Logic and Proof Hilary 2024

Normal Forms for First-Order Logic

James Worrell

In this lecture we show how to transform an arbitrary formula of first-order logic to an equisatisfiable formula in *Skolem form*. This translation is in preparation for our subsequent treatment of deduction using unification and resolution.

1 Equivalence and Substitution

Two first-order formulas F and G over a signature σ are *logically equivalent*, denoted $F \equiv G$, if for all σ-assignments $\mathcal{A}$ we have $\mathcal{A} \models F$ iff $\mathcal{A} \models G$.

All the propositional equivalences carry over to the first-order setting, e.g., we still have De Morgan's law $\neg(F \wedge G) \equiv (\neg F \vee \neg G)$, etc. Moreover logical equivalence remains a congruence with respect to the Boolean connectives $\wedge$, $\vee$ and $\neg$, that is, $(F_1 \wedge G_1) \equiv (F_2 \wedge G_2)$ if $F_1 \equiv G_1$ and $F_2 \equiv G_2$, etc. In addition we have that that if $F \equiv G$ then $\forall x\, F \equiv \forall x\, G$ and $\exists x\, F \equiv \exists x\, G$.

The following equivalences will play an important role in transforming formulas into Skolem form.

Proposition 1. Let F and G be arbitrary formulas. Then

(A) $\neg\forall x F \equiv \exists x \neg F$
$\neg\exists x F \equiv \forall x \neg F$

(B) If x does not occur free in G then:
$(\forall x F \wedge G) \equiv \forall x(F \wedge G)$
$(\forall x F \vee G) \equiv \forall x(F \vee G)$
$(\exists x F \wedge G) \equiv \exists x(F \wedge G)$
$(\exists x F \vee G) \equiv \exists x(F \vee G)$

(C) $(\forall x F \wedge \forall x G) \equiv \forall x(F \wedge G)$
$(\exists x F \vee \exists x G) \equiv \exists x(F \vee G)$

(D) $\forall x \forall y F \equiv \forall y \forall x F$
$\exists x \exists y F \equiv \exists y \exists x F$

Proof. As an example, we prove the first equivalences in (A) and (B). For the former we have

$$\begin{aligned}
\mathcal{A} \models \neg\forall x F \ &\text{iff } \mathcal{A} \not\models \forall x F \\
&\text{iff } \mathcal{A}_{[x\mapsto a]} \not\models F \ \text{ for some } a \in U_{\mathcal{A}} \\
&\text{iff } \mathcal{A}_{[x\mapsto a]} \models \neg F \ \text{ for some } a \in U_{\mathcal{A}} \\
&\text{iff } \mathcal{A} \models \exists x \neg F
\end{aligned}$$

For the first equivalence in (B) we have

$$\begin{aligned}
\mathcal{A} \models (\forall x F \wedge G) &\text{ iff } \mathcal{A} \models \forall x F \text{ and } \mathcal{A} \models G \\
&\text{ iff for all } a \in U_{\mathcal{A}},\ \mathcal{A}_{[x\mapsto a]} \models F \text{ and } \mathcal{A} \models G \\
&\text{ iff for all } a \in U_{\mathcal{A}},\ \mathcal{A}_{[x\mapsto a]} \models F \text{ and } \mathcal{A}_{[x\mapsto a]} \models G \text{ (by the Relevance Lemma)} \\
&\text{ iff for all } a \in U_{\mathcal{A}},\ \mathcal{A}_{[x\mapsto a]} \models F \wedge G \\
&\text{ iff } \mathcal{A} \models \forall x (F \wedge G)\,.
\end{aligned}$$

□

A formula is in *prenex form* if it can be written

$$Q_1 y_1\, Q_2 y_2 \ldots Q_n y_n\, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Example 2. We use Proposition 1 to transform the formula

$$\neg(\exists x\, P(x,y) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w) \tag{1}$$

to prenex form by the following chain of equivalences:

$$\begin{aligned}
\neg(\exists x\, P(x,y) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w) &\equiv (\neg\exists x\, P(x,y) \wedge \neg\forall z\, Q(z)) \wedge \exists w\, Q(w) \\
&\equiv (\forall x\, \neg P(x,y) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w) \\
&\equiv \forall x\, \exists z\, (\neg P(x,y) \wedge \neg Q(z)) \wedge \exists w\, Q(w) \\
&\equiv \forall x\, \exists z\, \exists w\, ((\neg P(x,y) \wedge \neg Q(z)) \wedge Q(w))\,.
\end{aligned}$$

Note that in the above equational reasoning we use the fact that logical equivalence is a congruence with respect to the Boolean operators (i.e., the Substitution Theorem).

