Graphs, foams, homology, polytopes, and abstract tensors

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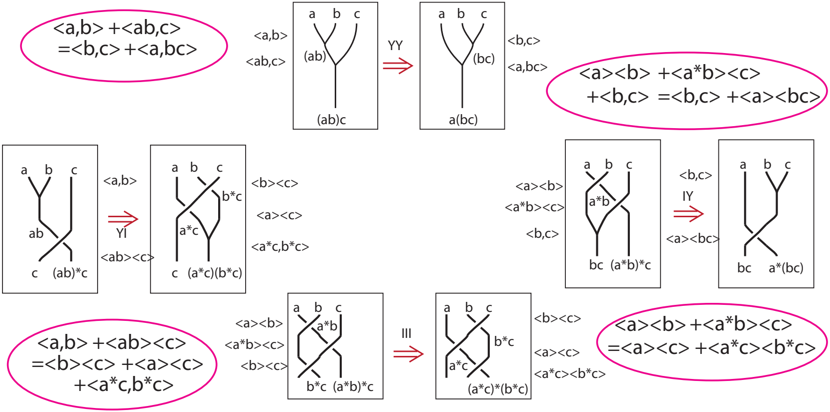
5. Current funding: Simons Foundation.

Section 1

in which systems of abstract tensor equations are formulated and a solution is proposed.

3D moves



<a,b> +<ab,c>
=<b,c> +<a,bc>
<a,b>
<ab,c>
a b c
(ab)
(ab)c
YY
a b c
(bc)
a(bc)
<b,c>
<a,bc>
<a><b> +<a*b><c>
+<b,c> =<b,c> +<a><bc>
a b c
ab
c (ab)*c
<a,b>
YI
<ab><c>
a b c
b*c
a*c
c (a*c)(b*c)
<b><c>
<a><c>
<a*c,b*c>
<a><b>
<a*b><c>
<b,c>
a b c
a*b
bc (a*b)*c
<b,c>
IY
<a><bc>
a b c
bc a*(bc)
<a,b> +<ab><c>
=<b><c> +<a><c>
+<a*c,b*c>
<a><b>
<a*b><c>
<b><c>
a b c
a*b
b*c (a*b)*c
III
a b c
b*c
a*c
(a*c)*(b*c)
<b><c>
<a><c>
<a*c><b*c>
<a><b> +<a*b><c>
=<a><c> +<a*c><b*c>

In 3-D equations

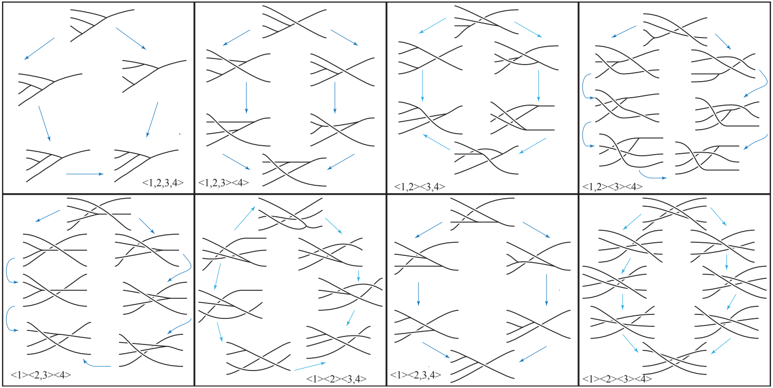
A	$\mathsf{Y}_j^{\ell,n}\mathsf{Y}_i^{j,k} = \mathsf{Y}_p^{n,k}\mathsf{Y}_i^{\ell,p}$
YI	$\mathsf{Y}_d^{a,b}\mathsf{X}_{e,f}^{d,c} = \mathsf{X}_{c,h}^{b,c}\mathsf{X}_{e,g}^{a,c}\mathsf{Y}_f^{g,h}.$
IY	$\mathsf{X}_{d,g}^{a,b}\mathsf{X}_{h,f}^{g,c}\mathsf{Y}_e^{d,h} = \mathsf{Y}_i^{b,c}\mathsf{X}_{e,f}^{a,i}.$
III	$\mathsf{X}_{d,e}^{a,b}\mathsf{X}_{i,h}^{e,c}\mathsf{X}_{f,g}^{d,i} = \mathsf{X}_{j,k}^{b,c}\mathsf{X}_{f,e}^{a,j}\mathsf{X}_{g,h}^{e,k}$

Easy Solution

$$\mathsf{Y}_{c}^{a,b} = \begin{cases} 1 & \text{if } c = ab, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathsf{X}_{c,d}^{a,b} = \begin{cases} 1 & \text{if } c = b \;\&\; d = b^{-1}ab, \\ 0 & \text{otherwise.} \end{cases}$$

4D moves



4-D Equations

YYY	$\mathsf{K}^{m,n}_{p,q}\mathsf{K}^{p,r}_{s,t} = \mathsf{K}^{n,r}_{u,v}\mathsf{K}^{m,u}_{s,w}\mathsf{K}^{w,v}_{t,q}$
YYI	$\mathsf{M}^{p,q}_{t,u,v}\mathsf{M}^{u,r}_{x,y,z}\mathsf{K}^{t,x}_{m,n} = \mathsf{K}^{q,r}_{s,i}\mathsf{M}^{p,s}_{m,y,j}\mathsf{M}^{j,i}_{n,z,v}$
YY	$\mathsf{W}^{p,q,r}_{i,j}\mathsf{M}^{i,s}_{m,n,y} = \mathsf{M}^{r,s}_{t,u,v}\mathsf{M}^{q,t}_{m,x,z}\mathsf{W}^{p,x,u}_{n,h}\mathsf{W}^{h,z,v}_{y,i}$
YII	$\text{Ж}^{p,q,r}_{t,x,v}\mathsf{M}^{x,s}_{y,u,v}\mathsf{M}^{t,i}_{m,n,z} = \mathsf{M}^{r,s}_{j,x,g}\mathsf{M}^{q,j}_{m,t,x}\text{Ж}^{p,t,x}_{n,y,f}\text{Ж}^{f,w,g}_{z,u,v}$
IYY	$\mathsf{K}^{p,q}_{u,v}\mathsf{W}^{v,r,s}_{i,j}\mathsf{W}^{u,i,t}_{y,z} = \mathsf{W}^{q,s,t}_{u,v}\mathsf{W}^{p,r,u}_{y,i}\mathsf{K}^{i,v}_{z,j}$
IYI	$\mathsf{M}^{p,q}_{u,v,x}\text{Ж}^{x,r,s}_{i,j,n}\mathsf{X}^{v,j,t}_{h,g,f}\mathsf{W}^{u,i,h}_{y,z} = \mathsf{W}^{q,s,t}_{u,v}\text{Ж}^{p,r,u}_{y,i,j}\mathsf{M}^{j,v}_{z,f,n}$
IIY	$\mathsf{W}^{p,q,r}_{h,j}\mathsf{W}^{j,s,t}_{i,n}\text{Ж}^{h,i,u}_{x,y,z} = \text{Ж}^{r,t,u}_{v,g,f}\text{Ж}^{q,s,v}_{x,h,j}\mathsf{W}^{p,h,g}_{y,i}\mathsf{W}^{i,j,f}_{z,n}$
IIII	$\text{Ж}^{p,q,r}_{v,x,y}\text{Ж}^{y,s,t}_{z,e,n}\text{Ж}^{x,e,u}_{k,h,j}\text{Ж}^{v,z,k}_{g,f,i} = \text{Ж}^{r,t,u}_{k,z,v}\text{Ж}^{q,s,k}_{g,e,x}\text{Ж}^{p,e,z}_{f,h,y}\text{Ж}^{y,x,v}_{i,j,n}$

Solutions part 1

$$\mathsf{K}_{r,s}^{p,q} = \begin{cases} 1 \text{ if } \exists\ a, b, c \text{ s.t.} \\ \begin{cases} p = ((ab, c), 0|0), \quad q = ((a, b), 0|0), \\ r = ((a, bc), 0|0), \,\&\, s = ((b, c), 0|0). \end{cases} \\ 0 \text{ otherwise;} \end{cases}$$

Solutions part 2

$$\mathsf{M}_{p,q,r}^{i,j} = \begin{cases} 1 \text{ if } \exists\ a, b, c \text{ s.t.} \\ \begin{cases} i = ((0,0), ab|c), & j = ((a,b), 0|0), \\ p = ((a,b) \triangleleft c, 0|0), & q = ((0,0), a|c), \\ \&\ r = ((0,0), b|c), \end{cases} \\ 0 \text{ otherwise;} \end{cases}$$

Solutions part 3

$$\mathsf{W}_{p,q}^{i,j,\ell} = \begin{cases} 1 \text{ if } \exists\ a, b, c \text{ s.t.} \\ \begin{cases} i = ((b,c), 0|0), \quad j = ((0,0), (a \lhd b)|c), \\ \ell = ((0,0), a|b), \\ p = ((0,0), a|(bc)), \quad \& \ q = ((b,c), 0|0), \end{cases} \\ 0 \text{ otherwise;} \end{cases}$$

Solutions part 4

$$Ж^{p,q,r}_{s,t,u} = \begin{cases} 1 \text{ if } \exists\ a, b, c \text{ s.t.} \\ \begin{cases} p = ((0,0), b|c), \quad q = ((0,0), (a \triangleleft b)|c), \\ r = ((0,0), a|b), \\ s = ((0,0), (a|b) \triangleleft c), \quad t = ((0,0), a|c), \\ \& \ u = ((0,0), b|c), \end{cases} \\ 0 \text{ otherwise;} \end{cases}$$

where, *e.g.*, $a \triangleleft c = c^{-1}ac$, $(a,b) \triangleleft c = (a \triangleleft c, b \triangleleft c)$, and $(a|b) \triangleleft c = (a \triangleleft c)|(b \triangleleft c)$

Algebraic setting

Let T denote a free module

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So when I write, *e.g.*

$$\mathsf{M}^{i,j}_{p,q,r} = \begin{cases} 1 \text{ if } \exists\ a, b, c \text{ s.t.} \\ \left\{ \begin{array}{ll} i = ((0,0), ab|c), & j = ((a,b), 0|0), \\ p = ((a,b) \triangleleft c, 0|0), & q = ((0,0), a|c), \\ \& \ r = ((0,0), b|c), & \end{array} \right. \\ 0 \text{ otherwise;} \end{cases}$$

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the tensor M is a morph.

$$V \otimes V \xleftarrow{\mathsf{M}} V \otimes V \otimes V.$$

In general,

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and

$$V \otimes V \otimes V \xleftarrow{\text{Ж}} V \otimes V \otimes V.$$

I soon want to given the geometric background for these equations, and point out that, I soon want to given the geometric background for these equations, and point out that, as usual, I soon want to given the geometric background for these equations, and point out that, as usual, these systems of equations generalize immediately to all dimensions, and in an appropriate algebraic context, we can find solutions.

Section 2

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The Space Y^n

Let $\Delta^{n+1} = \{\vec{x} \in \mathbb{R}^{n+2} : \sum x_i = 1 \ \& \ 0 \leq x_i\}$ denote the standard simplex.

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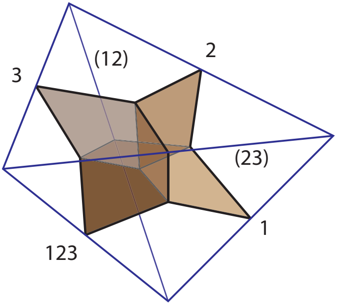
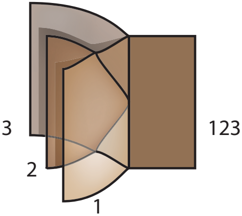
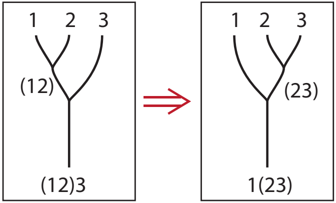
$$Y^n = C\left(\cup_{j=1}^{n+2} Y^{n-1}_j\right).$$

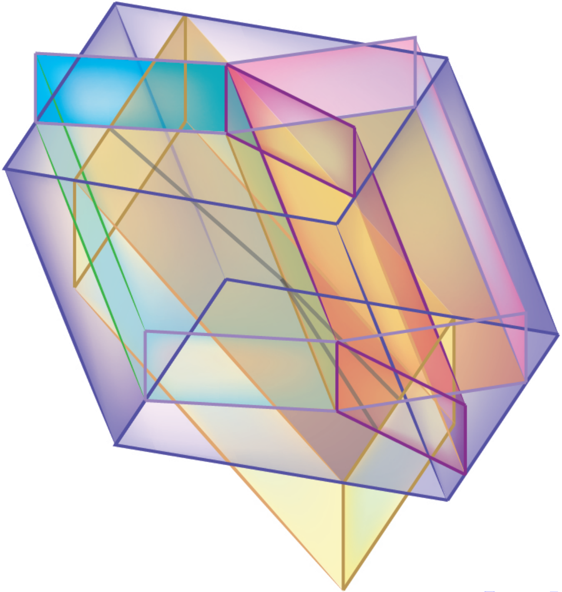
The space Y^2







The space Y^3



Foam Definition

An n-**dimensional foam** is a compact top. sp. X for which each point $x \in X$ has a nbhd. $N(x)$ that is homeom. to a nbhd. M of a point in Y^n.

Local pictures of crossings

Now take

$$\left[\cup_{\ell=1}^{k}\left(\Delta^{j_1}\times\cdots\times Y^{j_\ell-1}\times\cdots\times\Delta^{j_k}\right)\subset\mathbb{R}^{n+1}\times\{\ell\}\right],$$

Local pictures of crossings

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and project this into $\mathbb{R}^{n+1}$.

Local pictures of crossings

Now take

$$\left[\cup_{\ell=1}^{k} \left(\Delta^{j_1} \times \cdots \times Y^{j_\ell - 1} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{\ell\}\right],$$

and project this into $\mathbb{R}^{n+1}$. The factor ℓ is in the $(n+2)$nd coordinate and represents the relative height of each Y.

$$\left[\left(Y^{j_1-1} \times \Delta^{j_2} \cdots \times \Delta^{j_\ell} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{1\}\right],$$

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$$\left[\left(\Delta^{j_1} \times Y^{j_2-1} \cdots \times \Delta^{j_\ell} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{2\}\right],$$

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These project to a 0-dimensional multiple point in $\mathbb{R}^{n+1}$.

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Section 3

in which a discussion of categorification is used to justify the ideas presented in the previous slide.

New school categorification

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Great invariants for classical knots and knot cobordisms. Not so good for knotted closed surfaces.

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Instead of equality among morphisms, posit 2-morphisms that satisfy their own set of relations. Climb the dimension ladder.

Abstract Tensor Formalism

Suppose V, W, etc. f.dim'l vec. sp. over $\mathbb{F}$.

Abstract

Tensor

Formalism

Suppose V, W, etc. f.dim'l vec. sp. over $\mathbb{F}$.

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Tensor

Formalism

$$\vec{x} = \sum_{j=1}^{n} x^j e_j$$

Suppose V, W, etc. f.dim'l vec. sp. over $\mathbb{F}$.

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$$\vec{x} = \sum_{j=1}^{n} x^j e_j \quad \text{where} \quad e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow jth \\ \\ \\ \\ \end{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} n \text{ rows}$$

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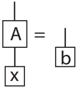
Write $A=$  . So Ax=b is written  .

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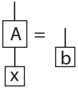
. So Ax=b is written 

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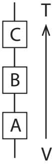
Composition of linear maps is denoted by vertical juxtaposition.

$$T \xleftarrow{C} U \xleftarrow{B} W \xleftarrow{A} V$$



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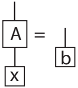
. So Ax=b is written 

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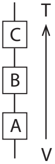

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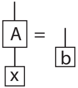
. So Ax=b is written 

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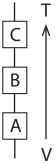

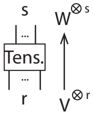

Example: A Frobenius alg. is a v. sp. V tog. w/ linear maps

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 $\begin{array}{c} V \\ \uparrow \\ \mathbb{F} \end{array}$ unit,

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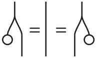

unital axiom

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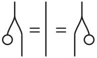 here | denotes the identity map $V \to V$

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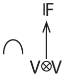
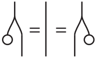


non deg. pairing

that satisfy:



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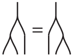

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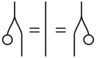
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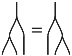

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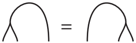

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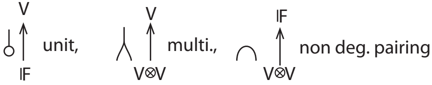

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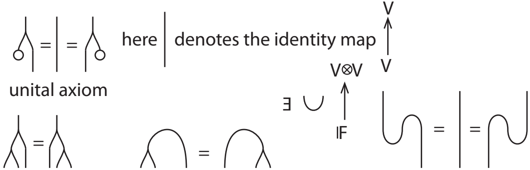

unital axiom

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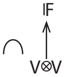

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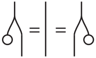

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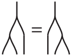
∃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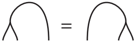
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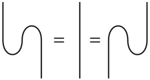
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We can use  and  to define comulti.

