

Parametric Datatype-Genericity

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Abstract

Datatype-generic programs are programs that are parametrized by a *datatype* or *type functor*: whereas polymorphic programs abstract from the ‘integers’ in ‘lists of integers’, datatype-generic programs abstract from the ‘lists of’. There are two main styles of datatype-generic programming: the *Algebra of Programming* approach, characterized by structured recursion operators arising from initial algebras and final coalgebras, and the *Generic Haskell* approach, characterized by case analysis over the structure of a datatype. We show that the former enjoys a kind of *higher-order naturality*, relating the behaviours of generic functions at different types; in contrast, the latter is more ad hoc, with no coherence required or provided between the various clauses of a definition. Moreover, the naturality properties arise ‘for free’, simply from the parametrized types of the generic functions: we present a *higher-order parametricity* theorem for datatype-generic operators.

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1. Introduction

Consider the following familiar datatype of lists, with a fold operator and a *length* function:

$$\begin{aligned} \mathbf{data} \text{ List } \alpha &= \text{Nil} \mid \text{Cons } \alpha \text{ (List } \alpha) \\ \text{foldL} :: \beta &\rightarrow (\alpha \rightarrow \beta \rightarrow \beta) \rightarrow \text{List } \alpha \rightarrow \beta \\ \text{foldL } e f \text{ Nil} &= e \\ \text{foldL } e f \text{ (Cons } a x) &= f a \text{ (foldL } e f x) \end{aligned}$$

$$\begin{aligned} \text{length} :: \text{List } \alpha &\rightarrow \text{Int} \\ \text{length} &= \text{foldL } 0 \text{ } (\lambda a n \rightarrow 1 + n) \end{aligned}$$

Here also is a datatype of binary trees, with its fold operator:

$$\begin{aligned} \mathbf{data} \text{ Tree } \alpha &= \text{Empty} \mid \text{Node } \alpha \text{ (Tree } \alpha) \text{ (Tree } \alpha) \\ \text{foldT} :: \beta &\rightarrow (\alpha \rightarrow \beta \rightarrow \beta \rightarrow \beta) \rightarrow \text{Tree } \alpha \rightarrow \beta \\ \text{foldT } e f \text{ Empty} &= e \\ \text{foldT } e f \text{ (Node } a x y) &= f a \text{ (foldT } e f x) \text{ (foldT } e f y) \end{aligned}$$

One can compute the ‘left spine’ of a binary tree as a list, using *foldT*:

$$\begin{aligned} \text{lspine} :: \text{Tree } \alpha &\rightarrow \text{List } \alpha \\ \text{lspine} &= \text{foldT Nil } (\lambda a x y \rightarrow \text{Cons } a x) \end{aligned}$$

The ‘left depth’ of a binary tree is defined to be the length of the left spine:

$$\begin{aligned} \text{ldepth} :: \text{Tree } \alpha &\rightarrow \text{Int} \\ \text{ldepth} &= \text{length} \circ \text{lspine} \end{aligned}$$

It should come as no surprise that the two steps encoded above can be fused into one, computing the left depth of the tree directly rather than via a list:

$$\begin{aligned} \text{ldepth} :: \text{Tree } \alpha &\rightarrow \text{Int} \\ \text{ldepth} &= \text{foldT } 0 \text{ } (\lambda a m n \rightarrow 1 + m) \end{aligned}$$

This result is a consequence of the well-known *fusion law* for *foldT*, which states that

$$\begin{aligned} h \circ \text{foldT } e f &= \text{foldT } e' f' \\ \iff h e = e' \wedge h (f a u v) &= f' a (h u) (h v) \end{aligned}$$

Importantly, there is nothing inherently specific to binary trees involved here. The ‘fold’ operator is *datatype-generic*, which is to say, parametrized by a datatype, such as *Tree* or *List*. Moreover, there is a datatype-generic fusion law for folds, of which this law for *foldT* is a datatype-specific instance.

But there is more to this particular application of fusion than meets the eye. It reveals some deep structure of the datatype-generic fold operator, relating folds on binary trees and on lists. Similar relationships hold between the folds on any two datatypes. Specifically, the central observation of this paper is that:

fold is a higher-order natural transformation.

That is, ‘fold’ is a rather special kind of datatype-generic operator, both enjoying and requiring coherence between its datatype-specific instances. This is in contrast to many other datatype-generic operators such as parsers, pretty-printers and marshallers, for which there need be no such coherence.

The situation is analogous to that between *parametric* and *ad hoc* polymorphism. For example, the parametrically polymorphic function (now with an explicit universal type quantification) *length* ::

$\forall \alpha. \text{List } \alpha \rightarrow \text{Int}$ amounts to a family of monomorphic functions $\text{length}_\alpha :: \text{List } \alpha \rightarrow \text{Int}$, one for each type α . In languages satisfying parametricity, such as Girard’s and Reynolds’ polymorphic lambda calculi [9, 30], there is a *naturality property*, relating different monomorphic instances of the polymorphic function length , namely:

$$\text{length}_\beta (\text{map } f \ x) = \text{length}_\alpha \ x \quad \text{for } f :: \alpha \rightarrow \beta$$

Informally, mapping over a list preserves its length. This naturality property follows from the type of the function length alone; one need not look at the function definition in order to prove it [35].

In contrast, languages that provide ad hoc polymorphism do not satisfy parametricity, and hence sacrifice these naturality properties. For instance, Harper and Morrisett [11] present a polymorphic lambda calculus extended with *intensional polymorphism*, which supports run-time type analysis through a ‘typecase’ construct. This allows one to define a function $\text{countInts} :: \text{List } \alpha \rightarrow \text{Int}$ that counts all the integers in a list, so that for example $\text{countInts } [4, 5, 6] = 3$, but $\text{countInts } [\text{True}, \text{False}, \text{True}] = 0$, and $\text{countInts } [(1, 2), (3, 4)] = 0$. This too can be considered as a family of monomorphic functions $\text{countInts}_\alpha :: \text{List } \alpha \rightarrow \text{Int}$, one for each type α . But the polymorphism is ad hoc, and there is no corresponding specific instances need not be coherent: the equation that would be the naturality property, namely

$$\text{countInts}_\beta (\text{map } f \ x) = \text{countInts}_\alpha \ x \quad \text{for } f :: \alpha \rightarrow \beta$$

does not hold for the intended definition of countInts , so certainly does not follow from its type.