Let F be a formula, x a variable, and t a term. Then $F[t/x]$ (read "F with t for x") denotes the formula with t substituted for every free occurrence of x in F. For example,

$$(\forall x\, P(x,y) \wedge Q(x))[t/x] = \forall x\, P(x,y) \wedge Q(t)\,.$$

Formally, we define $F[t/x]$ by induction on terms and formulas as follows. On terms we have:

$$\begin{aligned}
c[t/x] &= c \quad \text{for } c \text{ a constant symbol} \\
y[t/x] &= y \quad \text{for } y \neq x \text{ a variable} \\
x[t/x] &= t \\
f(t_1,\ldots,t_k)[t/x] &= f(t_1[t/x],\ldots,t_k[t/x]) \quad \text{for } f \text{ a } k\text{-ary function symbol}
\end{aligned}$$

We then extend the definition of $[t/x]$ to formulas as follows:

$$\begin{aligned}
P(t_1,\ldots,t_k)[t/x] &= P(t_1[t/x],\ldots,t_k[t/x]) \\
(\neg F)[t/x] &= \neg(F[t/x]) \\
(F \wedge G)[t/x] &= F[t/x] \wedge G[t/x] \\
(F \vee G)[t/x] &= F[t/x] \vee G[t/x] \\
(Qy\, F)[t/x] &= Qy\, (F[t/x]) \quad y \neq x \text{ a variable, } Q \in \{\forall, \exists\} \\
(Qx\, F)[t/x] &= Qx\, F \quad Q \in \{\forall, \exists\}\,.
\end{aligned}$$

Warning! The notation we use for the substitution is the reverse of that used in Schöning's book. The latter uses $[x/t]$ to denote the substitution of t for x. Our use is more standard.

A key fact about substitution is the following. The proof is in Appendix A.

Lemma 3 (Translation Lemma)**.** If t is term and F is a formula such that no variable in t occurs bound in F, then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}[\![t]\!]]} \models F$.

To illustrate the necessity of the side-condition in the Translation Lemma, let F be the formula $\forall y P(x)$ and let $\mathcal{A}$ be the assignment with $U_{\mathcal{A}} = \{1, 2\}$, $P_{\mathcal{A}} = \{1\}$, $x_{\mathcal{A}} = 1$, and $y_{\mathcal{A}} = 1$. Then $F[y/x] = \forall y\, P(y)$ and so $\mathcal{A} \not\models F[y/x]$. But $\mathcal{A}[\![y]\!] = 1$ and so $\mathcal{A}_{[x \mapsto \mathcal{A}[\![y]\!]]} \models F$. The reason we cannot apply the Translation Lemma in this case is that the variable y in the term to be substituted becomes bound by the quantifier $\forall y$ in F. This phenomenon is called *variable capture*.

In first-order logic we can rename bound variables in a formula while preserving logical equivalence. For example, we have $\forall x\, P(x) \equiv \forall y\, P(y)$. This is similar to the fact that the definite integral $\int_0^\infty f(s) ds$ denotes the same value as $\int_0^\infty f(t) dt$. We make this idea formal as follows:

Proposition 4. Let $F = Qx\, G$ be a formula where $Q \in \{\forall, \exists\}$. Let y be a variable that does not occur in G. Then $F \equiv Qy\, (G[y/x])$.

Proof. We prove the proposition in the case of $\forall$; the case for $\exists$ is similar. Let $\mathcal{A}$ be an assignment. Then

$$\begin{aligned}
\mathcal{A} \models \forall y\, (G[y/x]) \quad & \text{iff} \quad \mathcal{A}_{[y \mapsto a]} \models G[y/x] \text{ for all } a \in U_{\mathcal{A}} \\
& \text{iff} \quad \mathcal{A}_{[y \mapsto a][x \mapsto \mathcal{A}_{[y \mapsto a]}(y)]} \models G \text{ for all } a \in U_{\mathcal{A}} \text{ (Translation Lemma)} \\
& \text{iff} \quad \mathcal{A}_{[y \mapsto a][x \mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \\
& \text{iff} \quad \mathcal{A}_{[x \mapsto a][y \mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \\
& \text{iff} \quad \mathcal{A}_{[x \mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \text{ (Relevance Lemma)} \\
& \text{iff} \quad \mathcal{A} \models \forall x\, G\,.
\end{aligned}$$