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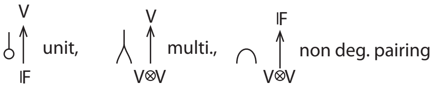

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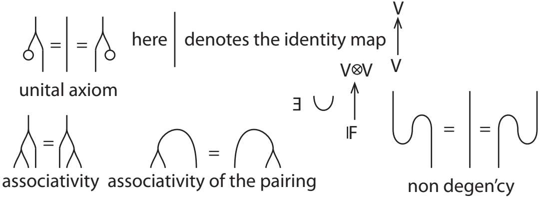

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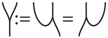

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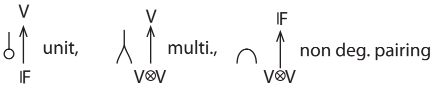

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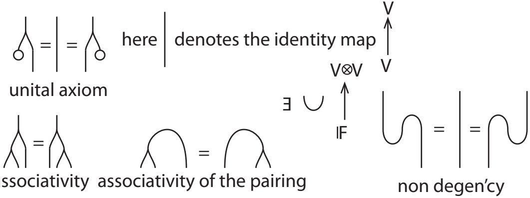

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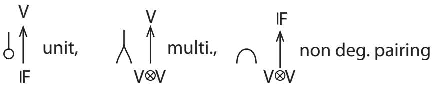

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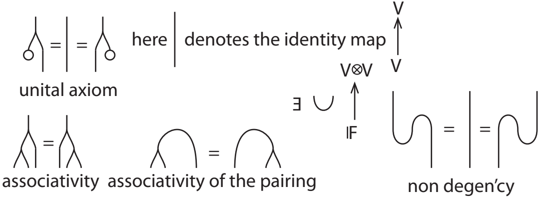

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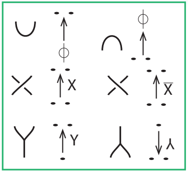
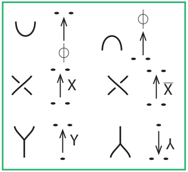


Generating 1-morphs



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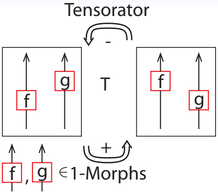

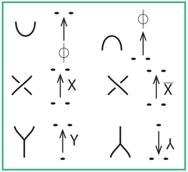

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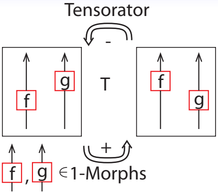

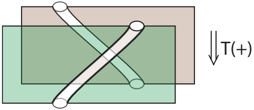

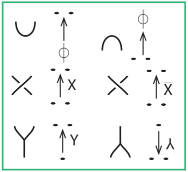

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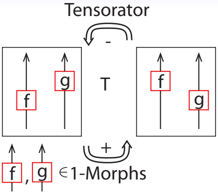

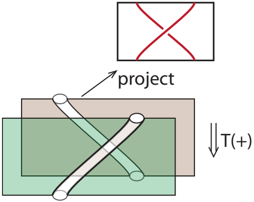

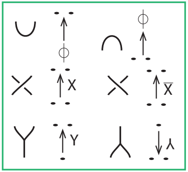

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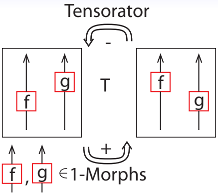
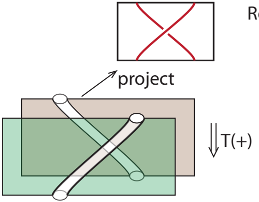
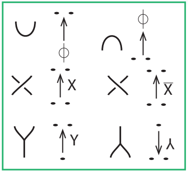
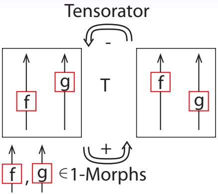


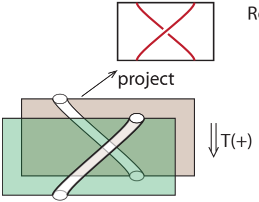
Relations (3-morphism)



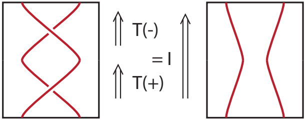


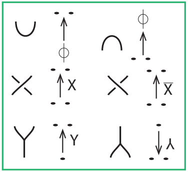
Generating 1-morphs





Relations (3-morphism)





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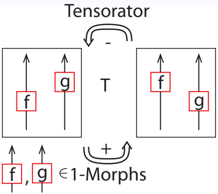

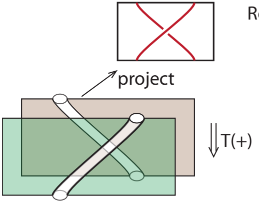

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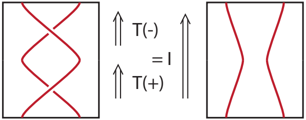
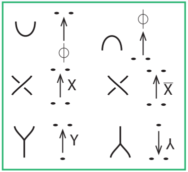
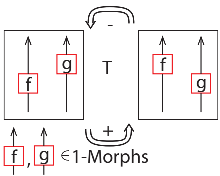
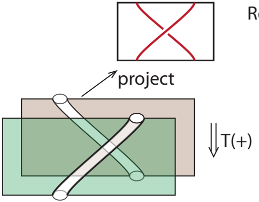


ditto for $T(+)\circ_2 T(-)$



Generating 1-morphs





T(±) is natural w.r.t.
any 2-morph.

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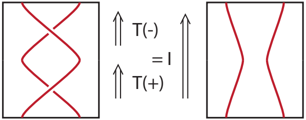

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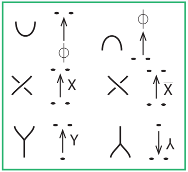

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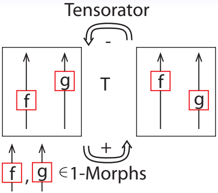

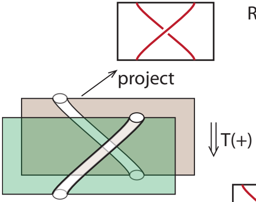

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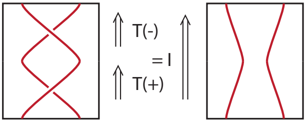
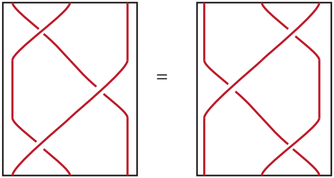
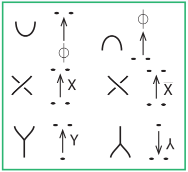
Relations (3-morphism)



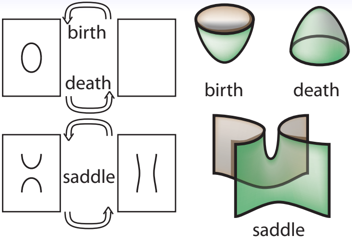
ditto for $T(+)\circ_2 T(-)$

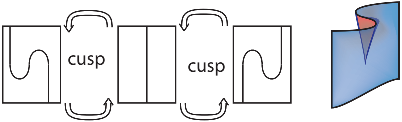
In part.:





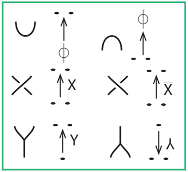
Generating 1-morphs



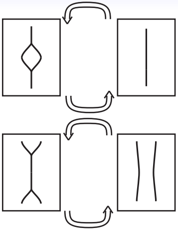


- The critical points evolve to be come folds.
  (co-oriented away from optimal points)

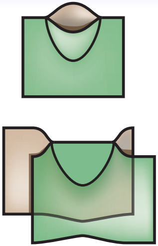
- Several obvious relations hold among these 2-morphisms.
  Including
  - canceling birth/saddle death/saddle
  - lips
  - beak-to-beak
  - swallow-tail
  - horizontal cusp



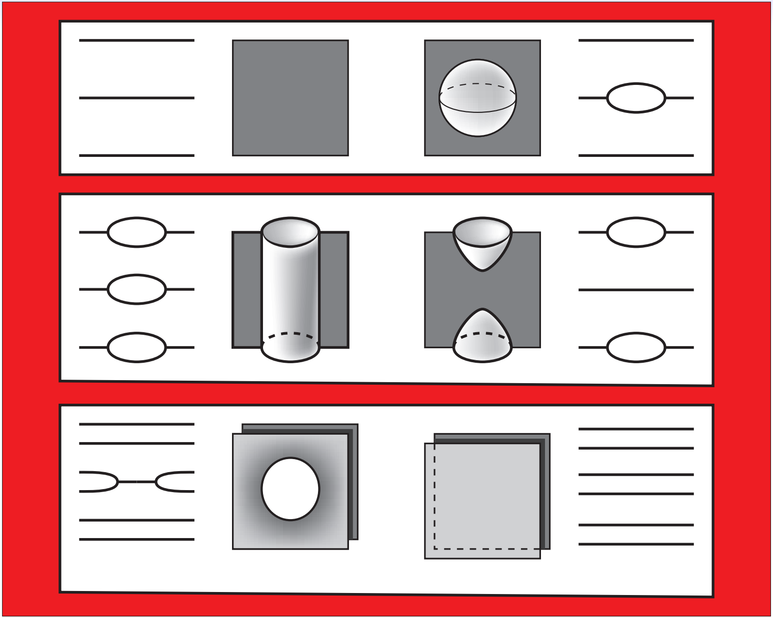
Generating 1-morphs

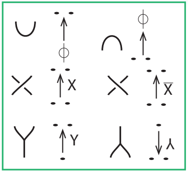


zippers

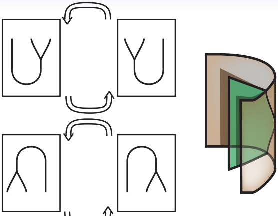


- The vertices of Y or ⅄ evolve to form seams of the foam.
  (co-oriented towards the single sheet)
- There are zig-zag moves that cancel a pair of zipper 2-morphisms.
- Under some circumstances, one might want to suppose
  bubble and saddle moves hold.
  But, as is the case with birth followd by death,
  or a pair of opposite saddles,
  it is better to suppose that the moves
  in the next slide are some type of 3-morphisms.

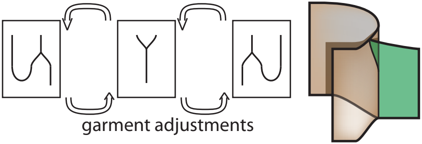




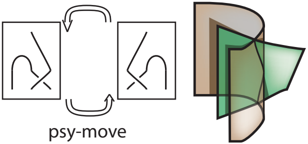
Generating 1-morphs



speedometers

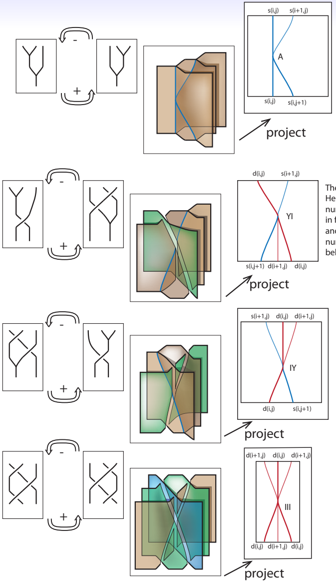


garment adjustments



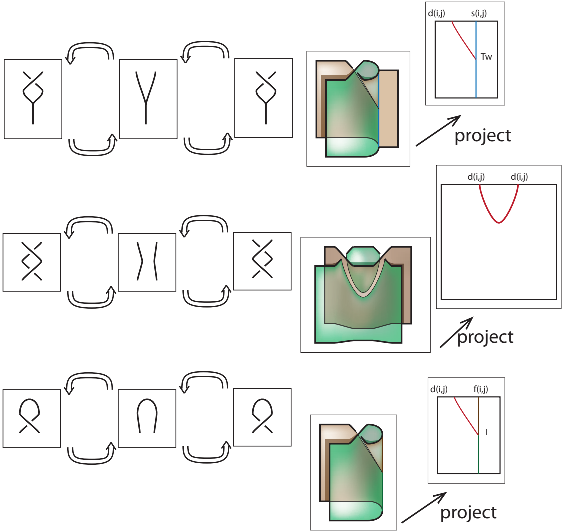
psy-move

- These 2-morphs indicate how folds, seams, double points can interact.
- Each move is invertible in both directions.



The chart depiction. Here j indicates the number of sheets in front of the point and i indicates the number of sheets behind the point.

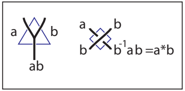
The broken surface representation of the resulting foam.


d(i,j)
s(i,j)
Tw
project
d(i,j)
d(i,j)
project
d(i,j)
f(i,j)
I
project

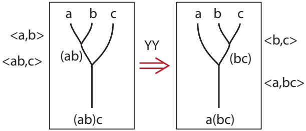
Section 4

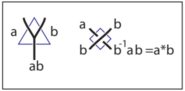
in which an homology theory that encompasses both quandle and group homology is outlined.

Fundamental group

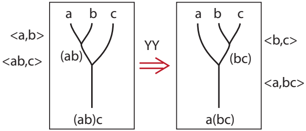


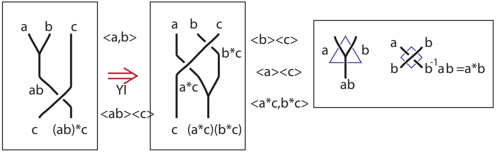
$$(ab)c = a(bc)$$





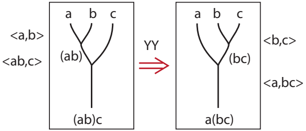
(ab)c = a(bc)



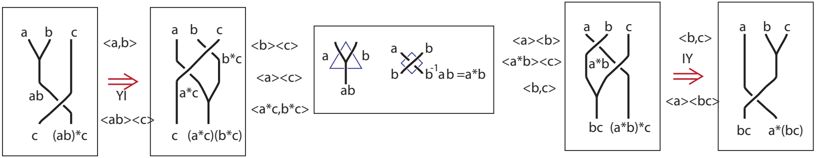


(ab)*c = (a*c)(b*c)

$(ab)c = a(bc)$



$(a*b)*c = a*(bc)$



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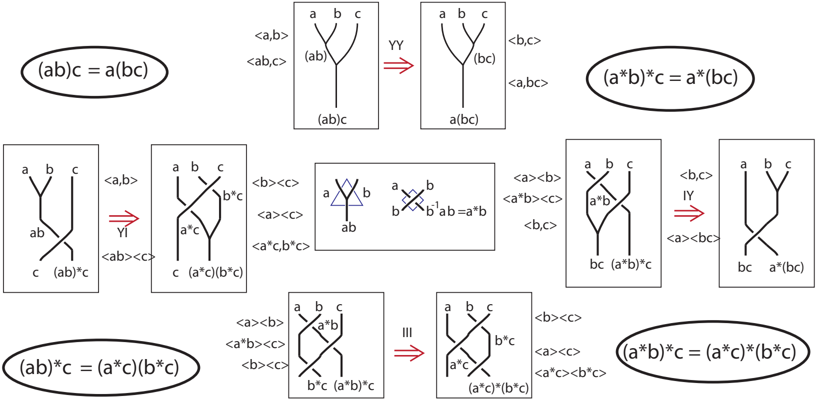
(ab)c = a(bc)
<a,b>
<ab,c>
a b c
(ab)
(ab)c
YY
a b c
(bc)
a(bc)
<b,c>
<a,bc>
(a*b)*c = a*(bc)
a b c
ab
c (ab)*c
<a,b>
YI
<ab><c>
a b c
b*c
a*c
c (a*c)(b*c)
<b><c>
<a><c>
<a*c,b*c>
a b
ab
a b
b b⁻¹a b =a*b
<a><b>
<a*b><c>
<b,c>
a b c
a*b
bc (a*b)*c
<b,c>
IY
<a><bc>
a b c
bc a*(bc)
(ab)*c = (a*c)(b*c)
<a><b>
<a*b><c>
<b><c>
a b c
a*b
b*c (a*b)*c
III
a b c
b*c
a*c
(a*c)*(b*c)
<b><c>
<a><c>
<a*c><b*c>
(a*b)*c = (a*c)*(b*c)

$$
\begin{aligned}
\text{YY} \quad & (ab)c &=& \ a(bc), \\
\text{IY}: \quad & (ab) \lhd c &=& \ (a \lhd c)(b \lhd c), \\
\text{YI}: \quad & (a \lhd b) \lhd c &=& \ a \lhd (bc), \\
\text{III}: \quad & (a \lhd b) \lhd c &=& \ (a \lhd c) \lhd (b \lhd c).
\end{aligned}
$$

A quandle

satisfies three axioms that correspond to the Reidemeister moves:

$$\begin{array}{lrrcl} I: & (\forall a): & a \triangleleft a & = & a \\ II: & (\forall a, b)(\exists c): & c \triangleleft b & = & a \\ III: & (\forall a, b, c): & (a \triangleleft b) \triangleleft c & = & (a \triangleleft c) \triangleleft (b \triangleleft c). \end{array}$$

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We are interested in how the group G and its associated quandle $\mathrm{Conj}(G)$ interact.

