An algebra for games of choice

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The simplest game of choice is one whose outcome is determined by a single pair of moves : the player makes the first move by selecting a set p of possible outcomes; and the second move is made by the adversary, who selects the actual outcome from p. The rules of the game are therefore given by a family of sets of outcomes, and the player's choice is restricted to members of this family.

$$\mathbf{G} \subseteq \mathbf{PP}U$$

where U is the universe of outcomes. Some example games, small and large:

$$\{\}, \{U\}, \mathbf{PP}U$$

A game is lost by the participant who is faced with the choice from an empty set. Thus a loss for the player is the empty family; and the player has the opportunity of winning the game whenever the family of choices contains the empty set. We assume that the player will take such an opportunity to win, whenever offered. In fact, it is always good strategy for the player to restrict as far as possible the range of choices available to the adversary. So a game Q is defined to be better than P if Q always allows the player to choose an equal or smaller set than P does

$$P \leq Q \quad \hat{=} \quad \forall p \in P. \exists q \in Q. \; q \subseteq p.$$

The bottom of this ordering is a loss for the player

$$\bot = \{\}$$

For simplicity, we want to ignore the distinction between a game and one which is both as good as it, and as bad. We therefore adopt a standard trick, and represent each game as the union of all games that are as bad or worse. This gives the upward closure ($\uparrow$) of the family of sets.

$$\uparrow P = \{p \mid \exists p' \in P. \; p' \subseteq p\}.$$
$$\mathbf{G} = \{P \mid P \in \mathbf{PP}U \;\; \wedge \;\; P = \; \uparrow P\}$$

Now a win for the player has to be represented as the universal set

$$\top = \; \uparrow \{\{\}\} \; = \; \mathbf{PP}U$$

This artificiality is compensated by a much simpler definition of the ordering of games

$$P \leq Q \;\text{ iff }\; P \subseteq Q.$$

The game $(P \cup Q)$ is one which offers the player a choice of playing either game P or game Q. The choice is exercised by selecting a set either from P or from Q. Alternatively, the player's choice can be restricted to the intersection of two games $(P \cap Q)$. Because of upward closure,

$$(P \cap Q) \; = \; \{p \cup q \mid p \in P \;\; \wedge \;\; q \in Q\}$$

This describes a game in which the player makes an independent selection of p from P and q from Q; it is now the adversary who effectively has the choice of which game to play, by selecting the actual outcome from $(p \cup q)$.

The algebra for games defined so far is a complete lattice.

A game which offers the player no choice is called *sequential*, and one which offers the adversary no choice is called *deterministic*.

$$P \in \text{sequential} \quad = \quad \exists u \subseteq U. P = \uparrow u$$
$$Q \in \text{deterministic} \quad = \quad \exists u \subseteq U. \; Q = \bigcup_{t \in u} \uparrow \{t\}$$

(I would like to explore alternative definitions of sequential = filter and deterministic = prime). So $\top$ is sequential and $\bot$ is deterministic. Sequentiality is preserved by $\cap$. Determinism is preserved by $\cup$.

All games other than $\top$ and $\bot$ are called *proper*. The worst and best of these are defined

$$\bot' = \{U\}$$
$$\top' = \top - \{\}$$

$\bot'$ is sequential and $\top'$ is deterministic.

$$P \text{ is proper iff } \bot' \subseteq P \subseteq \top'$$

Proper games form a complete lattice.

Another way of describing a game is as follows: the player can first choose a set of outcomes which are *not* wanted, and then the adversary chooses an outcome which is *not* a member of the chosen set. A game P played in this way is defined

$$P^o \;\; \widehat{=} \;\; \{\bar{p} \mid p \in \bar{P}\}$$

This operator is a linear negation. Also

$$P^o \text{ is deterministic iff } P \text{ is sequential}$$

Game-splitting by the player is even more favourable than the union of games. The player makes an independent choice from each game, but the adversary's choice has to be permitted by *both* of the player's choices

$$P \sqcup\!\!\!\!\;\lrcorner Q = \{(p \cap q) \mid p \in P \;\; \wedge \;\; q \in Q\}$$

$(\mathbb{G}, \sqcup\!\!\!\!\;\lrcorner, \bot')$ is a commutative monoid

$P \cup Q \subseteq P \sqcup\!\!\!\!\;\lrcorner Q$ if $P, Q \neq \bot$
$P \sqcup\!\!\!\!\;\lrcorner Q = P$ if P is improper
$\sqcup\!\!\!\!\;\lrcorner$ distributes through union.
$P = P \sqcup\!\!\!\!\;\lrcorner P$ if P is sequential
(For finite universes, I think the reverse is also true)
$\sqcup\!\!\!\!\;\lrcorner$ preserves sequentiality

If $P \sqcup\!\!\!\!\;\lrcorner Q \neq \top$, then P and Q are said to be *consistent*.