□

2 Skolem Form

A formula is *rectified* if no variable occurs both bound and free and if all quantifiers in the formula refer to different variables. For example, the formula

$$\forall x\, \exists y\, P(x, f(y)) \wedge \forall y\, (Q(x, y) \vee R(x))$$

is not rectified since y is bound on two separate occasions and x occurs both free and bound. By renaming the bound variables we obtain the following equivalent rectified formula:

$$\forall u\, \exists v\, P(u, f(v)) \wedge \forall y\, (Q(x, y) \vee R(x))\,.$$

In general we can always obtain an equivalent rectified formula by renaming bound variables using Proposition 4.

Lemma 5. Every formula is equivalent to a rectified formula.

Given a rectified formula F we can use the equivalences in Proposition 1 to convert F to an equivalent formula in rectified prenex form (RPF) by "pushing all quantifiers to the front" in the manner of Example 2.

Theorem 6. Every formula is equivalent to a rectified formula in prenex form.

We say that a formula in RPF is in *Skolem form* if it does not contain any occurrences of the existential quantifier. We can transform a formula in RPF to an equisatisfiable (though not necessarily logically equivalent) formula in Skolem form by using extra function symbols. For example, the formulas $\forall x \exists y P(x,y)$ and $\forall x P(x, f(x))$ are equisatisfiable. An assignment that satisfies the left-hand formula can be extended to an assignment satisfying the right-hand formula by interpreting f as a "selection function" that maps each x to some y such that $P(x,y)$ holds. More generally we have the following proposition.

Proposition 7. Let $F = \forall y_1 \forall y_2 \ldots \forall y_n \exists z\, G$ be a rectified formula. Given a function symbol f of arity n that does not occur in F,[1] write

$$F' = \forall y_1 \forall y_2 \ldots \forall y_n\, G[f(y_1, \ldots, y_n)/z]\,.$$

Then F and F' are equisatisfiable.

Proof. We prove that if F is satisfiable then so is F'. The reverse direction is left as an exercise.

Suppose that $\mathcal{A} \models F$ for some assignment $\mathcal{A}$. We define an assignment $\mathcal{A}'$ that extends $\mathcal{A}$ with an interpretation of the function symbol f such that $\mathcal{A}' \models F'$.

Given $a_1, \ldots, a_n \in U_{\mathcal{A}}$, pick $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[y_1 \mapsto a_n] \ldots [y_n \mapsto a_n][z \mapsto a]} \models G$ and define $f_{\mathcal{A}'}(a_1, \ldots, a_n) = a$. Since the function symbol f does not occur in G we have

$$\mathcal{A}'_{[y_1 \mapsto a_n] \ldots [y_n \mapsto a_n][z \mapsto f_{\mathcal{A}'}(a_1, \ldots, a_n)]} \models G\,,$$

and so, by the Translation Lemma,

$$\mathcal{A}'_{[y_1 \mapsto a_n] \ldots [y_n \mapsto a_n]} \models G[f(y_1, \ldots, y_n)/z]\,.$$

Since the above holds for all $a_1, \ldots, a_n \in U_{\mathcal{A}}$, we conclude that $\mathcal{A}' \models \forall y_1 \forall y_2 \ldots \forall y_n\, G[f(y_1, \ldots, y_n)/z]$. □

Example 8. We find an equisatifiable Skolem form of the formula

$$\forall x\, \exists y\, \forall z\, \exists w\, (\neg P(a, w) \vee Q(f(x), y))\,.$$

We apply Proposition 7, eliminating $\exists y$ and introducing a new function symbol g, yielding

$$\forall x\, \forall z\, \exists w\, (\neg P(a, w) \vee Q(f(x), g(x)))\,.$$

Then we eliminate $\exists w$ by introducing a new function symbol h, yielding

$$\forall x\, \forall z\, (\neg P(a, h(x, z)) \vee Q(f(x), g(x)))\,.$$

[1] In the case $n = 0$ we consider f as a constant symbol.