Remark

There are related concepts for which the homology sketched below applies, *e.g.*:

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- G-families of quandles (IIJO)

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- G-families of quandles (IIJO)
- MCQ multiple conjugation quandles (Ishii, See also CIST)
- Lebed's qualgebras

Homology of G-families of quandles was defined to study handle-body knots.

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Here we use, YY, YI, IY, and III to define the homological conditions.

Slicing

Cut the interval $[0, n]$ into integral pieces.

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$$\langle 1, 2, \ldots, \ell_1 \rangle \langle \ell_1 + 1, \ldots, \ell_1 + \ell_2 \rangle \cdots$$

$$\left\langle \sum_{i=1}^{j-1} \ell_i + 1, \ldots, \sum_{i=1}^{j} \ell_i \right\rangle \ldots \left\langle \sum_{i=1}^{k} \ell_i + 1, \ldots, n \right\rangle .$$

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$$\langle 1, 2, \ldots, \ell_1 \rangle \langle \ell_1 + 1, \ldots, \ell_1 + \ell_2 \rangle \cdots$$
$$\left\langle \sum_{i=1}^{j-1} \ell_i + 1, \ldots, \sum_{i=1}^{j} \ell_i \right\rangle \cdots \left\langle \sum_{i=1}^{k} \ell_i + 1, \ldots, n \right\rangle .$$

Such a slice corresponds to a decomposition of the n-ball into a product of simplices. There are 2^{n-1} ways to cut.

Boundaries

$\partial\langle j+1, j+2, \ldots, j+k\rangle$

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$$= \triangleleft(j+1)\langle j+2, \ldots, j+k\rangle$$

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$$\partial\langle j+1, j+2, \ldots, j+k\rangle$$
$$= \triangleleft(j+1)\langle j+2, \ldots, j+k\rangle$$
$$+ \sum_{\ell=1}^{k-1}(-1)^{\ell}\langle j+1, \ldots, (j+\ell)\cdot(j+\ell+1), \ldots, j+k\rangle$$

Boundaries

$$
\begin{aligned}
&\partial\langle j+1, j+2, \ldots, j+k\rangle \\
&\quad = \ \triangleleft(j+1)\langle j+2, \ldots, j+k\rangle \\
&\quad + \sum_{\ell=1}^{k-1}(-1)^{\ell}\langle j+1, \ldots, (j+\ell)\cdot(j+\ell+1), \ldots, j+k\rangle \\
&\qquad +(-1)^{k}\langle j+1, \ldots, j+k-1\rangle.
\end{aligned}
$$

Boundaries

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&\qquad +(-1)^{k}\langle j+1,\ldots,j+k-1\rangle.
\end{aligned}
$$

$$
\partial(PQ) = (\partial P)Q + (-1)^{\dim P} P(\partial Q).
$$

Boundaries

$$\begin{aligned}
&\partial\langle j+1, j+2, \ldots, j+k\rangle \\
&= \triangleleft(j+1)\langle j+2, \ldots, j+k\rangle \\
&+ \sum_{\ell=1}^{k-1}(-1)^{\ell}\langle j+1, \ldots, (j+\ell)\cdot(j+\ell+1), \ldots, j+k\rangle \\
&\quad +(-1)^{k}\langle j+1,\ldots,j+k-1\rangle.
\end{aligned}$$

$$\partial(PQ) = (\partial P)Q + (-1)^{\dim P} P(\partial Q).$$

In part,

$$\partial\langle j+1\rangle = \triangleleft(j+1)\sqcup - \sqcup.$$

Following Przytycki, one can observe that $\partial \circ \partial = 0$ in this context, if and only if

- $a(bc) = (ab)c$
- $a \triangleleft (bc) = (a \triangleleft b) \triangleleft c$
- $(ab) \triangleleft c = (a \triangleleft c)(b \triangleleft c)$
- $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

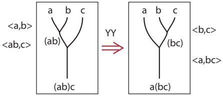
Sample computations 1

$$\begin{aligned}\partial\langle 1,2,3\rangle &= \langle 2,3\rangle - \langle 1\cdot 2,3\rangle \\ &\quad +\langle 1,2\cdot 3\rangle - \langle 1,2\rangle\end{aligned}$$

Sample computations 1

$$\partial\langle 1,2,3\rangle = \langle 2,3\rangle - \langle 1\cdot 2,3\rangle + \langle 1,2\cdot 3\rangle - \langle 1,2\rangle$$

<a,b> +<ab,c>
=<b,c> +<a,bc>

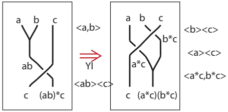


Sample computations 2

$$
\begin{aligned}
\partial\langle 1,2\rangle\langle 3\rangle &= \partial\langle 1,2\rangle\langle 3\rangle + \langle 1,2\rangle \triangleleft (3) - \langle 1,2\rangle \\
&= \langle 2\rangle\langle 3\rangle - \langle 1\cdot 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle \\
&\quad + \langle 1 \triangleleft 3, 2 \triangleleft 3\rangle - \langle 1,2\rangle
\end{aligned}
$$

Sample computations 2

$$
\begin{aligned}
\partial\langle 1,2\rangle\langle 3\rangle &= \partial\langle 1,2\rangle\langle 3\rangle + \langle 1,2\rangle \triangleleft (3) - \langle 1,2\rangle \\
&= \langle 2\rangle\langle 3\rangle - \langle 1\cdot 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle \\
&\quad + \langle 1 \triangleleft 3, 2 \triangleleft 3\rangle - \langle 1,2\rangle
\end{aligned}
$$



<a,b> +<ab><c>
=<b><c> +<a><c>
+<a*c,b*c>

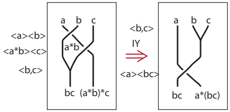
Sample computations 3

$$\begin{aligned}\partial\langle 1\rangle\langle 2,3\rangle &= \langle 2,3\rangle - \langle 2,3\rangle - \langle 1\rangle\partial\langle 2,3\rangle \\ &= -\langle 1 \triangleleft 2\rangle\langle 3\rangle + \langle 1\rangle\langle 2\cdot 3\rangle - \langle 1\rangle\langle 2\rangle\end{aligned}$$

Sample computations 3

$$
\begin{aligned}
\partial\langle 1\rangle\langle 2,3\rangle &= \langle 2,3\rangle - \langle 2,3\rangle - \langle 1\rangle\partial\langle 2,3\rangle \\
&= -\langle 1 \triangleleft 2\rangle\langle 3\rangle + \langle 1\rangle\langle 2\cdot 3\rangle - \langle 1\rangle\langle 2\rangle
\end{aligned}
$$

<a><b> +<a*b><c>
+<b,c> =<b,c> +<a><bc>

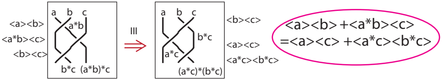


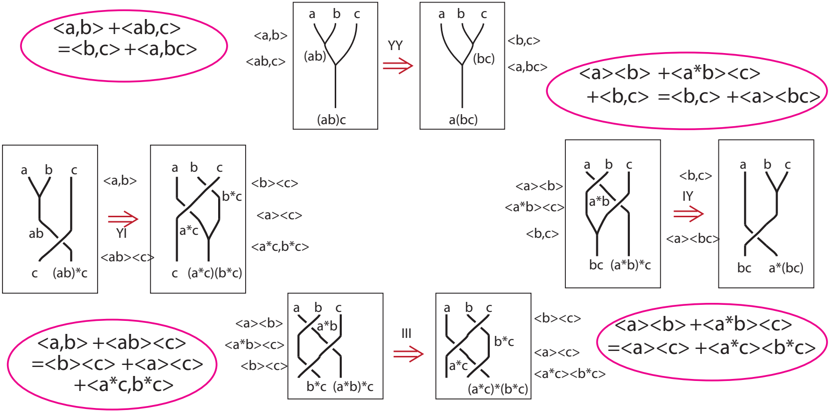
Sample computations

$$
\begin{aligned}
\partial\langle 1\rangle\langle 2\rangle\langle 3\rangle &= \langle 2\rangle\langle 3\rangle - \langle 2\rangle\langle 3\rangle - \langle 1\rangle\partial(\langle 2\rangle\langle 3\rangle) \\
&= -\langle 1 \triangleleft 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle \\
&\quad +\langle 1 \triangleleft 3\rangle\langle 2 \triangleleft 3\rangle - \langle 1\rangle\langle 2\rangle
\end{aligned}
$$

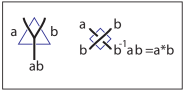
Sample computations

$$
\begin{aligned}
\partial\langle 1\rangle\langle 2\rangle\langle 3\rangle &= \langle 2\rangle\langle 3\rangle - \langle 2\rangle\langle 3\rangle - \langle 1\rangle\partial(\langle 2\rangle\langle 3\rangle) \\
&= -\langle 1 \triangleleft 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle \\
&\quad +\langle 1 \triangleleft 3\rangle\langle 2 \triangleleft 3\rangle - \langle 1\rangle\langle 2\rangle
\end{aligned}
$$

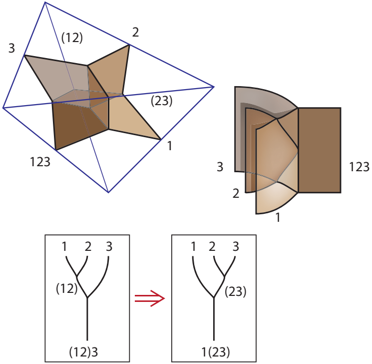



<a,b> +<ab,c>
=<b,c> +<a,bc>
<a,b>
<ab,c>
a b c
(ab)
(ab)c
YY
a b c
(bc)
a(bc)
<b,c>
<a,bc>
<a><b> +<a*b><c>
+<b,c> =<b,c> +<a><bc>
a b c
ab
c (ab)*c
<a,b>
YI
<ab><c>
a b c
b*c
a*c
c (a*c)(b*c)
<b><c>
<a><c>
<a*c,b*c>
<a><b>
<a*b><c>
<b,c>
a b c
a*b
bc (a*b)*c
<b,c>
IY
<a><bc>
a b c
bc a*(bc)
<a,b> +<ab><c>
=<b><c> +<a><c>
+<a*c,b*c>
<a><b>
<a*b><c>
<b><c>
a b c
a*b
b*c (a*b)*c
III
a b c
b*c
a*c
(a*c)*(b*c)
<b><c>
<a><c>
<a*c><b*c>
<a><b> +<a*b><c>
=<a><c> +<a*c><b*c>

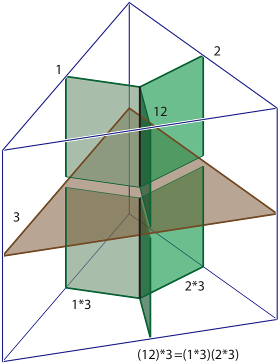
Triangles and Squares

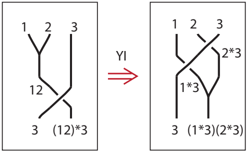


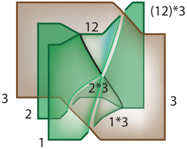
Tetrahedron



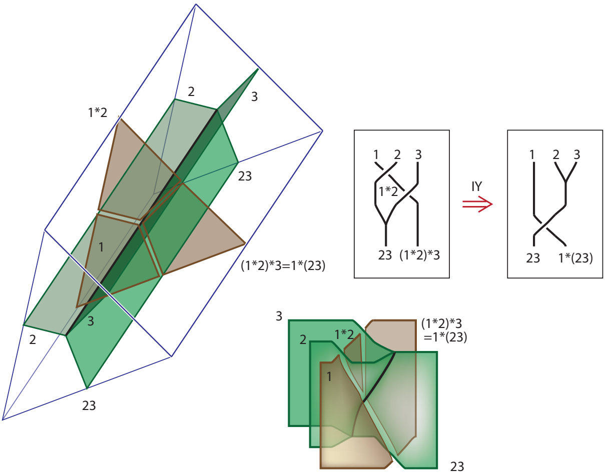
First Prism


1
2
12
3
1*3
2*3
(12)*3=(1*3)(2*3)

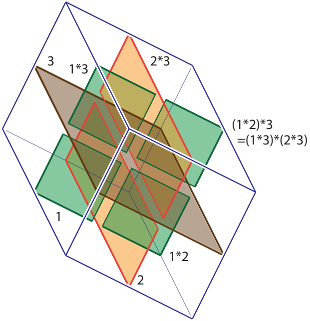

1 2 3
12
3 (12)*3
YI
1 2 3
2*3
1*3
3 (1*3)(2*3)


(12)*3
12
2*3
3
2
1
1*3
3

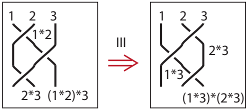
Second Prism



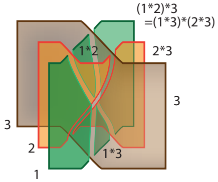
Cube



3
1*3
2*3
(1*2)*3
=(1*3)*(2*3)
1
1*2
2

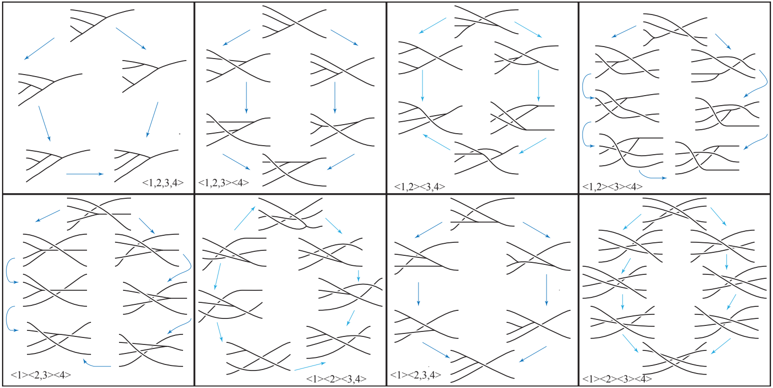


1 2 3
1*2
2*3 (1*2)*3
III
1 2 3
2*3
1*3
(1*3)*(2*3)

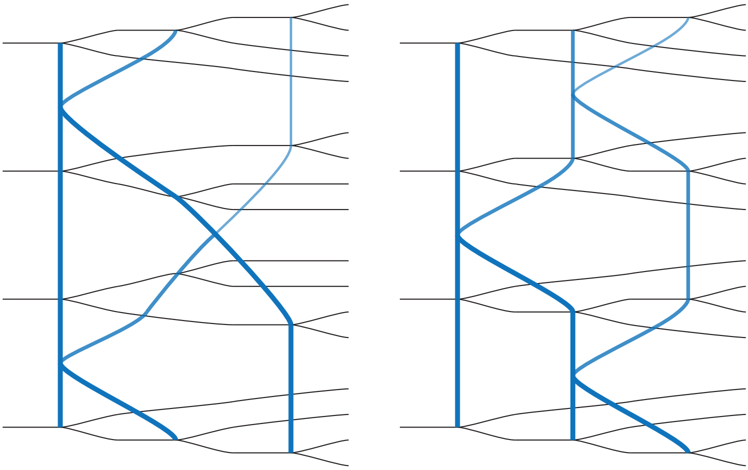


(1*2)*3
=(1*3)*(2*3)
1*2
2*3
3
3
2
1*3
1

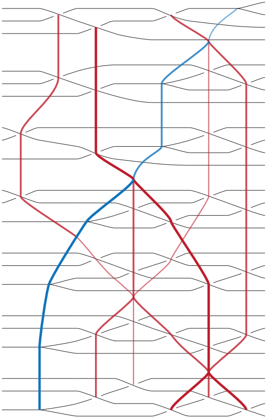
8 interesting moves

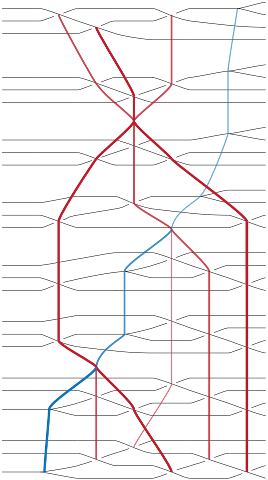


The YYY-move (Stasheff polytope)

