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Variational Convergence of IP-DGFEM

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In this paper, we develop the theory required to perform a variational convergence analysis for discontinuous Galerkin finite element methods when applied to minimization problems. For Sobolev indices in $[1, \infty)$, we prove generalizations of many techniques of classical analysis in Sobolev spaces and apply them to a typical energy minimization problem for which we prove convergence of a *variational interior penalty discontinuous Galerkin finite element method* (VIP-DGFEM). Our main tool in this analysis is a theorem which allows the extraction of a "weakly" converging subsequence of a family of discrete solutions and which shows that any "weak limit" is a Sobolev function.

Key words and phrases: discontinuous Galerkin method, compactness, Γ-convergence, embedding theorems

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1 Introduction

For many highly nonlinear problems the standard finite element analysis based on coercivity (or more generally on inf-sup conditions) cannot be applied in a straightforward fashion. It can be advantageous to apply variational convergence techniques based on compactness of discrete solutions and weakened notions of monotonicity. The field of nonlinear elasticity is a particularly fertile source for problems of this type. For example, minimizing the energy functional

$$\mathcal{I}(u) = \int_\Omega f(x, u, \nabla u) \,\mathrm{d}x, \tag{1.1}$$

where f is a *stored energy function*, over a suitable set of admissible deformations is a classical problem of hyperelasticity (see [3, 7]). The computation of minimizers to (1.1) is a largely unsolved problem. Only if it is assumed that such a minimizer is smooth (at least $\mathrm{C}^1(\bar{\Omega})$) can it be shown that a conforming Galerkin finite element discretization of (1.1) converges [4, 17]. Some positive results, using highly non-standard splitting techniques whose approximation properties are entirely unclear can be found in [19]. Our analysis in the present work only covers the case where f is convex in the third argument which is insufficient to cover physically realistic stored energies (where f is at most polyconvex) and it can therefore only be considered an exploratory first step towards the solution of (1.1). However, we hope that the flexibility of the discontinuous Galerkin method will allow us in the future to finally tackle problems in this class.

Energy minimization problems are by no means the only application of the techniques developed in this article. They can be used to provide additional insight into any nonlinear problem where classical analysis based on coercivity, or an inf-sup condition, fails but weaker notions of stability (such as lower semicontinuity) are available. They can furthermore be used to prove convergence of a numerical discretization without making any smoothness assumptions on exact solutions. For example, it becomes straightforward to prove the convergence of DGFEMs for variational inequalities. An application which is currently under investigation is the development of the DGFEM for the Allen–Cahn and Cahn–Hilliard equations with double-obstacle potential [13].

Finally, the tools we develop, a continuous reconstruction operator, broken Sobolev–Poincaré and Friedrichs inequalities and trace theorems for arbitrary Sobolev indices may be generally useful for analysing DGFEMs.

The outline of the paper is as follows. In Section 2 we provide the main definitions for discontinuous Galerkin spaces and norms, in Section 3 we state a continuous minimization problem and the corresponding DGFE discretization proposed. Sections 4 and 5 are devoted to the developement of the tools which we require to prove the compactness results of Section 6. In Section 7 we state and prove the convergence theorem for the VIP-DGFEM. Finally, Section 8 is devoted to an interesting application of our compactness results: we show that, for suitable choices of discrete norms, the embedding constants for broken spaces are the same as for the corresponding classical Sobolev space.

2 Discontinuous finite element spaces

Let $\Omega \subset \mathbb{R}^n$ be a polygonal Lipschitz domain and let $(\mathcal{T}_h)_{h\in(0,1]}$ be a family of partitions of $\bar{\Omega}$ into *polyhedral elements* which are C^∞ images of a set of "reference" polyhedra. More precisely, we assume that there exists a finite number of reference polyedra $\hat{\kappa}_1, \ldots, \hat{\kappa}_r$, and that for each $\kappa \in \mathcal{T}_h$ there exists a C^∞ invertible map F_κ and a reference element $\hat{\kappa}_i$ such that $\kappa = F_\kappa(\hat{\kappa}_i)$. We assume that elements are closed sets and that $\mathrm{diam}(\kappa) \le h$ for all $\kappa \in \mathcal{T}_h$, where h denotes here the global mesh size. With no loss of generality, we assume that $h \in (0, 1]$. We will provide further assumptions on the mesh regularity in the following section.

Throughout, we shall use the symbols $\approx$, $\lesssim$ and $\gtrsim$ to compare quantities which differ only up to positive constants depending only upon the domain Ω, or the mesh quality (i.e., on the constants appearing in the next Assumption 1), but not on the local or global mesh size.

2.1 Mesh regularity

Let $\mathcal{H}^{n-1}$ denote the $(n-1)$-dimensional Hausdorff measure and, for a Hausdorff-measurable set $A \subset \mathbb{R}^n$, let $\dim_H A$ denote the Hausdorff dimension of A.

In this section we propose a set of assumptions on the family of partitions $(\mathcal{T}_h)_{h\in(0,1]}$ which are required in order to apply the theory developed in this paper. As it is standard in the finite element literature, we define the set of $(n-1)$-dimensional faces $\mathcal{E}_h$ of the partition as follows:

$$\begin{aligned}\mathcal{E}_h &= \{\kappa \cap \kappa' : \kappa, \kappa' \in \mathcal{T}_h, \dim_H(\kappa \cap \kappa') = n-1\} \\ &\quad \cup \{\kappa \cap \partial\Omega : \kappa \in \mathcal{T}_h, \dim_H(\kappa \cap \partial\Omega) = n-1\}.\end{aligned}$$

Furthermore, we use Γ_{int} to denote the union of all faces $e \in \mathcal{E}_h$ such that $\dim_H(e \cap \partial\Omega) < n-1$.

Throughout this article, we make the following assumption on our family of meshes.

Assumption 1 (Mesh Quality) *Let* $h_\kappa = \mathrm{diam}(\kappa)$ *for all* $\kappa \in \mathcal{T}_h$. *The family of partitions* $(\mathcal{T}_h)_{h\in(0,1]}$ *satisfies:*

(a) Shape Regularity. *Each element* $\kappa \in \mathcal{T}_h$ *is shape regular, i.e., there exist* C_1, C_2*, independent of* h*, such that*

$$\mathrm{Lip}(F_\kappa) \le C_1 h_\kappa^n \quad \forall \kappa \in \mathcal{T}_h \quad \text{and} \quad \mathrm{Lip}(F_\kappa^{-1}) \le C_2 h_\kappa^{-n} \quad \forall \kappa \in \mathcal{T}_h.$$

where $\mathrm{Lip}(F_\kappa)$ *and* $\mathrm{Lip}(F_\kappa^{-1})$ *stand for the Lipschitz constants of the mappings* F_κ *and* F_κ^{-1}*, respectively.*

(b) Contact Regularity. *Each face $e \in \mathcal{E}_h$ is shape regular. More precisely, we assume that there exists a constant $C_1 > 0$ such that, for all $h \in (0,1]$ and for all faces $e \in \mathcal{E}_h$, there exist a point $x_e \in e$ and a radius $\rho_e \geq C_1 \mathrm{diam}(e)$ such that $B_e = B(x_e, \rho_e) \cap A_e \subset e$, where A_e is the affine hyperplane spanned by e. Moreover, we set $h_e := \mathrm{diam}(e)$, and there are positive constants such that:*

$$c_e h_\kappa \leq h_e \leq C_e h_\kappa \qquad c'_e h_{\kappa'} \leq h_e \leq C'_e h_{\kappa'}$$

where $e = \kappa \cap \kappa'$.

(c) Submesh Condition. *There exists a simplicial submesh $\widetilde{\mathcal{T}}_h$ such that*

1. *for each $\tilde{\kappa} \in \widetilde{\mathcal{T}}_h$ there exists $\kappa \in \mathcal{T}_h$ such that $\tilde{\kappa} \subset \kappa$.*
2. *the elements $\tilde{\kappa} \in \widetilde{\mathcal{T}}_h$ are uniformly shape regular, and*
3. *if $\tilde{\kappa} \subseteq \kappa$, $\kappa \in \mathcal{T}_h$, then $h_{\tilde{\kappa}} := \mathrm{diam}(\tilde{\kappa})$ satisfies: $\tilde{c} h_\kappa \leq h_{\tilde{\kappa}}$ for a positive constant $\tilde{c}$.*

We denote by $h(x)$ the piecewise constant function defined as $h(x) = h_\kappa$, $x \in \mathrm{int}(\kappa)$ and $h(x) = h_e$, $x \in e$. The global mesh size h can be chosen as $h = \max_{x \in \overline{\Omega}} h(x)$.

Remark 1 *The existence of a simplicial submesh is an entirely technical assumption which may be tedious to verify in practise. We have included it since it seemed the most general assumption under which we were able to prove the required results. We note also that in dimension $n = 2, 3$ such a submesh can be constructed under fairly mild assumptions on the partition $\mathcal{T}_h$ [5, Corollary 7.3]. In fact, it seems quite straightforward to generalize the proof to arbitrary dimensions.*

Lemma 2 *There exists a constant C, independent of the mesh size, such that*

$$\sharp\{e \in \mathcal{E}_h : e \subset \kappa\} \leq C \qquad \forall \kappa \in \mathcal{T}_h \quad \forall h \in (0,1].$$

Proof Let $\kappa \in \mathcal{T}_h$ and let $E \subset \mathcal{E}_h$ be the set of faces contained in κ. By Assumption 1b, we have

$$\begin{aligned} \sharp E\, h_\kappa^{n-1} &\approx \sum_{e \in E} h_e^{n-1} \approx \sum_{e \in E} \mathcal{H}^{n-1}(e) \\ &= \mathcal{H}^{n-1}(\partial \kappa) \approx h_\kappa^{n-1}. \end{aligned}$$

Upon dividing by h_κ^{n-1} we obtain $\sharp E \approx 1$. ■

2.2 Broken Sobolev spaces and DGFE spaces

Let $p \in [1, \infty)$. We will use standard Sobolev spaces $\mathrm{W}^{1,p}(\Omega)$ and L^p-spaces $\mathrm{L}^p(\Omega)$ with their corresponding norms, with a self-evident notation. The broken Sobolev space $\mathrm{W}^{1,p}(\mathcal{T}_h)$ is defined by

$$\mathrm{W}^{1,p}(\mathcal{T}_h) = \Big\{ u \in \mathrm{L}^1(\Omega) : u|_\kappa \in \mathrm{W}^{1,p}(\kappa) \text{ for all } \kappa \in \mathcal{T}_h \Big\}.$$

The dual index is denoted by $p' = p/(p-1)$. The Sobolev-dual index which is related to the Sobolev embedding theorems (see [2]) is denoted by $p^* = np/(n-p)$ if $p < n$ and $p^* = \infty$ if $p \geq n$. We recall that $\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^q(\Omega)$, $q \in [1, p^*] \setminus \{+\infty\}$ and that this embedding is compact for all $q < p^*$ (see [2] for further details).

