Report no. 07/13

The Bramble-Pasciak$^+$ preconditioner for saddle point problems

Martin Stoll
Oxford University Computing Laboratory, Numerical Analysis Group, Wolfson Building, Parks Road, Oxford OX1 3QD, U.K.,
martin.stoll@comlab.ox.ac.uk

Andy Wathen
Oxford University Computing Laboratory, Numerical Analysis Group, Wolfson Building, Parks Road, Oxford OX1 3QD, U.K.,
andy.wathen@comlab.ox.ac.uk

The Bramble-Pasciak Conjugate Gradient method is a well known tool to solve linear systems in saddle point form. A drawback of this method in order to ensure applicability of Conjugate Gradients is the need for scaling the preconditioner which typically involves the solution of an eigenvalue problem. Here, we introduce a modified preconditioner and inner product which without scaling enable the use of a MINRES variant and can be used for the simplified Lanczos process. Furthermore, the modified preconditioner and inner product can be combined with the original Bramble-Pasciak setup to give new preconditioners and inner products. We emphasize the new methods by showing numerical experiments for Stokes problems.

Key words and phrases: Linear systems, Krylov subspaces, Non-standard inner products

Oxford University Computing Laboratory
Numerical Analysis Group
Wolfson Building
Parks Road
Oxford, England OX1 3QD

July, 2007

1 Introduction

In 1988 Bramble and Pasciak [5] introduced a special Preconditioned Conjugate Gradients (CG) method to solve the symmetric saddle point problem

$$\underbrace{\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}}_{\mathcal{A}} \; x = b \tag{1.1}$$

where A is symmetric positive definite and C is positive semi-definite. A block triangular preconditioner of the form

$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ B & -I \end{bmatrix} \text{ where } \quad \mathcal{P}^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ BA_0^{-1} & -I \end{bmatrix} \tag{1.2}$$

is used such that the original matrix $\mathcal{A}$ preconditioned by $\mathcal{P}$ becomes

$$\widehat{\mathcal{A}} = \mathcal{P}^{-1}\mathcal{A} = \begin{bmatrix} A_0^{-1}A & A_0^{-1}B^T \\ BA_0^{-1}A - B & BA_0^{-1}B^T + C \end{bmatrix}. \tag{1.3}$$

which is clearly non-symmetric. However, the block A_0 is assumed to be positive definite but further assumptions are in fact necessary. A well known result of Faber and Manteuffel in [10] states that there exist no optimal short-term recurrence methods for general non-symmetric matrices. Amazingly, Bramble and Pasciak introduced the inner product

$$\langle x, y\rangle_{\mathcal{H}} = x^T\mathcal{H}y \tag{1.4}$$

where

$$\mathcal{H} = \begin{bmatrix} A - A_0 & 0 \\ 0 & I \end{bmatrix} \tag{1.5}$$

and showed that A is self-adjoint (symmetric) in this inner product. In fact, A_0 has to be such that $A - A_0$ is positive definite and the Preconditioned CG method can be used, i.e. the matrix $\widehat{\mathcal{A}}$ is symmetric positive definite in the $\mathcal{H}$-inner product. The Bramble-Pasciak CG method is widely used in practice and very well analyzed, see [1–3, 6, 7, 15, 19, 20, 25, 31, 33]. Extensions to the classical Bramble Pasciak case are known and carefully discussed, see [19, 22, 28]; a Schur complement preconditioner S_0 is introduced which under certain conditions still guarantees the positive definiteness of the system, i.e.

$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ B & -S_0 \end{bmatrix} \text{ and } \quad \mathcal{P}^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ S_0^{-1}BA_0^{-1} & -S_0^{-1} \end{bmatrix}. \tag{1.6}$$

In the original paper [5] some conditions are imposed on the matrix A_0 and similar requirements have to be made for the extensions (cf. [19, 22, 28]). Usually an eigenvalue problem has to be solved in order to ensure that the preconditioned matrix $\widehat{\mathcal{A}}$ is symmetric and positive definite and

$$\langle x, y\rangle_{\mathcal{H}}$$

defines an inner product. The eigenvalue computation can be quite expensive and it is therefore desired to avoid this. Simoncini recently analyzed the conditions imposed on A_0 and how they can be relaxed, see [28]. Under the assumption that the matrix $A - A_0$ doesn't have to be positive definite different it is still possible to use efficient Krylov subspace solvers. One typical choice would be the ideal transpose-free QMR (ITFQMR) which is based on the simplified Lanczos method, see [14] and Section 3.3 in this paper. The important point is that less effective general non-symmetric iterative methods do not have to be used with this structure. In this paper we introduce a new preconditioner $\mathcal{P}^+$ which is similar to the one proposed by Bramble and Pasciak. We analyze its behaviour for three different Krylov subspace solvers CG [18], MINRES [23] and ITFQMR [11]: all of these methods have fixed work per iteration. Furthermore, we study the possibility of combination preconditioning recently proposed by Stoll and Wathen [30] for the classical and the $\mathcal{P}^+$ preconditioners and their corresponding inner products. Numerical Experiments will be shown in Section 5 using examples from the IFISS package, see [8].

2 The modified preconditioner

Considering the relation for self-adjointness in an $\mathcal{H}$-inner product

$$x^T \widehat{\mathcal{A}}^T \mathcal{H} y = \langle \mathcal{H}\widehat{\mathcal{A}}x, y\rangle = \langle \widehat{\mathcal{A}}x, y\rangle_{\mathcal{H}} = \langle x, \widehat{\mathcal{A}}y\rangle_{\mathcal{H}} = x^T \mathcal{H}\widehat{\mathcal{A}} y \quad \forall x, y. \tag{2.1}$$

The key condition is then seen to be $\widehat{\mathcal{A}}^T\mathcal{H} = \mathcal{H}\widehat{\mathcal{A}}$ and this has to be satisfied in order to be able to employ efficient algorithms, see [21] where the relation to optimal short-term recurrences is made. The original Bramble-Pasciak CG method fulfills this condition for

$$\widehat{\mathcal{A}} = \begin{bmatrix} A_0^{-1}A & A_0^{-1}B^T \\ BA_0^{-1}A - B & BA_0^{-1}B^T + C \end{bmatrix}$$

and

$$\mathcal{H} = \begin{bmatrix} A - A_0 & 0 \\ 0 & I \end{bmatrix}.$$

The obvious drawback of this method is the necessity to scale the matrix A_0 such that the matrix $\mathcal{H}$ is not only symmetric but also positive definite. Usually an eigenvalue problem for $A_0^{-1}A$ has to be solved which can be very costly, see [16] for a survey of methods that could be applied.