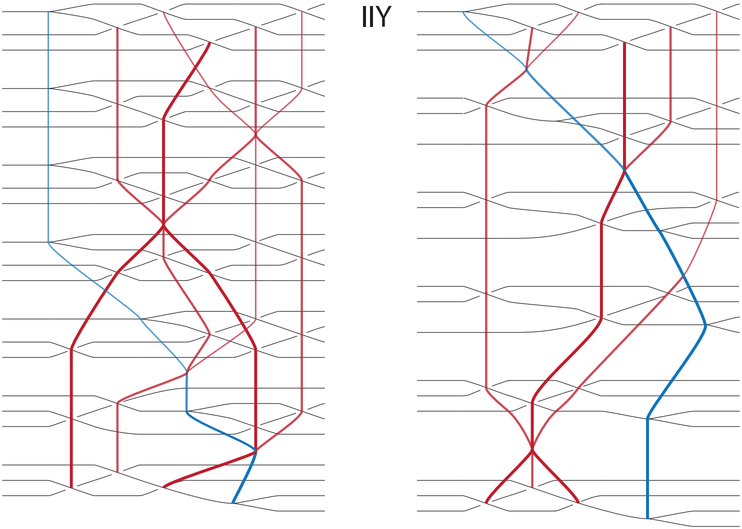


The YII-move

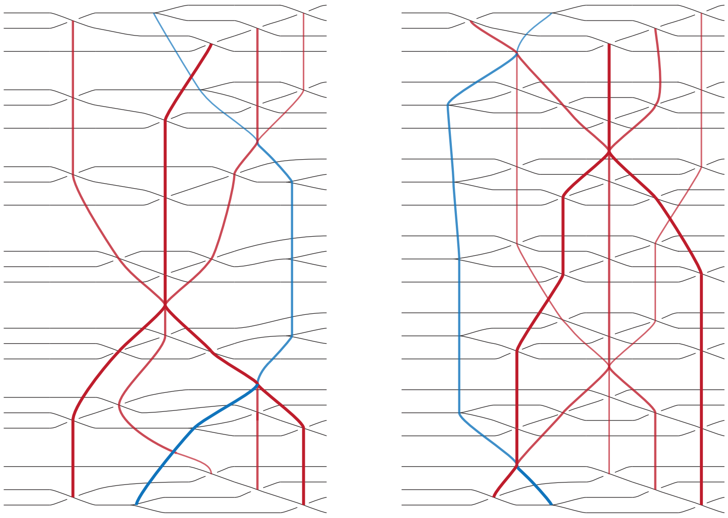




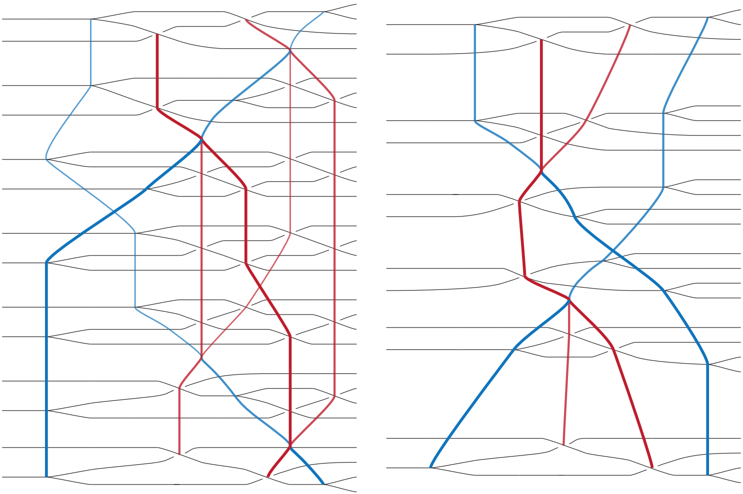
The IIY-move



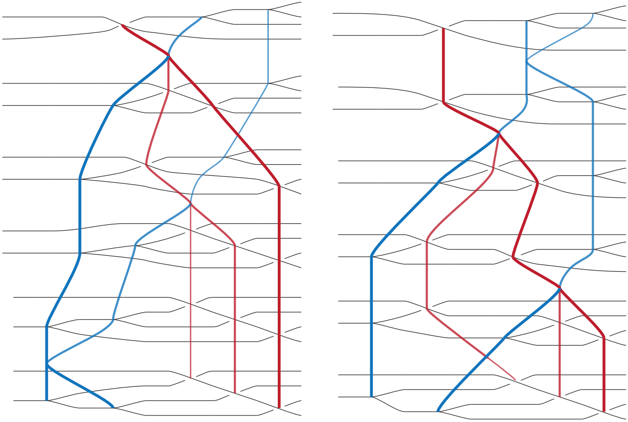
The IYI-move



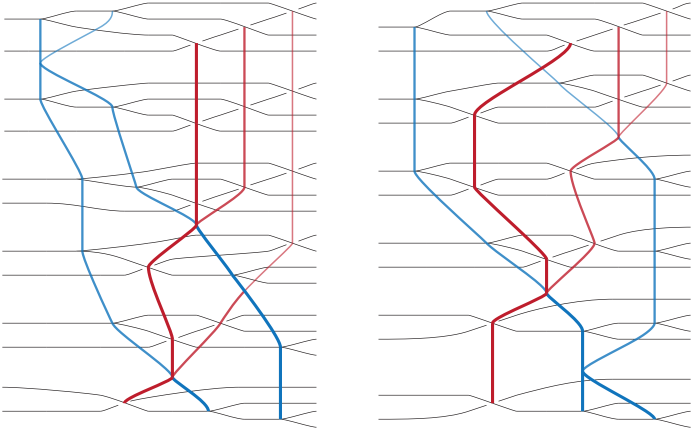
The YY-move



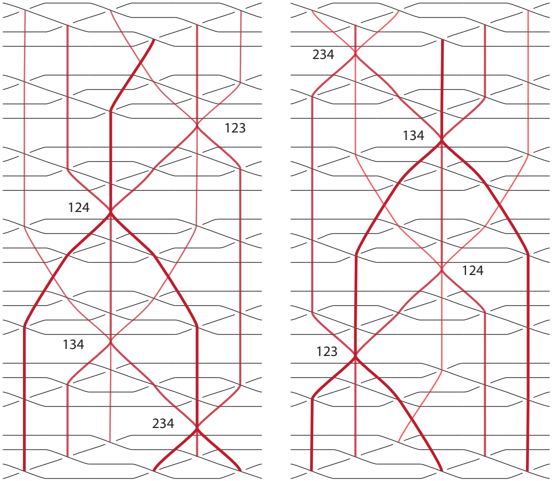
The YYI-move



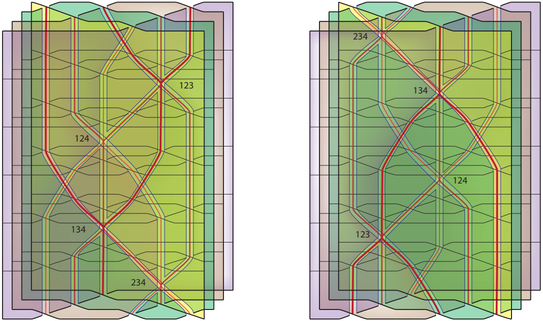
The IYY-move



The tetrahedral-move



The tetrahedral-move



Visual interlude

in which many (not all) of the Reidemeister/Roseman moves for 2-foams are indicated.

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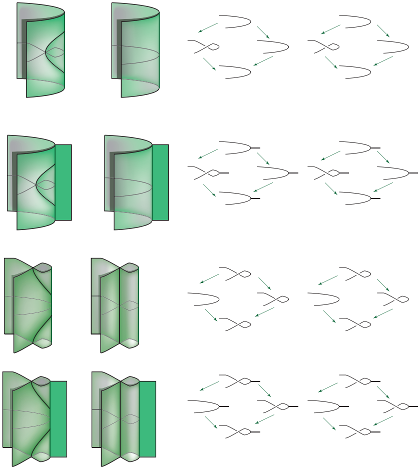
in which many (not all) of the Reidemeister/Roseman moves for 2-foams are indicated. Categorical hint:

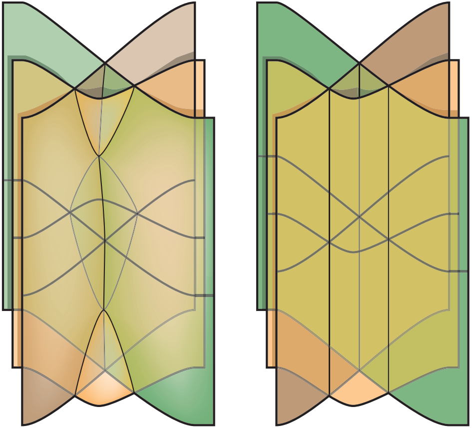
Visual interlude

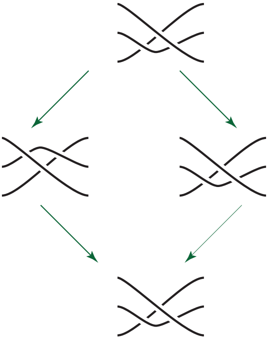
in which many (not all) of the Reidemeister/Roseman moves for 2-foams are indicated. Categorical hint: when certain 2-morphisms are defined to be invertible, these moves arise.

Critical points of the branch point set

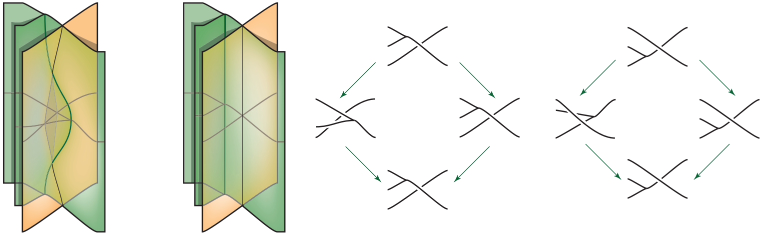


Critical points of the triple point set

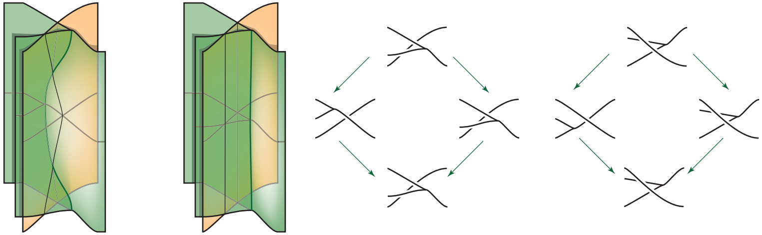




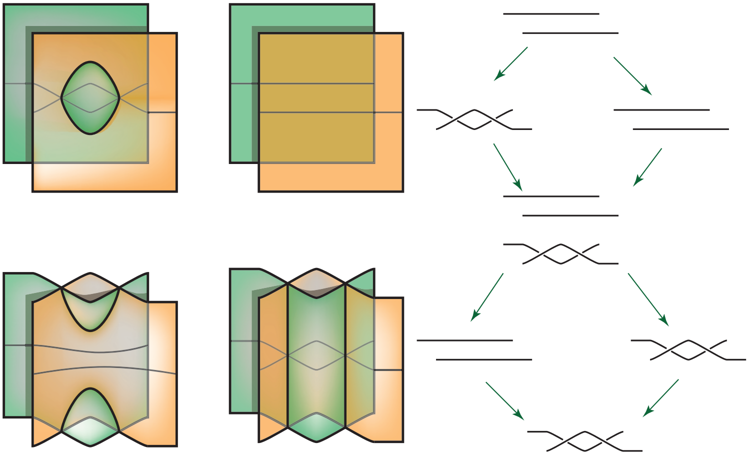
Critical points of the intersection set 1



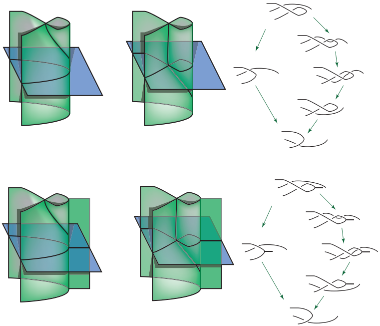
Critical points of the intersection set 2



Critical points of the double point set



Int. pts. b/2 branch/twist set and trnsvs. sheet



Not all 3-morphisms (or identities among 2 morphisms) are listed here.

Not all 3-morphisms (or identities among 2 morphisms) are listed here. Some missing moves are due to considerations on charts.

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Not all 3-morphisms (or identities among 2 morphisms) are listed here. Some missing moves are due to considerations on charts. I just haven't drawn them yet. Others are not listed here for spacial considerations.

The analogues in one higher dimensions

of the generating chains in homology.

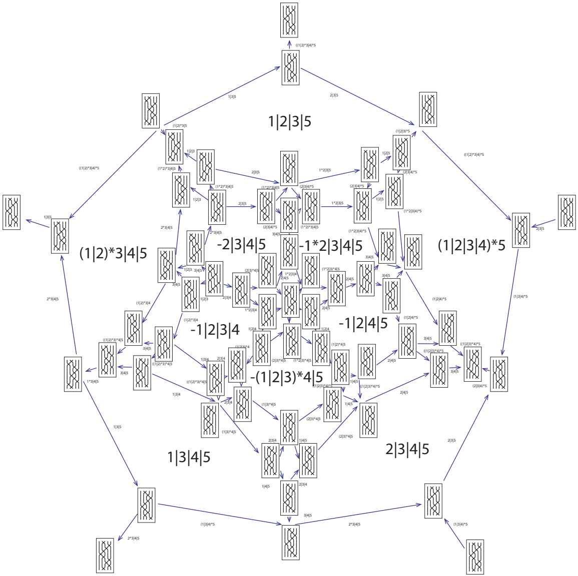
The analogues in one higher dimensions

of the generating chains in homology. One bit of the visual interlude is finished,

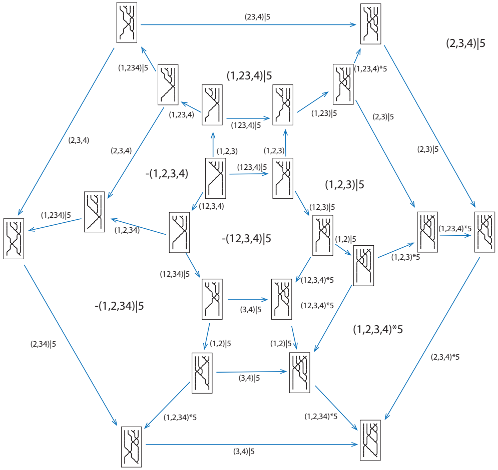
The analogues in one higher dimensions

of the generating chains in homology. One bit of the visual interlude is finished, but I want to astound you a little more.

