Quantum Proofs for Classical Theorems



Ronald de Wolf



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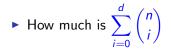
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5. But then there is a T with at least m/2 crossing edges!



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Unexpected proofs: Information theory

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5. Exponentiating both sides finishes the proof

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Pr[outcome i] = Tr($P_i | \phi \rangle \langle \phi |$) State $| \phi \rangle$ then collapses to $P_i | \phi \rangle / || P_i | \phi \rangle ||$ Special case: $P_i = |i\rangle \langle i|$, then Pr[outcome i] = $|\alpha_i|^2$ Example 1:

Lower bounds for locally decodable codes

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- Still the only superpolynomial bound known for LDCs

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Example 2:

Lower bounds for linear programs

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- FMPTW'12 show the same for all extended formulations

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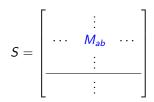
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: $\text{Tr}\left[(2\text{diag}(a) - aa^T)x\right] \leq 1$

Slack of this constraint w.r.t. vertex $bb^T \in COR(k)$: $S_{ab} = 1 - \text{Tr}\left[(2\text{diag}(a) - aa^T)bb^T\right] = (1 - a^Tb)^2 = M_{ab}$

► Take slack matrix S for COR, with 2^k vertices bb^T for columns, 2^k a-constraints for first 2^k rows, remaining inequalities for other rows $S = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$

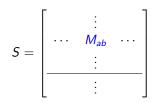


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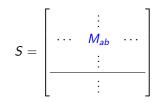
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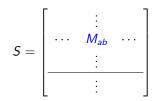
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- Wittgenstein: throw away the ladder after you climbed it

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- Results in functional analysis, other areas of math

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Good to have quantum techniques in your tool-box!