Conversion to Skolem Form: Summary

We convert an arbitrary first-order formula F to an equisatisfiable formula in Skolem form as follows:

1. Rectify F by systematically renaming its bound variables, yielding a logically equivalent formula F_1.
2. Using the equivalences in Proposition 1 move all the quantifiers in F_1 to the outside, yielding an equivalent formula F_2 in prenex form.
3. Repeatedly eliminate the outermost existential quantifier in F_2 until an equisatisfiable formula F_3 in Skolem form is obtained. (This process is called *Skolemisation*.)

A Proof of The Translation Lemma

In this section we give the proof of the Translation Lemma. The proof is very technical and can be regarded as optional.

Given an assignment $\mathcal{A}$ we first show by induction on terms s that $\mathcal{A}[\![s[t/x]]\!] = \mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![s]\!]$. The base cases are as follows:

$$\begin{array}{ll}
\mathcal{A}[\![c[t/x]]\!] = \mathcal{A}[\![c]\!] = \mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![c]\!] & c \text{ a constant symbol}\\
\mathcal{A}[\![y[t/x]]\!] = \mathcal{A}[\![y]\!] = \mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![y]\!] & y \neq x \text{ a variable}\\
\mathcal{A}[\![x[t/x]]\!] = \mathcal{A}[\![t]\!] = \mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![x]\!] &
\end{array}$$

For the induction step we have

$$\begin{array}{rcll}
\mathcal{A}[\![f(t_1,\ldots,t_k)[t/x]]\!] & = & \mathcal{A}[\![f(t_1[t/x],\ldots,t_k[t/x])]\!] & \\
& = & f_{\mathcal{A}}(\mathcal{A}[\![t_1[t/x]]\!],\ldots,\mathcal{A}[\![t_k[t/x]]\!]) & \\
& = & f_{\mathcal{A}}(\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![t_1]\!],\ldots,\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![t_k]\!]) & \text{(by induction hypothesis)}\\
& = & f_{\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}}(\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![t_1]\!],\ldots,\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![t_k]\!]) & \\
& = & \mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![f(t_1,\ldots,t_k)]\!]\,. &
\end{array}$$

Next we use induction on formulas to show that for all formulas F, $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]} \models F$. The base case is that F is an atomic formula $P(t_1,\ldots,t_k)$ for a k-ary predicate symbol P. Then

$$\begin{array}{rll}
\mathcal{A} \models P(t_1,\ldots,t_k)[t/x] & \text{iff} & \mathcal{A} \models P(t_1[t/x],\ldots,t_k[t/x])\\
& \text{iff} & (\mathcal{A}[\![t_1[t/x]]\!],\ldots,\mathcal{A}[\![t_k[t/x]]\!]) \in P_{\mathcal{A}}\\
& \text{iff} & (\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![t_1]\!],\ldots,\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![t_k]\!]) \in P_{\mathcal{A}}\\
& \text{iff} & (\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![t_1]\!],\ldots,\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}[\![t_k]\!]) \in P_{\mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]}}\\
& \text{iff} & \mathcal{A}_{[x\mapsto\mathcal{A}[\![t]\!]]} \models P(t_1,\ldots,t_k)\,.
\end{array}$$

The inductive cases for the propositional connectives are routine. The case for the universal quantifier $\forall y$, where $y \neq x$, is given below.

$$
\begin{aligned}
\mathcal{A} \models (\forall y F)[t/x] \quad & \text{iff} \quad \mathcal{A} \models \forall y (F[t/x]) \\
& \text{iff} \quad \mathcal{A}_{[y \mapsto d]} \models F[t/x] \text{ for all } d \in U_{\mathcal{A}} \\
& \text{iff} \quad \mathcal{A}_{[y \mapsto d][x \mapsto \mathcal{A}_{[y \mapsto d]}[\![t]\!]]} \models F \text{ for all } d \in U_{\mathcal{A}} \quad \text{(induction hypothesis)} \\
& \text{iff} \quad \mathcal{A}_{[y \mapsto d][x \mapsto \mathcal{A}[\![t]\!]]} \models F \text{ for all } d \in U_{\mathcal{A}} \quad (y \text{ does not occur in } t) \\
& \text{iff} \quad \mathcal{A}_{[x \mapsto \mathcal{A}[\![t]\!]][y \mapsto d]} \models F \text{ for all } d \in U_{\mathcal{A}} \quad (y \neq x) \\
& \text{iff} \quad \mathcal{A}_{[x \mapsto \mathcal{A}[\![t]\!]]} \models \forall y F\,.
\end{aligned}
$$

The case for the existential quantifier is similar to the above. This concludes the proof.