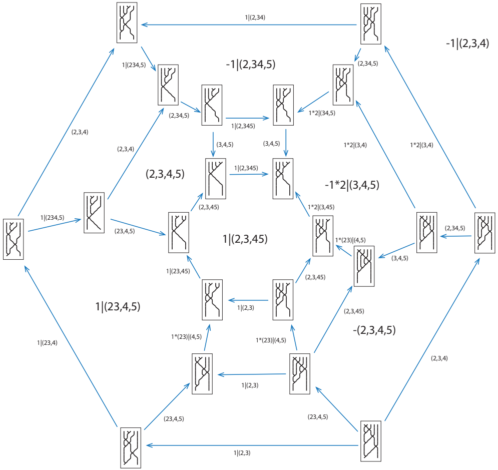
1|2|3|4|5



1234|5

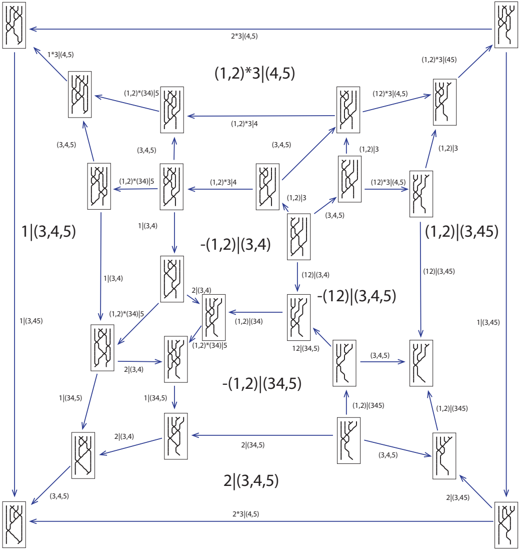

(23,4)|5
(2,3,4)|5
(1,234)|5
(1,23,4)|5
(1,23,4)*5
(1,23,4)
(123,4)|5
(1,23)|5
(2,3)|5
(2,3,4)
(2,3,4)
(1,2,3)
(1,2,3)
(2,3)|5
-(1,2,3,4)
(123,4)|5
(1,2,3)|5
(12,3,4)
(12,3)|5
(1,234)|5
(1,2,34)
-(12,3,4)|5
(1,2)|5
(1,23,4)*5
(1,2,3)*5
(12,34)|5
(12,3,4)*5
-(1,2,34)|5
(3,4)|5
(12,3,4)*5
(1,2,3,4)*5
(2,34)|5
(1,2)|5
(1,2)|5
(2,3,4)*5
(3,4)|5
(1,2,34)*5
(1,2,34)*5
(3,4)|5

1|2345

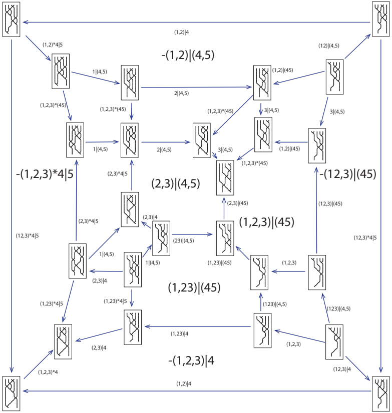


1|(2,34)
-1|(2,3,4)
1|(234,5)
-1|(2,34,5)
(2,34,5)
(2,34,5)
1|(2,345)
1*2|(34,5)
(2,3,4)
(2,3,4)
(3,4,5)
(3,4,5)
1*2|(3,4)
1*2|(3,4)
(2,3,4,5)
1|(2,345)
-1*2|(3,4,5)
1|(234,5)
(23,4,5)
(2,3,45)
1*2|(3,45)
(2,34,5)
1|(2,3,45)
1*(23)|(4,5)
(3,4,5)
1|(23,45)
(2,3,45)
1|(23,4,5)
1|(2,3)
(2,3,45)
-(2,3,4,5)
1*(23)|(4,5)
1*(23)|(4,5)
1|(23,4)
(2,3,4)
1|(2,3)
(23,4,5)
(23,4,5)
1|(2,3)

12|345

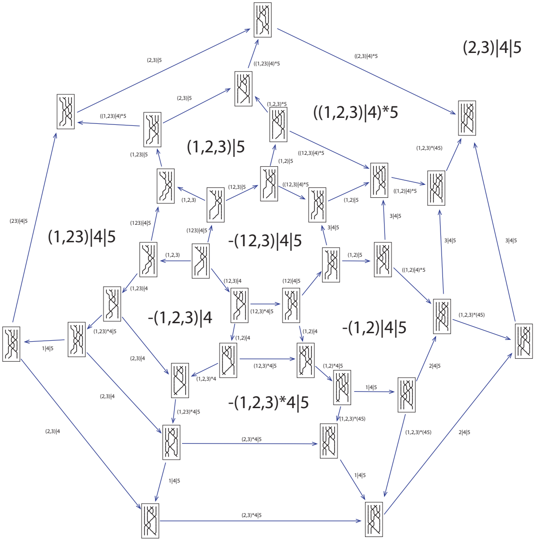

2*3|(4,5)
1*3|(4,5)
(1,2)*3|(45)
(1,2)*3|(4,5)
(1,2)*(34)|5
(12)*3|(4,5)
(1,2)*3|4
(3,4,5)
(3,4,5)
(3,4,5)
(1,2)|3
(1,2)|3
(1,2)*(34)|5
(1,2)*3|4
(12)*3|(4,5)
(1,2)|3
(3,4,5)
1|(3,4,5)
1|(3,4)
-(1,2)|(3,4)
(1,2)|(3,45)
1|(3,4)
(12)|(3,4)
(12)|(3,45)
2|(3,4)
-(12)|(3,4,5)
(1,2)*(34)|5
1|(3,45)
(1,2)|(34)
1|(3,45)
(1,2)*(34)|5
12|(34,5)
(3,4,5)
2|(3,4)
-(1,2)|(34,5)
1|(34,5)
1|(34,5)
(1,2)|(345)
(1,2)|(345)
2|(3,4)
2|(34,5)
(3,4,5)
2|(3,4,5)
(3,4,5)
2|(3,45)
2*3|(4,5)

123|45



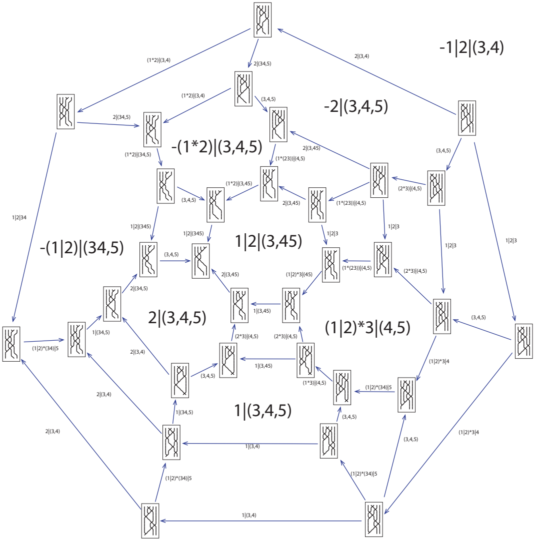
(1,2)|4
(1,2)*4|5
-(1,2)|(4,5)
(12)|(4,5)
1|(4,5)
(1,2)|(45)
2|(4,5)
(1,2,3)*(45)
(1,2,3)*(45)
(1,2,3)*(45)
3|(4,5)
3|(4,5)
1|(4,5)
2|(4,5)
3|(4,5)
(1,2)|(45)
(1,2,3)*(45)
-(1,2,3)*4|5
(2,3)*4|5
(2,3)|(4,5)
-(12,3)|(45)
(2,3)|(45)
(12,3)|(45)
(2,3)*4|5
(2,3)|4
(1,2,3)|(45)
(12,3)*4|5
(12,3)*4|5
(23)|(4,5)
1|(4,5)
1|(4,5)
(1,23)|(45)
(1,2,3)
(2,3)|4
(1,23)|(45)
(1,23)*4|5
(1,23)*4|5
(123)|(4,5)
(123)|(4,5)
(1,23)|4
(2,3)|4
(1,2,3)
-(1,2,3)|4
(1,2,3)*4
(12,3)|4
(1,2)|4

123|4|5

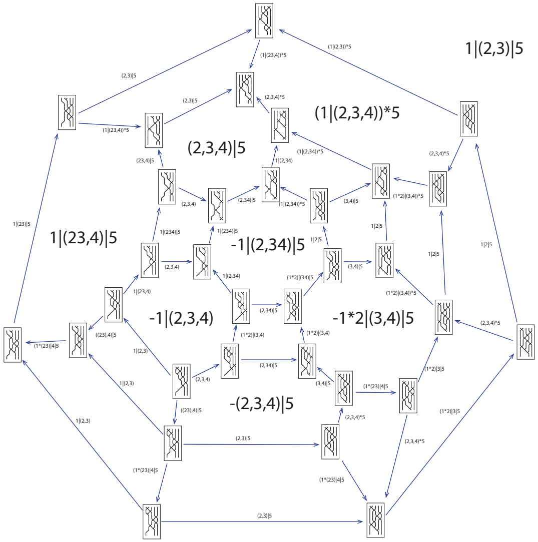


(2,3)|4|5
((1,2,3)|4)*5
(1,2,3)|5
(1,23)|4|5
-(12,3)|4|5
-(1,2,3)|4
-(1,2)|4|5
-(1,2,3)*4|5
(2,3)|5
((2,3)|4)*5
((1,23)|4)*5
(2,3)|5
((1,23)|4)*5
(1,2,3)*5
(1,23)|5
((12,3)|4)*5
(1,2)|5
(1,2,3)*(45)
(12,3)|5
((12,3)|4)*5
(1,2,3)
((1,2)|4)*5
(1,2)|5
(123)|4|5
3|4|5
(23)|4|5
(123)|4|5
3|4|5
3|4|5
3|4|5
(1,2,3)
(1,2)|5
((1,2)|4)*5
(12,3)|4
(12)|4|5
(1,23)|4
(12,3)*4|5
(1,2,3)*(45)
(1,23)*4|5
(1,2)|4
(1,2)|4
1|4|5
(2,3)|4
(12,3)*4|5
(1,2)*4|5
2|4|5
(1,2,3)*4
1|4|5
(2,3)|4
(1,23)*4|5
(1,2,3)*(45)
(2,3)|4
(1,2,3)*(45)
2|4|5
(2,3)*4|5
1|4|5
1|4|5
(2,3)*4|5

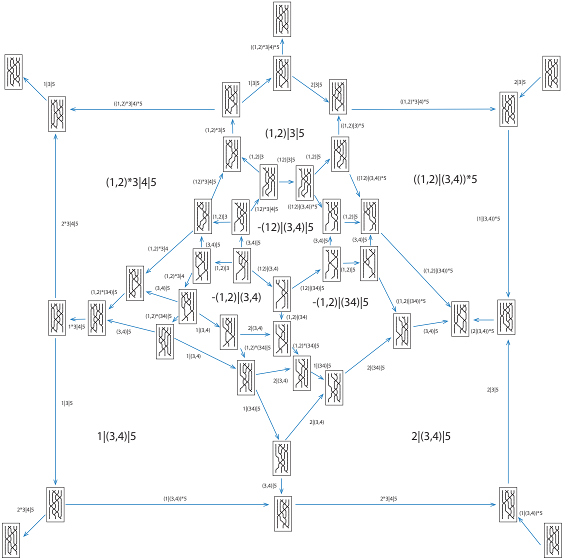
1|2|345


-1|2|(3,4)
-2|(3,4,5)
-(1*2)|(3,4,5)
1|2|(3,45)
-(1|2)|(34,5)
2|(3,4,5)
(1|2)*3|(4,5)
1|(3,4,5)
(1*2)|(3,4)
2|(34,5)
2|(3,4)
(1*2)|(3,4)
(3,4,5)
2|(34,5)
(1*2)|(34,5)
(3,4,5)
2|(3,45)
(1*(23))|(4,5)
(1*2)|(3,45)
(3,4,5)
2|(3,45)
(1*(23))|(4,5)
(2*3)|(4,5)
1|2|34
1|2|(345)
1|2|(345)
1|2|3
1|2|3
1|2|3
1|2|3
(3,4,5)
(1*(23))|(4,5)
(2*3)|(4,5)
2|(34,5)
2|(3,45)
(1|2)*3|(45)
1|(3,45)
(3,4,5)
(1|2)*(34)|5
1|(34,5)
2|(3,4)
(2*3)|(4,5)
(2*3)|(4,5)
1|(3,45)
(1|2)*3|4
(1*3)|(4,5)
(1|2)*(34)|5
2|(3,4)
2|(3,4)
1|(34,5)
(3,4,5)
1|(3,4)
(1|2)*(34)|5
(3,4,5)
(1|2)*3|4
(1|2)*(34)|5
1|(3,4)

1|234|5



12|34|5


((1,2)*3|4)*5
1|3|5
2|3|5
((1,2)*3|4)*5
((1,2)*3|4)*5
2|3|5
(1,2)*3|5
(1,2)|3|5
((1,2)|3)*5
(1,2)|3
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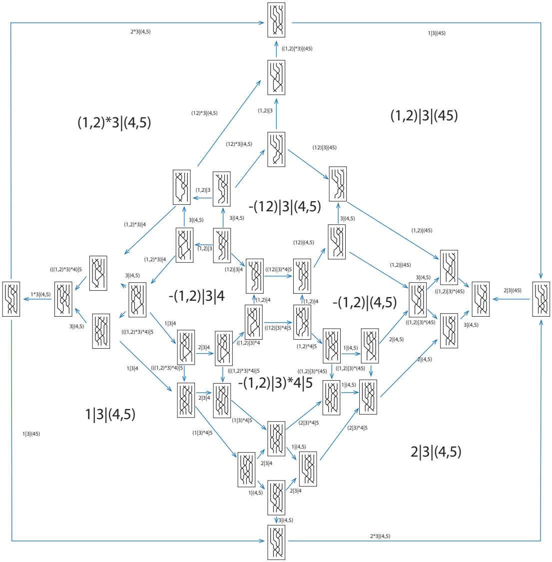
12|3|45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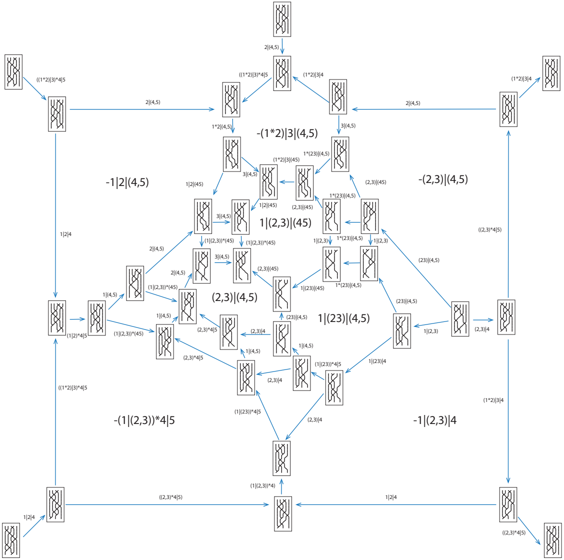
1|23|45



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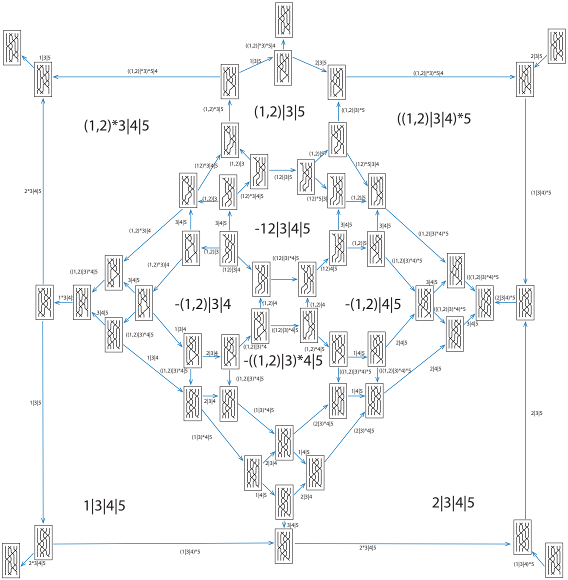

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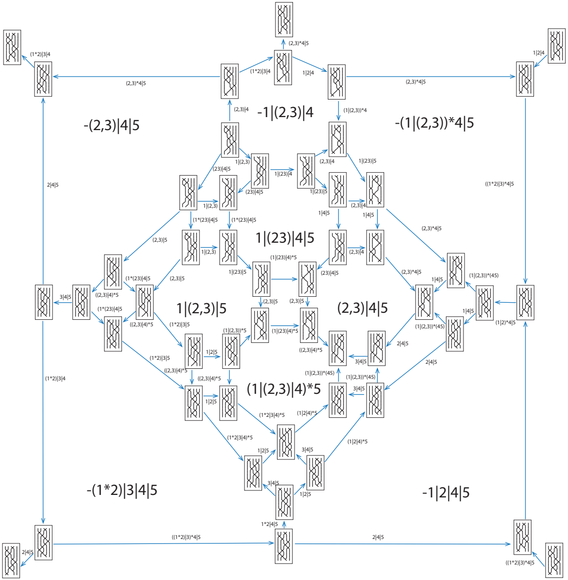

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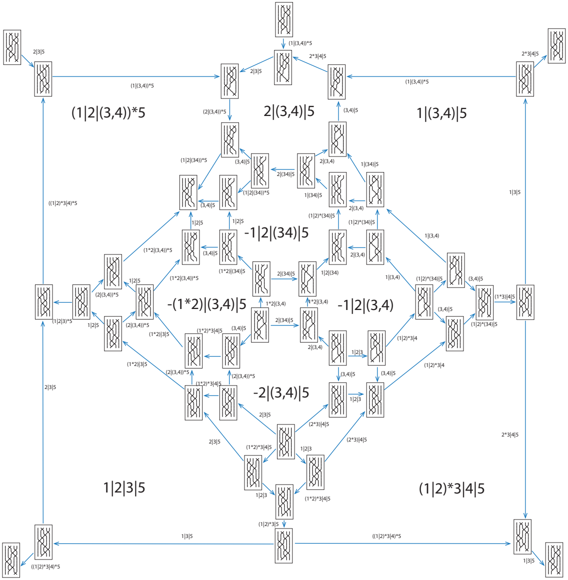

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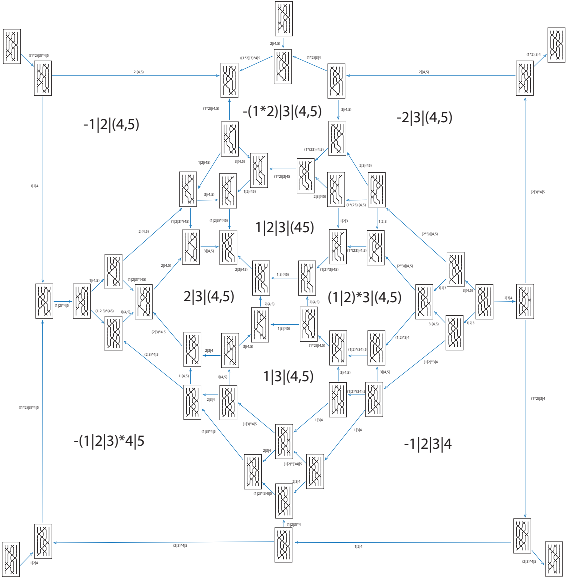

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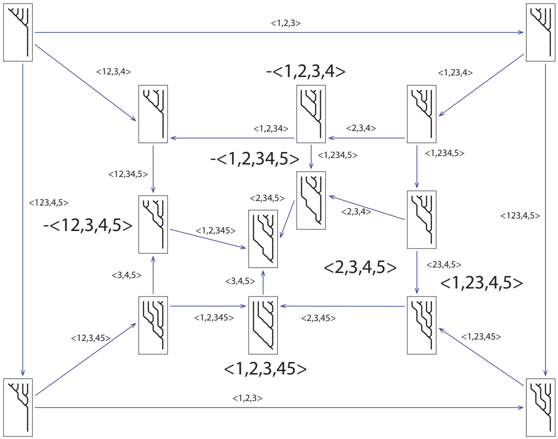
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Section 5

in which the polytopal duals to the generating chains are formulated.