The subspace of discontinuous finite element functions of polynomial degree no higher than k is defined as

$$S^k(\mathcal{T}_h) = \Big\{ u \in \mathrm{L}^1(\Omega) : u|_\kappa \circ F_\kappa^{-1} \in P^k \text{ for all } \kappa \in \mathcal{T}_h \Big\},$$

where P^k denotes the space of polynomials of degree k in $\mathbb{R}^n$. For each face $e \in \mathcal{E}_h$, $e \subset \Gamma_{\mathrm{int}}$ we denote by κ^+ and κ^- its neighbouring elements. We write ν^+, ν^- to denote the outward normal unit vectors to the boundaries $\partial\kappa^\pm$, respectively. The jump and average of a vector-valued function $\varphi \in \mathrm{W}^{1,1}(\mathcal{T}_h)^m$ with traces $\varphi = \varphi^\pm$ from $\kappa^\pm$ is defined as

$$[\![\varphi]\!] = \varphi^+ \otimes \nu^+ + \varphi^- \otimes \nu^- \quad \text{and}$$
$$\{\varphi\} = \tfrac{1}{2}(\varphi^+ + \varphi^-).$$

For $u \in \mathrm{W}^{1,p}(\mathcal{T}_h)$, we define the broken Sobolev semi-norms:

$$|u|^p_{\mathrm{W}^{1,p}(\mathcal{T}_h)} = \|\nabla u\|^p_{\mathrm{L}^p(\Omega)} + \int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s,$$
$$|u|^p_{\mathrm{W}^{1,p}_D(\mathcal{T}_h)} = |u|^p_{\mathrm{W}^{1,p}(\mathcal{T}_h)} + \int_{\Gamma_D} h^{1-p} |u|^p \,\mathrm{d}s$$

Next, we recall some important facts about the Banach space $\mathrm{BV}(\Omega)$ of functions of bounded variation which contains the spaces $\mathrm{W}^{1,p}(\mathcal{T}_h)$. To simplify the notation, our discussion is for scalar functions, but equivalent results for vector-valued spaces follow immediately from the scalar results. The space is equipped with the norm

$$\|u\|_{\mathrm{BV}} = \|u\|_{\mathrm{L}^1(\Omega)} + |Du|(\Omega),$$

where Du is the measure representing the distributional derivative of u and $|Du|(\Omega)$ is its total variation, defined by

$$|Du|(\Omega) = \sup_{\substack{\varphi \in \mathrm{C}^1_c(\Omega)^n \\ \|\varphi\|_{\mathrm{L}^\infty} \leq 1}} \int_\Omega u \,\mathrm{div}\varphi \,\mathrm{d}x.$$

The symbol $\mathrm{C}^1_\mathrm{c}(\Omega)$ denotes the space of continuously differential functions with compact support in Ω.

Compactness and many other properties of the space $\mathrm{BV}(\Omega)$ will play an important role in our analysis. The variation (distributional derivative) of a broken Sobolev function $u \in \mathrm{W}^{1,p}(\mathcal{T}_h)$ is given by the following formula which can be easily verified using integration by parts on every element of the mesh.

$$-\int_\Omega u \,\mathrm{div}\varphi \,\mathrm{d}x = \int_\Omega \nabla u \cdot \varphi \,\mathrm{d}x - \int_{\Gamma_{\mathrm{int}}} [\![u]\!] \cdot \varphi \,\mathrm{d}s \qquad \forall \varphi \in \mathrm{C}^1_\mathrm{c}(\Omega)^n. \tag{2.1}$$

We conclude this section with a result that highlights the correct scaling of the penalization for the $|\cdot|_{\mathrm{W}^{1,p}(\mathcal{T}_h)}$-semi-norm. This observation is the crucial starting point to lift results for the space BV to DGFE spaces.

Lemma 3 *There exists a constant C, independent of h and of p, such that, for all $p \in [1,\infty)$,*

$$|Du|(\Omega) \le C\,|u|_{\mathrm{W}^{1,p}(\mathcal{T}_h)} \qquad \forall u \in \mathrm{W}^{1,p}(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof The proof is a straightforward generalization of [18, Theorem 3.26] to the case $p \ne 2$. For the sake of completeness, we include a brief sketch.

The variation is bounded by

$$|Du|(\Omega) \le \|\nabla u\|_{\mathrm{L}^1(\Omega)} + \int_{\Gamma_{\mathrm{int}}} |[\![u]\!]| \,\mathrm{d}s.$$

Since $|\Omega| < +\infty$, we obviously have $\|\nabla u\|_{\mathrm{L}^1(\Omega)} \le |\Omega|^{1-1/p} \|\nabla u\|_{\mathrm{L}^p(\Omega)}$. We can use Hölder's inequality and Assumption 1 to estimate

$$\begin{aligned}
\int_{\Gamma_{\mathrm{int}}} |[\![u]\!]| \,\mathrm{d}s &= \int_{\Gamma_{\mathrm{int}}} h^{1/p'} h^{(1-p)/p} |[\![u]\!]| \,\mathrm{d}s \\
&\le \left(\int_{\Gamma_{\mathrm{int}}} h \,\mathrm{d}s \right)^{1/p'} \left(\int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s \right)^{1/p} \\
&\lesssim \left(\sum_{e \subset \Gamma_{\mathrm{int}}} h_e^n \right)^{1/p'} \left(\int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s \right)^{1/p}
\end{aligned}$$

By Assumption 1 as well as Lemma 2, we have

$$\sum_{e \subset \Gamma_{\mathrm{int}}} h_e^n \lesssim \sum_{e \subset \Gamma_{\mathrm{int}}} \sum_{\substack{\kappa \in \mathcal{T}_h \\ e \subset \kappa}} h_\kappa^n \lesssim \sum_{\kappa \in \mathcal{T}_h} h_\kappa^n \approx |\Omega|,$$

which gives the result. ∎

3 Problem statement

Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ where Γ_D has positive surface measure. Let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be a Carathéodory function, i.e., measurable in its first and continuous in its second and third argument. Suppose that it also satisfies the p-growth condition

$$c_0(|F|^p - |u|^r + a_0(x)) \le f(x, u, F) \le c_1(|F|^p + |u|^q + a_1(x)) \tag{3.1}$$

where $a_i \in \mathrm{L}^1(\Omega)$. We furthermore require that $p \in (1, \infty)$, that $r < p$, and that $r \le q < p^*$. Let $g\colon \Gamma_N \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function which satisfies the growth condition

$$|g(x, u)| \le c_2(|u|^r + a_2(x)), \tag{3.2}$$

where $a_2 \in \mathrm{L}^1(\Gamma_N)$ and r is the same index as in (3.1).

We define the functional $\mathcal{I} : \mathrm{W}^{1,p}(\Omega)^m \to \mathbb{R}$ by

$$\mathcal{I}(u) = \int_\Omega f(x, u, \nabla u)\,\mathrm{d}x + \int_{\Gamma_N} g(x, u)\,\mathrm{d}s, \qquad u \in \mathrm{W}^{1,p}(\Omega)^m. \tag{3.3}$$

Fix $u_D \in \mathrm{W}^{1,p}(\Omega)^m$ and let $\mathcal{A}$ be the closed, affine subspace of $\mathrm{W}^{1,p}(\Omega)^m$ defined by

$$\mathcal{A} = \big\{u \in \mathrm{W}^{1,p}(\Omega)^m : u|_{\Gamma_D} = u_D\big\},$$

the set of admissible trial functions. We consider the problem of finding a minimizer of $\mathcal{I}$ in $\mathcal{A}$. The existence of minimizers follows from the direct method of the calculus of variations; see for example Theorems 3.1, 3.4 and 4.1 in [8]. Note in particular that, if $m = 1$ or $n = 1$, then Theorem 3.1 in [8] shows that convexity of f in its third argument is a necessary and sufficient condition for $\mathcal{I}$ to be sequentially weakly lower semicontinuous (which is a necessary condition for the direct method to apply to our problem).

Before proposing a discretization strategy, we summarize the most important technical facts about (3.3) which we use in the convergence proof in Section 7.

Lemma 4 *Let f and g be Carathéodory functions which respectively satisfy the growth conditions* (3.1) *and* (3.2)*.*

(i) If $u_j \to u$ strongly in $\mathrm{L}^q(\Omega)^m$ and $F_j \to F$ strongly in $\mathrm{L}^p(\Omega)^{m\times n}$ then

$$\int_\Omega f(x, u_j, F_j)\,\mathrm{d}x \to \int_\Omega f(x, u, F)\,\mathrm{d}x, \qquad \text{as } j \to \infty.$$

(ii) If $u_j \to u$ strongly in $\mathrm{L}^r(\Gamma_N)^m$ then

$$\int_{\Gamma_N} g(x, u_j)\,\mathrm{d}s \to \int_{\Gamma_N} g(x, u)\,\mathrm{d}s \qquad \text{as } j \to \infty.$$

(iii) If $u_j \to u$ strongly in $\mathrm{L}^1(\Omega)^m$, $F_j \rightharpoonup F$ weakly in $\mathrm{L}^p(\Omega)^{m\times n}$, and if f is convex in the third argument, then

$$\int_\Omega f(x,u,F)\,\mathrm{d}x \le \liminf_{j\to\infty} \int_\Omega f(x,u_j,F_j)\,\mathrm{d}x.$$

Items (i) and (ii) follow immediately from Fatou's Lemma while item (iii) is an application of [8, Theorem 3.4].

We turn now to the discretization of the functional (3.3). To this end, we first define the lifting operator $R\colon \mathrm{W}^{1,p}(\mathcal{T}_h)^m \to (S^l(\mathcal{T}_h))^{m\times n}$ via

$$\int_\Omega R(u)\cdot\varphi\,\mathrm{d}x = -\int_{\Gamma_{\mathrm{int}}} [\![u]\!]\cdot\{\varphi\}\,\mathrm{d}s \qquad \forall\varphi\in S^l(\mathcal{T}_h)^{m\times n}. \tag{3.4}$$

The lifting $R(u)$ is a bulk representation of the jump contribution to the distributional gradient of u. The polynomial degree l is a method parameter and can be chosen arbitrarily.

We propose the following discrete functional

$$\begin{aligned}\mathcal{I}_h(u_h) = &\int_\Omega f\big(x,u_h,\nabla u_h + R(u_h)\big)\,\mathrm{d}x + \int_{\Gamma_N} g(x,u_h)\,\mathrm{d}s \\ &+ \int_{\Gamma_D} h^{1-p}|u_h - u_D|^p\,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} h^{1-p}|[\![u_h]\!]|^p\,\mathrm{d}s,\end{aligned} \tag{3.5}$$

and our discrete problem is to find a minimizer of (3.5) among all possible vector fields in $S^k(\mathcal{T}_h)^m$. In the tradition of the literature on discontinuous Galerkin finite element methods, we chose to label the variational method in (3.5) as VIP-DGFEM (variational interior penalty discontinuous Galerkin finite element method).

Essentially the same DGFE discretization (with $p = 2$) was defined by Eyck and Lew [22] for applications in finite elasticity. We refer to their paper for a linearized stability analysis and very promising numerical results.

Note that despite its appearance, (3.5) is in fact straightforward to implement. The definition of the lifting operator (3.4) allows the construction of $R(u_h)$ locally in each element, taking into account only the degrees of freedom on the edges of the element. For example, if $R(u_h)$ is chosen to be piecewise constant (which is sufficient to obtain convergence) then

$$R(u_h)|_\kappa = |\kappa|^{-1}\sum_{e\subset\partial\kappa}\int_e [\![u_h]\!]\,\mathrm{d}s \qquad \forall\kappa\in\mathcal{T}_h. \tag{3.6}$$

In order to analyze the discretization (3.5) of (3.3), we need to prove several results about DGFE spaces which are collected in the next three sections.

4 Reconstruction operator

As is the case in many works on discontinuous Galerkin methods, ranging from *a posteriori* error estimation [12] to the proof of broken Poincaré type inequalities [5, 6, 20], we require at several points a continuous reconstruction operator. In this section we will make use of the assumption that there exists a simplicial submesh of $\mathcal{T}_h$ (see Assumption 1c).

Our goal is to define a family of quasi-interpolant operators $Q_h : S^k(\mathcal{T}_h) \to \mathrm{W}^{1,\infty}(\Omega)$ and to provide localized error estimates for $Q_h u - u$ in L^q norms, $q \in [1, \infty)$. Our results are more general than previous ones in that we consider arbitrary Sobolev indices but weaker than those in [5], for example, since we restrict ourselves to a fixed polynomial degree. In fact, our proofs do not carry over to arbitrary $\mathrm{W}^{1,p}(\mathcal{T}_h)$ functions in an obvious way. The idea of using quasi-interpolant operators was inspired by [16].

As we have said earlier, in order to simplify the notation, our discussion here and in the next section is for scalar functions only. The corresponding results for vector-valued functions follow immediately.