Game-splitting by the adversary is defined as the conjugate of game-splitting by the player

$$P \sqcap\!\!\!\!\;\ulcorner Q = (P^o \sqcup\!\!\!\!\;\lrcorner Q^o)^o$$

Its properties are also easily calculated by duality from those of $\sqcup\!\!\!\!\;\lrcorner$

So far, we have considered only a single-move game, which is played from a fixed (and therefore unmentioned) starting position. A more interesting game is one with a set S of possible starting positions; it is specified as a function

$$F : S \rightarrow \mathbb{G}$$

which maps each $s \in S$ to the family of moves that are legal for the player in that position. All the operations defined so far are easily lifted to this new kind of game

$$\top \;=\; \lambda s.\top \qquad f \cup g \;\hat{=}\; \lambda s.(f.s \cup g.s).$$

f is sequential if $f.s$ is sequential for all $s \in S$
Equally obvious is the fact that these lifted operations have exactly the same algebraic properties as before.

If f and g are games, we can define a new game by a conditional which tests the initial state. If R is any subset of initial states

$$f \lhd R \rhd g \;\hat{=}\; \lambda s.(f.s \text{ if } s \in R \text{ otherwise } g.s)$$

For example, among the simplest possible games is one which effectively defines some subset W of initial states as winning states for the player, and $\overline{W}$ as winning states for the adversary

$$f \;=\; \top \lhd W \rhd \bot$$

The laws for conditionals are quite familiar. As a binary operator, it distributes through all idempotent operators, and admits distribution by all operators, for example

$$\begin{aligned} f \lhd W \rhd (g \cap h) \;&=\; (f \lhd W \rhd g) \cap (f \lhd W \rhd h) \\ f \sqcup\!\!\!\perp (g \lhd W \rhd h) \;&=\; (f \sqcup\!\!\!\perp g) \lhd W \rhd (f \sqcup\!\!\!\perp h) \end{aligned}$$

Suppose now that we define a subset $W \subseteq U$ of *outcomes* as winning positions for the player of the game. It is then possible to calculate which *initial* positions are winnable by the player. They are exactly the positions that permit the set W itself as a legal move for the player; because then every selection by the adversary is a win. For any game f, there is a function $\overleftarrow{f}$ which maps any set of winning final positions to the set of winning initial positions

$$s \in \overleftarrow{f}.W \;\text{ iff }\; W \in f.s$$

Because $f.s$ is upward closed, this is a monotonic function, traditionally called a predicate transformer.

Conversely, let p be *any* monotonic function from $\mathbf{P}U$ to $\mathbf{P}S$. From this we can define a game

$$\overrightarrow{p}.s \;\hat{=}\; \{V \mid s \in p.V\}$$

Because p is monotonic, $\overrightarrow{p}.s$ is always upward closed. In fact $\overrightarrow{p}$ is just the game whose winning initial states can be computed by p.

$$\overleftarrow{\overrightarrow{p}} = p \;\text{ and }\; \overrightarrow{\overleftarrow{f}} = f.$$

So any game can be equally well defined by its predicate transformer, whenever it is more convenient to do so. Similarly, the operations on games can be defined in either way, as shown by the following laws

$$\overleftarrow{\bot}.W = \{\} \qquad \overleftarrow{\bot'}.W = (S \lhd W = U \rhd \{\})$$

$$\overleftarrow{(f \cup G)}.W = \overleftarrow{f}.W \cup \overleftarrow{g}.W \text{ etc.}$$

$$\overleftarrow{(f^{o})}.W = \neg(\overleftarrow{f}.\overline{W})$$

$$\overleftarrow{(f \sqcup\!\!\!\perp g)}.W = \bigcup_{x \cap y = W} \overleftarrow{f}.x \cap \overleftarrow{g}.y$$

$f \subseteq g$ iff $\overleftarrow{f} \subseteq \overleftarrow{g}$ (where ordering of predicate transformers is pointwise).

For defining sequential composition of games, it is certainly more convenient to use predicate transformers. Let

$$p : \mathbf{P}S \to \mathbf{P}T \text{ and } q : \mathbf{P}U \to \mathbf{P}S$$

Then $p; q : \mathbf{P}U \to \mathbf{P}T$ is defined simply as composition of these two functions

$$(p; q).W \;=\; p(q.W)$$

From this definition, it is easy to see that sequential composition distributes leftward through all the lattice operations on games, e.g.

$$\top; p \;=\; \top \qquad (p \cap q); r \;=\; (p; r) \cap (q; r)$$

Furthermore, if p is sequential

$$p; (q \cap r) \;=\; (p; q) \cap (p; r)$$

and if p is deterministic

$$p; (q \cup r) \;=\; (p; q) \cup (p; r)$$

The definition of composition without predicate transformers would be

$$\overrightarrow{p; q} \;=\; \lambda t \bigcap_{s \in \vec{p}.t} \Big(\bigcup_{u \in \bar{s}} (\vec{q}.u) \Big)$$

which is much more difficult to work with.