Note that Haskell’s type class mechanism [10] allows one to solve many of the problems for which ad hoc polymorphism seems attractive, without sacrificing parametricity. It is still not possible — or at least, not straightforward — to write something like countInts above, but one can write instead a function $\text{count} :: \text{Eq } \alpha \Rightarrow \alpha \rightarrow \text{List } \alpha \rightarrow \text{Int}$ that counts the occurrences of a given element in a list (in which the notation ‘ $\text{Eq } \alpha \Rightarrow$ ’ denotes a type class context, essentially a bounded quantification over the type class Eq of types α possessing an equality operation). Using the standard compilation technique by translation into dictionary-passing style, count will be implemented with an additional parameter giving the equality operation, effectively having the type

$$\text{count}' :: (\alpha \rightarrow \alpha \rightarrow \text{Bool}) \rightarrow \alpha \rightarrow \text{List } \alpha \rightarrow \text{Int}$$

This turns the apparent ad hoc polymorphism into parametric polymorphism; the naturality property for this type, when specialized to functions, turns out to be

$$\begin{aligned} g \ a \ d' &= h \ (f \ a) \ (f \ d') \implies \\ \text{count}'_\beta \ h \ (f \ a) \ (\text{map } f \ x) &= \text{count}'_\alpha \ g \ a \ x \end{aligned}$$

for any $f :: \alpha \rightarrow \beta$, $g :: \alpha \rightarrow \alpha \rightarrow \text{Bool}$, $h :: \beta \rightarrow \beta \rightarrow \text{Bool}$. This property is valid; among other consequences, when g and h are specialized to the equality functions on α and β respectively, it states that mapping with an injective function preserves element counts.

This paper is concerned with *parametric datatype-generic* operators such as fold, which obey a naturality property analogous to that of length above, and contrasts them with *ad hoc* (or *intensionally*) datatype-generic operators such as parsers, pretty printers and marshallers, which do not. Cardelli and Wegner [5] consider parametric polymorphism to be ‘true polymorphism, whereas ad hoc polymorphism is some kind of apparent polymorphism whose polymorphic character disappears at close range’. We refrain from making quite so strong a statement in the context of datatype-generic programming; however, we do consider parametric datatype-genericity to be the ‘gold standard’ of datatype-generic programming, to be preferred over ad hoc datatype-genericity when it is available.

2. Datatype-generic programming

Computer science uses a small number of highly overloaded terms, corresponding to characteristics that recur in many areas of the field. The distinction between *static* and *dynamic* aspects is one example, applying to typing, binding, IP addresses, web pages, loading of libraries, memory allocation, and program analyses, among many other things.

The term *generic* is another example, although in this case particularly within the programming languages community rather than computer science as a whole. ‘Generic programming’ means different things to different people. We have been using it to refer to programs parametrized by a datatype such as ‘lists of’ or ‘trees of’; but to reduce confusion, we have coined a new term *datatype-generic programming* [7] for this specific usage. Examples of datatype-generic programs are the map and fold higher-order traversal patterns of the *origami programming* style [6], and data processing tools such as pretty printers, parsers, encoders, marshallers, comparators, and so on that form the main applications of Generic Haskell [14].

All of the operations named above can be defined once and for all, for all datatypes, rather than again and again for each specific datatype. However, the two families of operations differ in the style of definition that can be provided. Roughly speaking, the origami family of datatype-generic operations have definitions that are *parametric* in the datatype parameter, whereas those of the data processor family are in general *ad hoc* in the parameter. That is to say, in the former case the datatype parameter is passed around and applied but not analysed, whereas the latter case relies on a case analysis of the parameter. As a consequence, different specific instances of a datatype-generic operation from the origami family are related, whereas there is no such constraint on instances of a datatype-generic data processor — the origami operations are *natural* in the datatype parameter.

That is not to say that the data processing operations are non-natural. But it does mean that the definition by case analysis does not provide naturality naturally: ensuring naturality requires very careful consideration of the interaction between the different cases of the definition. By analogy with Cardelli and Wegner’s observations about polymorphism, we claim that naturality is at least a useful healthiness condition on datatype-generic definitions.

2.1 Origami programming

The *Origami* [6] or *Algebra of Programming* [3] style is based around the idea that datatypes are *fixpoints* of *functors*.

We assume a cartesian closed category \mathbb{C} , with initial and final objects, and all ω -colimits and ω -limits; the objects of \mathbb{C} model ‘types’, and the arrows model ‘programs’ between those types. We use ω -cocontinuous and ω -continuous functors (that is, those that preserve ω -colimits and ω -limits respectively) to describe the shape of recursive datatypes; among others, this includes all the polynomial functors (that is, those constructed from some base objects using products and coproducts). An example of a polynomial functor is the operation L whose action on objects is given by $L\alpha = 1 + \text{Int} \times \alpha$, and whose action on arrows Lf behaves as the identity on a unit (in the left of the sum), and applies f to the second component of a pair (in the right of the sum).

An F-algebra is a pair (α, f) with $f : F\alpha \rightarrow \alpha$. A standard result [32] is that an ω -cocontinuous functor F possesses an initial F-algebra, which we denote $(\mu F, \text{IN}_F)$. We write $\text{FOLD}_F f$ for the witness to the initiality, the unique homomorphism from the initial algebra to an F-algebra (α, f) ; thus, for particular F , the type of FOLD_F itself is

$$(F\alpha \rightarrow \alpha) \rightarrow (\mu F \rightarrow \alpha)$$

The uniqueness of the witness to initiality is expressed in the universal property

$$h = \text{FOLD}_F f \iff h \cdot \text{IN}_F = f \cdot Fh$$

For example, $\mu\mathbb{L}$ (where \mathbb{L} is as defined above) is the datatype of finite lists of integers; an integer-specific version of the function *length* from Section 1 can be written

$$\text{length} = \text{FOLD}_{\mathbb{L}} (\text{zero} \nabla \text{inc2}) : \text{List } \text{Int} \rightarrow \text{Int}$$

where $\text{zero} : 1 \rightarrow \text{Int}$ always yields zero, $\text{inc2} : \text{Int} \times \text{Int} \rightarrow \text{Int}$ returns the successor of the second component of a pair of integers, and ∇ is the morphism combining the two branches of a sum.

One simple consequence of the universal property, obtained by letting $h = \text{FOLD}_F f$, is an *evaluation rule* showing how a data structure is consumed:

$$\text{FOLD}_F f \cdot \text{IN}_F = f \cdot F(\text{FOLD}_F f)$$

A more interesting consequence is a *fusion law*, for combining a FOLD with a following function:

$$h \cdot \text{FOLD}_F f = \text{FOLD}_F g \iff h \cdot f = g \cdot Fh$$

Dually, an F-coalgebra for an ω -continuous functor F is a pair (α, f) with $f : \alpha \rightarrow F\alpha$; the final F-coalgebra is $(\nu F, \text{OUT}_F)$; the witness to finality, the unique homomorphism from an F-coalgebra (α, f) to the final coalgebra, is denoted $\text{UNFOLD}_F f$, whose uniqueness is expressed in the universal property

$$h = \text{UNFOLD}_F f \iff \text{OUT}_F \cdot h = Fh \cdot f$$

Consequences of the universal property include an evaluation rule:

$$\text{OUT}_F \cdot \text{UNFOLD}_F f = F(\text{UNFOLD}_F f) \cdot f$$

and a fusion law for combining an UNFOLD with a preceding function:

$$\text{UNFOLD}_F f \cdot h = \text{UNFOLD}_F g \iff f \cdot h = Fh \cdot g$$

For example, the ω -cocontinuous functor \mathbb{S} acting on objects as $\mathbb{S}\alpha = \text{Int} \times \alpha$ and on arrows as $\mathbb{S}f = \text{id}_{\text{Int}} \times f$ induces a datatype $\text{IStream} = \nu \mathbb{S}$ of infinite streams of integers. A step function $f : \alpha \rightarrow \mathbb{S}\alpha$ induces a stream producer $\text{UNFOLD}_{\mathbb{S}} f : \alpha \rightarrow \text{IStream}$; so $\text{repeat} = \text{UNFOLD}_{\mathbb{S}} (\text{id} \Delta \text{id}) : \text{Int} \rightarrow \text{IStream}$ yields a stream of copies of a given integer, where Δ is the morphism making a pair using two functions, and

$$\text{zipAdd} = \text{UNFOLD}_{\mathbb{S}} \text{addHeads} : \text{IStream} \times \text{IStream} \rightarrow \text{IStream}$$

adds two integer streams element-wise, where

$$\text{addHeads} (\text{SCons } x \text{ xs}, \text{SCons } y \text{ ys}) = (x + y, (xs, ys))$$

(pattern-matching on a stream constructor ‘SCons’ for brevity).