In this section, I am going to review something that is

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Permutahedron

In $\mathbb{R}^n$ consider the convex hull of

$$\{(\sigma(1), \sigma(2), \ldots, \sigma(n)) : \sigma \in \Sigma_n\}.$$

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$\boldsymbol{i} = (i, i+1)$.

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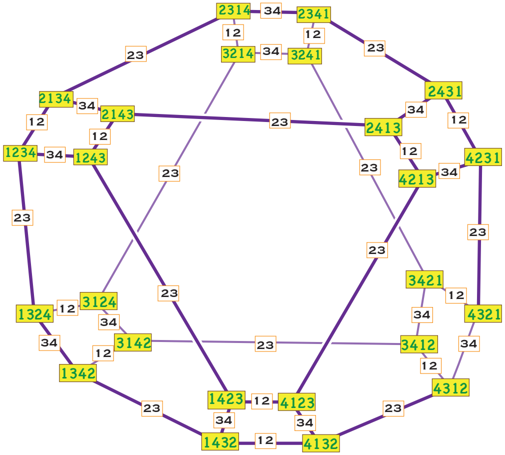
$\boldsymbol{i} = (i, i+1)$. Hexagonal faces are
$\boldsymbol{i}(\boldsymbol{i}+1)\boldsymbol{i}(\boldsymbol{i}+1)\boldsymbol{i}(\boldsymbol{i}+1)$. *etc.*



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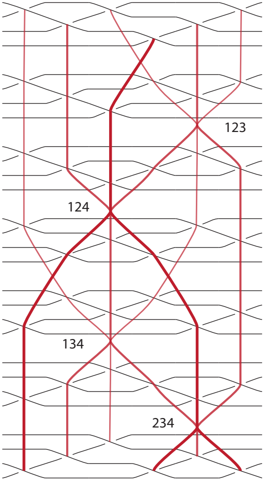

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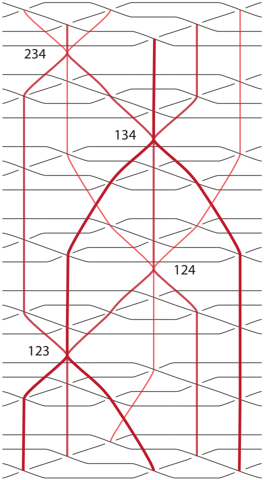

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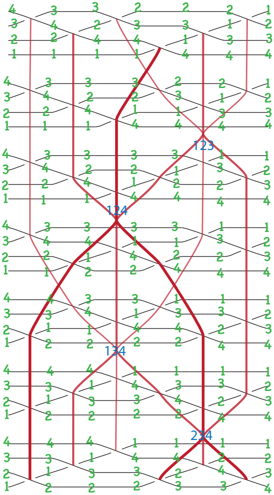
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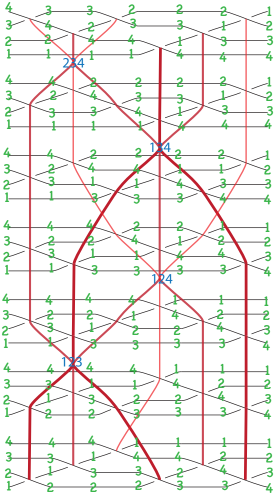
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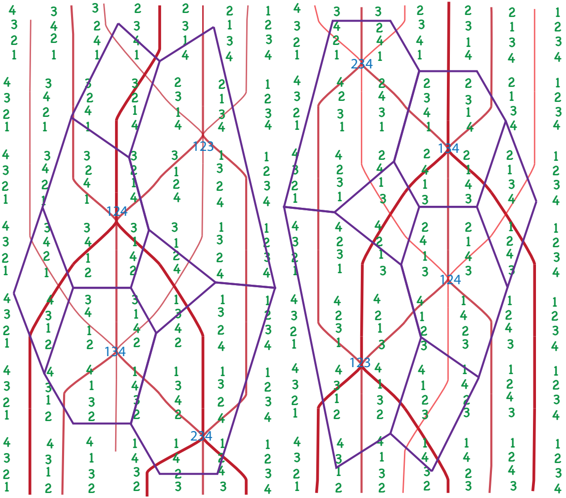

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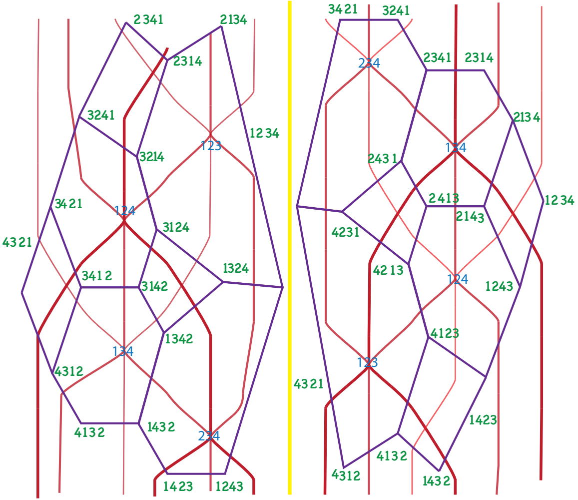
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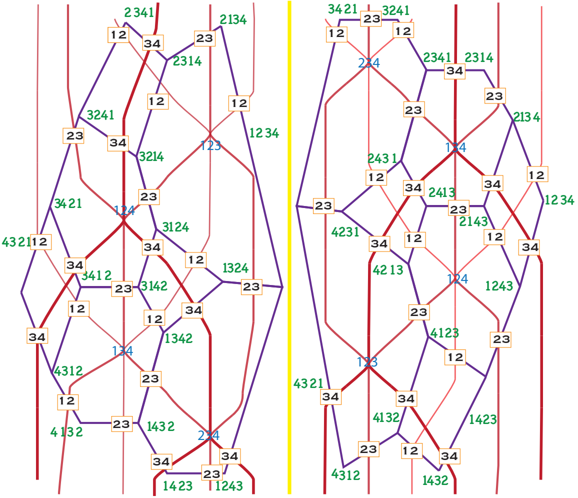

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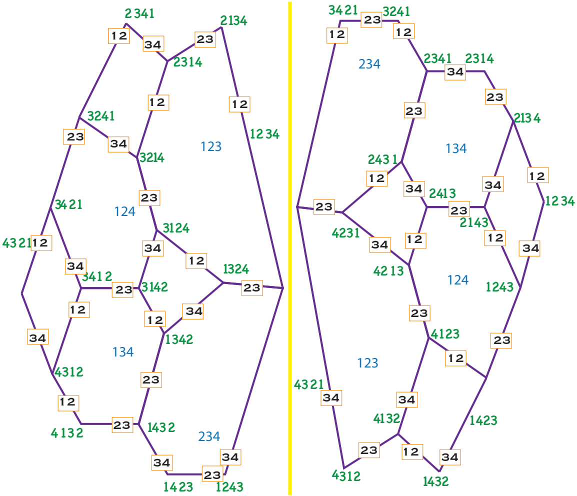

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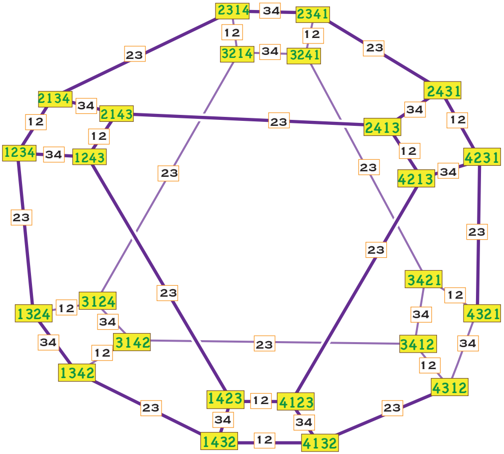
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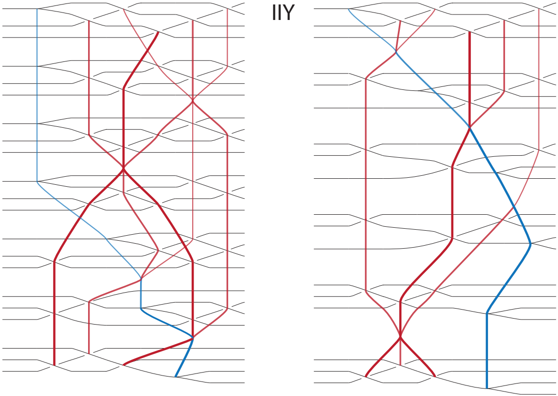

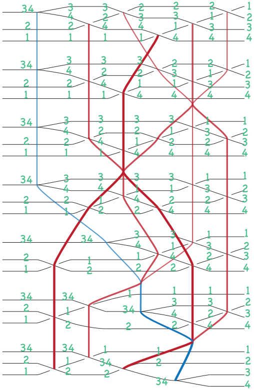

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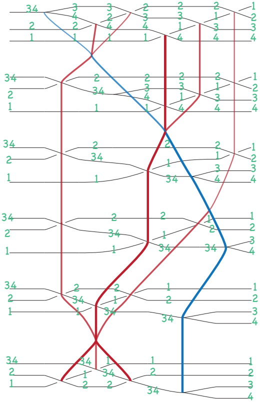

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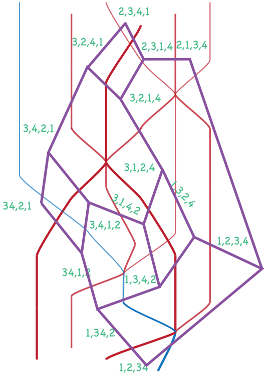
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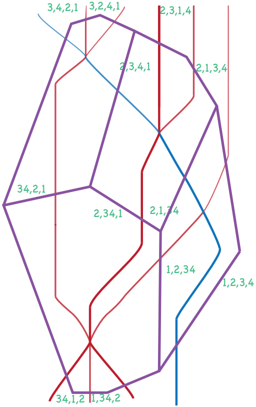
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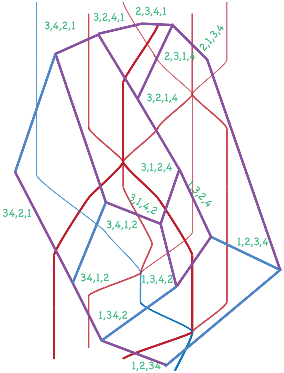

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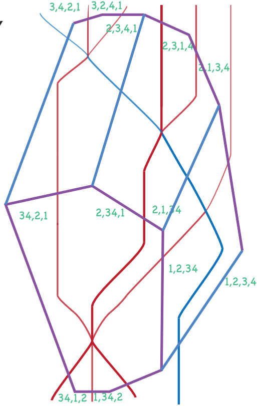

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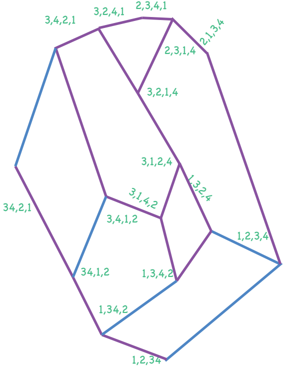
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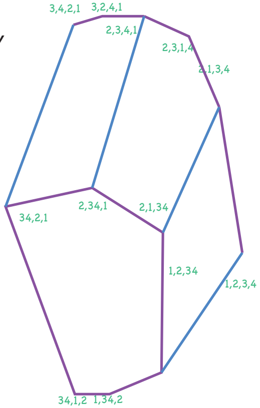
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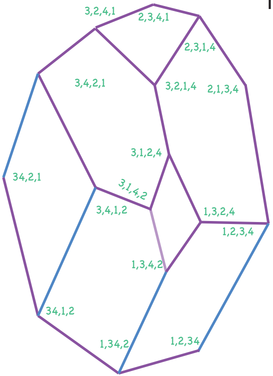

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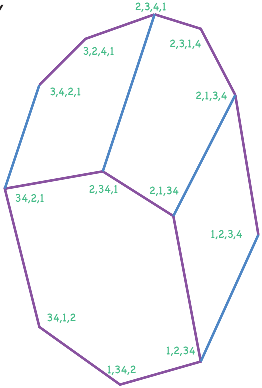

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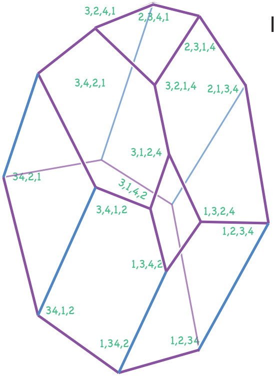
IIY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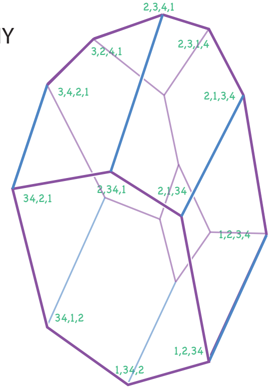
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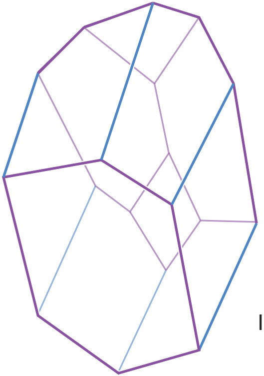

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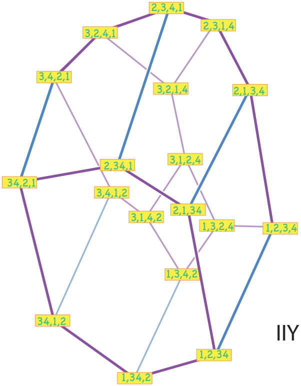
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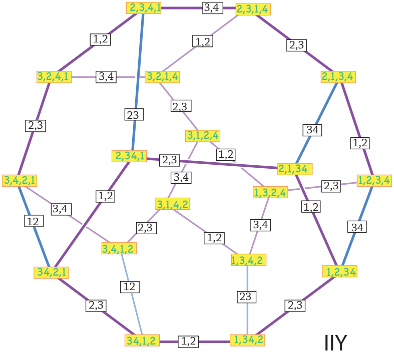
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IIY

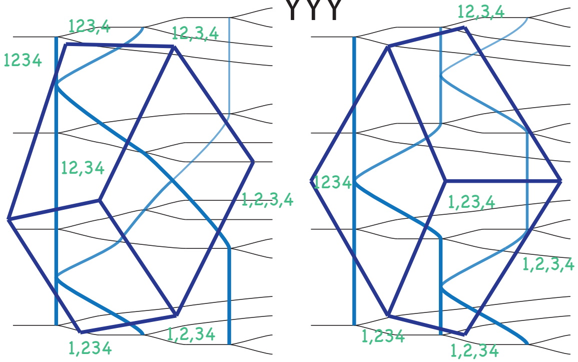
Before being in Asia, I thought that these polytopes were new.