In contrast, we introduce the preconditioner

$$\mathcal{P}^+ = \begin{bmatrix} A_0 & 0 \\ -B & I \end{bmatrix} \text{ and } \quad \left(\mathcal{P}^+\right)^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ BA_0^{-1} & I \end{bmatrix} \tag{2.2}$$

and obtain by left preconditioning with $\mathcal{P}^+$

$$\widehat{\mathcal{A}} = \left(\mathcal{P}^+\right)^{-1}\mathcal{A} = \begin{bmatrix} A_0^{-1}A & A_0^{-1}B^T \\ BA_0^{-1}A + B & BA_0^{-1}B^T - C \end{bmatrix}. \tag{2.3}$$

Simple algebra shows that $\widehat{\mathcal{A}}$ is self-adjoint in the inner product induced by

$$\mathcal{H}^+ = \begin{bmatrix} A + A_0 & 0 \\ 0 & I \end{bmatrix}. \tag{2.4}$$

Note that for a positive definite preconditioner A_0 the matrix $\mathcal{H}^+$ is always positive definite, i.e.

$$x^T(A + A_0)x = x^T Ax + x^T A_0 x > 0$$

due to the positive definiteness of the matrices A and A_0. Thus, we are in this case always equipped with an inner product and not just a symmetric bilinear form whatever symmetric and positive definite A_0 is chosen, and so the appropriate Krylov subspace method can be used in this inner product. In the context of particular methods, we will discuss the eigenvalue properties of the preconditioned matrix. We also want to mention that the Bramble-Pasciak$^+$ configuration is applicable in the presence of a Schur-complement preconditioner S_0. The modified preconditioner then becomes

$$\mathcal{P}^+ = \begin{bmatrix} A_0 & 0 \\ -B & S_0 \end{bmatrix} \text{ and } \quad \left(\mathcal{P}^+\right)^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ S_0^{-1}BA_0^{-1} & S_0^{-1} \end{bmatrix} \tag{2.5}$$

and the inner product matrix is defined by

$$\mathcal{H}^+ = \begin{bmatrix} A + A_0 & 0 \\ 0 & S_0 \end{bmatrix}. \tag{2.6}$$

3 Methods for solving the $\mathcal{P}^+$-preconditioned system

3.1 The Conjugate Gradients method

The CG algorithm is one of the most powerful Krylov subspace methods to solve a linear system. In [5] Bramble-Pasciak introduce the preconditioning by a block-triangular matrix that results in a non-symmetric matrix which is symmetric and positive definite in a non-standard inner product; hence, the CG algorithm is applicable. To justify this, Klawonn shows in [19] that the matrix

$$\widehat{\mathcal{A}}^T\mathcal{H} = \begin{bmatrix} AA_0^{-1}A - A & AA_0^{-1}B^T - B^T \\ BA_0^{-1}A - B & BA_0^{-1}B^T + C \end{bmatrix} \tag{3.1}$$

can be split into

$$\begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} AA_0^{-1}A - A & 0 \\ 0 & BA^{-1}B^T + C \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ 0 & I \end{bmatrix}. \tag{3.2}$$

Now note that because of the positive definiteness of A we can rewrite $AA_0^{-1}A - A$ as

$$A^{1/2}(A^{1/2}A_0^{-1}A^{1/2} - I)A^{1/2}.$$

The positivity is guaranteed if

$$x^T A^{1/2} A_0^{-1} A^{1/2} x > x^T x$$

which for $y = A^{1/2}x$ further reduces to

$$y^T A_0^{-1} y > y^T A^{-1} y.$$

Using the last result, we obtain

$$y^T A_0 y < y^T A y. \tag{3.3}$$

Due to the congruence transformation (3.2) and Equation 3.3 we see that all eigenvalues are positive. The same has to be shown for the new preconditioner $\mathcal{P}^+$. Using Klawonn's approach we get for the matrix

$$\widehat{\mathcal{A}}^T \mathcal{H}^+ = \begin{bmatrix} AA_0^{-1}A + A & AA_0^{-1}B^T + B^T \\ BA_0^{-1}A + B & BA_0^{-1}B^T - C \end{bmatrix} \tag{3.4}$$

a decomposition of the following type

$$\begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} AA_0^{-1}A + A & 0 \\ 0 & -BA^{-1}B^T - C \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ 0 & I \end{bmatrix}. \tag{3.5}$$

This shows that the matrix has negative eigenvalues since $-BA_0^{-1}B^T - C$ is always negative definite and therefore the reliable applicability of the CG method cannot be guaranteed. Nevertheless, in practice the results of the CG method applied to this problem show good convergence behaviour. One remedy could be to use augmented Krylov subspace methods, see [29]. Another alternative is to use a special implementation of MINRES which will be introduced in Section 3.2.

3.2 MINRES for the saddle point problem

In Section 2 we showed that for a positive definite preconditioner A_0 the inner product matrix $\mathcal{H}^+$ will always be symmetric positive definite. MINRES (Minimal Residual Method) was introduced in 1975 by Paige and Saunders in [23] as a method for minimizing the residual over the current Krylov subspace based on the symmetric Lanczos method. It is typically the method of choice for symmetric indefinite systems. Since the preconditioned matrix $\widehat{\mathcal{A}}$ is symmetric in the inner product induced by $\mathcal{H}^+$, we can use a version of the classical Lanczos method to generate a basis for the Krylov subspace and then minimize the $\mathcal{H}^+$-norm of the residual. The $\mathcal{H}^+$-Lanczos method (cf. Algorithm 1) generates an $\mathcal{H}^+$-orthonormal basis for the Krylov subspace which can be expressed in matrix terms as

$$AV_k = V_k T_k + \beta_k v_{k+1} e_k^T \tag{3.6}$$

with

$$T_k = \begin{bmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{k-1} \\ & & \beta_{k-1} & \alpha_k \end{bmatrix} \text{ and } V_k = [v_1, v_2, \ldots, v_k]$$

as well as $V_k^T \mathcal{H}^+ V_k = I$. Using the basis V_k and the coefficients stored in T_k we can

Algorithm 1 Algorithm for $\mathcal{H}^+$-Lanczos

INPUT : $A \in \mathbb{R}^{n,n}$ and $b \in \mathbb{R}^n$
OUTPUT : $v_1, v_2, \ldots$ an orthonormal basis for the Krylov subspace

Choose start vector $v_1 \mathbb{R}^n$ with $\|v_1\| = 1$.
Set $\beta_0 = 0$
for $k = 1, 2, \ldots$ **do**
  $\tilde{v}_{k+1} = \widehat{\mathcal{A}} v_k - \beta_{k-1} v_{k-1}$
  Compute $\alpha_k = \langle \tilde{v}_{k+1}, v_k \rangle_{\mathcal{H}^+}$
  $\tilde{v}_{k+1} = \tilde{v}_{k+1} - \alpha_k v_k$
  Set $\beta_k = \|\tilde{v}_{k+1}\|_{\mathcal{H}^+}$
  Set $v_{k+1} = \tilde{v}_{k+1} / \beta_k$
end for

implement a MINRES procedure, see Algorithm 2. The following condition holds for

Algorithm 2 Algorithm for $\mathcal{H}^+$-MINRES

INPUT : $A \in \mathbb{R}^{n,n}$ and $b \in \mathbb{R}^n$
OUTPUT : $x \in \mathbb{R}^n$