2.2 Generic Haskell

A different approach to datatype-generic programming involves case analyses on the structure of datatypes: ‘a generic program is defined by induction on structure-representation types’ [16]. We take as representative of this approach the Generic Haskell extension [12, 15, 23] of Haskell [29], based on the notion of type-indexed functions with kind-indexed types [13], in which the family of type indexes is the polynomial types (sums and products of the unit type and some basic types such as *Int*); however, our remarks apply just as well to a number of related techniques, such as the ‘Scrap Your Boilerplate’ series [20, 21, 22].

Consider the datatype-generic function *enc*, encoding a data structure as a list of bits. In Generic Haskell, this is defined roughly as follows. (We have made some simplifications, such as omitting cases for labels and constructors, for brevity. We have also adapted the Generic Haskell syntax slightly, for consistency with the rest of

this paper; in particular, we use the list constructors from Section 1, so that we can use ‘:’ for type or kind judgements.)

$$\begin{aligned} \text{encode} \{ \alpha : \star \} & : (\text{encode} \{ \alpha \}) \Rightarrow \alpha \rightarrow \text{List } \text{Bool} \\ \text{encode} \{ \text{Unit} \} () & = \text{Nil} \\ \text{encode} \{ \text{Int} \} n & = \text{encodeInt } n \\ \text{encode} \{ \alpha :+ \beta \} (\text{Inl } x) & = \text{Cons } (\text{False}, \text{encode} \{ \alpha \} x) \\ \text{encode} \{ \alpha :+ \beta \} (\text{Inr } y) & = \text{Cons } (\text{True}, \text{encode} \{ \beta \} y) \\ \text{encode} \{ \alpha : \times \beta \} (x, y) & = \text{encode} \{ \alpha \} x ++ \text{encode} \{ \beta \} y \end{aligned}$$

The first line gives a type declaration, as usual; it declares that *encode* specialized to a type α of kind \star has type $\alpha \rightarrow \text{List } \text{Bool}$. (This does not mean that *encode* can be applied only to types of kind \star . Rather, the type of *encode* at a type index of another kind is derived automatically from this. For example, $\text{encode} \{ \text{List} \}$ has type $(\alpha \rightarrow \text{List } \text{Bool}) \rightarrow (\text{List } \alpha \rightarrow \text{List } \text{Bool})$; hence the slogan that ‘type-indexed functions have kind-indexed types’ [13].) Moreover, the context ‘ $(\text{encode} \{ \alpha \}) \Rightarrow$ ’ indicates that *encode* depends on itself, that is, it is defined inductively. There are cases for each possible top-level structure of the type index; the base case for integers assumes a primitive function $\text{encodeInt} : \text{Int} \rightarrow \text{List } \text{Bool}$. The Generic Haskell compiler uses these five cases to derive a specialization of *encode* for any polynomial datatype, such as for the types of integer lists and binary trees introduced in Section 1.

The important point is that the behaviour of *encode* is dispersed across five separate cases, and any desired coherence between different instances has to be very carefully engineered. Recall, for example, the relationship from Section 1 between folds on binary trees and on their left spines. Is there a corresponding relationship between the encodings of binary trees and their left spines? It turns out that there is some relationship between specializations of *encode* for binary trees and for their left spines; but that relationship depends on carefully engineered interaction between the behaviours in different branches, and depends non-trivially on prefix-freeness of the encoding (a happy consequence of this particular definition) — and therefore is not a naturality property in the same sense.

To restate our point: ad hoc datatype-generic programs may in fact be natural, and perhaps naturality is a useful healthiness condition for datatype genericity; but with ad hoc techniques, this naturality requires careful design, rather than arising automatically.

3. Higher-order functors and natural transformations

It is standard to note that one can define a category whose objects are functors from a category \mathbb{D} to a category \mathbb{E} , with natural transformations as morphisms. We shall consider initial algebras, so we limit our attention to ω -cocontinuous functors, writing $\mathbb{D} \rightsquigarrow \mathbb{E}$ for the category of such functors. This category inherits colimits from \mathbb{E} , constructed pointwise.

The next step is to consider functors to and from these functor categories, i.e. *higher-order functors* (or *hofunctors* for short). Consider μ which maps functors of $\mathbb{C} \rightsquigarrow \mathbb{C}$ to objects of \mathbb{C} . A natural transformation $\phi : F \rightarrow G$ between ω -cocontinuous functors F, G induces a corresponding mapping $\mu\phi : \mu F \rightarrow \mu G$ between the initial algebras of those functors, defined by:

$$\mu\phi = \text{FOLD}_F (\text{IN}_G \cdot \phi_{\mu G})$$

Preservation of the identity, $\mu(\text{id}_F) = \text{id}_{\mu F} : \mu F \rightarrow \mu F$, follows from the reflection rule, and preservation of composition

$$\mu(\phi \cdot \psi) = \mu\phi \cdot \mu\psi : \mu F \rightarrow \mu H$$

follows from fold fusion. Moreover μ also preserves ω -colimits. We denote arbitrary hofunctors with calligraphic capitals (such as \mathcal{H}).

There is a notion of ‘application’ appropriate for the cartesian-closed structure on Cat [24, IV.6], namely the functor $*$ from $(\mathbb{D} \rightsquigarrow \mathbb{E}) \times \mathbb{D}$ to \mathbb{E} defined by:

$$\begin{aligned} F * \alpha &= F\alpha \\ \phi * f &= Gf \cdot \phi_\alpha = \phi_\beta \cdot Ff \end{aligned}$$

for $\phi : F \rightarrow G$ and $f : \alpha \rightarrow \beta$. (For later convenience, we make the convention that ‘ $*$ ’ binds tighter than ‘ \cdot ’.)

