Logic and Proof Hilary 2024

Exercise Sheet 5

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1. Consider the theory $\boldsymbol{T}$ of the structure $\mathcal{A} = (\mathbb{N}, 0, s, <)$, where s is the unary function given by $s(n) = n + 1$.

(a) Suppose that F is a conjunction of atomic formulas, all of which mention the variable x. Show that there is a quantifier-free formula G such that $\boldsymbol{T} \models \exists x F \leftrightarrow G$.

(b) By following the reasoning in the lecture notes, conclude from (i) that $\boldsymbol{T}$ has quantifier elimination.

(c) Say that $S \subseteq \mathbb{N}$ is *definable* if there is a formula F with one free variable x such that $\mathcal{A}_{[x \mapsto a]} \models F$ if and only if $a \in S$. Given that $\boldsymbol{T}$ has quantifier elimination, show that the definable subsets of $\mathbb{N}$ are the finite and cofinite subsets of $\mathbb{N}$.

2. This question concerns the theory $\boldsymbol{T}$ of the structure $\mathcal{A} = (\mathbb{N}, 0, 1, +, <, \{P_k\}_k)$, where for each integer $k > 1$, the unary predicate $P_k(n)$ holds if and only if n is divisible by k. You are given that $\boldsymbol{T}$ has quantifier elimination.

(a) Say that a set $S \subseteq \mathbb{N}$ is *ultimately periodic* if there exist positive integers n_0 and p such that for all $n \geq n_0$, $n \in S$ iff $n + p \in S$. Show that any quantifier-free formula that mentions a single variable x defines an ultimately periodic subset of $\mathbb{N}$.

(b) Using your answer to part (a), or otherwise, show that there is no formula on free variables x, y and z that defines the multiplication relation $M = \{(a, b, c) \in \mathbb{N}^3 : ab = c\}$ on the structure $\mathcal{A}$.