4.1 Local projection operators

Let us first introduce some notation for the submesh $\widetilde{\mathcal{T}}_h$ (see Assumption 1c). We denote by $\widetilde{\mathcal{N}}_h$ the set of nodes of $\widetilde{\mathcal{T}}_h$ and by $\widetilde{\mathcal{N}}_h^0$ the subset of internal nodes. For every $z \in \widetilde{\mathcal{N}}_h$, we define the star-shaped patch

$$\widetilde{T}_z = \bigcup \{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h : z \in \widetilde{\kappa}\}, \tag{4.1}$$

and we set $h_z = \mathrm{diam}(\widetilde{T}_z)$. Due to the assumptions on the submesh $\widetilde{\mathcal{T}}_h$, it is clear that $\widetilde{T}_z$ contains a finite number of elements which is independent of the mesh size.

Next, we establish the existence of linear maps $\pi_z : \mathrm{BV}(\Omega) \to \mathbb{R}$, $z \in \widetilde{\mathcal{N}}_h$, such that

$$\|u - \pi_z(u)\|_{\mathrm{L}^1(\widetilde{T}_z)} \le C h_z |Du|(\widetilde{T}_z) \qquad \forall z \in \widetilde{\mathcal{N}}_h \quad \forall u \in \mathrm{BV}(\Omega), \tag{4.2}$$

where C is independent of h and z. To achieve this, we have to distinguish between the cases when z lies on the boundary $\partial\Omega$ and in the interior of the domain Ω. If $z \in \widetilde{\mathcal{N}}_h^0$, i.e., $z \in \mathrm{int}(\Omega)$, let $B_z = B(z, \rho_z)$, where $\rho_z \approx h_z$ such that $B_z \subset \widetilde{T}_z$. The existence of such radii follows immediately from Assumption 1c. Setting $\pi_z(u) = (u)_{B_z}$ (i.e., the mean value over the ball B_z), we shall first prove (4.2) for interior vertices which will also give us an intuition how to proceed for boundary vertices. We note that our construction as well as the proofs of the estimates are only minor modifications of the L^2 case treated by Verfürth [23].

Lemma 5 *Let $K \subset \mathbb{R}^n$ be star-shaped with respect to the point $x_0 \in K$ and define*

$$\rho = \inf_{x \in \partial K} |x - x_0|_2 \quad \text{and} \quad h = \sup_{x \in \partial K} |x - x_0|_2,$$

as well as the chunkiness parameter $\gamma = h/\rho$. *There exists a constant C, depending only on γ and on n such that*

$$\|u\|_{\mathrm{L}^1(K)} \leq C(\|u\|_{\mathrm{L}^1(B)} + \rho |Du|(K)) \qquad \forall u \in \mathrm{BV}(K), \tag{4.3}$$

where $B = B(x_0, \rho)$, and

$$\|u - (u)_B\|_{\mathrm{L}^1(K)} \leq C\rho |Du|(K) \qquad \forall u \in \mathrm{BV}(K). \tag{4.4}$$

The proof of Lemma 5, which is merely a modification of the proof of [23, Lemma 4.1], is given in the Appendix.

We note immediately that Lemma 5 together with shape regularity assumption on the submesh $\widetilde{\mathcal{T}}_h$ implies (4.2) for interior nodes.

If z lies at the boundary, we define h_z as before but we now set

$$\rho_z = \inf_{x \in \partial \widetilde{T}_z \setminus \partial\Omega} |z - x|_2.$$

Let $\tilde{B}_z = B(z, \rho_z) \cap \widetilde{T}_z = B(z, \rho_z) \cap \bar{\Omega}$. Repeating the proof of Lemma 5 verbatim we obtain

$$\|v\|_{\mathrm{L}^1(\Omega)} \leq C\big(\|v\|_{\mathrm{L}^1(\tilde{B}_z)} + h_z |Dv|(\widetilde{T}_z)\big) \qquad \forall v \in \mathrm{BV}(\widetilde{T}_z). \tag{4.5}$$

Since $\tilde{B}_z$ is not necessarily convex, we apply a further reduction to the first term on the right-hand side of (4.5). Since $\partial\Omega$ is Lipschitz continuous, there exists a cone C with positive opening angle α, which can be chosen independently of z, and apex 0 such that $(z + C) \cap B(z, \varepsilon) \subset \mathbb{R}^n \setminus \widetilde{T}_z$ for some $\varepsilon > 0$. Let $a \in \mathbb{R}^n$, $|a|_2 = \rho_z/2$, be the direction of the axis of the cone C pointing into $\widetilde{T}_z$ and define $z' = z + a$. It can be easily seen that $\widetilde{B}_z$ is star-shaped with respect to z' and that there exists a value $r_0 \in (0, 1/2]$ which depends only on α, such that $B(z', r_0\rho_z) \subset \widetilde{B}_z \subset \widetilde{T}_z$. Hence, we may define $\pi_z(u) = (u)_{B_z}$ again (but note that B_z is defined differently now) to obtain the following result.

Lemma 6 *For $z \in \widetilde{\mathcal{N}}_h$ and $u \in \mathrm{BV}(\Omega)$ let $\pi_z(u) = (u)_{B_z}$ where B_z is defined as in the above discussion; then,* (4.2) *holds with a constant C independent of the mesh size.*

Proof For interior vertices, we have already shown that (4.2) holds with a constant depending only on γ_z, which measures mesh quality, and it remains to prove a similar bound for boundary vertices.

Using (4.5) with $v = u - \pi_z(u)$, we have

$$\|u - \pi_z(u)\|_{\mathrm{L}^1(\widetilde{T}_z)} \lesssim \|u - \pi_z(u)\|_{\mathrm{L}^1(\widetilde{B}_z)} + h_z|Du|(\widetilde{T}_z).$$

We now apply Lemma 5 with $K = \widetilde{B}_z$, $B = B_z$, $h = \rho_z$ and $\rho = r_0\rho_z$ to obtain

$$\|u - \pi_z(u)\|_{\mathrm{L}^1(\widetilde{B}_z)} \lesssim h_z|Du|(\widetilde{B}_z).$$

Combining this estimate with the previous formula, we obtain

$$\|u - \pi_z(u)\|_{\mathrm{L}^1(\widetilde{T}_z)} \lesssim h_z|Du|(\widetilde{T}_z).$$

■

4.2 Construction and analysis of Q_h

Finally, we are in a position to define and analyze the reconstruction operator. For each $h \in (0, 1]$ let $Q_h : S^k(\mathcal{T}_h) \to \mathrm{W}^{1,\infty}(\Omega)$ be the linear operator defined by

$$Q_h u = \sum_{z\in\widetilde{\mathcal{N}}_h} \pi_z(u)\lambda_z \tag{4.6}$$

where λ_z is the standard P^1 nodal basis function on the mesh $\widetilde{\mathcal{T}}_h$ associated with the vertex z.

For later use we define for each $z \in \widetilde{\mathcal{N}}_h$, $\kappa \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$:

$$T_z = \bigcup\{\kappa \in \mathcal{T}_h \ : \ z \in \kappa\},\ T_\kappa = \bigcup\{T_z \ : \ z \in \kappa\},\ \text{and}\ T_e = \bigcup\{T_\kappa \ : \ e \in \kappa\}.$$

Since $\widetilde{\mathcal{T}}_h$ is a submesh of $\mathcal{T}_h$, we have that $T_z \supseteq \widetilde{T}_z$, where $\widetilde{T}_z$ was defined in (4.1). If we denote by $\mathcal{K}_\kappa$ the number of elements $\kappa' \in \mathcal{T}_h \cap T_\kappa$, due to Assumption 1 (Contact regularity), it follows that $\mathcal{K}_\kappa$ is bounded independent of h and of κ. Together with Assumption 1c this implies that

$$h_z = \mathrm{diam}(\widetilde{T}_z) \approx \mathrm{diam}(T_z) \approx \max_{\kappa,\ z\in\kappa} \mathrm{diam}(T_\kappa)$$

and also

$$\mathrm{diam}(T_\kappa) \approx \min_{\kappa'\in T_\kappa} h_{\kappa'} \approx h_\kappa.$$

We use the labels $\mathrm{W}^{1,p}(\mathcal{T}_h \cap T_z)$, $\mathrm{W}^{1,p}(\mathcal{T}_h \cap T_\kappa)$ and $\mathrm{W}^{1,p}(\mathcal{T}_h \cap T_e)$ to denote the restriction of the broken Sobolev semi-norm to the sets T_z, T_κ and T_e, respectively.

Theorem 7 *Let $p, q \in [1, \infty)$. The reconstruction operator Q_h defined in* (4.6) *satisfies the local estimates*

$$\|u - Q_h u\|_{\mathrm{L}^q(\kappa)} \lesssim h_\kappa^{\frac{n}{q}-\frac{n}{p}+1}|u|_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_\kappa)} \quad \forall\kappa \in \mathcal{T}_h \tag{4.7}$$

$$\|u - Q_h(u)\|_{\mathrm{L}^q(e)} \lesssim h_e^{\frac{(n-1)}{q}-\frac{n}{p}+1}|u|_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_e)} \quad \forall e \in \mathcal{E}_h \cap \partial\Omega \tag{4.8}$$

$$\|\nabla Q_h u\|_{\mathrm{L}^p(\kappa)} \lesssim |u|_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_\kappa)} \quad \forall\kappa \in \mathcal{T}_h. \tag{4.9}$$

Furthermore, for $q \in [p, p^*] \setminus \{\infty\}$, we have the global estimates

$$\|u - Q_h u\|_{\mathrm{L}^q(\Omega)} \lesssim h^{\frac{n}{q}-\frac{n}{p}+1}|u|_{\mathrm{W}^{1,p}(\mathcal{T}_h)} \quad \text{and} \tag{4.10}$$

$$\|\nabla Q_h u\|_{\mathrm{L}^p(\Omega)} \lesssim |u|_{\mathrm{W}^{1,p}(\mathcal{T}_h)}. \tag{4.11}$$

Proof Fix $q \in [1, \infty)$. For each $z \in \widetilde{\mathcal{N}}_h$ we use Lemma 21 to obtain

$$\|u - \pi_z(u)\|_{\mathrm{L}^q(\widetilde{T}_z)} \approx h_z^{\frac{n}{q}-n}\|u - \pi_z(u)\|_{\mathrm{L}^1(\widetilde{T}_z)}.$$

Our local projection result Lemma 6 gives

$$\begin{aligned}\|u - \pi_z(u)\|_{\mathrm{L}^q(\widetilde{T}_z)} &\lesssim h_z^{\frac{n}{q}-n+1}|Du|(\widetilde{T}_z)\\ &\lesssim h_z^{\frac{n}{q}-n+1}\|\nabla u\|_{\mathrm{L}^1(T_z)} + h_z^{\frac{n}{q}-n+1}\sum_{e\subset T_z}\int_e |[\![u]\!]|\,\mathrm{d}s.\end{aligned}$$

For the bulk term $\|\nabla u\|_{\mathrm{L}^1(T_z)}$ we use Lemma 21 and for the surface term we use Hölder's inequality (as in the proof of Lemma 3) to deduce

$$\begin{aligned}\|u - \pi_z(u)\|_{\mathrm{L}^q(\widetilde{T}_z)} &\lesssim h_z^{\frac{n}{q}-\frac{n}{p}+1}\|\nabla u\|_{\mathrm{L}^p(T_z)} + h_z^{\frac{n}{q}-\frac{n}{p}+1}\left(\sum_{e\subset T_z} h_e^{1-p}\int_e |[\![u]\!]|^p\,\mathrm{d}s\right)^{1/p}\\ &\lesssim h_z^{\frac{n}{q}-\frac{n}{p}+1}|u|_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_z)}.\end{aligned} \tag{4.12}$$

We now prove the local estimate (4.7). Using the fact that the hat functions $\{\lambda_z\}_{z\in\widetilde{\mathcal{N}}_h}$ form a partition of unity, we have:

$$\|u - Q_h(u)\|^q_{\mathrm{L}^q(\kappa)} = \Big\|\sum_{z\in\widetilde{\mathcal{N}}_h\cap\kappa}(u - \pi_z(u))\lambda_z\Big\|^q_{\mathrm{L}^q(\kappa)}.$$