Compute $r_0 = b - A x_0$ with given x_0.
Set $v_1 = r_0 / \|r_0\|$
Set $\beta_0 = 0$
for $k = 1, 2, \ldots$ **do**
  Compute v_{k+1}, α_k and β_k by using the $\mathcal{H}^+$-Lanczos method.
  Update the matrix T_k.
  Modify last column of T_k by using the last two Givens rotations.
  Compute and apply kth Givens rotation.
  Compute p_{k-1}.
  Update solution $x_k = x_{k-1} + a_{k-1} p_{k-1}$ where a_{k-1} is the kth entry of $\|r_0\| (Q_k e_1)_{k \times 1}$ with Q_k from the updated QR factorization.
end for

the residual

$$\begin{aligned} \|r_k\|_{\mathcal{H}^+} &= \|b - Ax_k\|_{\mathcal{H}^+} \\ &= \|b - Ax_0 - AV_k y_k\|_{\mathcal{H}^+} \\ &= \|r_0 - V_{k+1}T_{k+1}y_k\|_{\mathcal{H}^+} \\ &= \left\|V_{k+1}(V_{k+1}^T\mathcal{H}^+ r_0 - T_{k+1}y_k)\right\|_{\mathcal{H}^+} \\ &= \left\|V_{k+1}^T\mathcal{H}^+ r_0 - T_{k+1}y_k\right\|_{\mathcal{H}^+} \\ &= \|\|r_0\|\, e_1 - T_{k+1}y_k\|_{\mathcal{H}^+} \end{aligned} \tag{3.7}$$

using that V_{k+1} is orthogonal in the $\mathcal{H}^+$-inner product. Implementation details for MINRES can be found in [17]. Applying the $\mathcal{H}^+$-MINRES method can also be justified by studying the problem in the context of the Faber-Manteuffel theorem. In 1984 Faber and Manteuffel [10] proved only matrices which are normal(s) in some inner product admit a (s+2)-term recurrence which minimizes some relevant quantity at each iteration. In the most common case of 3-term recurrence methods such as CG or MINRES the normal(1) condition implies that the eigenvalues of the problem matrix lie on a straight line in the complex plane. A survey paper by Liesen and Strakos, see [21], gives a description of the Faber and Manteuffel paper in more accessible linear algebra terms. Essentially, a matrix M admits an $(s+2)$-term recurrence method if the B-adjoint[1], $M^+ = BM^TB^{-1}$, can be expressed as a polynomial of degree s in M, i.e.

$$M^+ = p_s(M).$$

The self adjointness relation $\widehat{\mathcal{A}}^T\mathcal{H}^+ = \mathcal{H}^+\widehat{\mathcal{A}}$ obviously gives that a polynomial of degree $s = 1$ must exist such that the $\mathcal{H}^+$-adjoint of $\widehat{\mathcal{A}}^+ = p_1(\widehat{\mathcal{A}})$ for the simple polynomial $p_1(z) = z$, i.e. use $M = \widehat{\mathcal{A}}$ and $B = (\mathcal{H}^+)^{-1}$.

3.3 The simplified Lanczos method

The non-symmetric Lanczos process (cf. [11, 13, 14, 17, 26]) generates two sequences of vectors v_k and w_k that are orthogonal to each other and are generated by

$$\rho_{k+1}v_{k+1} = Av_k - \mu_k v_k - \nu_{k-1}v_{k-1} \tag{3.8}$$

for the first sequence and

$$\zeta_{k+1}w_{k+1} = A^Tv_k - \mu_k w_k - \tfrac{\nu_{k-1}\rho_k}{\zeta_k}w_{k-1} \tag{3.9}$$

for the second sequence with $\mu_k = w_k^TAv_k/w_k^Tv_k$ and $\nu_k = \zeta_k w_k^Tv_k/w_{k-1}^Tv_{k-1}$. There is more than one way to scale the two vectors in every iteration step. Here, we use $\|v_j\| = 1$ and $\|w_j\| = 1$. The biorthogonality condition between W_k and V_k gives

$$D_k = diag(\delta_1, \delta_2, \ldots, \delta_k) \text{ where } \delta_j = w_j^Tv_j. \tag{3.10}$$

[1] B is a symmetric positive definite inner product matrix

Furthermore, we can now write the recursions in terms of matrices and get

$$AV_k = V_{k+1}H_k \tag{3.11}$$

as well was

$$A^T W_k = W_{k+1}\Gamma_{k+1}^{-1}H_k\Gamma_{k+1} \tag{3.12}$$

where the matrix Γ_k is defined as follows

$$\Gamma_k = diag(\gamma_1, \gamma_2, \ldots, \gamma_k) \text{ with } \gamma_j = \begin{cases} 1 & if \quad j = 1 \\ \gamma_{j-1}\rho_j/\zeta_j & if \quad j > 1. \end{cases} \tag{3.13}$$

One advantage of the non-symmetric Lanczos process is that H_k is a tridiagonal matrix which is typically non-symmetric. There are cases where the non-symmetric Lanczos process can break down which can have different implications. The first case the so-called *lucky breakdown* indicates that the solution lies already in the current Krylov space. In the case of $w_j^T v_j = 0$ and neither v_{j+1} nor w_{j+1} are zero the so-called *serious breakdown* occurs. A remedy is to use look-ahead strategies, see [12,24] for more details.

The non-symmetric Lanczos process can now be simplified using the self-adjointness of $\widehat{\mathcal{A}}$ in the $\mathcal{H}^+$-inner product, i.e.

$$A^T\mathcal{H}^+ = \mathcal{H}^+A.$$

In [13] Freund and Nachtigal observe that for the Lanczos vectors the relation

$$v_j = \phi_j(A)v_1 \text{ and } w_j = \gamma_j\phi_j(A^T)w_1 \tag{3.14}$$

holds where ϕ is a polynomial of degree $j-1$ the so-called *Lanczos polynomial*. Using Equation (3.14) and setting $w_1 = \mathcal{H}^+v_1$, we get

$$w_j = \gamma_j\phi_j(A^T)w_1 = \gamma_j\phi_j(A^T)\mathcal{H}^+v_1 = \gamma_j\mathcal{H}^+\phi_j(A)v_1 = \gamma_j\mathcal{H}^+v_j.$$

Hence, we can compute the vector w_j without multiplying by A^T. Instead,

$$w_{j+1} = \gamma_{j+1}\mathcal{H}^+v_{j+1} \tag{3.15}$$

can be used. The parameter $\gamma_{j+1} = \gamma_j\rho_{j+1}/\zeta_{j+1}$ involves ζ_{j+1} which cannot be computed at that time. Thus the relation (3.15) has to be reformulated to

$$\tilde{w}_{j+1} = \zeta_{j+1}w_{j+1} = \gamma_j\rho_{j+1}\mathcal{H}^+v_{j+1} = \gamma_j\mathcal{H}^+\tilde{v}_{j+1}$$

which gives us now a computable version of the simplified Lanczos method, see Algorithm 3.