Before being in Asia, I thought that these polytopes were new. But now I see that they are sold in the grocery store in Japan.

Before being in Asia, I thought that these polytopes were new. But now I see that they are sold in the grocery store in Japan.

YYY

YYY



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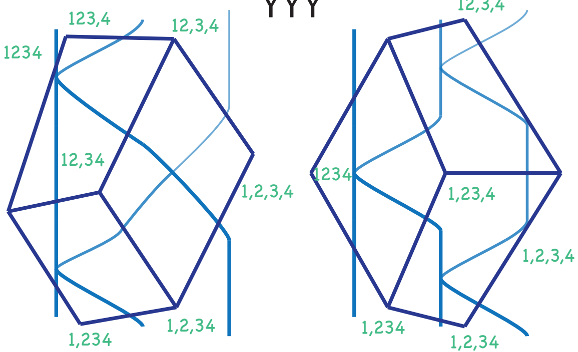

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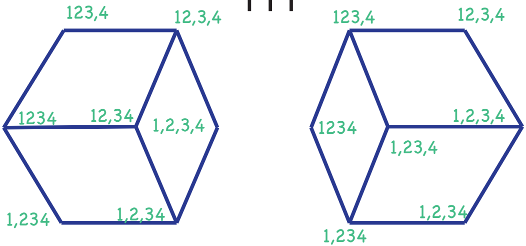

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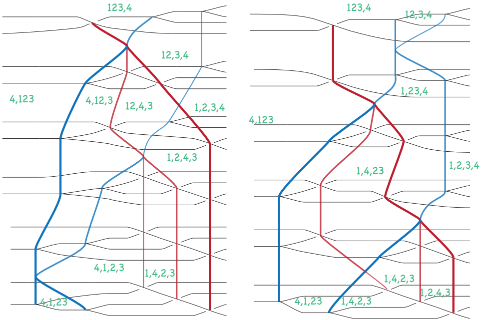

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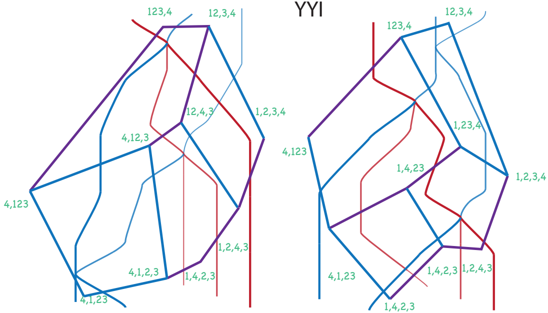
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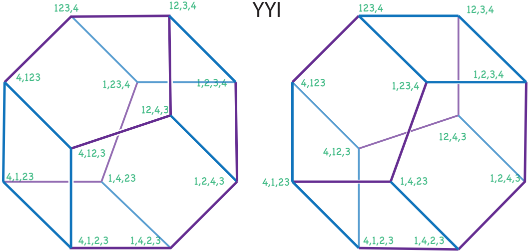
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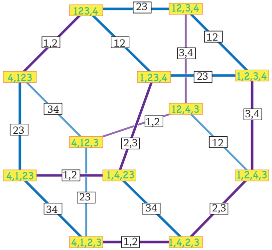

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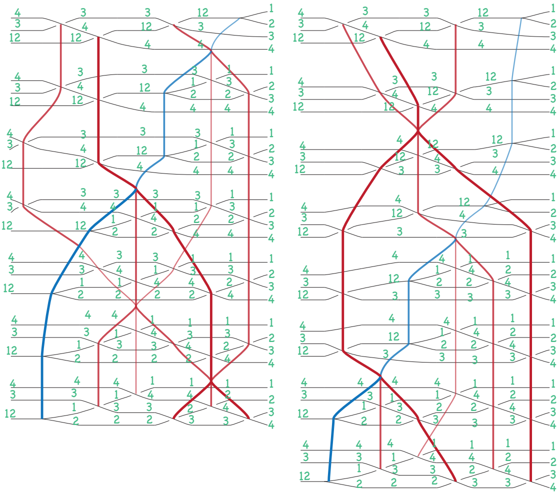
YII

YII



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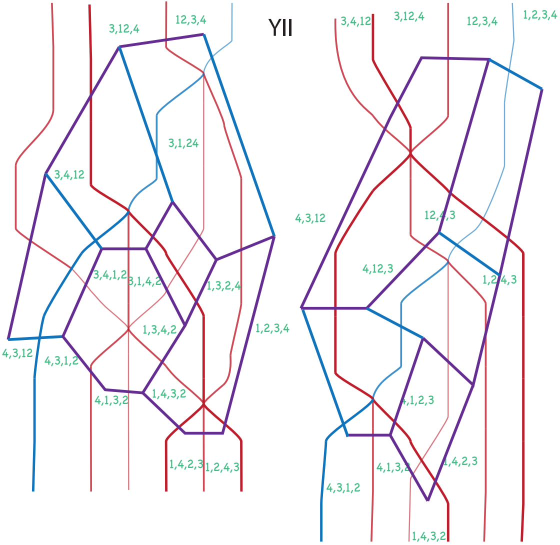

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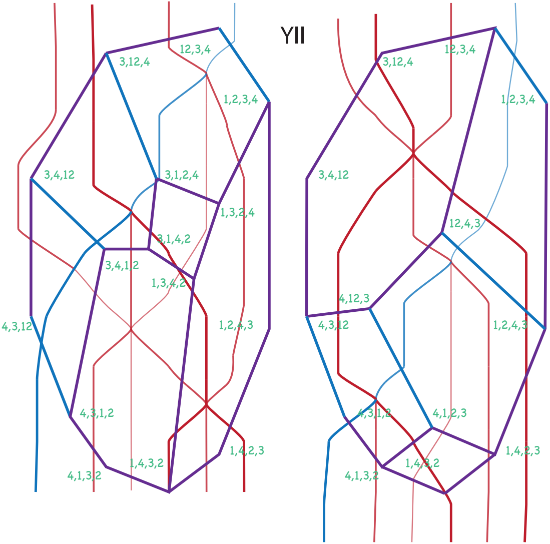

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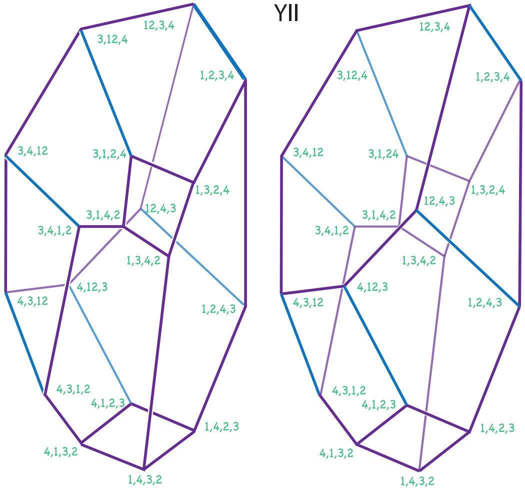
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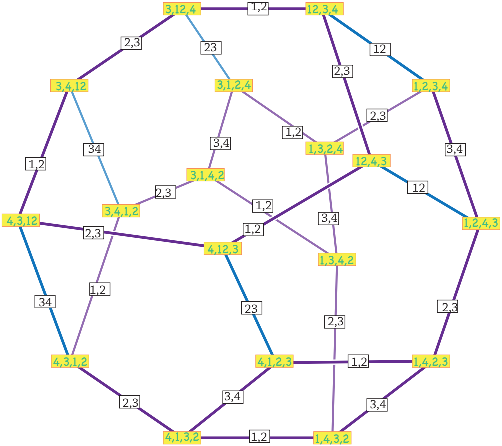
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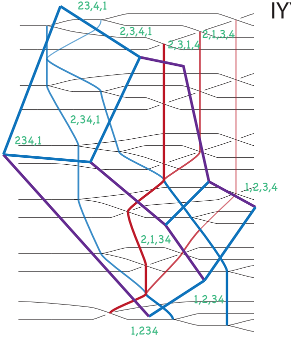

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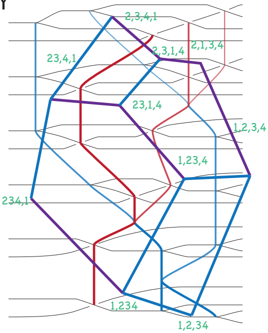

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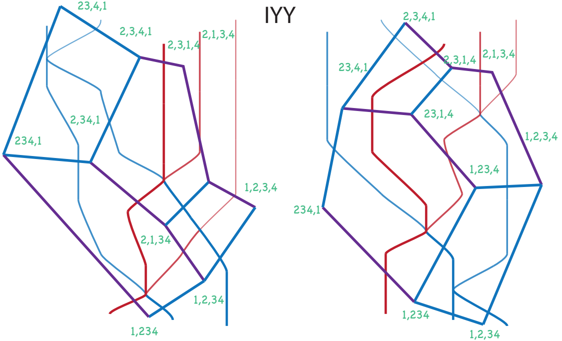

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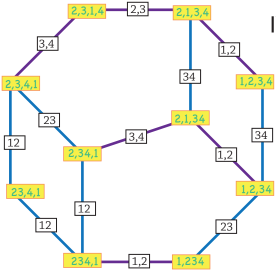
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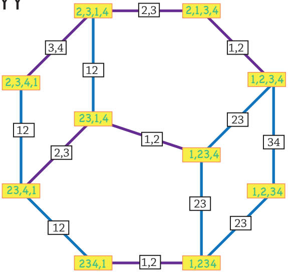
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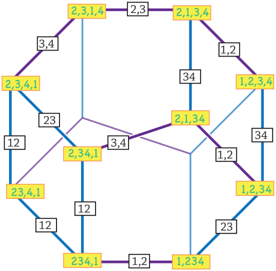

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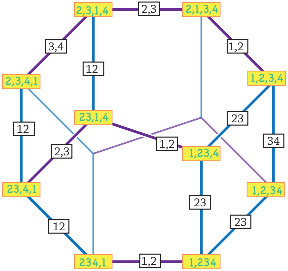

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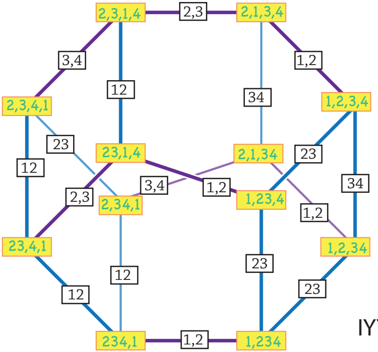
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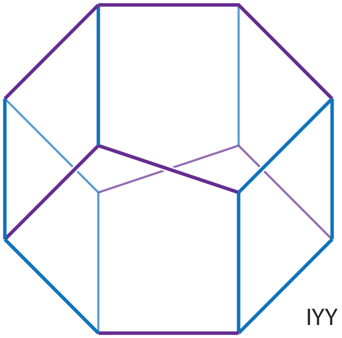
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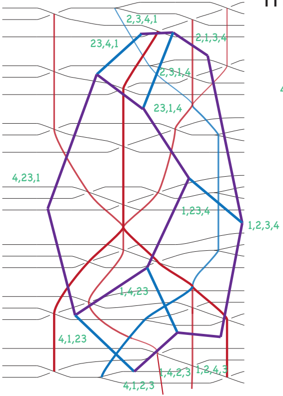

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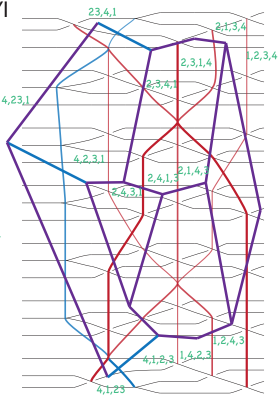

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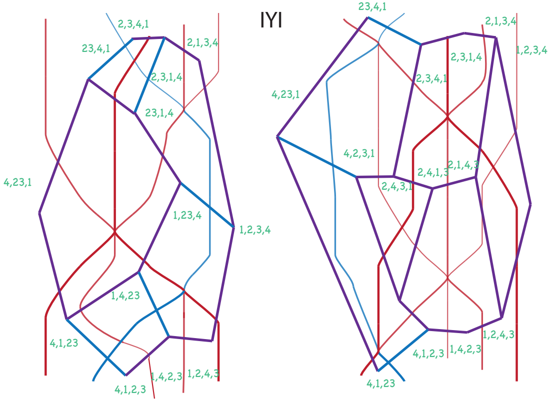

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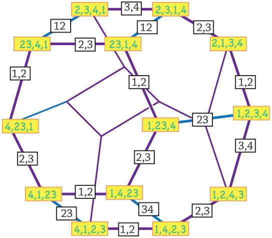
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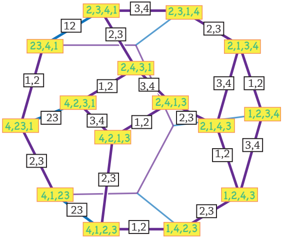
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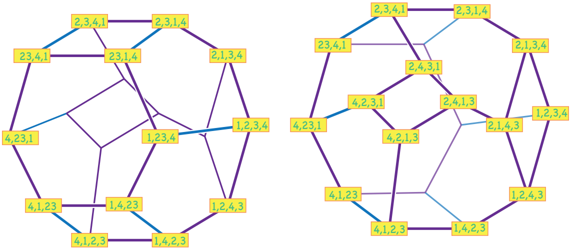
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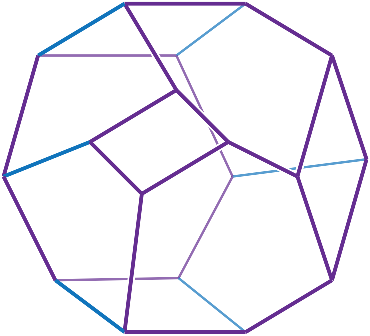
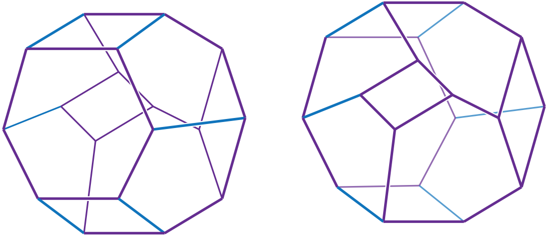


|Y|

IYI



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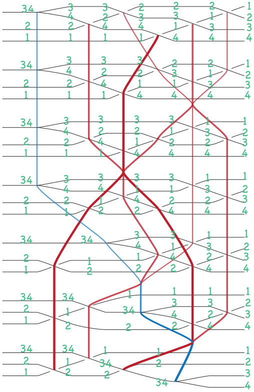
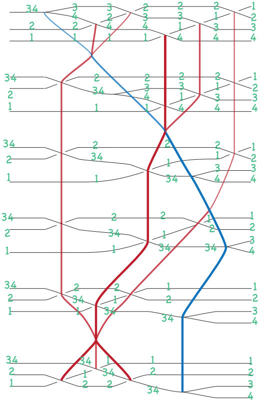
IIY





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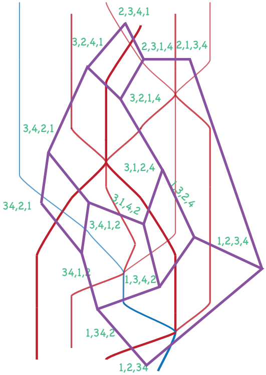

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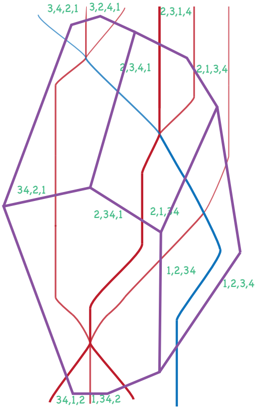

