Programming Research Group

SCALAR INVERSES IN QUANTUM STRUCTURALISM

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1 Scalar inverses via calculus of fractions

In a †-compact category [1, 4] the commutative monoid $\mathbf{C}(\mathrm{I.I})$ does not admit (multiplicative) *inverses*, but sometimes it is useful to have them. The naive way to proceed is to construct *fractions* by considering pairs (f, s), with $f : A \to B$ any morphism and $s : \mathrm{I} \to \mathrm{I}$ a scalar, subject to some congruence, such that $(-, s)$ ends up playing the role of $s^{-1} \bullet -$ i.e. $\frac{f}{s} := (f, s)$. Hence we require

$$(f, s) \sim (g, t) \Rightarrow t \bullet f = s \bullet g\,,$$

but only requiring this constraint does not yield transitivity, and we also need to take the potential existence of a *zero scalar* and/or *zero divisors* into account. The required congruence is displayed in the statement of Theorem 1.2, together with a statement about its universality. A scalar s is said to be *positive* if there is an *element* $\psi \in \mathbf{C}(\mathrm{I}, A)$ such that $s = \psi^\dagger \circ \psi$, is *zero* if for all scalars t holds $s \circ t = s$, and is a *divisor of zero* if there is a scalar t such that st is zero. Clearly, there is at most one zero scalar in a category, which we denote by o.

Definition 1.1 A †-compact category is called *local*[1] iff all of its positive scalars are either divisors of zero, or are invertible.

Theorem 1.2 *Every †-compact category* $\mathbf{C}$ *has a universal localisation* $L\mathbf{C}$ *equipped with a †-compact functor* $\mathbf{C} \to L\mathbf{C}$*, which is initial for all local †-compact categories with a †-compact functor from* $\mathbf{C}$*. In particular, the objects of* $L\mathbf{C}$ *are those of* $\mathbf{C}$*, and a morphism in* $L\mathbf{C}(A, B)$ *is in the form* $\frac{f}{s}$*, where*

$$s \in \Sigma = \{s \in \mathbf{C}(\mathrm{I}, \mathrm{I}) \mid \forall t \in \mathbf{C}(\mathrm{I}, \mathrm{I}) : s \circ t \neq o\}\,,$$

and these fractions are taken modulo the congruence

$$\frac{f}{s} = \frac{g}{t} \qquad \Longleftrightarrow \qquad \exists u, v \in \Sigma :\ u \circ s = v \circ t \ \ \&\ \ u \bullet f = v \bullet g\,.$$

Proof: It is easy to see that Σ is a multiplicative system allowing calculus of fractions in the sense of [3], and we took $L\mathbf{C}$ to be the 'category of fractions' $\mathbf{C}[\Sigma]$. The †-compact structure on the fractions is defined pointwise:

$$\frac{f}{s} \otimes \frac{g}{t} = \frac{f \otimes g}{s \circ t} \qquad\qquad \left(\frac{f}{s}\right)^\dagger = \frac{f^\dagger}{s^\dagger} \qquad\qquad \text{etc.}$$

The universal property is proven in [3]. □

Although of less importance for this paper, let us mention that the results in [3] also imply that the limits and colimits of $L\mathbf{C}$ are created in $\mathbf{C}$, including any biproducts, and, that every †-compact category $\mathbf{C}$ also admits a couniversal localisation $\ell\mathbf{C}$ and a †-compact functor $\ell\mathbf{C} \to \mathbf{C}$ which is final for all local †-compact categories with a †-compact functor to $\mathbf{C}$, where $\ell\mathbf{C}$ is the subcategory of $\mathbf{C}$ generated by the elements $\psi \in \mathbf{C}(\mathrm{I}, A)$ which are such that $\psi^\dagger \circ \psi = 1_\mathrm{I}$.

[1]We borrow this terminology from ring-theory: ring is local if it has a unique maximal ideal i.e. if all of its elements are either divisors of zero or or invertible.

2 Categories of pure states

We can use the above together with the WP-construction [2] to produce categories of pure states in the Birkhoff-von Neumann sense i.e. without any redundancies. We define equivalent morphisms within the universal localization:

$$f \sim g \iff \frac{f \otimes f^\dagger}{Tr(f \circ f^\dagger)} = \frac{g \otimes g^\dagger}{Tr(g \circ g^\dagger)}\,.$$

The action of the congruence $\sim$ defines a monad **P** for which the **P**-algebras are exactly those local †-compact categories where the elements are the projectors with unique representing vectors i.e.

$$\frac{f \otimes f^\dagger}{Tr(f \circ f^\dagger)} = \frac{g \otimes g^\dagger}{Tr(g \circ g^\dagger)} \implies f \sim g\,.$$

References

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