Algorithm 3 Algorithm for the simplified Lanczos method

INPUT : $A \in \mathbb{R}^{n,n}$ and $b \in \mathbb{R}^n$
OUTPUT : $x \in \mathbb{R}^n$

Choose v_1 and compute $w_1 = \mathcal{H}^+ v_1$
Compute $\rho_1 = \|v_1\|$ and $\zeta_1 = \|w_1\|$
Set $\gamma_1 = \frac{\rho_1}{\zeta_1}$
for $k = 1, 2, \ldots$ **do**
  Compute $\mu_k = (w_k^T A v_k)/(w_k^T v_K)$
  Set $\nu_k = \zeta_k (w_k^T v_k)/(w_{k-1}^T v_{k-1})$
  $v_{k+1} = A v_k - \mu_k v_k - \nu_k v_{k-1}$
  $w_{k+1} = \gamma_k v_{k+1}$
  Compute $\rho_{k+1} = \|v_{k+1}\|$ and $\zeta_{k+1} = \|w_{k+1}\|$
  Set $\gamma_{k+1} = \gamma_k \rho_{k+1}/\zeta_{k+1}$.
end for

3.4 The ideal transpose-free QMR method

In [11] Freund introduced the ideal transpose-free QMR method (ITFQMR) by using the simplification of the Lanczos method. Freund's implementation is based on a QMR -from-BICG procedure and coupled two term recurrence relations, details can be found in [11, 14]. For simplicity reasons, we introduce the algorithm based on the simplified Lanczos method and the standard updated QR technique using Equation 3.7. In more detail, we have

$$AV_k = V_{k+1} H_k \tag{3.16}$$

from the non-symmetric Lanczos process and get as a result

$$r_k = V_{k+1}(\|r_0\| e_1 - H_k y_k) \tag{3.17}$$

for the residual. The term $(\|r_0\| e_1 - H_k y_k)$ is called *quasi-residual*. Minimizing the quasi-residual $(\|r_0\| e_1 - H_k y_k)$ by applying the standard QR technique known from MINRES we obtain the Quasi-Minimal Residual (QMR) method. If we can in the process of minimizing the quasi-residual omit multiplication with the transpose in the non-symmetric Lanczos by employing the simplified version we get the ideal transpose-free QMR method (ITFQMR), see Algorithm 4.

For a better understanding of the convergence behaviour of ITFQMR we analyze the eigenvalues of

$$\widehat{\mathcal{A}} = \left(\mathcal{P}^+\right)^{-1} \mathcal{A} = \begin{bmatrix} A_0^{-1} A & A_0^{-1} B^T \\ B A_0^{-1} A + B & B A_0^{-1} B^T \end{bmatrix}$$

for the case $A_0 = A$ which gives

$$\widehat{\mathcal{A}} = \left(\mathcal{P}^+\right)^{-1} \mathcal{A} = \begin{bmatrix} I & A^{-1} B^T \\ 2B & B A^{-1} B^T \end{bmatrix}. \tag{3.18}$$

Algorithm 4 Algorithm for ideal transpose-free QMR

INPUT : $A \in \mathbb{R}^{n,n}$ and $b \in \mathbb{R}^n$
OUTPUT : $x \in \mathbb{R}^n$

Compute $r_0 = B - Ax_0$
Set $v_1 = r_0$ and compute $w_1 = \mathcal{H}^+ v_1$
Compute $\rho_1 = \|v_1\|$ and $\zeta_1 = \|w_1\|$
Set $\gamma_1 = \frac{\rho_1}{\zeta_1}$
for $k = 1, 2, \ldots$ **do**
  Compute one step of the non-symmetric simplified Lanczos process
  Update matrix T_k
  Modify last column of T_k by using the last two Givens rotations.
  Compute and apply kth Givens rotation.
  Compute p_{k-1}.
  Update solution $x_k = x_{k-1} + a_{k-1}p_{k-1}$ where a_{k-1} is the kth entry of $\|r_0\| (Q_k e_1)_{k\times 1}$ with Q_k from the updated QR factorization..
end for

Given an eigenpair $(\lambda, \begin{bmatrix} x \\ y \end{bmatrix})$ of $\widehat{\mathcal{A}}$ we know that

$$\begin{bmatrix} I & A^{-1}B^T \\ 2B & BA^{-1}B^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + A^{-1}B^T y \\ 2Bx + BA^{-1}B^T y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.19}$$

holds. Equation 3.19 shows that

$$x + A^{-1}B^T y = \lambda x \Longleftrightarrow Ax + B^T y = \lambda Ax.$$

For $\lambda = 1$ we get

$$Ax + B^T y = Ax$$

which gives $B^T y = 0$ and therefore $y = 0$ under the condition that $Bx = 0$. Since the kernel of B is $n - m$ dimensional we have $\lambda = 1$ with multiplicity $n - m$.

For $\lambda \neq 1$, we conclude from Equation 3.19 that $x = \frac{1}{\lambda - 1} A^{-1}B^T y$. Using this

$$BA^{-1}B^T y = \frac{\lambda(\lambda - 1)}{\lambda + 1} y \tag{3.20}$$

can be obtained. This shows that for an eigenvalue σ of $BA^{-1}B^T$ we get

$$\sigma = \frac{\lambda(\lambda - 1)}{\lambda + 1}. \tag{3.21}$$

Hence, the eigenvalues of $\widehat{\mathcal{A}}$ become

$$\lambda_{1,2} = \frac{1 + \sigma}{2} \pm \sqrt{\frac{(1 + \sigma)^2}{4} + \sigma}. \tag{3.22}$$

Obviously, $\sigma > 0$ and therefore we have m negative eigenvalues given by Equation 3.22. This shows that there are $2m + 1$ different eigenvalues and we expect the method to converge after $2m+1$ steps. This also extends the analysis made in Section 3.1 by giving exact formulas for the eigenvalues of $(\mathcal{P}^+)^{-1}\mathcal{A}$.

A similar analysis for the classical Bramble-Pasciak case was made by Simoncini in [28].

The indefiniteness of $(\mathcal{P}^+)^{-1}\mathcal{A}$ indicates that methods such as $\mathcal{H}^+$-MINRES or IT-FQMR should be used. We will illustrate their convergence behaviour in Section 5 by applying them to Stokes examples from IFISS .

4 Combination preconditioning with the Bramble-Pasciak$^+$ setup

In 2007 Stoll and Wathen (cf. [30]) show that different preconditioners $\mathcal{P}_1, \mathcal{P}_2$ and bilinear forms $\mathcal{H}_1, \mathcal{H}_2$ can be combined to give a new preconditioner $\mathcal{P}_3$ and a bilinear form $\mathcal{H}_3$. We quote Lemma 3.5 from [30]

Lemma 4.1 *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are left preconditioners for the symmetric matrix $\mathcal{A}$ for which symmetric matrices $\mathcal{H}_1$ and $\mathcal{H}_2$ exist with $\mathcal{P}_1^{-T}\mathcal{A}$ self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$ and $\mathcal{P}_2^{-T}\mathcal{A}$ self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}_2}$ and if*

$$\alpha\mathcal{P}_1^{-T}\mathcal{H}_1 + \beta\mathcal{P}_2^{-T}\mathcal{H}_2 = \mathcal{P}_3^{-T}\mathcal{H}_3$$

for some matrix $\mathcal{P}_3$ and some symmetric matrix $\mathcal{H}_3$ then $\mathcal{P}_3^{-1}\mathcal{A}$ is self-adjoint in $\langle\cdot,\cdot\rangle_{\mathcal{H}_3}$.

