposed and proved now-standard theorems to the effect that "method of lines" discretizations of PDEs are stable if and only if the pseudospectra of the spatial discretization operator lie in the stability region of the time discretization formula [II.15]. Trefethen and D. J. Higham showed that related mathematics governs the phenomenon of stiffness of ODEs [II.16].

Trefethen's interest in PDEs is wide-ranging, and at Oxford he is coordinating a project to produce a book called *The PDE Coffee Table Book*. This book will contain 2-page presentations of each of 100 PDEs, illustrating the key features of their mathematics and behaviour.

III. Numerical linear algebra

Numerical linear algebra has been one of Trefethen's main research areas since his graduate days at Stanford and is the subject of another one of his textbooks, a SIAM bestseller. One problem he has been involved with concerns the algorithm of Gaussian elimination, which has been known for forty years to be catastrophically unstable for some matrices and yet is "always" stable in practice. There was a popular view that the cause of this phenomenon was that the matrices arising in practice are somehow special, but Trefethen and Schreiber showed that there is no need for such a hypothesis: the bad matrices are exponentially rare among various classes of random matrices [III.2].

Trefethen's broader activities in numerical linear algebra have concerned nonsymmetric matrix iterations such as GMRES, BiCGSTAB and QMR. In strongly nonsymmetric cases he and coauthors showed that eigenvalues can be arbitrarily unreliable as a basis on which to construct an algorithm, and proposed the first eigenvalue-free hybrid matrix iteration [III.4]. They also investigated the behaviour of random matrices and random recurrence relations. Whereas random dense matrices are well-conditioned, Trefethen and Viswanath showed that random triangular matrices are exponentially ill-conditioned, having condition numbers that grow with the dimension N (with probability 1) at precise rates such as 2^N or $e^{N/2}$, depending on the class of matrices [III.10].

IV. Numerical conformal mapping and complex analysis

Since working with Birkhoff and Henrici as a student Trefethen has been a well-known figure in computational complex analysis. In particular, beginning in 1978 he has worked on the problem of computing Schwarz-Christoffel conformal maps onto polygons; such maps present challenging problems of numerical integration and solution of systems of equations. His first work proposed what became the standard algorithm in this field [IV.1], and later papers investigated applications in electrical engineering, fluid dynamics, queuing theory, and complex approximation. His Fortran package SCPACK became the standard software tool for such problems until it was supplanted by the MATLAB SC Toolbox written by his former student Driscoll. In 2002 he and Driscoll published a monograph, the only book on Schwarz-Christoffel mapping, which covers a wide range of applications and variations on the theme [I.5].

Recently Trefethen and his student Banjai made a contribution in another area of complex analysis. In 1949 Goodman posed the *omitted area problem* in univalent function theory: what is the maximal area A^* of the unit disk that the image of a univalent function, normalized in the standard fashion, can omit? Subsequent work in the decades since then led to little progress, the best bounds as of 2001 being $0.240005\pi < A^* \leq 0.31\pi$. Trefethen and Banjai devised a complex spectral method to compute the maximum to high accuracy, and to ten digits they found (contradicting the earlier publications) $A = 0.2385813248\pi$ [IV.11].

V. Approximation theory

Trefethen's earliest research interest, beginning with his undergraduate thesis at Harvard, was approximation theory. That thesis discovered that error curves in complex approximations tend to be exponentially close to perfect circles. Eventually he was able to analyze this phenomenon in terms of a theorem of Carathéodory and Féjér [V.1], and in work with Gutknecht, this led to a new "Carathéodory-Féjér method", based on singular values of Hankel matrices, for computing near-best approximations [V.8]. This seemingly specialized set of results turned out to be an independent discovery of certain aspects of what is now the large field known as H^∞ control. Trefethen and Gutknecht also made other contributions to rational approximation, showing that approximants on the unit disk need not be unique and that complex approximants can be arbitrarily better than real ones [V.5]. Recently Trefethen has become involved with barycentric interpolation and related approximation matters closely linked to computation [V.19,VIII.9].

VI. Eigenvalues, pseudospectra and dynamics

Trefethen is known as the person who coined the term "pseudospectra" and he has advocated attention to effects of nonnormality in a wide variety of applications [VI.3,VI.5,VI.11]. This activity began in the late 1980s, when he encountered related phenomena in the disparate areas of spectral methods for PDEs, eigenvalue theory for Toeplitz matrices, and convergence of nonsymmetric matrix iterations. It became clear that there were widespread misconceptions about such effects among mathematicians and scientists. He has been advocating the computation of pseudospectra to shed light on such problems, and the computational side of this subject has developed since then very far, to the point where one can now routinely calculate these sets for virtually any problem for which one can compute eigenvalues [VI.19]. The widely used software system EigTool was developed at Oxford by Trefethen's student Wright.

Trefethen has pursued the theme of nonnormality in applications as diverse as shuffling of cards, transition to turbulence, the design of lasers, and nonhermitian quantum mechanics [VI.20]. He and his former post-doc Embree will publish a major book in 2004 summarizing this work [I.6]. To date, hundreds of papers have been published on pseudospectra by authors around the world.

VII. Fluid dynamics

Trefethen is not a leading figure in the field of fluid mechanics, but he has made one set of contributions that has had considerable impact. This concerns the century-old problem of transition to turbulence of high-speed flows, notably in pipes and channels. For many years, it was assumed that the eigenmodes of the linearized problem must have something to do with the process by which these flows become unstable and turbulent. This point of view was largely supplanted in the early 1990s by work of a number of people including Butler and Farrell, Schmid and Henningson, Gustavsson, Boberg and Brosa, Grossmann, and Trefethen. The new view emphasizes the role of linear, nonnormal effects in transition to turbulence, notably the vortex tilting mechanism whose roots go back as far as Orr and Rayleigh. Trefethen's paper [VII.3], which has become a standard reference in this area, was the first to compute pseudospectra for the "full" linearized Navier-Stokes operators for such problems, and showed that a linear nonmodal amplification mechanism of size $O(R)$ may lead, through nonlinear interactions, to the possibility of a flow becoming destabilized by perturbations of size not just $O(R^{-1})$ but smaller. Several of his papers concern these linear-nonlinear interactions. Subsequent work by Chapman has confirmed that such an effect occurs in planar Poiseuille flow.

VIII. Other

Finally Trefethen has a serious interest in the foundations and in the popularization of numerical computation—a field at the heart of the scientific enterprise that is largely invisible not only to laymen but even, often, to scientists. He has published essays on such matters and made them the subject of his Inaugural Lecture at Oxford. In 2002 he organized a "100-Digit, 100-Dollar Challenge" that engaged the efforts of hundreds of contestants worldwide [VIII.8].