Rearranging terms, and recalling that $\|\lambda_z\|_{\mathrm{L}^\infty(\Omega)} = 1$ and that $\lambda_z = 0$ outside $\widetilde{T}_z$, we compute:

$$\|u - Q_h(u)\|^q_{\mathrm{L}^q(\kappa)} \lesssim \sum_{z\in\widetilde{\mathcal{N}}_h\cap\kappa}\|u - \pi_z(u)\|^q_{\mathrm{L}^q(\kappa\cap\widetilde{T}_z)} \lesssim \sum_{z\in\widetilde{\mathcal{N}}_h\cap\kappa}\|u - \pi_z(u)\|^q_{\mathrm{L}^q(\widetilde{T}_z)}.$$

Using (4.12), we obtain:

$$\|u - Q_h(u)\|^q_{\mathrm{L}^q(\kappa)} \lesssim \sum_{z\in\widetilde{\mathcal{N}}_h\cap\kappa} h_z^{q(\frac{n}{q}-\frac{n}{p}+1)}|u|^q_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_z)}.$$

Rearranging terms, using the definition on T_κ and recalling that the cardinality of $\widetilde{\mathcal{N}}_h \cap \kappa$ is uniformly bounded:

$$\begin{aligned}\|u - Q_h(u)\|_{\mathrm{L}^q(\kappa)} &\lesssim h_\kappa^{\frac{n}{q}-\frac{n}{p}+1}\left(\sum_{z\in\widetilde{\mathcal{N}}_h\cap\kappa}|u|^q_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_z)}\right)^{1/q}\\ &\lesssim h_\kappa^{\frac{n}{q}-\frac{n}{p}+1}|u|_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_\kappa)}.\end{aligned}$$

If $e \in \mathcal{E}_h \cap \partial\Omega$, then

$$\|u - Q_h u\|_{\mathrm{L}^q(e)} \le \sum_{z \in \widetilde{\mathcal{N}}_h \cap e} \|u - \pi_z(u)\|_{\mathrm{L}^q(e \cap \widetilde{T}_z)}.$$

The set $e \cap \widetilde{T}_z$ is a union of faces of elements in $\widetilde{\mathcal{T}}_h$. We can therefore use the local inverse estimate

$$\|u - \pi_z(u)\|^q_{\mathrm{L}^q(e \cap \widetilde{T}_z)} \lesssim h_\kappa^{-1} \|u - \pi_z(u)\|^q_{\mathrm{L}^q(\kappa \cap \widetilde{T}_z)},$$

after which proceed as above to obtain (4.8).

The proof of the third local estimate (4.9) follows along the same lines.

To prove the first global estimate (4.10), we assume $q \in [p, p^*]$, $q \neq \infty$. It then holds that $\frac{n}{q} - \frac{n}{p} + 1 > 0$, and we set $h^* = h^{\frac{n}{q} - \frac{n}{p} + 1}$ (here h is the global mesh size). We sum (4.7) (to power q) over $\kappa \in \mathcal{T}_h$, to obtain

$$\begin{aligned} \|u - Q_h u\|^q_{\mathrm{L}^q(\Omega)} &\lesssim (h^*)^q \sum_{\kappa \in \mathcal{T}_h} \left(\|\nabla u\|^p_{\mathrm{L}^p(T_\kappa)} + \int_{\Gamma_{\mathrm{int}} \cap T_\kappa} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s \right)^{q/p} \\ &\lesssim (h^*)^q \left(\sum_{\kappa \in \mathcal{T}_h} \left[\|\nabla u\|^p_{\mathrm{L}^p(T_\kappa)} + \int_{\Gamma_{\mathrm{int}} \cap T_\kappa} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s \right] \right)^{q/p}, \end{aligned}$$

where we used the fact $\sum |a_i|^\alpha \le (\sum |a_i|)^\alpha$ for $\alpha \ge 1$. Finally, we note that due to Lemma 2, each element κ appears only in finitely many sets $T_{\kappa'}$ and thus, taking the qth root, we obtain the result.

The second global estimate can be proved in the same way. ∎

5 Broken embedding theorems

5.1 Poincaré inequalities

In this section, we prove broken Sobolev–Poincaré inequalities for any $p \in [1, n)$. Similar results were previously derived by Lasis and Süli[15] for $p = 2$. The idea in our proof is the same as in the proof of Theorem 7, to use the known results in $\mathrm{BV}(\Omega)$ and in the Sobolev spaces $\mathrm{W}^{1,p}(\Omega)$ together with local norm-equivalence and the reconstruction operator.

Theorem 8 (Sobolev–Poincaré Inequalities) *Let $p < n$ and let $p^* = np/(n-p)$. There exists a constant C_s such that*

$$\|u - (u)_\Omega\|_{\mathrm{L}^{p^*}(\Omega)} \le C_s \, |u|_{\mathrm{W}^{1,p}(\mathcal{T}_h)} \qquad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0,1].$$

In particular, it holds that

$$\|u\|_{\mathrm{L}^{p^*}(\Omega)} \le C_s \left(\|u\|_{\mathrm{L}^1(\Omega)} + |u|_{\mathrm{W}^{1,p}(\mathcal{T}_h)} \right) \qquad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof Let $v = u-(u)_\Omega$. It is easy to see that Q_h respects constants. Hence, it follows that $Q_h v = Q_h u - (u)_\Omega$ and

$$\|v\|_{L^{p^*}(\Omega)} \le \|v - Q_h v\|_{L^{p^*}(\Omega)} + \|Q_h v - (Q_h v)_\Omega\|_{L^{p^*}(\Omega)} + \|(Q_h v)_\Omega\|_{L^{p^*}(\Omega)}. \tag{5.1}$$

For the first term on the right-hand side of (5.1) we use Theorem 7 to estimate

$$\|v - Q_h v\|_{L^{p^*}(\Omega)} \le C_r |v|_{W^{1,p}(\mathcal{T}_h)}.$$

For the second term on the right-hand side of (5.1), we employ the Poincaré–Sobolev inequality for $W^{1,p}(\Omega)$, and (4.11), to obtain

$$\|Q_h v - (Q_h v)_\Omega\|_{L^{p^*}(\Omega)} \le C(p,\Omega)\|\nabla Q_h v\|_{L^p(\Omega)} \le C_r C(p,\Omega)|v|_{W^{1,p}(\mathcal{T}_h)}.$$

For the last term, we note that $\|(Q_h v)_\Omega\|_{L^{p^*}(\Omega)} \lesssim \|Q_h v\|_{L^1(\Omega)}$ and

$$\begin{aligned} \|Q_h v\|_{L^1(\Omega)} &\le \|Q_h v - v\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)} \\ &\le C_r h |v|_{W^{1,1}(\mathcal{T}_h)} + C(\Omega)|Du|(\Omega), \end{aligned}$$

where we used Theorem 7 on the first term and the Poincaré inequality for $BV(\Omega)$ on the second term on the right-hand side.

Using our estimate in Lemma 3, we deduce that $|Dv|(\Omega) = |Du|(\Omega) \lesssim |u|_{W^{1,p}(\mathcal{T}_h)}$, and we can combine our estimates to yield the first result.

The second result follows immediately from $\|(u)_\Omega\|_{L^{p^*}(\Omega)} \lesssim \|u\|_{L^1(\Omega)}$. ∎

5.2 Trace theorem

We first recall some facts about traces of functions of bounded variation. The following result summarizes Theorems 1 and 2 in [11, Sec. 5.3].

Theorem 9 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. There exists a bounded, linear operator $T\colon BV(\Omega) \to L^1(\partial\Omega)$ (we write $Tu = u$) such that*

$$\int_\Omega u \,\mathrm{div}\varphi \,\mathrm{d}x = -\int_\Omega \varphi\cdot \mathrm{d}Du + \int_{\partial\Omega} (\varphi\cdot\nu) u \,\mathrm{d}s \qquad \forall u \in BV(\Omega) \quad \forall \varphi \in C^1(\mathbb{R}^n)^n,$$

where ν is the unit outward normal to $\partial\Omega$.

If $u \in BV(\Omega)$ then, for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega$, the identity

$$Tu(x) = \lim_{r\to 0} \fint_{B(x,r)\cap\Omega} u \,\mathrm{d}x \tag{5.2}$$

holds.

First, we notice that identity (5.2) immediately implies a Friedrichs inequality for $BV(\Omega)$, and therefore, by Theorem 8, a broken Sobolev–Poincaré inequality with respect to the norm $|\cdot|_{W^{1,p}_D(\mathcal{T}_h)}$ which penalizes boundary values.

Lemma 10 (Friedrichs Inequality for BV) *Let $u \in \mathrm{BV}(\Omega)$ and let Γ_D be a subset of $\partial\Omega$ with positive surface measure. Then, there exists a constant C_F such that*

$$\|u\|_{\mathrm{L}^1(\Omega)} \le C_F\bigg(|Du|(\Omega) + \int_{\Gamma_D} |u|\,\mathrm{d}s\bigg) \qquad \forall u \in \mathrm{BV}(\Omega).$$

Proof We use the usual compactness technique to prove this result. For contradiction, suppose that no such constant C_F exists. Then, there exists a sequence $u_j \in \mathrm{BV}(\Omega)$ such that $\|u_j\|_{\mathrm{L}^1(\Omega)} = 1$ and $|Du_j|(\Omega) + \|u_j\|_{\mathrm{L}^1(\Gamma_D)} \to 0$ as $j \to \infty$. Since $\|u_j\|_{\mathrm{BV}}$ is bounded, there exists a subsequence (not relabelled) and a $u \in \mathrm{BV}(\Omega)$ such that $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$. Since this implies $u_j \to u$ strongly in $\mathrm{L}^1(\Omega)$ it follows that $\|u\|_{\mathrm{L}^1(\Omega)} = 1$. Since the functional $v \mapsto |Dv|(\Omega) + \|v\|_{\mathrm{L}^1(\Gamma_D)}$ is convex and strongly continuous, it is also lower semicontinuous with respect to weak-$*$ convergence. Therefore, $|Du|(\Omega) = 0$, which implies that u is constant in Ω. Since $\|u\|_{\mathrm{L}^1(\Gamma_D)} = 0$ the trace of u at Γ_D vanishes which means that $u = 0$ and contradicts the assumption that $\|u\|_{\mathrm{L}^1(\Omega)} = 1$. ■

Corollary 10A (Broken Friedrichs Inequality) *Let $p \in [1, n)$ and suppose that $\Gamma_D \subset \partial\Omega$ with positive surface measure. Then there exists a constant C, independent of h, such that,*

$$\|u\|_{\mathrm{L}^{p^*}(\Omega)} \le C|u|_{\mathrm{W}^{1,p}_D(\mathcal{T}_h)} \qquad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof First, we estimate the boundary penalization, following the proof of Lemma 3:

$$\begin{aligned}\int_{\Gamma_D} |u|\,\mathrm{d}s &\lesssim \bigg(\sum_{e\subset\Gamma_D} h_e^n\bigg)^{1/p'} \bigg(\int_{\Gamma_D} h^{1-p}|u|^p\,\mathrm{d}s\bigg)^{1/p} \\ &\lesssim h^{1/p'}\mathcal{H}^{n-1}(\Gamma_D)^{1/p'}\bigg(\int_{\Gamma_D} h^{1-p}|u|^p\,\mathrm{d}s\bigg)^{1/p}.\end{aligned}$$

The result now follows immediately by combining Theorem 8 and Lemma 10. ■

Theorem 11 (Broken Trace Theorem) *Let $p < n$ and set $q = p(n-1)/(n-p)$ (i.e., $(n-1)/p - (n-1)/q = 1 - 1/p$). There exists a constant C, independent of h, such that*

$$\|u\|_{\mathrm{L}^q(\partial\Omega)} \le C\Big(\|u\|_{\mathrm{L}^1(\Omega)} + |u|_{\mathrm{W}^{1,p}(\mathcal{T}_h)}\Big) \qquad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0,1]. \tag{5.3}$$