3.1 Higher-order naturality

We have seen that μ is a hofunctor on \mathbb{C} ; another such hofunctor is the \mathcal{H} defined by

$$\mathcal{H}x = x * \mu x$$

The action of \mathcal{H} on objects of $\mathbb{C} \rightsquigarrow \mathbb{C}$ (which are functors) is

$$\mathcal{H}F = F(\mu F)$$

The action of this hofunctor on arrows of $\mathbb{C} \rightsquigarrow \mathbb{C}$ (which are natural transformations) is to take $\phi : F \rightarrow G$ to an arrow

$$\mathcal{H}\phi = \phi * \mu\phi : F(\mu F) \rightarrow G(\mu G)$$

of \mathbb{C} . Expanding the above definition of the application functor, this becomes

$$\mathcal{H}\phi = G(\mu\phi) \cdot \phi_{\mu F} = \phi_{\mu G} \cdot F(\mu\phi)$$

The lifting of the notion of functor to the functor category prompts the consideration of similarly lifting the notion of natural transformation. A *higher-order natural transformation* (or *hont* for short) on a category \mathbb{C} is just the normal notion of natural transformation, specialized to the functor category $\mathbb{C} \rightsquigarrow \mathbb{C}$: for hofunctors $\mathcal{H}, \mathcal{K} : (\mathbb{C} \rightsquigarrow \mathbb{C}) \rightsquigarrow \mathbb{D}$, a hont $\text{OP} : \mathcal{H} \rightarrow \mathcal{K}$ is a family of arrows $\text{OP}_F \in \mathbb{D}(\mathcal{H}F, \mathcal{K}F)$ for $F : \mathbb{C} \rightsquigarrow \mathbb{C}$, such that $\text{OP}_G \cdot \mathcal{H}\phi = \mathcal{K}\phi \cdot \text{OP}_F$ for $\phi : F \rightarrow G$. Diagrammatically:

$$\begin{array}{ccc} \mathcal{H}F & \xrightarrow{\text{OP}_F} & \mathcal{K}F \\ \mathcal{H}\phi \downarrow & & \downarrow \mathcal{K}\phi \\ \mathcal{H}G & \xrightarrow{\text{OP}_G} & \mathcal{K}G \end{array}$$

It turns out that the constructor IN of initial algebras is a hont $\mathcal{H} \rightarrow \mu$, where the hofunctor \mathcal{H} is as defined above. That is,

$$\text{IN}_G \cdot \phi_{\mu G} \cdot F(\mu\phi) = \mu\phi \cdot \text{IN}_F$$

This condition is straightforward to verify:

$$\begin{aligned} & \mu\phi \cdot \text{IN}_F \\ &= \{ \text{characterization } \mu\phi = \text{FOLD}_F(\text{IN}_G \cdot \phi_{\mu G}) \} \\ & \text{FOLD}_F(\text{IN}_G \cdot \phi_{\mu G}) \cdot \text{IN}_F \\ &= \{ \text{FOLD evaluation} \} \\ & \text{IN}_G \cdot \phi_{\mu G} \cdot F(\text{IN}_G \cdot \phi_{\mu G}) \\ &= \{ \text{characterization of } \mu\phi \text{ as a FOLD again} \} \\ & \text{IN}_G \cdot \phi_{\mu G} \cdot F(\mu\phi) \end{aligned}$$

3.2 Fold

Our key example is provided by FOLD . Specifically, FOLD is a hont $\mathcal{H} \rightarrow \mathcal{K}$, where α is a fixed object of \mathbb{C} , and contravariant hofunctors \mathcal{H}, \mathcal{K} are defined by:

$$\begin{aligned} \mathcal{H}F &= (F\alpha \rightarrow \alpha) \\ \mathcal{H}\phi &= (\cdot\phi_\alpha) : (G\alpha \rightarrow \alpha) \rightarrow (F\alpha \rightarrow \alpha) \quad \text{for } \phi : F \rightarrow G \\ \mathcal{K}F &= (\mu F \rightarrow \alpha) \\ \mathcal{K}\phi &= (\cdot\mu\phi) : (\mu G \rightarrow \alpha) \rightarrow (\mu F \rightarrow \alpha) \quad \text{for } \phi : F \rightarrow G \end{aligned}$$

Because the hofunctors are contravariant, some of the arrows in the higher-order naturality property get reversed:

$$\begin{array}{ccc} \mathcal{H}G & \xrightarrow{\text{OP}_G} & \mathcal{K}G \\ \mathcal{H}\phi \downarrow & & \downarrow \mathcal{K}\phi \\ \mathcal{H}F & \xrightarrow{\text{OP}_F} & \mathcal{K}F \end{array}$$

Thus, our claim induces a proof obligation

$$\text{FOLD}_F(f \cdot \phi_\alpha) = \text{FOLD}_G f \cdot \mu\phi$$

for $\phi : F \rightarrow G$ and $f : G\alpha \rightarrow \alpha$, which, since $\mu\phi$ is itself an instance of FOLD , can be discharged using fold fusion:

$$\begin{aligned} & \text{FOLD}_G f \cdot \text{IN}_G \cdot \phi_{\mu G} \\ &= \{ \text{FOLD evaluation} \} \\ & f \cdot G(\text{FOLD}_G f) \cdot \phi_{\mu G} \\ &= \{ \text{naturality of } \phi \} \\ & f \cdot \phi_\alpha \cdot F(\text{FOLD}_G f) \end{aligned}$$

In particular, an integer-specific instance of the example in Section 1 has $\text{T}(\alpha) = 1 + \text{Int} \times (\alpha \times \alpha)$, so that μT is binary trees of integers, and $\text{L}\alpha = 1 + \text{Int} \times \alpha$ as before, so that μL is lists of integers. Choose $\phi : \text{T} \rightarrow \text{L} = \text{id} + \text{id} \times \text{fst}$, which discards right children. Then $\text{lspine} = \mu\phi$, and the naturality property implies that $\text{ldepth} = \text{length} \cdot \text{lspine}$ as required. (For a polymorphic datatype $\text{Tree } \alpha$, one would need to use a *bifunctor* such as $\text{T}_\alpha(\beta) = 1 + \alpha \times (\beta \times \beta)$; the theory generalises smoothly in this way.)

3.3 Paramorphism

We have shown that both the datatype-generic constructor IN of initial algebras and the FOLD operator enjoy a rather special property, being parametric in their shape parameters. However, there is nothing inherently specific about FOLD and IN in this regard; many other datatype-generic operators enjoy similar properties. For example, there are generalizations of FOLD , from so-called iteration to primitive recursion; these too are honts.

Meertens [25] captures the familiar pattern of primitive recursion by defining the *paramorphism*:

$$\text{PARA}_F : (F(\alpha \times \mu F) \rightarrow \alpha) \rightarrow (\mu F \rightarrow \alpha)$$

This too is a hont $\text{PARA} : \mathcal{H} \rightarrow \mathcal{K}$ where contravariant hofunctor $\mathcal{H} : (\mathbb{C} \rightsquigarrow \mathbb{C}) \rightsquigarrow \mathbb{C}^{\text{OP}}$ is given by:

$$\begin{aligned} \mathcal{H}F &= (F(\alpha \times \mu F) \rightarrow \alpha) \\ \mathcal{H}\phi &= (\cdot(\phi * (\text{id} \times \mu\phi))) : \mathcal{H}G \rightarrow \mathcal{H}F \quad \text{for } \phi : F \rightarrow G \end{aligned}$$

and \mathcal{K} is the hofunctor used for FOLD above.