In [30] the theory is illustrated by showing the application to saddle point problems. The saddle point problem preconditioned by $\mathcal{P}_3$ is self-adjoint in the bilinear form defined by $\mathcal{H}_3$ and under some conditions for the combination parameters α and β also defines an inner product. It is also possible to determine these parameters such that the preconditioned matrix is positive definite in the inner product induced by $\mathcal{H}_3$.

Using the results given in [30] we want to analyze the possibility of combining the classical Bramble-Pasciak configuration with the newly created Bramble-Pasciak$^+$ setup. Therefore, we have the preconditioners

$$\mathcal{P}_1 = \begin{bmatrix} A_0 & 0 \\ B & -I \end{bmatrix} \text{ and } \quad \mathcal{P}_2 = \begin{bmatrix} A_0 & 0 \\ -B & I \end{bmatrix}$$

and for the inner products

$$\mathcal{H}_1 = \begin{bmatrix} A - A_0 & 0 \\ 0 & I \end{bmatrix} \text{ and } \quad \mathcal{H}_2 = \begin{bmatrix} A + A_0 & 0 \\ 0 & I \end{bmatrix}.$$

Instead of $\alpha, \beta \in \mathbb{R}$ we use the combination parameters α and $1 - \alpha$ and get

$$\alpha\mathcal{P}_1^{-T}\mathcal{H}_1 + (1-\alpha)\mathcal{P}_2^{-T}\mathcal{H}_2 = \begin{bmatrix} A_0^{-1}A + (1-2\alpha)I & A_0^{-1}B^T \\ 0 & (1-2\alpha)I \end{bmatrix}.$$

If we find a splitting as described in Lemma 4.1 than a new preconditioner and bilinear form are given. One splitting possibility would be

$$P_3^{-T} = \begin{bmatrix} A_0^{-1} & A_0^{-1}B^T \\ 0 & (1-2\alpha)I \end{bmatrix} \Longrightarrow P_3 = \begin{bmatrix} A_0 & 0 \\ \frac{1}{(2\alpha-1)}B & \frac{1}{1-2\alpha}I \end{bmatrix}$$

as the new preconditioner and the bilinear form is then defined by

$$\mathcal{H}_3 = \begin{bmatrix} A + (1-2\alpha)A_0 & 0 \\ 0 & I \end{bmatrix}.$$

Note, that for $\alpha = 1$ we obtain the classical Bramble-Pasciak configuration and $\alpha = 0$ gives the Bramble-Pasciak$^+$ setup.

We now have to analyze if positivity in the new bilinear form can be achieved and if the bilinear form is an inner product which can be exploited for short-term recurrence methods. Hence, the matrix

$$\widehat{\mathcal{A}}^T\mathcal{H}_3$$

with $\widehat{\mathcal{A}} = \mathcal{P}_3^{-1}\mathcal{A}$ has to be analyzed. The matrix

$$\widehat{\mathcal{A}}^T\mathcal{H}_3 = \begin{bmatrix} AA_0^{-1}A + (1-2\alpha)A & AA_0^{-1}B^T + (1-2\alpha)B^T \\ BA_0^{-1}A + (1-2\alpha)B & BA_0^{-1}B^T - (1-2\alpha)C \end{bmatrix}$$

can be split as

$$\widehat{\mathcal{A}}^T\mathcal{H}_3 = \begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} AA_0^{-1}A + (1-2\alpha)A & 0 \\ 0 & (2\alpha-1)BA^{-1}B^T - (1-2\alpha)C \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ 0 & I \end{bmatrix}.$$

The number of positive and negative eigenvalues is determined by the eigenvalues of the matrix

$$\begin{bmatrix} AA_0^{-1}A + (1-2\alpha)A & 0 \\ 0 & (2\alpha-1)(BA_0^{-1}B^T + C) \end{bmatrix}$$

which we analyze in more detail. We first consider the case where $C = 0$ and then it is easy to see that the block $(2\alpha-1)BA_0^{-1}B^T$ is positive for $\alpha > 1/2$. With this choice for α we have to find conditions such that the block $AA_0^{-1}A + (1-2\alpha)A$ is also positive definite. Similar to the analysis made in Section 3.1, we note the equivalence

$$A^{\frac{1}{2}}\left(A^{\frac{1}{2}}A_0^{-1}A^{\frac{1}{2}} + (1-2\alpha)I\right)A^{\frac{1}{2}}$$

and use the field of values so that positive definiteness is expressed by the relation

$$x^TA^{\frac{1}{2}}A_0^{-1}A^{\frac{1}{2}}x + (1-2\alpha)x^Tx > 0 \quad \forall x.$$

The last equation can be rewritten as

$$x^TA^{\frac{1}{2}}A_0^{-1}A^{\frac{1}{2}}x > (2\alpha-1)x^Tx.$$

and if we introduce $y = A^{\frac{1}{2}}x$ we get

$$y^T A_0^{-1} y > (2\alpha - 1) y^T A^{-1} y.$$

Thus, positivity is given if $y^T A_0 y < (2\alpha - 1) y^T A y$ which can also be written as

$$A_0 < (2\alpha - 1) A.$$

In addition we want the matrix $\mathcal{H}_3$ to define an inner product which will be satisfied if the block $A + (1 - 2\alpha)A_0 > 0$ which is equivalent to

$$\frac{1}{2\alpha - 1} A > A_0.$$

Again, the case $\alpha = 1$ gives the Bramble Pasciak configuration and $\alpha = 0$ shows that there is no configuration that makes the Bramble-Pasciak$^+$ setup positive definite and CG reliable applicable. It is still possible to obtain a reliable CG method in the combination preconditioning case, ie. if

$$A_0 < \min\left((2\alpha - 1)A, \frac{1}{2\alpha - 1} A \right)$$

which imposes more restrictions on A_0 then Bramble and Pasciak did in [5]. The case $C \neq 0$ can be treated equivalently since the block $(2\alpha - 1)(BA_0^{-1}B^T + C)$ will be positive for all $\alpha > 1/2$ and the above analysis applies.