Proof Summing qth powers of (4.8) over the faces on $\partial\Omega$, we obtain:

$$\|u\|^q_{\mathrm{L}^q(\partial\Omega)} \lesssim \|Q_h u\|^q_{\mathrm{L}^q(\partial\Omega)} + \sum_{e\in\mathcal{E}_h, e\subset\partial\Omega} h_\kappa^{n-1-\frac{nq}{p}+q} |u|^q_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_e)}.$$

For the choice $q = p(n-1)/(n-p)$ we have $n - 1 - nq/p + q = 0$ and furthermore, $q/p \geq 1$. The latter property can be used to estimate

$$\sum_{i=1}^{J} |a_j|^{q/p} \leq \Big(\sum_{i=1}^{J} |a_j|\Big)^{q/p}$$

Hence, we can estimate further,

$$\begin{aligned}
\|u\|^q_{\mathrm{L}^q(\partial\Omega)} &\lesssim \|Q_h u\|^q_{\mathrm{L}^q(\partial\Omega)} + \sum_{e\in\mathcal{E}_h, e\subset\partial\Omega} |u|^q_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_e)} \\
&\lesssim \|Q_h u\|^q_{\mathrm{L}^q(\partial\Omega)} + \Big(\sum_{e\in\mathcal{E}_h, e\subset\partial\Omega} |u|^p_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_e)}\Big)^{q/p} \\
&\lesssim \|Q_h u\|^q_{\mathrm{L}^q(\partial\Omega)} + |u|^q_{\mathrm{W}^{1,p}(\mathcal{T}_h)}.
\end{aligned}$$

The trace inequality (5.3) is obtained by employing the trace theorem (see for instance Theorem 6.4.1 in [14]) for $Q_h u$, the continuity property of Q_h and the estimate (4.11) of Theorem 8. ■

6 Compactness in $\mathrm{W}^{1,p}(\mathcal{T}_h)$

We are finally in a position to give the first main results of this work. As we have already mentioned in Section 3, the lifting operator defined in (3.4) provides a bulk representation of the jump contribution to the gradients. We first analyze the main features of the lifting operator. The right-hand side in (3.4) is an inner product on a finite-dimensional space (cf. also Lemma 22) while the left-hand side, for $u \in \mathrm{W}^{1,p}(\mathcal{T}_h)^m$ fixed, is a linear functional on $S^k(\mathcal{T}_h)^{m\times n}$ and hence R is well-defined.

Lemma 12 *Let $p \in [1,\infty)$. There exists a constant C_R such that*

$$\|R(u)\|_{\mathrm{L}^p(\Omega)} \leq C_R \left(\int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s \right)^{1/p} \qquad \forall u \in \mathrm{W}^{1,p}(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof For each $u \in \mathrm{W}^{1,p}(\mathcal{T}_h)$ and for each $\varphi \in S^l(\mathcal{T}_h)^n$ we have

$$\begin{aligned}
\int_{\Gamma_{\mathrm{int}}} [\![u]\!] \cdot \{\varphi\} \,\mathrm{d}s &\leq \int_{\Gamma_{\mathrm{int}}} \big|h^{-1/p'} [\![u]\!]\big| \, \big|h^{1/p'} \{\varphi\}\big| \,\mathrm{d}s \\
&\leq \Big(\int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s \Big)^{1/p} \Big(\frac{1}{2} \int_{\Gamma_{\mathrm{int}}} h \big(|\varphi^+| + |\varphi^-|\big)^{p'} \,\mathrm{d}s \Big)^{1/p'}.
\end{aligned}$$

We can further bound the second term in the last estimate by

$$\begin{aligned}\int_{\Gamma_{\rm int}} h\big(|\varphi^+| + |\varphi^-|\big)^{p'} \,\mathrm{d}s &\le 2^{p'-1} \int_{\Gamma_{\rm int}} h\big(|\varphi^+|^{p'} + |\varphi^-|^{p'}\big) \,\mathrm{d}s \\ &\lesssim \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} h|\varphi|^{p'} \,\mathrm{d}s \\ &\lesssim \sum_{\kappa\in\mathcal{T}_h} \int_{\kappa} |\varphi|^{p'} \,\mathrm{d}x.\end{aligned}$$

Thus, we have shown that

$$\begin{aligned}\int_{\Gamma_{\rm int}} [\![u]\!] \cdot \{\varphi\} \,\mathrm{d}s &\le C\Big(\int_{\Gamma_{\rm int}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s\Big)^{1/p} \|\varphi\|_{\mathrm{L}^{p'}(\Omega)} \qquad (6.1)\\ &\qquad \forall u \in \mathrm{W}^{1,p}(\mathcal{T}_h) \quad \forall \varphi \in S^l(\mathcal{T}_h) \quad \forall p \in [1,\infty),\end{aligned}$$

where C depends only on the mesh quality and on p. Using the inf-sup condition of Lemma 22 in the Appendix 9.2, we obtain the result. ■

Theorem 13 (Compactness in $\mathrm{W}^{1,p}(\mathcal{T}_h)$) *Let $p \in (1,\infty)$. For each $h \in (0,1]$ let $u_h \in \mathrm{W}^{1,p}(\mathcal{T}_h)$ be such that*

$$\sup_{h\in(0,1]} \big[\|u_h\|_{\mathrm{L}^1(\Omega)} + |u_h|_{\mathrm{W}^{1,p}(\mathcal{T}_h)}\big] < +\infty. \qquad (6.2)$$

Then, there exists a sequence $h_j \downarrow 0$ and a function $u \in \mathrm{W}^{1,p}(\Omega)$ such that

$$\begin{aligned} u_{h_j} &\overset{*}{\rightharpoonup} u \qquad \text{in } \mathrm{BV}(\Omega), \text{ and}\\ \nabla u_{h_j} + R(u_{h_j}) &\rightharpoonup \nabla u \qquad \text{in } \mathrm{L}^p(\Omega).\end{aligned}$$

Proof From Lemma 3 it follows that $\|u_h\|_{\mathrm{BV}}$ is bounded. Hence, there exists a subsequence (which is not relabelled for notational convenience) and a function $u \in \mathrm{BV}(\Omega)$ such that $u_h \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$. Using the boundedness of the penalty term and applying Lemma 12 we also see that ∇u_h and $R(u_h)$ are bounded in L^p which implies their weak compactness. Upon extracting a further subsequence (again not relabelled), we obtain

$$\nabla u_h \rightharpoonup F_a \quad \text{and} \quad R(u_h) \rightharpoonup F_j,$$

as $h \to 0$, where $F_a, F_j \in \mathrm{L}^p(\Omega)^n$. We show now that Du_h converges to $F_a + F_j$ in the sense of distributions. Since $\nabla u_h \rightharpoonup F_a$, we only need to show that the jumps generate F_j in the limit, i.e., that

$$\int_{\Gamma_{\rm int}} [\![u_h]\!] \cdot \varphi \,\mathrm{d}s \to \int_\Omega F_j \cdot \varphi \,\mathrm{d}x \qquad \forall \varphi \in \mathrm{C}^1_{\rm c}(\Omega)^n. \qquad (6.3)$$

To this end, we add and subtract a function $\varphi_h \in S^l(\mathcal{T}_h)^n$, then use the definition of $R(u_h)$ and subtract φ again. This procedure gives

$$\begin{aligned}\int_{\Gamma_{\text{int}}} [\![u_h]\!] \cdot \varphi \, \mathrm{d}s &= \int_{\Gamma_{\text{int}}} [\![u_h]\!] \cdot \{\varphi - \varphi_h\} \, \mathrm{d}s + \int_{\Gamma_{\text{int}}} [\![u_h]\!] \cdot \{\varphi_h\} \, \mathrm{d}s \\ &= \int_{\Gamma_{\text{int}}} [\![u_h]\!] \cdot \{\varphi - \varphi_h\} \, \mathrm{d}s + \int_{\Omega} R(u_h) \cdot \varphi_h \, \mathrm{d}x \\ &= \int_{\Gamma_{\text{int}}} [\![u_h]\!] \cdot \{\varphi - \varphi_h\} \, \mathrm{d}s + \int_{\Omega} R(u_h) \cdot (\varphi_h - \varphi) \, \mathrm{d}x \\ &\quad + \int_{\Omega} R(u_h) \cdot \varphi \, \mathrm{d}x.\end{aligned}$$

Using Lemma 12 it follows immediately that, if we choose φ_h in such a way that $\|\varphi - \varphi_h\|_{\mathrm{L}^\infty} \to 0$ then the first and second term tend to zero as $h \to 0$. Since $R(u_h)$ converges weakly to F_j, it follows that Du_h converges to $F_a + F_j$ in the sense of distributions. Since Du_h converges also to Du in the sense of distribution, it follows that $Du = F_a + F_j \in \mathrm{L}^p(\Omega)^n$ in the space of distributions. Therefore, the singular part of Du is zero, whereby u has weak derivative $\nabla u = F_a + F_j \in \mathrm{L}^p(\Omega)^n$. Poincaré's inequality implies that $u \in \mathrm{W}^{1,p}(\Omega)$. ■

Lemma 14 (Compact Embeddings) *Under the conditions of Theorem 13 it also holds that*

$$u_{h_j} \to u \qquad \text{in } \mathrm{L}^q(\Omega) \qquad \forall\, q \;:\; 1 \le q < p^*, \quad \text{and} \tag{6.4}$$

$$u_{h_j} \to u \qquad \text{in } \mathrm{L}^q(\partial\Omega) \qquad \forall\, q \;:\; 1 \le q < q^*, \tag{6.5}$$

where $q^* = (n-1)p/(n-p)$ *if* $p < n$ *and* $q^* = \infty$ *if* $p \ge n$.

Proof For the proof of strong $\mathrm{L}^q(\Omega)$ convergence it suffices to use the compactness of the embedding $\mathrm{BV}(\Omega) \subset \mathrm{L}^1(\Omega)$ and use an interpolation theorem to lift the strong convergence to the L^q spaces indicated.

Unfortunately, the trace operator presented in Theorem 9 is not compact and thus, we must revert to using the continuous reconstruction operator Q_h to prove the second result.

From (4.8) is follows that, for each face $e \subset \partial\Omega \cap \mathcal{E}_h$,

$$\|u_h - Q_h u_h\|^q_{\mathrm{L}^q(e)} \lesssim h_e^{n-1-\frac{nq}{p}+q} |u|^q_{\mathrm{W}^{1,p}(\mathcal{T}_h \cap T_e)}. \tag{6.6}$$

We prove (6.5) only for $q \in [p, (n-1)p/(n-p))$, the other cases being an immediate consequence of the statement for e.g., $q = p$. Set $\alpha = n - 1 - nq/p + q > 0$. Summing (6.6) over the faces on the boundary, we obtain:

$$\|u_h - Q_h u_h\|^q_{\mathrm{L}^q(\partial\Omega)} \lesssim h^\alpha \sum_{e \subset \partial\Omega} |u|^q_{\mathrm{W}^{1,p}(\mathcal{T}_h \cap T_e)}.$$

Since $q \geq p$ we can use $\|\cdot\|_{\ell^q} \leq \|\cdot\|_{\mathrm{L}^p}$, and Assumption 1b, to deduce that

$$\begin{aligned}\|u_h - Q_h u_h\|^q_{\mathrm{L}^q(\partial\Omega)} &\lesssim h^\alpha \sum_{e\subset\partial\Omega} |u|^q_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_e)} \\ &\lesssim h^\alpha \left(\sum_{e\subset\partial\Omega} |u|^p_{\mathrm{W}^{1,p}(\mathcal{T}_h\cap T_e)}\right)^{q/p} \lesssim h^\alpha |u|^q_{\mathrm{W}^{1,p}(\mathcal{T}_h)}.\end{aligned}$$

This implies that

$$\|u_h - Q_h u_h\|_{\mathrm{L}^q(\partial\Omega)} \to 0 \qquad \text{as } h \to 0. \tag{6.7}$$

Since the trace operator from $\mathrm{W}^{1,p}(\Omega)$ to $\mathrm{L}^q(\Omega)$ is compact (see Theorem 6.3 in [2]) and $Q_h u_h$ is bounded in $\mathrm{W}^{1,p}(\Omega)$, it follows that $Q_h u_h \to u$ in $\mathrm{L}^q(\partial\Omega)$ and therefore, by virtue of (6.7), $u_h \to u$ in $\mathrm{L}^q(\partial\Omega)$. ■

7 Convergence for VIP-DGFEM

We begin by studying the coercivity properties of the functional (3.5) as a discretization of (3.3).