4. Co-algebraic honts

We can obtain another class of honts by dualizing the constructions of the previous section, using ω -continuous functors. In this case, we obtain an ω -continuous hofunctor $\nu : (\mathbb{C} \rightsquigarrow \mathbb{C}) \rightsquigarrow \mathbb{C}$ dualizing μ , with action on natural transformation $\phi : F \rightarrow G$ defined by:

$$\nu\phi = \text{UNFOLD}_G(\phi_{\nu F} \cdot \text{OUT}_F) : \nu F \rightarrow \nu G$$

(preservation of identities and compositions is straightforward to check). We then have a family of dual natural transformations.

4.1 Out

Dually to IN , the destructor OUT of final co-algebras is natural in its functor parameter too: $\text{OUT} : \nu \rightarrow \mathcal{K}$ where $\nu : (\mathbb{C} \rightsquigarrow \mathbb{C}) \rightsquigarrow \mathbb{C}$ is as defined above, and covariant hofunctor $\mathcal{K} : (\mathbb{C} \rightsquigarrow \mathbb{C}) \rightsquigarrow \mathbb{C}$ is given by:

$$\begin{aligned}\mathcal{H}F &= F(\nu F) \\ \mathcal{H}\phi &= \phi * \nu\phi : F(\nu F) \rightarrow G(\nu G) \quad \text{for } \phi : F \rightarrow G\end{aligned}$$

That is,

$$G(\nu\phi) \cdot \phi_{\nu F} \cdot \text{OUT}_F = \text{OUT}_G \cdot \nu\phi$$

which may again easily be verified using the definition of $\nu\phi$ and the evaluation rule of UNFOLD.

4.2 Unfold

Similarly, the UNFOLD operator dualizing FOLD is a hont $\mathcal{H} \xrightarrow{\dot{}} \mathcal{K}$ where covariant hofunctors $\mathcal{H}, \mathcal{K} : (\mathbb{C} \rightsquigarrow \mathbb{C}) \rightsquigarrow \mathbb{C}$ are given by:

$$\begin{aligned}\mathcal{H}F &= (\alpha \rightarrow F\alpha) \\ \mathcal{H}\phi &= (\phi_\alpha \cdot) : (\alpha \rightarrow F\alpha) \rightarrow (\alpha \rightarrow G\alpha) \quad \text{for } \phi : F \rightarrow G \\ \mathcal{H}F &= (\alpha \rightarrow \nu F) \\ \mathcal{H}\phi &= (\nu\phi \cdot) : (\alpha \rightarrow \nu F) \rightarrow (\alpha \rightarrow \nu G) \quad \text{for } \phi : F \rightarrow G\end{aligned}$$

The higher-order naturality amounts to the claim that

$$\text{UNFOLD}_G(\phi_\alpha \cdot f) = \nu\phi \cdot \text{UNFOLD}_F f$$

for $f : \alpha \rightarrow F\alpha$; by definition of $\nu\phi$ and unfold fusion, it suffices to show

$$\phi_{\nu F} \cdot \text{OUT}_F \cdot \text{UNFOLD}_F f = G(\text{UNFOLD}_F f) \cdot \phi_\alpha \cdot f$$

which follows from the naturality of ϕ and the evaluation rule for UNFOLD:

$$\begin{aligned}& \phi_{\nu F} \cdot \text{OUT}_F \cdot \text{UNFOLD}_F f \\ &= \{ \text{UNFOLD evaluation} \} \\ & \phi_{\nu F} \cdot F(\text{UNFOLD}_F f) \cdot f \\ &= \{ \text{naturality of } \phi \} \\ & G(\text{UNFOLD}_F f) \cdot \phi_\alpha \cdot f\end{aligned}$$

(Note that the higher-order naturality of UNFOLD is simpler than that of FOLD, because it does not involve contravariance. Perhaps UNFOLD should be better appreciated [8] — even considered the ‘ordinary’ case, and FOLD its dual?)

As a simple application of the higher-order naturality of UNFOLD, consider Pascal’s Triangle:

$$\begin{array}{ccccccc} & & & & 1 & & & & \\ & & & & 1 & & 1 & & \\ & & & 1 & 2 & & 1 & & \\ & & 1 & 3 & 3 & & 1 & & \\ & 1 & 4 & 6 & 4 & & 1 & & \\ \dots & & & & & & & & \end{array}$$

The triangular shape can be expressed as the final coalgebra of the bifunctor U defined by $U_\alpha = \text{Int} \times \text{IStream} \times \text{IStream} \times \alpha$, giving a corner, two infinite edges and an inner structure. Pascal’s Triangle itself is $\text{UNFOLD}_U \text{step} (\text{repeat } 1)$, where

$$\begin{aligned}\text{step} (SCons\ x\ (SCons\ y\ xs)) \\ &= \mathbf{let}\ zs = SCons\ (2 \times y)\ (\text{zipAdd}\ (xs, zs)) \\ & \mathbf{in}\ (x, SCons\ y\ xs, SCons\ y\ xs, zs)\end{aligned}$$

Pascal’s Triangle has many nice properties. One of them is that the n th element of the central column 1, 2, 6, 20... is the number of: non-decreasing sequences of n integers drawn from $0 \dots n$; direct routes on a grid making n steps East and n steps North in total; directed, convex polyominoes having semiperimeter $n + 2$; and so on [31, Sequence A000984]. Extraction of this middle column is achieved by $\nu\phi$, where the natural transformation $\phi : U \rightarrow S$ is defined by $\phi(x, ys, zs, u) = (x, u)$. By higher-order naturality of UNFOLD,

$$\nu\phi \cdot \text{UNFOLD}_U \text{step} = \text{UNFOLD}_S \text{step}'$$

where

$$\begin{aligned}\text{step}' (SCons\ x\ (SCons\ y\ xs)) \\ &= \mathbf{let}\ zs = SCons\ (2 \times y)\ (\text{zipAdd}\ (xs, zs)) \\ & \mathbf{in}\ (x, zs)\end{aligned}$$

This yields a direct method of computing the sequence, without having to generate Pascal’s Triangle first.

4.3 Anamorphism

As in the initial algebra case, there is a further family of operators expressing various forms of co-iteration and primitive co-recursion, all of which are also honts.

Uustalu and Vene [34] dualize Meertens’ paramorphisms by defining the *apomorphism*:

$$\text{APO}_F : (\alpha \rightarrow F(\alpha + \nu F)) \rightarrow (\alpha \rightarrow \nu F)$$

This is another hont $\text{APO} : \mathcal{H} \xrightarrow{\dot{}} \mathcal{K}$ where covariant hofunctor $\mathcal{H} : (\mathbb{C} \rightsquigarrow \mathbb{C}) \rightsquigarrow \mathbb{C}$ is given by:

$$\begin{aligned}\mathcal{H}F &= (\alpha \rightarrow F(\alpha + \nu F)) \\ \mathcal{H}\phi &= ((\phi * (\text{id} + \nu\phi)) \cdot) : \mathcal{H}F \rightarrow \mathcal{H}G \quad \text{for } \phi : F \rightarrow G\end{aligned}$$

and \mathcal{K} is the hofunctor used for UNFOLD above.