5 Numerical Experiments

In this section we will show the results of our numerical experiments. The matrices are coming from the Stokes problem and in particular were generated using the IFISS[2] package. The Stokes equation

$$\begin{aligned} -\nabla^2 u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned}$$

can be transformed using a weak formulation which can than be treated using the finite element method, see [9] for details. The linear system governing the finite element method for the Stokes problem is a saddle point problem

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}$$

where $C \neq 0$ for stabilized elements. This matrix is symmetric but indefinite and could be treated with MINRES in the first place. But in order to improve the convergence we have to compare our Bramble-Pasciak$^+$ preconditioner to other suited methods. One

[2] http://www.maths.manchester.ac.uk/~djs/ifiss/

candidate would be the block diagonal preconditioning introduced by Silvester and Wathen in [27, 32] where a preconditioner

$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ 0 & S_0 \end{bmatrix} \tag{5.1}$$

where A_0 is a preconditioner for A and S_0 is a Schur complement preconditioner and we can use MINRES with this type of preconditioner. More details can be found in [4]. In the Bramble-Pasciak$^+$ setup the preconditioned matrix is symmetric in the $\mathcal{H}^+$-inner product. This enables us to use the $\mathcal{H}^+$-MINRES method introduced in Section 3.2. We will compare this method to the classical MINRES algorithm for the block-diagonal preconditioner. In the IFISS implementation the preconditioner S_0 is chosen to be the Gramian matrix of the basis functions, see Chapter 6.2 in [9]. As the right hand side we chose the row sum of $\mathcal{A}$ such that for the solution x the condition $x_j = 1$ should hold.

Example 5.1 *The first example comes from the* IFISS *package where the Stokes problem for the channel domain is considered. The size of the system matrix $\mathcal{A}$ is given by* 9539×9539 *with* $m = 1089$ *and* $n = 8450$. *The results shown in Figure 1 are obtained by using the* $\mathcal{H}^+$-MINRES *method and the classical Preconditioned* MINRES *as given in [27,32], ie.* $A_0 = A$ *and* S_0 *the Gramian. The red (dashed) curve is showing the results of the Preconditioned* MINRES *with a block-diagonal preconditioner. The corresponding residual is given in the* 2*-norm. The blue (solid) line shows the* 2*-norm residuals computed by the* $\mathcal{H}^+$-MINRES *algorithm. Figure 2 shows the same results as Figure 1*

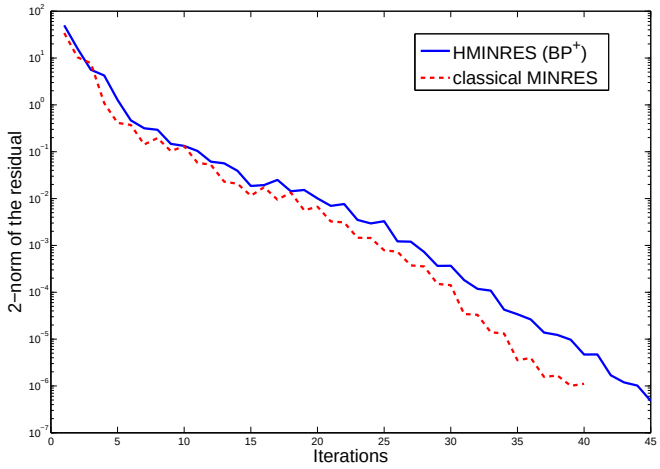

Figure 1: Results for $\mathcal{H}^+$-MINRES and classical Preconditioned MINRES .

but with the ITFQMR *method in addition. The results for the* ITFQMR *with Bramble-Pasciak$^+$ preconditioning and $\mathcal{H}^+$-inner product are shown as black (dash–dotted) line in Figure 2.*

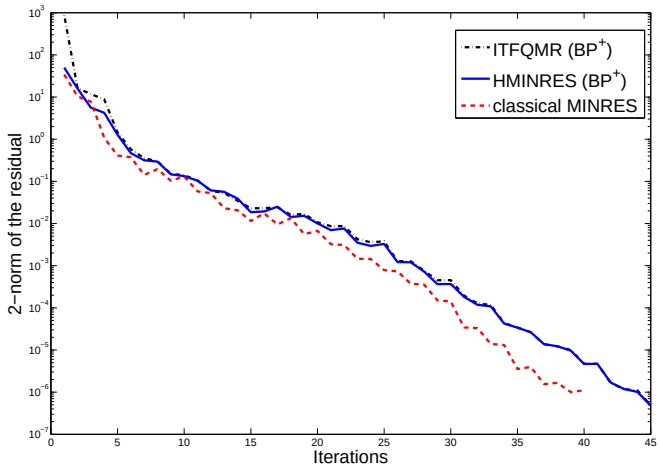

Figure 2: Additional ITFQMR results with Bramble-Pasciak$^+$ preconditioning.

Example 5.2 *In this example we use the same matrices as in Example 5.1 where the dimension of* $\mathcal{A}$ *is* 9539*. Instead of the exact preconditioner* $A_0 = A$ *we introduce the Incomplete Cholesky Preconditioner*[3] *where* $A_0 = R^T R$ *with* R *coming from the Incomplete Cholesky factorization of* $\mathcal{A}$*. Again in Figure 3, the blue (solid) line shows the convergence for the* $\mathcal{H}^+$-MINRES *method and the red (dashed) curve represents the behaviour for the Preconditioned* MINRES *method in the* 2*-norm. Again the results for the block-diagonal preconditioned system and the Bramble-Pasciak*$^+$ $\mathcal{H}^+$-MINRES *method are quite similar. In Figure 4 we added the* 2*-norm results for the* CG *method with classical Bramble-Pasciak preconditioner and inner product as the black (dash–dotted) line without checking whether the conditions necessary for* A_0 *are fulfilled.*

Example 5.3 *The problem matrix* $\mathcal{A}$ *is again of dimension* 9539 *and comes from the* IFISS *package. Furthermore, we are working with setting of the original Bramble-Pasciak paper, see [5] where only the preconditioner* A_0 *is used without introducing a Schur-complement preconditioner, ie.*

$$\mathcal{P} = \begin{bmatrix} A_0 & 0 \\ B & -I \end{bmatrix}.$$

Due to the structure of the inner product matrix

$$\mathcal{H} = \begin{bmatrix} A - A_0 & 0 \\ 0 & I \end{bmatrix}$$

where the block $A - A_0$ *can be found on the main diagonal, the exact preconditioner* $A_0 = A$ *cannot be applied. Thus, we again use a Incomplete Cholesky decomposition of*

[3]For more details see the MATLAB function *cholinc* and the references mentioned there.

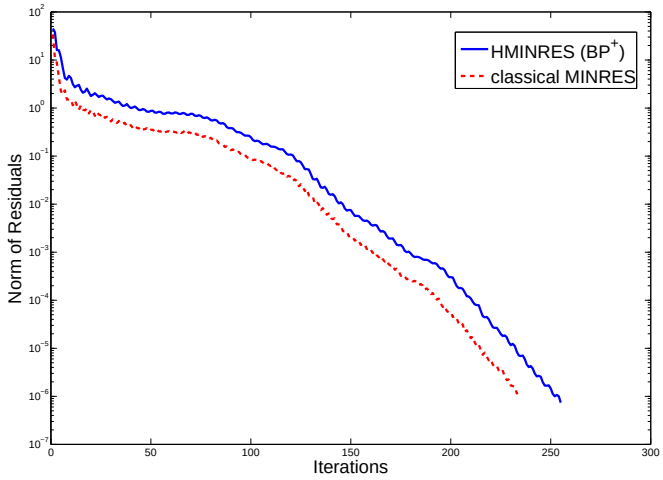

Figure 3: Results for MINRES and $\mathcal{H}^+$-MINRES with Incomplete Cholesky preconditioning.