Lemma 15 (Coercivity) *Suppose that the potentials f and g satisfy respectively* (3.1) *and* (3.2). *Then there exists a constant C, independent of h such that*

$$|u|^p_{\mathrm{W}^{1,p}_D(\mathcal{T}_h)} \leq C\left(\mathcal{I}_h(u) + 1\right) \qquad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof By the growth hypotheses (3.1) and (3.2) and the Trace Theorem 11, we have

$$\begin{aligned}\mathcal{I}_h(u) \;\geq\;& c_0\Big(\|\nabla u + R(u_h)\|^p_{\mathrm{L}^p(\Omega)} - \|u\|^r_{\mathrm{L}^r(\Omega)} - \|a_1\|_{\mathrm{L}^1(\Omega)}\Big) \\ &- c_2\Big(\|u\|^r_{\mathrm{L}^r(\Omega)} + |u|^r_{\mathrm{W}^{1,r}(\mathcal{T}_h)} + \|a_2\|_{\mathrm{L}^1(\Gamma_N)}\Big) \\ &+ \int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s + \int_{\Gamma_D} h^{1-p}|u - u_D|^p \,\mathrm{d}s.\end{aligned}$$

Since $r < p$, we can estimate

$$\|u\|^r_{\mathrm{L}^r(\Omega)} \lesssim \|u\|^r_{\mathrm{L}^p(\Omega)} \leq \frac{\varepsilon}{p/r}\|u\|^p_{\mathrm{L}^p(\Omega)} + \frac{1}{\varepsilon(p/r)'} \leq C(\varepsilon^{-1} + \varepsilon\|u\|^p_{\mathrm{L}^p(\Omega)}).$$

Treating the term $|u|^r_{\mathrm{W}^{1,r}(\mathcal{T}_h)}$ in a similar fashion, we obtain

$$\begin{aligned}\mathcal{I}_h(u) + C(\varepsilon) \;\geq\;& c_0\Big(\|\nabla u + R(u_h)\|^p_{\mathrm{L}^p(\Omega)} - \varepsilon\|u\|^p_{\mathrm{L}^p(\Omega)} - \varepsilon|u|^p_{\mathrm{W}^{1,p}(\mathcal{T}_h)}\Big) \\ &+ \int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s + \int_{\Gamma_D} h^{1-p}|u - u_D|^p \,\mathrm{d}s.\end{aligned}$$

An application of the broken Friedrichs inequality 10A gives

$$\|\nabla u + R(u)\|^p_{\mathrm{L}^p(\Omega)} - \varepsilon\|\nabla u\|^p_{\mathrm{L}^p(\Omega)} + \int_{\Gamma_{\mathrm{int}}} h^{1-p}|[\![u]\!]|^p \,\mathrm{d}s + \int_{\Gamma_D} h^{1-p}|u - u_D|^p \,\mathrm{d}s \lesssim \mathcal{I}_h(u) + C(\varepsilon).$$

The first term on the left-hand side can be multiplied by a factor $\delta \in (0,1)$ without changing the validity of the estimate. Thus, for $\delta \in (0,1)$, we have

$$\delta\|\nabla u + R(u)\|^p_{\mathrm{L}^p(\Omega)} - \varepsilon\|\nabla u\|^p_{\mathrm{L}^p(\Omega)} + \int_{\Gamma_{\mathrm{int}}} h^{1-p}|[\![u]\!]|^p \,\mathrm{d}s + \int_{\Gamma_D} h^{1-p}|u-u_D|^p \,\mathrm{d}s \lesssim \mathcal{I}_h(u) + C(\varepsilon).$$

Using an inverse triangle inequality, we have

$$\|\nabla u + R(u)\|^p_{\mathrm{L}^p(\Omega)} \geq 2^{1-p}\|\nabla u\|^p_{\mathrm{L}^p} - \|R(u)\|^p_{\mathrm{L}^p(\Omega)}.$$

If δ is sufficiently small then the penalty integral dominates $\delta\|R(u)\|^p_{\mathrm{L}^p(\Omega)}$. Furthermore, setting $\varepsilon = \frac{1}{2}\delta 2^{1-p}$, we obtain

$$\|\nabla u\|^p_{\mathrm{L}^p(\Omega)} + \int_{\Gamma_{\mathrm{int}}} h^{1-p}|[\![u]\!]|^p \,\mathrm{d}s + \int_{\Gamma_D} h^{1-p}|u-u_D|^p \,\mathrm{d}s \lesssim \mathcal{I}_h(u) + C(\varepsilon),$$

which is the required bound. ■

Lemma 15 together with Theorem 13 establishes the compactness of any family of DGFEM functions u_h for which $\mathcal{I}_h(u_h)$ is bounded. This allows us to use a direct method related technique (namely Γ-convergence; see [10, 9]) to prove the convergence of discrete minimizers to a minimizer of $\mathcal{I}$ in $\mathcal{A}$.

Theorem 16 (Convergence) *Suppose that f and g are Carathéodory functions which respectively satisfy* (3.1) *and* (3.2) *and that f is convex in its third argument.*

For each $h \in (0,1]$, let $u_h \in \operatorname{argmin}_{S^k(\mathcal{T}_h)}\mathcal{I}_h$. Then, there exists a subsequence $h_j \downarrow 0$ and $u \in \mathcal{A}$ such that $u_{h_j} \overset{*}{\rightharpoonup} u$. Any such accumulation point u is a minimizer of $\mathcal{I}$ in $\mathcal{A}$ and we have, as $j \to \infty$,

$$u_{h_j} \to u \quad \text{in } \mathrm{L}^q(\Omega) \quad \forall q < p^*, \tag{7.1}$$

$$\nabla u_{h_j} \rightharpoonup \nabla u \quad \text{in } \mathrm{L}^p(\Omega), \tag{7.2}$$

$$\mathcal{I}_{h_j}(u_{h_j}) \to \mathcal{I}(u) \quad \text{and} \tag{7.3}$$

$$\int_{\Gamma_D} h^{1-p}|u_{h_j} - u_D|^p \,\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} h^{1-p}|[u_{h_j}]|^p \,\mathrm{d}s \to 0. \tag{7.4}$$

If f is strictly convex in its third argument then, in addition,

$$|u - u_{h_j}|_{\mathrm{W}^{1,p}(\mathcal{T}_h)} \to 0 \qquad \text{as } j \to \infty.$$

If the minimizer is unique, then the entire family u_h converges.

Proof By the growth condition (3.1) any family (u_h) which is bounded in $\mathrm{W}^{1,p}(\mathcal{T}_h)$ has bounded energy $\mathcal{I}_h(u_h)$ and conversely, by Lemma 15, if $\mathcal{I}_h(u_h)$ is bounded then $|u_h|_{\mathrm{W}^{1,p}_D(\mathcal{T}_h)}$ is uniformly bounded in h.

From the broken Friedrichs inequality, Corollary 10A, and the compactness result, Theorem 13, we therefore deduce the existence of a subsequence $h_j \downarrow 0$ and of a limit point $u \in \mathrm{W}^{1,p}(\Omega)^m$ such that $\nabla u_{h_j} + R(u_{h_j}) \rightharpoonup \nabla u$ weakly in $\mathrm{L}^p(\Omega)^{m\times n}$ and $u_{h_j} \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)^m$. Lemma 14 implies (7.1).

Assume now that (u_{h_j}) is any minimizing sequence for $\mathcal{I}_{h_j}$ converging weakly-$*$ to some $u \in \mathrm{BV}(\Omega)^m$. From the boundedness of $|u_{h_j}|_{\mathrm{W}^{1,p}(\mathcal{T}_{h_j})}$ and a standard compactness and uniqueness argument, using Theorem 13, we deduce again that $u \in \mathrm{W}^{1,p}(\Omega)$ and that $\nabla u_{h_j} + R(u_{h_j}) \rightharpoonup \nabla u$ in L^p. Since the boundary penalty terms,

$$\int_{\Gamma_D} h^{1-p} |u_h - u_D|^p \,\mathrm{d}s$$

are bounded, using also Lemma 14, it follows that

$$\|u - u_D\|_{\mathrm{L}^p(\Gamma_\mathrm{D})} \;\leq\; \|u - u_{h_j}\|_{\mathrm{L}^p(\Gamma_\mathrm{D})} + \|u_{h_j} - u_D\|_{\mathrm{L}^p(\Gamma_\mathrm{D})} \to 0$$

as $j \to \infty$ and hence $u \in \mathcal{A}$.

Lemma 14 also implies the strong convergence of u_{h_j} to u in $\mathrm{L}^r(\partial\Omega)$, and therefore, it follows from Lemma 4 (ii) that the surface integral converges, i.e.,

$$\int_{\Gamma_\mathrm{N}} g(x, u_{h_j}) \,\mathrm{d}s \to \int_{\Gamma_\mathrm{N}} g(x, u) \,\mathrm{d}s \qquad \text{as } j \to \infty.$$

As a consequence, using Lemma 4 (iii), we deduce that

$$\mathcal{I}(u) \leq \liminf_{j\to\infty} \left[\int_\Omega f(x, u_{h_j}, \nabla u_{h_j} + R(u_{h_j})) \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g(x, u_{h_j}) \,\mathrm{d}s \right].$$

To see that $u \in \mathrm{argmin}_{\mathcal{A}} \mathcal{I}$, fix $v \in \mathcal{A}$ and let $v_h \in S^k(\mathcal{T}_h)$ converge strongly to v in the $\|\cdot\|_{\mathrm{L}^p(\Omega)}$ as well as the $|\cdot|_{\mathrm{W}^{1,p}(\mathcal{T}_h)}$-norm. The existence of such a sequence follows from standard approximation theory. From Lemma 4 (using also the Trace Theorem 11) we therefore obtain $\mathcal{I}_h(v_h) \to \mathcal{I}(v)$, which allows us to estimate

$$\begin{aligned} \mathcal{I}(u) \;&\leq\; \liminf_{j\to\infty} \left[\int_\Omega f(x, u_{h_j}, \nabla u_{h_j} + R(u_{h_j})) \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g(x, u_{h_j}) \,\mathrm{d}s \right] \\ &\leq\; \limsup_{j\to\infty} \mathcal{I}_{h_j}(u_{h_j}) \;\leq\; \limsup_{j\to\infty} \mathcal{I}_{h_j}(v_{h_j}) \;\leq\; \mathcal{I}(v). \end{aligned}$$

Since v was arbitrary it follows that $\mathcal{I}(u) \in \mathrm{argmin}_{\mathcal{A}} \mathcal{I}$. By choosing $v = u$ we find that all inequalities are equalities from which we can infer that $\mathcal{I}_h(u_h) \to \mathcal{I}(u)$ and that the penalty terms converge to zero as $h_j \to 0$, i.e. that (7.4) holds. As a consequence we also have $R(u_h) \to 0$ strongly which implies (7.2).

If f is strictly convex in its third argument then the theory of Young measures (see, for instance, [21]) shows that weak convergence together with convergence of the energy implies strong convergence.