5. Non-honts

This large collection of higher-order natural datatype-generic operations begs a question: is every datatype-generic operator a hont? Certainly not. In fact, the Generic Haskell function *encode* of Section 2.2 is a counterexample. If it were a hont, it would be of the form

$$\text{encode}_F : \mu F \rightarrow \text{List Bool}$$

This has the right type to be a hont $\mathcal{H} \xrightarrow{\dot{}} \mathcal{K}$, where \mathcal{H} is as used for IN, and \mathcal{K} is the constant hofunctor defined by $\mathcal{K}F = \text{List Bool}$ and $\mathcal{K}\phi = \text{id}_{\text{List Bool}}$. However, the naturality property corresponding to $\text{OP} : \mathcal{H} \xrightarrow{\dot{}} \mathcal{K}$ reduces to

$$\text{OP}_G \cdot \mu\phi = \text{OP}_F$$

for $\phi : F \rightarrow G$ — that is, all naturally related data structures (such as a binary tree and its left spine) are equivalent under OP . In particular, a binary tree and its left spine would have to have the same encoding.

Similarly, consider the generic size operation, another standard example in the Generic Haskell literature [15]:

$$\begin{aligned}\text{size}\{\alpha : \star\} & : (\text{size}\{\alpha\}) \Rightarrow \alpha \rightarrow \text{Int} \\ \text{size}\{\text{Unit}\} () & = 0 \\ \text{size}\{\text{Int}\} n & = 1 \\ \text{size}\{\alpha :+ \beta\} (\text{Inl } x) & = \text{size}\{\alpha\} x \\ \text{size}\{\alpha :+ \beta\} (\text{Inr } y) & = \text{size}\{\beta\} y \\ \text{size}\{\alpha : \times \beta\} (x, y) & = \text{size}\{\alpha\} x + \text{size}\{\beta\} y\end{aligned}$$

As a hont, this would have to be of the form $\text{SIZE} : \mathcal{H} \xrightarrow{\dot{}} \mathcal{K}$, where \mathcal{H} is as above and \mathcal{K} is the constant hofunctor $\mathcal{K}F = \text{Int}$ and $\mathcal{K}\phi = \text{id}_{\text{Int}}$. Again, the corresponding naturality property would imply that

$$\text{SIZE}_G \cdot \mu\phi = \text{SIZE}_F$$

for $\phi : F \rightarrow G$ — that is, all naturally related data structures would have to have the same size.

The problem here is that Generic Haskell is based on *case analysis* of the shape, rather than blind *application* of the shape parameter; it therefore allows completely different behaviours in different branches of the analysis. This gives greater flexibility (as a corollary to the above, generic *encode* and generic *SIZE* cannot be defined in the Algebra of Programming style), but with that greater power comes greater responsibility.

Raw syntax

Kinds	κ	$\star \mid \kappa_1 \Rightarrow \kappa_2$
Datatypes	T	$K \mid X \mid \Lambda X :: \kappa. T \mid T_1 T_2$
Type contexts	Δ	$X_1 :: \kappa_1, \dots, X_n :: \kappa_n$

Constants

1	$:: \star$
$(+)$	$:: \star \Rightarrow \star \Rightarrow \star$
(\times)	$:: \star \Rightarrow \star \Rightarrow \star$
μ	$:: (\star \Rightarrow \star) \Rightarrow \star$

Datatype judgements $\Delta \vdash T :: \kappa$

$\Delta \vdash K :: \kappa$	$(K :: \kappa \in \text{Sig})$	$\Delta, X :: \kappa \vdash X :: \kappa$
$\frac{\Delta, X :: \kappa_1 \vdash T :: \kappa_2}{\Delta \vdash \Lambda X :: \kappa_1. T :: \kappa_1 \Rightarrow \kappa_2}$		$\frac{\Delta \vdash T_1 :: \kappa_1 \Rightarrow \kappa_2 \quad \Delta \vdash T_2 :: \kappa_1}{\Delta \vdash T_1 T_2 :: \kappa_2}$

Figure 1. Lambda-calculus of datatypes

6. Higher-order theorems for free

The statement that PARA is a hont can be proven from first principles, or using the higher-order naturality of FOLD and IN, in terms of which PARA is defined. However, by analogy with the first-order case [30, 35], one might expect to be able to deduce the result ‘for free’ from the type of PARA, using a kind of second-order parametricity theorem [27]. In this section, we present such a theorem.

Our method is to define a small language in which initial algebra operators like PARA may be defined, and to prove a parametricity theorem for terms in this language. Though we present only the initial algebra case, the same method can also be applied to define a language for final co-algebra operators.

Our language has two parts.

6.1 Lambda-calculus of datatypes

The first part is a conventional typed λ -calculus of Figure 1. We shall interpret this calculus using the cartesian-closed structure on ω -cocomplete categories and ω -cocontinuous functors. Thus kinds are interpreted as categories:

$$\begin{aligned} \mathbb{C}_\star &= \mathbb{C} \\ \mathbb{C}_{\kappa_1 \Rightarrow \kappa_2} &= \mathbb{C}_{\kappa_1} \rightsquigarrow \mathbb{C}_{\kappa_2} \end{aligned}$$

This interpretation is readily extended to datatype contexts:

$$\mathbb{C}_{X_1 :: \kappa_1, \dots, X_n :: \kappa_n} = \mathbb{C}_{\kappa_1} \times \dots \times \mathbb{C}_{\kappa_n}$$

The objects of the category \mathbb{C}_Δ are type environments δ . That is, each δ in \mathbb{C}_Δ is a family of objects, i.e. a function mapping type variables $X :: \kappa$ to objects of \mathbb{C}_κ . The arrows of this category are indexed transformations $\tau \in \mathbb{C}_\Delta(\delta, \delta')$, so that $\tau X : \mathbb{C}_\kappa(\delta X, \delta' X)$.

We assume for each constant $K :: \kappa$ an interpretation $\llbracket K \rrbracket \in \mathbb{C}_\kappa$. Indeed we have used same names for the constants in Figure 1 as the corresponding objects in the metalanguage. Thus $\llbracket 1 \rrbracket$ is the object of \mathbb{C} , while $\llbracket + \rrbracket$ and $\llbracket \times \rrbracket$ are the corresponding binary functors $\mathbb{C} \rightsquigarrow \mathbb{C} \rightsquigarrow \mathbb{C}$ and $\llbracket \mu \rrbracket$ is the functor $\mu : (\mathbb{C} \rightsquigarrow \mathbb{C}) \rightsquigarrow \mathbb{C}$. That is, they map functors to objects, and natural transformations to ordinary morphisms.