A to get the preconditioner $A_0 = R^T R$. *In [28] Simoncini recommends the use of the simplified Lanczos procedure combined with the* QMR *method which gives the* ITFQMR . *Hence, we compare the* ITFQMR *method for the classical Bramble-Pasciak case and for the new Bramble-Pasciak*$^+$ *preconditioner with inner product* $\mathcal{H}^+$. *Figure 5 shows the* 2*-norm results of our computations. The black (dashed) line represents the convergence behaviour for the* ITFQMR *method with the Bramble-Pasciak preconditioner* $\mathcal{P}$ *and inner product matrix* $\mathcal{H}$. *The blue (solid) curve shows the norm of the residual coming from the* ITFQMR *method with the Bramble-Pasciak*$^+$ *preconditioner* $\mathcal{P}^+$ *and inner product matrix* $\mathcal{H}^+$.

We now show results for the combination preconditioning with the Bramble-Pasciak and the Bramble-Pasciak$^+$ setup. As shown in Section 4 the parameter α has to be adjusted in order to guarantee positive definiteness of the preconditioned matrix and also to define an inner product rather than a bilinear form. For different choices of α different methods are applicable. In the case of α giving the positivity of $\widehat{\mathcal{A}}$ in an inner product we will use CG to compute the solution to Equation 5.1. If only the bilinear form becomes an inner product and $\widehat{\mathcal{A}}$ is not positive definite $\mathcal{H}^+$-MINRES can be applied as explained in Section 3.2. In the case that neither the matrix $\widehat{\mathcal{A}}$ is positive nor $\mathcal{H}$ defines an inner product we are still able to apply the ITFQMR method based on the simplified Lanczos method.

Example 5.4 *In this example the matrix represents the flow over a channel domain and is of size* 2467×2467. *Our choice for* A_0 *is again the Incomplete Cholesky decomposition and* S_0*the Gramian. Figure 6 shows the results for different values of* α, *where the fastest convergence is given for* $\alpha = 2/3$ *in the blue (solid) curve. The choice for* $\alpha = 2/3$ *performs better than original Bramble-Pasciak method reflected by* $\alpha = 1$. *For*

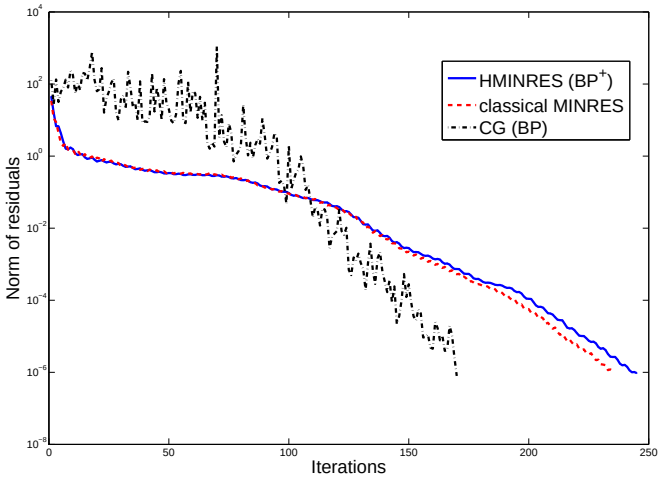

Figure 4: Additional classical BP results

comparison we also show the results for the preconditioned MINRES *in the red (dashed) line. Further values of* α *are shown, where* $\alpha = 1$ *represents the classical Bramble-Pasciak configuration.*

6 Conclusions

In this paper we presented a new preconditioner $\mathcal{P}^+$ based on the theory introduced by Bramble and Pasciak. We introduced an inner product in which the preconditioned matrix is self-adjoint. We discussed the applicability of different Krylov subspace solver including a thorough eigenvalue analysis of the preconditioned matrix in the case of $A_0 = A$. Furthermore, a short proof of the classical Bramble-Pasciak setup was given based on techniques introduced by Klawonn. The same techniques were used to study the definiteness of the Bramble-Pasciak$^+$ configuration. A MINRES algorithm based on the $\mathcal{H}^+$-inner product was introduced. We studied the simplified Lanczos method and based on it we developed an implementation of ITFQMR . We also analyzed the possibilities of combining the classical and the new Bramble-Pasciak setup to obtain new preconditioners as shown in [30]. Finally, numerical experiments based upon the IFISS package were shown and the comparison of the new methods to some well-known problems was made. The new methods showed similar behaviour to the classical block-diagonal preconditioned MINRES approach and in the case where no Schur-complement preconditioner was used the classical Bramble-Pasciak method in the ITFQMR setting showed very similar behaviour to the new Bramble-Pasciak$^+$ setup implemented with the ITFQMR process. Furthermore, we presented result for the combination preconditioning technique applied to Bramble-Pasciak$^+$.

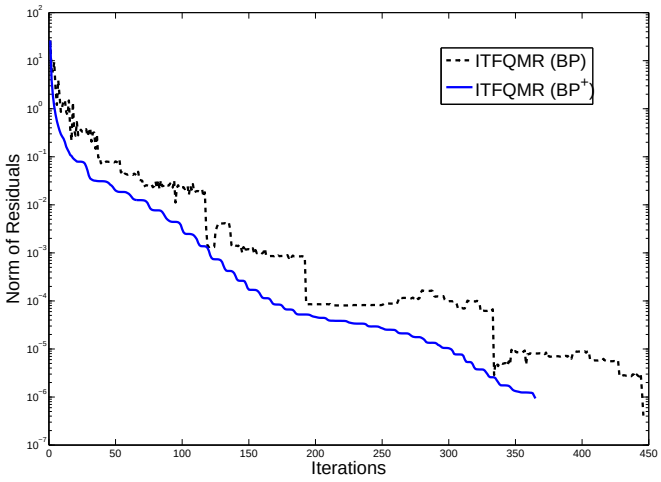

Figure 5: ITFQMR results for both classical Bramble-Pasciak and Bramble-Pasciak$^+$ setup.

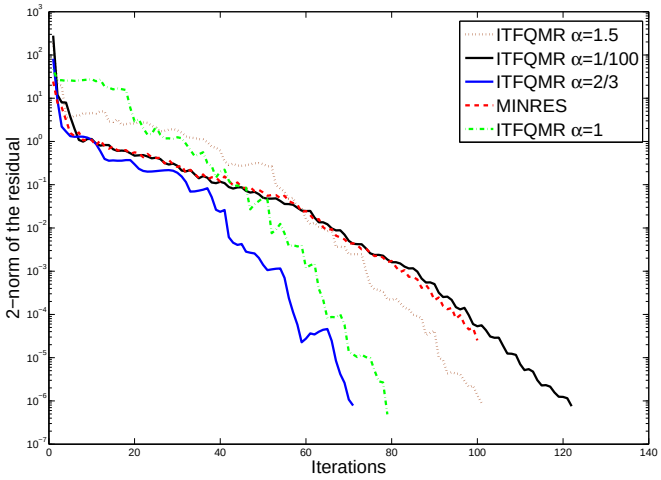

Figure 6: ITFQMR results for combination preconditioning with different values for α.