The last point follows from a standard and straightforward uniqueness argument.
■

8 Optimal embedding constants

In this final section, we present an interesting application of the compactness results of Section 6. Namely, we shall deduce that, in the limit as $h \to 0$, the optimal "embedding constant" in the Sobolev–Poincaré inequality (8) is the same as the embedding constant for the classical Sobolev space. We demonstrate the technique only on the example of the Sobolev–Poincaré inequality, but we believe that it should apply to any compact embedding of a Sobolev space.

Unfortunately, our results are incomplete for the particular broken semi-norm which we have chosen. We therefore present several equivalent norms and discuss the broken embedding constant in each case. First, we redefine $|\cdot|_{\mathrm{W}^{1,p}(\mathcal{T}_h)}$, by adding a penalty parameter,

$$|u|_{\mathrm{W}^{1,p}(\mathcal{T}_h)} = \Big(\|\nabla u\|_{\mathrm{L}^p(\Omega)}^p + \alpha \int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s\Big)^{1/p}. \tag{8.1}$$

In addition, we define the semi-norms

$$\begin{aligned}
|u|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)} &= \|\nabla u\|_{\mathrm{L}^p(\Omega)} + \alpha_1 \Big(\int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s\Big)^{1/p}, \\
|u|_{\mathrm{W}_R^{1,p}(\mathcal{T}_h)} &= \Big(\|\nabla u + R(u)\|_{\mathrm{L}^p(\Omega)}^p + \alpha_R \int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s\Big)^{1/p}, \quad \text{and} \\
|u|_{\mathrm{W}_{1,R}^{1,p}(\mathcal{T}_h)} &= \|\nabla u + R(u)\|_{\mathrm{L}^p(\Omega)} + \alpha_{1,R} \Big(\int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u]\!]|^p \,\mathrm{d}s\Big)^{1/p}
\end{aligned} \tag{8.2}$$

Proposition 17 *If the constants $\alpha, \alpha_1, \alpha_R, \alpha_{1,R}$ are all positive then the semi-norms defined in* (8.1) *and* (8.2) *are equivalent on the spaces $S^k(\mathcal{T}_h)$, with constants that are independent of h.*

Proof The uniform equivalences of $|\cdot|_{\mathrm{W}^{1,p}(\mathcal{T}_h)}$ and $|\cdot|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)}$ and that of $|\cdot|_{\mathrm{W}_R^{1,p}(\mathcal{T}_h)}$ and $|\cdot|_{\mathrm{W}_{1,R}^{1,p}(\mathcal{T}_h)}$ follow from norm-equivalence in $\mathbb{R}^2$. We now show that also $|\cdot|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)}$ and $|\cdot|_{\mathrm{W}_{1,R}^{1,p}(\mathcal{T}_h)}$ are equivalent. Clearly, for $u \in S^k(\mathcal{T}_h)$, it holds that

$$|u|_{\mathrm{W}_{1,R}^{1,p}(\mathcal{T}_h)} \le \|\nabla u\|_{\mathrm{L}^p(\Omega)} + \|R(u)\|_{\mathrm{L}^p(\Omega)} + \alpha_{1,R} \Big(\int_{\Gamma_{\mathrm{int}}} h^{1,p} |[\![u]\!]|^p \,\mathrm{d}s\Big)^{1/p}.$$

Using Lemma 12 this implies that $|\cdot|_{\mathrm{W}_{1,R}^{1,p}(\mathcal{T}_h)} \lesssim |\cdot|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)}$ with a constant which is independent of h.

To see that the reverse holds as well, assume for contradiction that $h_j \in (0,1]$ and that $u_j \in S^k(\mathcal{T}_{h_j})$, $j = 1, 2, \dots$, such that $|u_j|_{\mathrm{W}_1^{1,p}(\mathcal{T}_{h_j})} = 1$ and $|u_j|_{\mathrm{W}_{1,R}^{1,p}(\mathcal{T}_{h_j})} \to 0$ as $j \to \infty$. This implies that

$$\int_{\Gamma_{\mathrm{int}}} h^{1-p} |[\![u_j]\!]|^p \,\mathrm{d}s \to 0$$

which, by Lemma 12, implies that $\|R(u_{h_j})\|_{\mathrm{L}^p(\Omega)} \to 0$ and therefore also $\|\nabla u_j\|_{\mathrm{L}^p(\Omega)} \to 0$ as $j \to \infty$. However, this means that $|u_j|_{\mathrm{W}_1^{1,p}(\mathcal{T}_{h_j})} \to 0$ which contradicts the assumption. ■

We can now study the Poincaré constants of the various broken semi-norms. Let $p \in (1,\infty)$, $q \in [1,p^*)$ and let $V = \{v \in \mathrm{L}^1(\Omega) : (v)_\Omega = 0\}$. We begin by noting that the optimal constant $C(p,q)$ for the Sobolev–Poincaré inequality $\|u\|_{\mathrm{L}^q(\Omega)} \le C(p,q)\|\nabla u\|_{\mathrm{L}^p(\Omega)}$ for $u \in \mathrm{W}^{1,p}(\Omega) \cap V$ is given by

$$(C(p,q))^{-1} = \inf_{\substack{u \in \mathrm{W}^{1,p}(\Omega)\setminus V \\ \|u\|_{\mathrm{L}^q(\Omega)}=1}} \|\nabla u\|_{\mathrm{L}^p(\Omega)}.$$

Similarly, we can define the constants $C_h(p,q)$, $C_h^q(p,q)$, $C_h^R(p,q)$ and $C_h^{1,R}(p,q)$ upon replacing $\mathrm{W}^{1,p}(\Omega)$ by $S^k(\mathcal{T}_h)$ and $\|\nabla u\|_{\mathrm{L}^p(\Omega)}$ by the respective seminorms.

Proposition 18 *If $p \in (1,\infty)$ and $q \in [1,p^*)$, then $C_h^R(p,q) \to C(p,q)$ and $C_h^{1,R}(p,q) \to C(p,q)$ as $h \to 0$.*

Proof The technique of proof is the same as the proof of Theorem 16. We only note that, due to Lemma 14, any accumulation point of a family $(u_h)_{h\in(0,1]}$ satisfying $\|u_h\|_{\mathrm{L}^q(\Omega)} = 1$ must also have unit L^q-norm. ■

The case of the seminorms $|\cdot|_{\mathrm{W}^{1,p}(\mathcal{T}_h)}$ and $|\cdot|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)}$ is more interesting since the asymptotic behaviour of the Poincaré constants depends on the magnitude of the constants α and α_1.

Proposition 19 *There exists a constant $\hat\alpha_1 > 0$ such that*

$$\begin{cases} \lim_{h\downarrow 0} C_h^1(p,q) = C(p,q), & \text{if } \hat\alpha_1 \ge \alpha_1, \\ \liminf_{h\downarrow 0} C_h^1(p,q) > C(p,q), & \text{if } 0 < \alpha_1 < \hat\alpha_1. \end{cases}$$

Proof We begin by investigating the case where α_1 is large. Suppose that $u_h \in S^k(\mathcal{T}_h) \cap V$, $h \in (0,1]$, that $\|u_h\|_{\mathrm{L}^q(\Omega)} = 1$ and that $|u_h|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)} = C_h^1(p,q)^{-1}$. By standard approximation results and norm-equivalence it follows that $|u_h|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)}$ is bounded and hence we can extract a subsequence u_{h_j} converging weakly-$*$ in $\mathrm{BV}(\Omega)$ and strongly in $\mathrm{L}^q(\Omega)$ to a function $u \in \mathrm{W}^{1,p}(\Omega)$. In particular, $\|u\|_{\mathrm{L}^q(\Omega)} = 1$ and we have

$$\begin{aligned} \|\nabla u\|_{\mathrm{L}^p(\Omega)} &\le \liminf_{j\to\infty} \|\nabla u_{h_j} + R(u_{h_j})\|_{\mathrm{L}^p(\Omega)} \\ &\le \liminf_{j\to\infty} \Big(\|\nabla u_{h_j}\|_{\mathrm{L}^p(\Omega)} + \|R(u_{h_j})\|_{\mathrm{L}^p(\Omega)} \Big). \end{aligned}$$

If α_1 is sufficiently large it follows from Lemma 12 that

$$\|\nabla u\|_{\mathrm{L}^p(\Omega)} \le \liminf_{j\to\infty} |u_{h_j}|_{\mathrm{W}_1^{1,p}(\mathcal{T}_{h_j})}$$

which implies that $\liminf_{h\downarrow 0} C_h^1(p,q)^{-1} \ge C(p,q)^{-1}$. By standard approximation results we therefore obtain $\lim_{h\downarrow 0} C_h^1(p,q) = C(p,q)$.

Now assume that α_1 is small. Let $u \in \mathrm{W}^{1,p}(\Omega) \cap V$ such that $\|u\|_{\mathrm{L}^q(\Omega)} = 1$ and such that $\|\nabla u\|_{\mathrm{L}^p(\Omega)} = C(p,q)^{-1}$. For each $h \in (0,1]$ let u_h be defined by

$$u_h(x) = (u)_\kappa \qquad \forall x \in \kappa \qquad \forall \kappa \in \mathcal{T}_h.$$

Clearly, $u_h \in S^k(\mathcal{T}_h) \cap V$ and $\|u_h - u\|_{\mathrm{L}^q(\Omega)} \to 0$ as $h \downarrow 0$. Furthermore, we can bound the seminorm $|u_h|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)}$ in terms of $\|\nabla u\|_{\mathrm{L}^p(\Omega)}$ as follows.

$$\begin{aligned} \alpha_1^{-p} |u_h|^p_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)} &= \sum_{e \subset \Gamma_{\mathrm{int}}} h_e^{1-p} \mathcal{H}^{n-1}(e) |(u)_\kappa - (u)_{\kappa'}|^p \\ &\lesssim \sum_{e \subset \Gamma_{\mathrm{int}}} h_e^{n-p} \big[|(u)_\kappa - \pi| + |(u)_{\kappa'} - \pi| \big]^p, \end{aligned} \tag{8.3}$$

where $\pi \in \mathbb{R}$.

We construct π in a similar fashion as the local projection operators in Section 4.1. Let z be the center of the ball given in Assumption 1b; then $K = \kappa \cup \kappa'$ is star-shaped with respect to z the mesh-regularity assumptions imply the existence of a ball $B = B(z,\rho) \subset K$ such that $\rho \approx h_\kappa \approx h'_\kappa$. Hence, we can set $\pi = (u)_B$ and use Lemma 5 to deduce that

$$\begin{aligned} |(u)_\kappa - \pi| + |(u)_{\kappa'} - \pi| &\le h_\kappa^{-n} \|u - \pi\|_{\mathrm{L}^1(\kappa)} + h_{\kappa'}^{-n} \|u - \pi\|_{\mathrm{L}^1(\kappa')} \\ &\lesssim h_e^{-n+1} \|\nabla u\|_{\mathrm{L}^1(K)}. \end{aligned}$$

Upon taking p-th powers, and applying Jensen's inequality, we obtain

$$\big[|(u)_\kappa - \pi| + |(u)_{\kappa'} - \pi| \big]^p \lesssim h_e^p h_e^{-np} \|\nabla u\|^p_{\mathrm{L}^1(K)} \lesssim h_e^{p-n} \|\nabla u\|^p_{\mathrm{L}^p(K)}.$$

Combined with (8.3) and the contact regularity assumptions, this gives

$$\alpha_1^{-p} |u_h|^p_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)} \lesssim \|\nabla u\|^p_{\mathrm{L}^p(\Omega)}.$$

In summary, we have obtained that there exists a constant $\tilde{\alpha}$ which is independent of h such that,

$$\alpha_1^{-1} |u_h|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)} \le \tilde{\alpha} C(p,q)^{-1}.$$

Hence, for $\alpha_1 < 1/\tilde{\alpha}$ it follows that

$$|u_h|_{\mathrm{W}^{1,p}(\mathcal{T}_h)} \le \alpha_1 \tilde{\alpha} C(p,q)^{-1} < C(p,q)^{-1}.$$

as a consequence, we obtain that $\liminf_{h \downarrow 0} C_h^1(p,q) > C(p,q)$.