The interpretation of a datatype judgement $\Delta \vdash T :: \kappa$ is standard, but specialised to the cartesian closed structure of this category of categories, so that it defines a functor:

$$\llbracket \Delta \vdash T :: \kappa \rrbracket \in \mathbb{C}_\Delta \rightsquigarrow \mathbb{C}_\kappa$$

Raw syntax

Function types	S	$T \mid S_1 \rightarrow S_2$
Expressions	e	$k_{S_1, \dots, S_n} \mid x \mid \lambda x : S. e \mid e_1 e_2$
Expression contexts	Γ	$x_1 : S_1, \dots, x_n : S_n$

Function type judgements $\Delta \vdash S$

$\frac{\Delta \vdash T :: \star}{\Delta \vdash T}$	$\frac{\Delta \vdash S_1 \quad \Delta \vdash S_2}{\Delta \vdash S_1 \rightarrow S_2}$
--	---

Expression judgements $\Delta; \Gamma \vdash e : T$

$\frac{\Delta \vdash \Gamma \quad \Delta \vdash T_i :: \kappa_i}{\Delta; \Gamma \vdash k_{T_1, \dots, T_n} : S[T_i/X_i]} \quad (k : \forall X_i :: \kappa_i. S \in \text{Sig})$	
$\frac{\Delta \vdash \Gamma \quad \Delta \vdash S}{\Delta; \Gamma, x : S \vdash x : S}$	$\frac{\Delta; \Gamma, x : S_1 \vdash e : S_2}{\Delta; \Gamma \vdash \lambda x : S_1. e : S_1 \rightarrow S_2}$
$\frac{\Delta; \Gamma \vdash e_1 : S_1 \rightarrow S_2 \quad \Delta; \Gamma \vdash e_2 : S_1}{\Delta; \Gamma \vdash e_1 e_2 : S_2}$	

Figure 2. λ -calculus of operators

In particular, for $\tau \in \mathbb{C}_\Delta(\delta, \delta')$,

$$\begin{aligned} \llbracket \Delta \vdash T :: \kappa \rrbracket \delta &\in \mathbb{C}_\kappa \\ \llbracket \Delta \vdash T :: \kappa \rrbracket \tau &\in \mathbb{C}_\kappa(\llbracket \Delta \vdash T :: \kappa \rrbracket \delta, \llbracket \Delta \vdash T :: \kappa \rrbracket \delta') \end{aligned}$$

The interpretation of application is the functor $*$ introduced in Section 3.

6.2 A language for initial algebra combinators

The second part of the language allows us to express categorical combinators. The syntax and type rules of our language are given in Figure 2.

The signature Sig consists of the kinded type constants $T :: \kappa$ of Figure 1, plus polymorphic constants $k : \forall X_i :: \kappa_i. S$, where $X_1 :: \kappa_1, \dots, X_n :: \kappa_n \vdash S$. We assume a fixed signature, containing the usual polymorphic constants for 1 , $+$, and \times , as well as higher-order polymorphic constants:

$$\begin{aligned} \text{IN} &: \forall F :: \star \Rightarrow \star. F(\mu F) \rightarrow \mu F \\ \text{FOLD} &: \forall F :: \star \Rightarrow \star, A :: \star. (FA \rightarrow A) \rightarrow (\mu F \rightarrow A) \\ \text{MAP} &: \forall F :: \star \Rightarrow \star, A :: \star, B :: \star. (A \rightarrow B) \rightarrow (FA \rightarrow FB) \end{aligned}$$

Our language is somewhat restrictive; in particular it can express higher-order functions on data structures, but cannot express data structures containing functions. However it is sufficient for defining the various initial algebra operators. For example, the PARA operator of Section 3.3 can be expressed as

$$\begin{aligned} F :: \star \Rightarrow \star, A :: \star; \vdash \\ \lambda f : F(A \times \mu F) \rightarrow A. \\ fst \cdot \text{FOLD}_F(\lambda t : F(A \times \mu F). \\ (f t, \text{IN}_F(\text{MAP}_F \text{snd } t))) : \\ (F(A \times \mu F) \rightarrow A) \rightarrow (\mu F \rightarrow A) \end{aligned}$$

Next we shall define a parametric interpretation of our language, through which polymorphic types like the above yield parametricity properties, including higher-order naturality.

6.3 Interpretation of function types and expressions

We can interpret function type judgements $\Delta \vdash S$ straightforwardly as mappings of objects:

$$\begin{aligned} \mathcal{D}[\Delta \vdash S] &\in |\mathbb{C}_\Delta| \rightarrow |\mathbb{C}| \\ \mathcal{D}[\Delta \vdash T] \delta &= [[\Delta \vdash T :: \star]] \delta \\ \mathcal{D}[\Delta \vdash S_1 \rightarrow S_2] \delta &= [[\Delta \vdash S_1]] \delta \rightarrow [[\Delta \vdash S_2]] \delta \end{aligned}$$

However we cannot define a mapping of arrows, because the function type constructor \rightarrow is not covariant in both arguments. We shall instead interpret a judgement $\Delta \vdash S$ as mapping arrows of \mathbb{C}_Δ to relations between the corresponding objects of \mathbb{C} . Although relations support composition, this composition is not preserved by the semantic mapping, so we shall not use a category of relations, and the mapping will not be a functor.

To define relations, we need to identify a set of points of each object of \mathbb{C} . We define the functor $\text{Pt} : \mathbb{C} \rightsquigarrow \text{Set}$ by $\text{Pt} = \mathbb{C}(1, -)$. The corresponding notion of application takes $f \in \text{Pt}(A \rightarrow B)$ and $a \in \text{Pt} A$ and yields $f @ a \in \text{Pt} B$. We assume that \mathbb{C} is well-pointed, that is, equality of arrows, $f = g$, reduces to equality on points, $\forall a \in \mathbb{C}(1, A). f @ a = g @ a$.

Then the relational interpretation follows the usual definition. For $\tau \in \mathbb{C}(\delta, \delta')$, we define

$$\begin{aligned} \mathfrak{R}[\Delta \vdash S] \tau &\in \text{Rel}(\text{Pt}(\mathcal{D}[\Delta \vdash S] \delta), \text{Pt}(\mathcal{D}[\Delta \vdash S] \delta')) \\ \mathfrak{R}[\Delta \vdash T] \tau(p, p') &\equiv [[\Delta \vdash T :: \star]] \tau \cdot p = p' \\ \mathfrak{R}[\Delta \vdash S_1 \rightarrow S_2] \tau(f, f') &\equiv \forall p, p'. \mathfrak{R}[\Delta \vdash S_1] \tau(p, p') \implies \\ &\quad \mathfrak{R}[\Delta \vdash S_2] \tau(f @ p, f' @ p') \end{aligned}$$

For each constant $k : \forall \Delta. S$, we assume for each $\delta \in |\mathbb{C}_\Delta|$ a point

$$[[k]] \delta \in \text{Pt}(\mathcal{D}[\Delta \vdash S])$$

such that for $\tau \in \mathbb{C}(\delta, \delta')$ we have $\mathfrak{R}[\Delta \vdash S] \tau ([[k]] \delta, [[k]] \delta')$.

For example, for the constant IN , the type

$$\text{IN} : \forall F :: \star \Rightarrow \star. F(\mu F) \rightarrow \mu F$$

implies the property

$$\forall t, t'. \phi * \mu \phi \cdot t = t' \implies \mu \phi \cdot \text{IN}_F \cdot t = \text{IN}_G \cdot t'$$

(recall that $'\cdot'$ binds tighter than $'*'$), or equivalently

$$\forall t. \mu \phi \cdot \text{IN}_F \cdot t = \text{IN}_G \cdot \phi * \mu \phi \cdot t$$

As \mathbb{C} is well-pointed, this is equivalent to

$$\mu \phi \cdot \text{IN}_F = \text{IN}_G \cdot \phi * \mu \phi$$

which is exactly the higher-order naturality property of IN .