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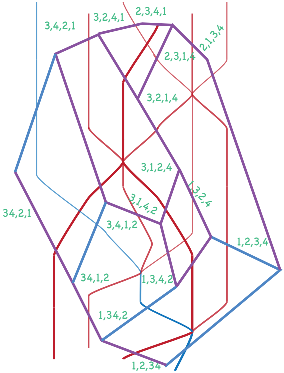
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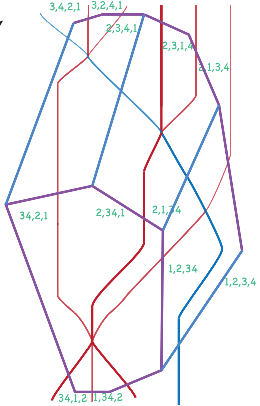
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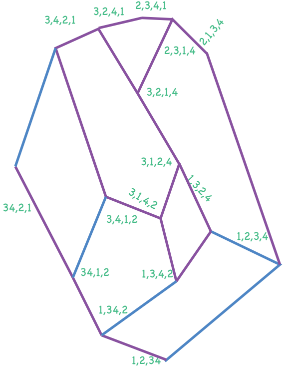
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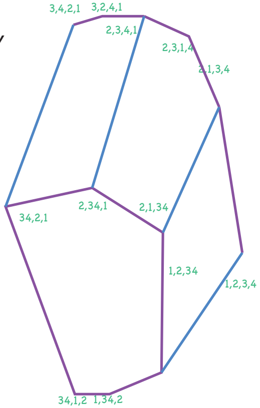
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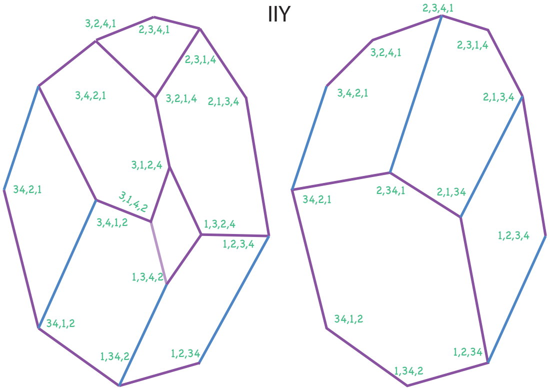
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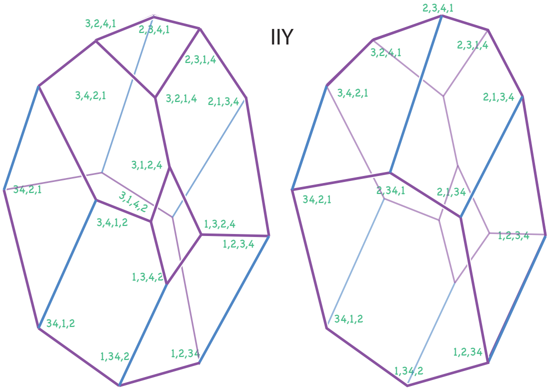
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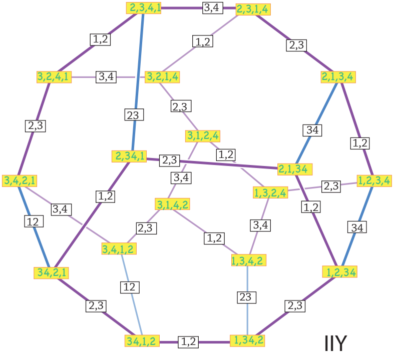

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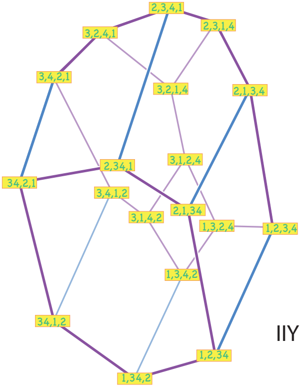
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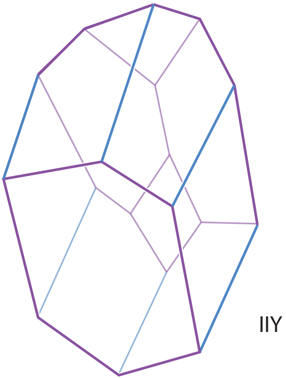
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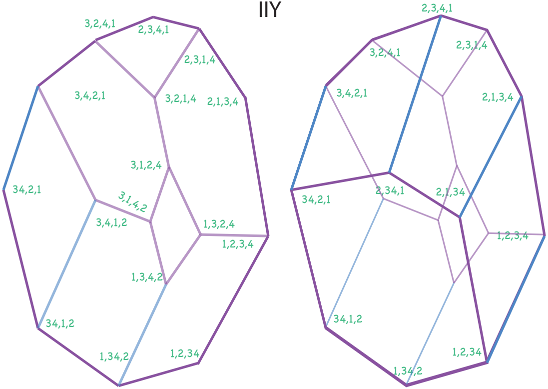

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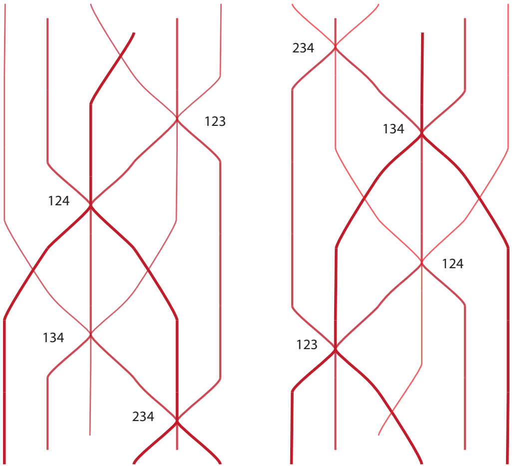
Next we'll use the graphical structure to formulate a series of Abstract tensor equations.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124
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234
234
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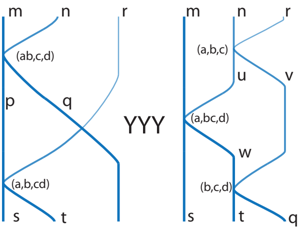
124
123

One of these equations is the Zamolodchikov tetrahedral equation.

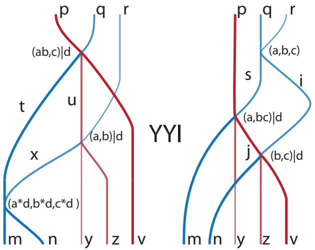
$$S_{123}S_{124}S_{134}S_{234} = S_{234}S_{134}S_{124}S_{123}.$$

One of these equations is the Zamolodchikov tetrahedral equation.

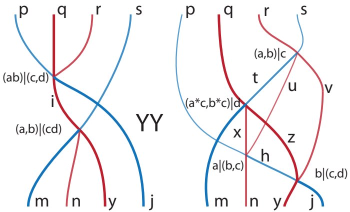
$$S_{123}S_{124}S_{134}S_{234} = S_{234}S_{134}S_{124}S_{123}.$$



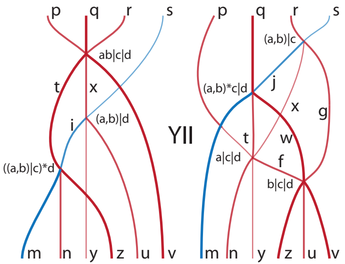
$$K^{m,n}_{p,q}\,K^{p,r}_{s,t} = K^{n,r}_{u,v}\;K^{m,u}_{s,w}\;K^{w,v}_{t,q}$$



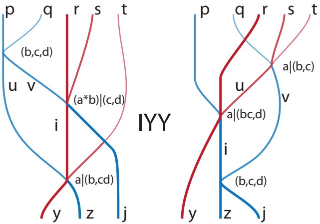
$$M^{p,q}_{t,u,v}\;M^{u,r}_{x,y,z}\;K^{t,x}_{m,n} = K^{q,r}_{s,i}\;M^{p,s}_{m,y,j}\;M^{j,i}_{n,z,v}$$



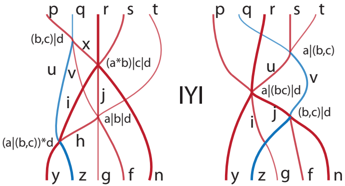
$$W^{p,q,r}_{i,j} M^{i,s}_{m,n,y} = M^{r,s}_{t,u,v} M^{q,t}_{m,x,z} W^{p,x,u}_{n,h} W^{h,z,v}_{y,i}$$



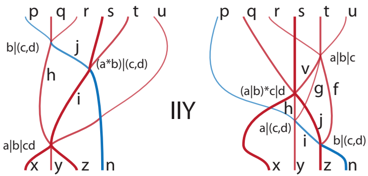
$$\mathbb{X}^{p,q,r}_{t,x,u} M^{x,s}_{i,y,u} M^{t,i}_{m,n,z} = M^{r,s}_{j,x,g} M^{q,j}_{m,t,w} \mathbb{X}^{p,t,x}_{n,y,f} \mathbb{X}^{f,w,g}_{z,u,v}$$



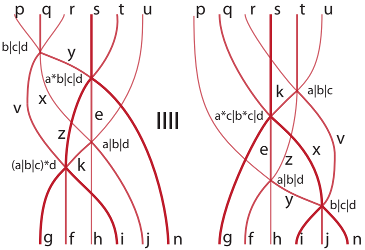
$$K^{p,q}_{u,v} W^{v,r,s}_{i,j} \; W^{u,i,t}_{y,z} = W^{q,s,t}_{u,v} W^{p,r,u}_{y,i} K^{i,v}_{z,j}$$



$$M^{p,q}_{u,v,x} X^{x,r,s}_{i,j,n} X^{v,j,t}_{h,g,f} W^{u,i,h}_{y,z} = W^{q,s,t}_{u,v} X^{p,r,u}_{y,i,j} M^{j,v}_{z,f,n}$$



$$W^{p,q,r}_{h,j}\, W^{j,s,t}_{i,n}\, X^{h,i,u}_{x,y,z} = X^{r,t,u}_{v,g,f}\, X^{q,s,v}_{x,h,j}\, W^{p,h,g}_{y,i}\, W^{i,j,f}_{z,n}$$



$$X^{p,q,r}_{v,x,y}\, X^{y,s,t}_{z,e,n}\, X^{x,e,u}_{k,h,j}\, X^{v,z,k}_{g,f,i} = X^{r,t,u}_{k,z,v}\, X^{q,s,k}_{g,e,x}\, X^{p,e,z}_{f,h,y}\, X^{y,x,v}_{i,j,n}$$

In 3-D

A	$\mathsf{Y}_j^{\ell,n}\mathsf{Y}_i^{j,k} = \mathsf{Y}_p^{n,k}\mathsf{Y}_i^{\ell,p}$
YI	$\mathsf{Y}_d^{a,b}\mathsf{X}_{e,f}^{d,c} = \mathsf{X}_{c,h}^{b,c}\mathsf{X}_{e,g}^{a,c}\mathsf{Y}_f^{g,h}.$
IY	$\mathsf{X}_{d,g}^{a,b}\mathsf{X}_{h,f}^{g,c}\mathsf{Y}_e^{d,h} = \mathsf{Y}_i^{b,c}\mathsf{X}_{e,f}^{a,i}.$
III	$\mathsf{X}_{d,e}^{a,b}\mathsf{X}_{i,h}^{e,c}\mathsf{X}_{f,g}^{d,i} = \mathsf{X}_{j,k}^{b,c}\mathsf{X}_{f,e}^{a,j}\mathsf{X}_{g,h}^{e,k}$

$$\mathsf{Y}_{c}^{a,b} = \begin{cases} 1 & \text{if } c = ab, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathsf{X}_{c,d}^{a,b} = \begin{cases} 1 & \text{if } c = b \;\&\; d = b^{-1}ab, \\ 0 & \text{otherwise.} \end{cases}$$

In 4-D

YYY	$\mathsf{K}^{m,n}_{p,q}\mathsf{K}^{p,r}_{s,t} = \mathsf{K}^{n,r}_{u,v}\mathsf{K}^{m,u}_{s,w}\mathsf{K}^{w,v}_{t,q}$
YYI	$\mathsf{M}^{p,q}_{t,u,v}\mathsf{M}^{u,r}_{x,y,z}\mathsf{K}^{t,x}_{m,n} = \mathsf{K}^{q,r}_{s,i}\mathsf{M}^{p,s}_{m,y,j}\mathsf{M}^{j,i}_{n,z,v}$
YY	$\mathsf{W}^{p,q,r}_{i,j}\mathsf{M}^{i,s}_{m,n,y} = \mathsf{M}^{r,s}_{t,u,v}\mathsf{M}^{q,t}_{m,x,z}\mathsf{W}^{p,x,u}_{n,h}\mathsf{W}^{h,z,v}_{y,i}$
YII	$\text{Ж}^{p,q,r}_{t,x,v}\mathsf{M}^{x,s}_{y,u,v}\mathsf{M}^{t,i}_{m,n,z} = \mathsf{M}^{r,s}_{j,x,g}\mathsf{M}^{q,j}_{m,t,x}\text{Ж}^{p,t,x}_{n,y,f}\text{Ж}^{f,w,g}_{z,u,v}$
IYY	$\mathsf{K}^{p,q}_{u,v}\mathsf{W}^{v,r,s}_{i,j}\mathsf{W}^{u,i,t}_{y,z} = \mathsf{W}^{q,s,t}_{u,v}\mathsf{W}^{p,r,u}_{y,i}\mathsf{K}^{i,v}_{z,j}$
IYI	$\mathsf{M}^{p,q}_{u,v,x}\text{Ж}^{x,r,s}_{i,j,n}\mathsf{X}^{v,j,t}_{h,g,f}\mathsf{W}^{u,i,h}_{y,z} = \mathsf{W}^{q,s,t}_{u,v}\text{Ж}^{p,r,u}_{y,i,j}\mathsf{M}^{j,v}_{z,f,n}$
IIY	$\mathsf{W}^{p,q,r}_{h,j}\mathsf{W}^{j,s,t}_{i,n}\text{Ж}^{h,i,u}_{x,y,z} = \text{Ж}^{r,t,u}_{v,g,f}\text{Ж}^{q,s,v}_{x,h,j}\mathsf{W}^{p,h,g}_{y,i}\mathsf{W}^{i,j,f}_{z,n}$
IIII	$\text{Ж}^{p,q,r}_{v,x,y}\text{Ж}^{y,s,t}_{z,e,n}\text{Ж}^{x,e,u}_{k,h,j}\text{Ж}^{v,z,k}_{g,f,i} = \text{Ж}^{r,t,u}_{k,z,v}\text{Ж}^{q,s,k}_{g,e,x}\text{Ж}^{p,e,z}_{f,h,y}\text{Ж}^{y,x,v}_{i,j,n}$

$$\mathsf{K}^{p,q}_{r,s} = \begin{cases} 1 \text{ if } \exists\ a, b, c \text{ s.t.} \\ \begin{cases} p = ((ab, c), 0|0), \quad q = ((a, b), 0|0), \\ r = ((a, bc), 0|0), \&\ s = ((b, c), 0|0). \end{cases} \\ 0 \text{ otherwise;} \end{cases}$$

$$
\mathsf{M}^{i,j}_{p,q,r} = \begin{cases} 1 \text{ if } \exists\ a, b, c \text{ s.t.} \\ \begin{cases} i = ((0,0), ab|c), & j = ((a,b), 0|0), \\ p = ((a,b) \triangleleft c, 0|0), & q = ((0,0), a|c), \\ \&\ r = ((0,0), b|c), \end{cases} \\ 0 \text{ otherwise;} \end{cases}
$$

$$
\mathsf{W}_{p,q}^{i,j,\ell} = \begin{cases} 1 \text{ if } \exists\ a, b, c \text{ s.t.} \\ \begin{cases} \ i = ((b,c), 0|0), \quad\ j = ((0,0), (a \triangleleft b)|c), \\ \ell = ((0,0), a|b), \\ \ p = ((0,0), a|(bc)), \ \ \&\ q = ((b,c), 0|0), \end{cases} \\ 0 \text{ otherwise;} \end{cases}
$$

$$\text{Ж}^{p,q,r}_{s,t,u} = \begin{cases} 1 \text{ if } \exists\ a, b, c \text{ s.t.} \\ \begin{cases} p = ((0,0), b|c), \quad q = ((0,0), (a \triangleleft b)|c), \\ r = ((0,0), a|b), \\ s = ((0,0), (a|b) \triangleleft c), \quad t = ((0,0), a|c), \\ \& \ u = ((0,0), b|c), \end{cases} \\ 0 \text{ otherwise;} \end{cases}$$

where, *e.g.*, $a \triangleleft c = c^{-1}ac$, $(a, b) \triangleleft c = (a \triangleleft c, b \triangleleft c)$, and $(a|b) \triangleleft c = (a \triangleleft c)|(b \triangleleft c)$

These tensors form a solution set over a module whose basis is determined by the underlying algebraic structure precisely because $\partial \circ \partial = 0$ in the chain complex.

Thanks

Thank you for your attention!