References

[1] Mark Ainsworth and Spencer Sherwin, 1999. Domain decomposition preconditioners for p and hp finite element approximation of Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, **175**(3-4): 243–266.

[2] Owe Axelsson and Maya Neytcheva, 2003. Preconditioning methods for linear systems arising in constrained optimization problems. *Numer. Linear Algebra Appl.*, **10**(1-2): 3–31. Dedicated to the 60th birthday of Raytcho Lazarov.

[3] Owe Axelsson and Maya Neytcheva, 2003. Robust preconditioners for saddle point problems. In *Numerical methods and applications*, Volume 2542 of *Lecture Notes in Comput. Sci.*, pages 158–166. Springer, Berlin.

[4] Michele Benzi, Gene H. Golub and Jörg Liesen, 2005. Numerical solution of saddle point problems. *Acta Numer.*, **14**: 1–137.

[5] James H. Bramble and Joseph E. Pasciak, 1988. A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. *Math. Comp.*, **50**(181): 1–17.

[6] C. Carstensen, M. Kuhn and U. Langer, 1998. Fast parallel solvers for symmetric boundary element domain decomposition equations. *Numer. Math.*, **79**(3): 321–347.

[7] H. C. Elman, 1996. Multigrid and Krylov subspace methods for the discrete Stokes equations. *In* N. D. Melson, T. A. Manteuffel, S. F. McCormick and C. C. Douglas, editors, *Seventh Copper Mountain Conference on Multigrid Methods*, Volume CP 3339, pages 283–299, Hampton, VA. NASA.

[8] Howard C. Elman, Alsison Ramage and David J. Silvester, June 2007. Algorithm xxx: IFISS, a Matlab toolbox for modelling incompressible flow. *ACM Transactions on Mathematical Software*, **33**(2).

[9] Howard C. Elman, David J. Silvester and Andrew J. Wathen, 2005. *Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics.* Numerical Mathematics and Scientific Computation. Oxford University Press, New York.

[10] Vance Faber and Thomas Manteuffel, 1984. Necessary and sufficient conditions for the existence of a conjugate gradient method. *SIAM J. Numer. Anal.*, **21**(2): 352–362.

[11] Roland W. Freund, 1994. Transpose-free quasi-minimal residual methods for non-Hermitian linear systems. In *Recent advances in iterative methods*, Volume 60 of *IMA Vol. Math. Appl.*, pages 69–94. Springer, New York.

[12] Roland W. Freund, Martin H. Gutknecht and Noël M. Nachtigal, 1993. An implementation of the look-ahead Lanczos algorithm for non-Hermitian matrices. *SIAM J. Sci. Comput.*, **14**(1): 137–158.

[13] Roland W. Freund and Noël M. Nachtigal, 1994. An implementation of the QMR method based on coupled two-term recurrences. *SIAM J. Sci. Comput.*, **15**(2): 313–337. Iterative methods in numerical linear algebra (Copper Mountain Resort, CO, 1992).

[14] Roland W. Freund and Noël M. Nachtigal, 1995. Software for simplified Lanczos and QMR algorithms. *Appl. Numer. Math.*, **19**(3): 319–341. Special issue on iterative methods for linear equations (Atlanta, GA, 1994).

[15] Gabriel N. Gatica and Norbert Heuer, 2002. Conjugate gradient method for dual-dual mixed formulations. *Math. Comp.*, **71**(240): 1455–1472.

[16] Gene H. Golub and Charles F. Van Loan, 1996. *Matrix computations*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, third edition.

[17] Anne Greenbaum, 1997. *Iterative methods for solving linear systems*, Volume 17 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

[18] Magnus R. Hestenes and Eduard Stiefel, 1952. Methods of conjugate gradients for solving linear systems. *J. Research Nat. Bur. Standards*, **49**: 409–436 (1953).

[19] Axel Klawonn, 1998. Block-triangular preconditioners for saddle point problems with a penalty term. *SIAM J. Sci. Comput.*, **19**(1): 172–184. Special issue on iterative methods (Copper Mountain, CO, 1996).

[20] U. Langer and M. Kuhn. Parallel solvers for large-scale, coupled finite and boundary element equations.

[21] Jörg Liesen and Zdeněk Strakoš, 2006. On optimal short-term recurrences for generating orthogonal Krylov subspace bases. submitted to SIAM review.

[22] Arnd Meyer and Torsten Steidten, 2001. Improvements and experiments on the Bramble-Pasciak Type CG for mixed problems in elasticity. Technical report, TU Chemnitz.

[23] C. C. Paige and M. A. Saunders, 1975. Solutions of sparse indefinite systems of linear equations. *SIAM J. Numer. Anal.*, **12**(4): 617–629.

[24] B. N. Parlett, D. R. Taylor and Z. S. Liu, 1984. The look ahead Lánczos algorithm for large unsymmetric eigenproblems. In *Computing methods in applied sciences and engineering, VI (Versailles, 1983)*, pages 87–96. North-Holland, Amsterdam.

[25] Jörg Peters, Volker Reichelt and Arnold Reusken, 2005. Fast iterative solvers for discrete Stokes equations. *SIAM J. Sci. Comput.*, **27**(2): 646–666.

[26] Y. Saad, 2003. *Iterative methods for sparse linear systems*. Society for Industrial and Applied Mathematics, Philadelphia, PA.

[27] David Silvester and Andrew Wathen, 1994. Fast iterative solution of stabilised Stokes systems. II. Using general block preconditioners. *SIAM J. Numer. Anal.*, **31**(5): 1352–1367.

[28] V. Simoncini, 2004. Block triangular preconditioners for symmetric saddle-point problems. *Appl. Numer. Math.*, **49**(1): 63–80.

[29] V. Simoncini and D.B. Szyld, February 2007. Recent computational developments in Krylov subspace methods for linear systems. *Numer. Linear Algebra Appl.*, **14**(1): 1–61.

[30] Martin Stoll and Andy Wathen, 2007. Combination preconditioning and self-adjointness for saddle point problems. Technical Report 07/09, Oxford University Computing Laboratory.

[31] Andrew Wathen and David Silvester, 1993. Fast iterative solution of stabilised Stokes systems. I. Using simple diagonal preconditioners. *SIAM J. Numer. Anal.*, **30**(3): 630–649.

[32] Andrew Wathen and David Silvester, 1993. Fast iterative solution of stabilised Stokes systems. I. Using simple diagonal preconditioners. *SIAM J. Numer. Anal.*, **30**(3): 630–649.

[33] Walter Zulehner, 2002. Analysis of iterative methods for saddle point problems: a unified approach. *Math. Comp.*, **71**(238): 479–505.