As a second example, for the constant

$$\text{FOLD} : \forall F :: \star \Rightarrow \star, A :: \star. (FA \rightarrow A) \rightarrow (\mu F \rightarrow A)$$

the property implied by the type reduces similarly to

$$\forall f, f'. a \cdot f = f' \cdot \phi * a \implies a \cdot \text{FOLD}_F f = \text{FOLD}_G f' \cdot \mu \phi$$

which combines the higher-order naturality property of FOLD with the first-order fusion law.

As a third example, for the constant

$$\text{MAP} : \forall F :: \star \Rightarrow \star, A :: \star, B :: \star. (A \rightarrow B) \rightarrow (FA \rightarrow FB)$$

the property implied by the type reduces similarly to

$$\forall f, f'. b \cdot f = f' \cdot a \implies \phi * b \cdot \text{MAP}_F f = \text{MAP}_G f' \cdot \phi * a$$

Since MAP_F represents the action of F on arrows, this reduces to

$$\forall f. \phi_{A'} \cdot Ff = Gf \cdot \phi_A$$

which is the statement of the naturality of ϕ . Note that as F occurs in both positive and negative positions in the type of MAP , the

parametricity property does not correspond to naturality in F , just as FOLD is parametric but not natural in A .

For each $\Gamma; \Delta \vdash e : S$, we have an interpretation

$$[[\Gamma; \Delta \vdash e : S]] \delta \in \mathbb{C}(\mathcal{D}[\Delta \vdash \Gamma] \delta, \mathcal{D}[\Delta \vdash S] \delta)$$

using the cartesian closed structure of \mathbb{C} in the usual way. In the special case where the expression e is closed, the expression context Γ will be empty, so that the semantics is an arrow in $\mathbb{C}(1, \mathcal{D}[\Delta \vdash S] \delta)$, that is, a point.

6.4 Consequences of the interpretation

Assuming these properties for the primitive constants, our aim is to infer similar properties for other combinators that we can define in our language.

This is expressed by our main theorem: if τ is a transformation between type environments δ and δ' , such that the corresponding relation holds between value environments γ and γ' , then the interpretation of $\Delta; \Gamma \vdash e : S$ with respect to δ and γ is related to the interpretation with respect to δ' and γ' . This is established in the usual way by a straightforward induction over the derivation of $\Delta; \Gamma \vdash e : S$.

Theorem 1 Given

$$\begin{aligned} \tau &\in \mathbb{C}_\Delta(\delta, \delta') \\ \gamma &\in \text{Pt}(\mathcal{D}[\Delta \vdash \Gamma] \delta) \\ \gamma' &\in \text{Pt}(\mathcal{D}[\Delta \vdash \Gamma] \delta') \end{aligned}$$

we have

$$\begin{aligned} \mathfrak{R}[\Delta \vdash \Gamma] \tau(\gamma, \gamma') &\implies \\ \mathfrak{R}[\Delta \vdash S] \tau &([[\Delta; \Gamma \vdash e : S]] \delta \cdot \gamma, [[\Delta; \Gamma \vdash e : S]] \delta' \cdot \gamma') \end{aligned}$$

□

When e is closed, things are much simpler. Γ will be empty, $\mathcal{D}[\Delta \vdash \Gamma] \delta$ and $\mathcal{D}[\Delta \vdash \Gamma] \delta'$ will both be 1 so the relation between them trivially holds.

Corollary 2 For any closed term $\Delta; \vdash e : S$ and $\tau \in \mathbb{C}_\Delta(\delta, \delta')$, we have

$$\mathfrak{R}[\Delta \vdash S] \tau ([[\Delta; \vdash e : S]] \delta, [[\Delta; \vdash e : S]] \delta')$$

□

For example, the PARA operator defined above had the type

$$F :: \star \Rightarrow \star, A :: \star; \vdash \dots : (F(A \times \mu F) \rightarrow A) \rightarrow (\mu F \rightarrow A)$$

From this type we can infer the property

$$\forall f, f'. a \cdot f = f' \cdot \phi * (a \times \mu \phi) \implies a \cdot \text{PARA}_F f = \text{PARA}_G f' \cdot \mu \phi$$

That is, we get for free the higher-order naturality property presented in Section 3.3, combined with a first-order fusion law; specializing to $a = \text{id}$ yields the higher-order naturality alone.

7. Conclusions

We have shown that the datatype-generic FOLD operator enjoys a kind of higher-order naturality property, relating different instances connected by a natural transformation; moreover, that naturality property arises for free from the type of the operator. Similar results apply to many other datatype-generic operators in the origami programming style. Moreover, those results may be derived for free from the higher-kinded type of the operator. In contrast, in approaches to datatype-generic programming that rely on case analysis on the shape of data, such properties have to be much more carefully engineered.

7.1 Inspiration

Our inspiration for the higher-order natural transformations described in this paper is Paul Hoogendijk’s very elegant work with Roland Backhouse [17, 18] on generic ‘zip’ functions $\text{ZIP}_{F,G} : F \circ G \rightarrow G \circ F$ that commute or transpose two functors F and G . The general case requires a relational setting, because such a transposition might be partial (for instance, yielding no result on mismatched data structures, such as when zipping a pair of lists of differing length) and non-deterministic (for instance, yielding results of arbitrary shape, such as when zipping with a constant functor F). However, the essence of the idea can be seen from considering the special case when G corresponds to a fixed-shape datatype such as Pair . In this case, the partially-parametrized remainder is conventionally called a ‘generic unzip’ $\text{UNZIP}_F : F \circ \text{Pair} \rightarrow \text{Pair} \circ F$, and can easily be given an explicit definition:

$$\text{UNZIP}_F = \text{fst} \Delta \text{snd}$$

where fst and snd are the projections from Pair , and Δ generates a Pair using two element-generating functions. It is not hard to show that UNZIP is a hont $(\circ \text{Pair}) \rightarrow (\text{Pair} \circ)$. The higher-order naturality property arising from this observation was used in [4] to transform an $O(n \log n)$ -time algorithm for computing bit-reversal permutations to $O(n)$ -time.

7.2 Related work

The equation capturing the higher-order naturality of FOLD has appeared before; for example, it is Theorem 6.12 of the second author’s PhD thesis [28], Equation 29 in the influential ‘bananas paper’ [26], and Equation (4) of [33]. However, to the best of our knowledge, the fact that this property is a naturality property has not been noted previously. Although Hoogendijk [17] coins the term ‘parametric polytypism’ (for what we call ‘parametric datatype-genericity’), and contrasts it with ‘ad hoc polytypism’, the only hont he considers is the generic ZIP described above. In particular, although Hoogendijk certainly makes use of FOLD , there is no evidence from his writing that he realised that it too was a hont. The closest observation we are aware of in the literature is an offhand remark (‘catamorphisms on different types can be related, but the precise details are not clear to us’) by Jeuring and Jansson [19]. The extension to other datatype-generic operators such as UNFOLD , and the higher-order parametricity result itself, appear to be novel, albeit perhaps not very surprising with the benefit of hindsight.

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