Finally, we note that if the latter property holds for a specific $\alpha_1 = \alpha_1'$ then it also holds for all $\alpha_1 < \alpha_1'$ and hence the proposition follows. ■

Remark 20 *We conclude our analysis of optimal Sobolev–Poincaré imbedding constants with a remark on the seminorm* $|\cdot|_{\mathrm{W}^{1,p}(\mathcal{T}_h)}$. *We can obviously use the construction of a "recovery sequence" for the* $|\cdot|_{\mathrm{W}_1^{1,p}(\mathcal{T}_h)}$*-seminorm to deduce that, if* α *is sufficiently small, then* $\liminf_{h \downarrow 0} C_h(p,q) > C(p,q)$. *However, we have a* *gap for large α. From the corresponding result for $C_h^1(p,q)$ it is easy to deduce from the following variation of Minkovski's inequality,*

$$(|a| + |b|)^p \le (1+\varepsilon)|a|^p + B_\varepsilon |b|^p,$$

where B_ε depends only on ε and on p, that

$$\lim_{\alpha\to\infty} \limsup_{h\downarrow 0} C_h(p,q) = C(p,q).$$

However, we are unable to prove that $\limsup_{h\downarrow 0} C_h(p,q) = C(p,q)$ for any sufficiently large (but fixed) α. In fact, our numerical experiments in one dimension suggest that this is not the case.

9 Appendix

9.1 Proof of Lemma 5

This proof is a modification of the proof of [23, Lemma 4.1].

Using the local approximation of BV functions by smooth functions (cf. [11, Sec. 5.2.2]), there exists a sequence $u_j \in \mathrm{BV}(K) \cap \mathrm{C}^\infty(K)$ such that $u_j \to u$ strictly in BV, i.e., $u_j \to u$ strongly in L^1 and $|Du_j|(K) \to |Du|(K)$ as $j \to \infty$. Hence, we can assume without loss of generality that $u \in \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,1}(\Omega)$.

We write

$$\|u\|_{\mathrm{L}^1(K)} = \|u\|_{\mathrm{L}^1(B)} + \|u\|_{\mathrm{L}^1(K\setminus B)}.$$

Let Σ be the unit sphere in $\mathbb{R}^n$ and, for each $\sigma \in \Sigma$, let $x_0 + r(\sigma)\sigma \in \partial K$. For the second term, we compute

$$\begin{aligned}
\|u\|_{\mathrm{L}^1(K\setminus B)} &= \int_\Sigma \int_\rho^{r(\sigma)} t^{n-1} |u(t\sigma)| \,\mathrm{d}t \,\mathrm{d}s(\sigma) \\
&\le \int_\Sigma \int_\rho^{r(\sigma)} t^{n-1} \big|u(t\sigma) - u(\rho\sigma)\big| \,\mathrm{d}t + \int_\Sigma \int_\rho^{r(\sigma)} t^{n-1} \big|u(\rho\sigma)\big| \,\mathrm{d}t \,\mathrm{d}s(\sigma) \\
&=: S_1 + S_2.
\end{aligned}$$

To obtain a bound on S_1, consider

$$\begin{aligned}
S_1 &= \int_\Sigma \int_\rho^{r(\sigma)} t^{n-1} \Big| \int_\rho^t \partial_r u(r\sigma) \,\mathrm{d}r \Big| \,\mathrm{d}t \,\mathrm{d}s(\sigma) \\
&\le \rho^{1-n} \int_\Sigma \int_\rho^{r(\sigma)} t^{n-1} \int_\rho^t r^{n-1} |\partial_r u(r\sigma)| \,\mathrm{d}r \,\mathrm{d}t \,\mathrm{d}s(\sigma) \\
&\le \frac{1}{n}\rho^{1-n}(h^n - \rho^n) \int_\Sigma \int_\rho^{r(\sigma)} r^{n-1} |\partial_r u(r\sigma)| \,\mathrm{d}r \,\mathrm{d}s(\sigma) \\
&\le \frac{\rho}{n}(\gamma^n - 1) \|\nabla u\|_{\mathrm{L}^1(K\setminus B)}.
\end{aligned}$$

For S_2, we estimate

$$\begin{aligned} S_2 &= \frac{1}{n}\int_\Sigma \big(r(\sigma)^n - \rho^n\big)|u(\rho\sigma)|\,\mathrm{d}s(\sigma) \\ &\le \frac{\rho}{n}\int_\Sigma \Big[\frac{h^n}{\rho^n} - 1\Big]\rho^{n-1}|u(\rho\sigma)|\,\mathrm{d}s(\sigma) \\ &= \frac{\rho}{n}(\gamma^n - 1)\int_\Sigma \rho^{n-1}|u(\rho\sigma)|\,\mathrm{d}s(\sigma) \\ &= \frac{\rho}{n}(\gamma^n - 1)\|u\|_{\mathrm{L}^1(\partial B)}. \end{aligned}$$

We bound $\|u\|_{\mathrm{L}^1(\partial B)}$ as follows:

$$\begin{aligned} \|u\|_{\mathrm{L}^1(\partial B)} &= \int_\Sigma \rho^{n-1}|u(\rho\sigma)|\,\mathrm{d}s(\sigma) \\ &= \int_\Sigma \rho^{n-1}\Big|\int_0^\rho \partial_r\Big[\Big(\frac{r}{\rho}\Big)^n u(r\sigma)\Big]\,\mathrm{d}r\Big|\,\mathrm{d}s(\sigma) \\ &= \int_\Sigma \rho^{n-1}\Big|\int_0^\rho \Big[\Big(\frac{r}{\rho}\Big)^n \partial_r u(r\sigma) + \frac{nr^{n-1}}{\rho^n}u(r\sigma)\Big]\,\mathrm{d}r\Big|\,\mathrm{d}s(\sigma) \\ &\le \int_\Sigma \rho^{-1}\int_0^\rho r^n\big|\partial_r u(r\sigma)\big|\,\mathrm{d}r\,\mathrm{d}s(\sigma) + n\int_\Sigma \rho^{-1}\int_0^\rho r^{n-1}|u(r\rho)|\,\mathrm{d}r\,\mathrm{d}s(\sigma) \\ &\le \|\nabla u\|_{\mathrm{L}^1(B)} + \frac{n}{\rho}\|u\|_{\mathrm{L}^1(B)}. \end{aligned}$$

Combining all our estimates, we obtain

$$\begin{aligned} \|u\|_{\mathrm{L}^1(K)} &\le \|u\|_{\mathrm{L}^1(B)} + \frac{\rho}{n}(\gamma^n - 1)\|\nabla u\|_{\mathrm{L}^1(K\setminus B)} \\ &\quad + \frac{\rho}{n}(\gamma^n - 1)\|\nabla u\|_{\mathrm{L}^1(B)} + (\gamma^n - 1)\|u\|_{\mathrm{L}^1(B)} \\ &= \gamma^n\|u\|_{\mathrm{L}^1(B)} + \frac{\rho}{n}(\gamma^n - 1)\|\nabla u\|_{\mathrm{L}^1(K)} \end{aligned}$$

which gives (4.3).

To obtain the second result, we note that the Poincaré inequality on balls takes the form (see [1], where this is proved for arbitrary convex sets)

$$\|u\|_{\mathrm{L}^1(B)} \le \rho\|\nabla u\|_{\mathrm{L}^1(B)} \qquad \forall u \in \mathrm{W}^{1,1}(B), (u)_B = 0. \tag{9.1}$$

Thus, (4.4) follows immediately from (4.3).

9.2 Auxiliary results

Lemma 21 *Let $(\mathcal{T}_h)_{h\in(0,1]}$ be a family of partitions of Ω satisfying Assumption 1. Then, for each $p, q \in [1,\infty]$, there exists a constant $C > 0$, independent of h, such that for any $\kappa \in \mathcal{T}_h$*

$$h_\kappa^{-\frac{n}{p}}\|v\|_{\mathrm{L}^p(\kappa)} \le C h_\kappa^{-\frac{n}{q}}\|v\|_{\mathrm{L}^q(\kappa)} \qquad \forall v \in S^k(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Moreover, for any $\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h$

$$h_{\widetilde{\kappa}}^{-\frac{n}{p}} \|v\|_{\mathrm{L}^p(\widetilde{\kappa})} \leq C h_{\widetilde{\kappa}}^{-\frac{n}{q}} \|v\|_{\mathrm{L}^q(\widetilde{\kappa})} \qquad \forall v \in S^1(\widetilde{\mathcal{T}}_h) + S^k(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof Let $\kappa \in \mathcal{T}_h$, $\hat{\kappa}$ its corresponding reference element and $F_\kappa : \hat{\kappa} \to \kappa$ the associated mapping. We set $J = |\det \nabla F_\kappa|$. Since F_κ is bi-Lipschitz we have $C^{-1} h_\kappa^n \leq J \leq C h_\kappa^n$ for some constant C which is independent of κ. From the area formula (cf. [11]), we have

$$\int_\kappa |u|^p \, \mathrm{d}x = \int_{\hat{\kappa}} J |u \circ F_\kappa|^p \, \mathrm{d}x \approx h_\kappa^n \int_{\hat{\kappa}} |u \circ F_\kappa|^p \, \mathrm{d}x.$$

Using norm-equivalence in finite-dimensional spaces, we obtain

$$\int_\kappa |u|^p \, \mathrm{d}x \approx h_\kappa^n \left(\int_{\hat{\kappa}} |u \circ F|^q \, \mathrm{d}x \right)^{p/q} \approx h^{n - np/q} \left(\int_\kappa |u|^q \, \mathrm{d}x \right)^{p/q}.$$

The first equivalence follows by taking the p-root.

The second equivalence is proved with the same technique, after noting that, given $v \in S^1(\widetilde{\mathcal{T}}_h) + S^k(\mathcal{T}_h)$ then $v|_{\widetilde{\kappa}}$ is a polynomial of degree k. Thus the previous reasoning applies. ■

Lemma 22 *Let $S^k(\mathcal{T}_h)$ be defined as in Section 2 and let the mesh-family satisfy Assumption 1. Then, there exists a constant C, independent of h, such that*

$$\inf_{u \in S^k(\mathcal{T}_h)} \sup_{v \in S^k(\mathcal{T}_h)} \frac{\int_\Omega uv \, \mathrm{d}x}{\|u\|_{\mathrm{L}^p(\Omega)} \|v\|_{\mathrm{L}^{p'}}} \geq C > 0.$$

Proof For a given $u \in \mathrm{L}^p(\Omega)$ set $v = |u|^{p-2} u$ so that $\int_\Omega uv = \|u\|_{\mathrm{L}^p(\Omega)} \|v\|_{\mathrm{L}^{p'}}$. At the discrete level, if $u \in S^k(\mathcal{T}_h)$, the choice $v = |u|^{p-2} u$ is not allowed, in general. Instead we set $v = \Pi_k(|u|^{p-2} u)$, where Π_k denotes the L^2-projection onto $S^k(\mathcal{T}_h)$ (note that this is a projection element by element) and therefore

$$\|\Pi_k u\|_{\mathrm{L}^2(\kappa)}^2 = \int_\kappa u \, \Pi_k u \, \mathrm{d}x \leq \|u\|_{\mathrm{L}^{p'}(\kappa)} \|\Pi_k u\|_{\mathrm{L}^p(\kappa)} \qquad \forall \kappa \in \mathcal{T}_h.$$

Using Lemma 21, we obtain

$$\|\Pi_k u\|_{\mathrm{L}^{p'}(\kappa)} \leq C_\Pi \|u\|_{\mathrm{L}^{p'}(\kappa)} \qquad \forall \kappa \in \mathcal{T}_h,$$

where C_Π is independent of h and κ. Moreover, by the definition of Π_k, it holds that $\int_\Omega u \, \Pi_k v \, \mathrm{d}x = \int_\Omega uv \, \mathrm{d}x$ for all $u \in S^k(\mathcal{T}_h)$. A possible value for the constant C in the statement is therefore given by $1/C_\Pi$. ■

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