Category Theory notes.

for Marktoberdorf

monoids. $(A, ;, id)$ | $id : A$, $; : A \times A \to A$ | A for arrows, $;$ for composition.

examples

$Bool, \wedge, t$ — $f;id = f = id;f$ — identity

$Prog, ;, skip$ — $f \times g \times h =$ — assoc

$Fun, \circ, \lambda x.x$

$Nat, *, 1$

Dual is $(A, \circ, id)^{op}$ where $x \circ y =_{df} y;x$

preorders $(B, \leq)$ | $\leq \subseteq \mathcal{P}(B \times B)$ | B objects, $\leq$ weakens

$(Bool, \vdash)$

$(Prog, \sqsubseteq)$

$(Nat, \leq)$

$x \leq x$ reflex

$x \leq y \;\&\; y \leq z$ transitive

antisymmetry (partial)

$x \leq y \wedge y \leq x \Rightarrow x = y$

symmetry (equivalence

$x \leq y \Rightarrow y \leq x$

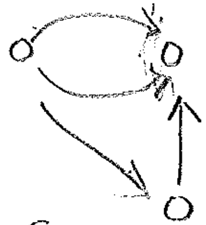
If A is a monoid, define $\leq$ s.t $(A, \leq)$ is a preorder

e.g

Dual is $B, \{(y,x) \mid x \leq y\}$

graphs $(B, A, \text{source}, \text{target})$ $\quad$ source, target: $A \to B$

examples



dominoes $\{\boxed{}, \boxed{o}, \dots\}$ $\{\boxed{o\ o \mid \ }, \dots\}$, $\overleftarrow{\boxed{o \mid \ }} = \boxed{o}$

automata states, transitions

data (entity, relation, domain, range)

$\square \xrightarrow{a} \square, \dots$

heap (object, reference, tail, head)

$\leq$ $(\mathrm{nat}, \{(x,y) \mid x \leq y\}, \mathrm{proj}_1, \mathrm{proj}_2)$ $\quad$ Dual $(B, A, \text{target}, \text{source})$

succ $(\mathrm{nat}, \{(0,1), (1,2), \dots\},$

$\{(x, x+1) \mid x \in \mathrm{nat}\}$ $\mathrm{proj}_1, \mathrm{proj}_2$

pathgraph $(B, \{\langle a_0, f_0, a_1, f_1, \dots, a_n\rangle \mid a_i = \overleftarrow{f_i} \ \& \ \overrightarrow{f_i} = a_{i+1}\}, \text{first}, \text{last}\})$

~~every graph is~~

every order $(B, \leq) = (B, \{(x,y) \mid x \leq y\}, \overset{\mathrm{proj}_1}{\cancel{\text{left}}}, \overset{\mathrm{proj}_2}{\cancel{\text{right}}})$ e.g Nat, $\leq$

0.4

Categories $(B, A, \overleftarrow{\ }, \overrightarrow{\ }, ;, id)$

$; : A \times A \not\to A \qquad id : B \to A$

$\overleftarrow{id_x} = x = \overrightarrow{id_x}$

$id_{\overleftarrow{f}} ; f = f = f ; id_{\overrightarrow{f}}$

$f;g$ defined iff $\overrightarrow{f} = \overleftarrow{g}$

$\overleftarrow{f;g} = \overleftarrow{f}, \quad \overrightarrow{f;g} = \overrightarrow{g}$

$;$ associative

Examples

Haskell (types, partial functions, , , $\circ$, ($\lambda a.a$))

Logic (propositions, proofs, first, last, cat, $\lambda x.x$)

Set (sets, functions, . . .

Fin (finite sets, . . .

Inj (sets, injections, . . .

. . . Nat, matrices

Pathgraphs $; = cat$

$(B, A, first, last, cat, \langle\rangle)$

where $(s \frown \langle a\rangle) ; (\langle a\rangle \frown t) = (s \frown t)$

A Monoids is a category with only one object

A preorder is a category with at most one arrow between any pair.

$f : x \to y$ means $\overleftarrow{f} = x$ & $\overrightarrow{f} = y$

$\xrightarrow{f}\xrightarrow{g}$ commutes means $\longrightarrow = f;g$

(Homo)morphisms

relate one structure to another

monoid : $\phi : (A, ;, id) \longrightarrow (A', ;', id')$ implements one structure to another

$\phi : A \to A$

iff.

$\phi(id) = id'$

$\phi(f;g) = (\phi f) ;' (\phi g)$

compositionality

respects composition

E.g.

$mod_2 : (Nat, +, 0) \longrightarrow (\{0,1\}, \not\equiv, 0)$ surjective

$inj : (\{0,1\}, \not\equiv, 0) \hookrightarrow (Nat, +, 0)$ injective

$id_M : M \hookrightarrow M$

$fold : (A^*, \wedge, \langle\rangle) \longrightarrow (A, ;, id)$

Mon is category of monoids and monoid morphisms

$post : \{A, ;, id\} \longrightarrow ((A \to A), \circ, \lambda x . x)$ pre similar

$post\ f = \lambda g . g;f$

order $\phi : (B, \leq) \rightarrow (B', \leq')$ $\phi : B \rightarrow B$

$x \leq y \Rightarrow (\phi x) \leq' (\phi y)$

$\times 3 : (Nat^+, \leq) \rightarrow (Nat^+, \leq)$ injective

$\div 3 : (Nat^+, \leq) \rightarrow (Nat, \leq)$ surjective

NB $x \leq (y \times 3)$ iff $x \div 3 \leq y$

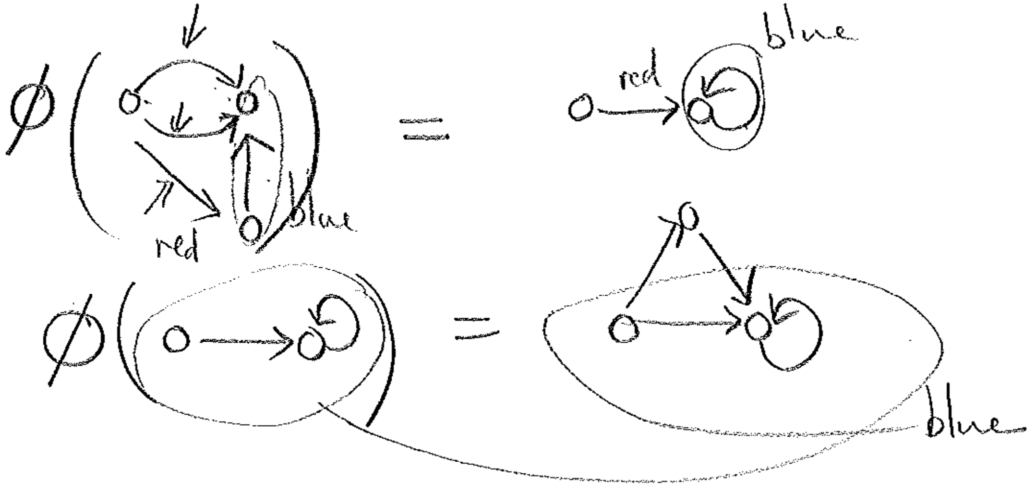
Pre is category of preorders & monotonic functions

Graphs

Graphs $\quad \phi : (B, A, \overleftarrow{src}, \rightarrow) \longrightarrow (B', A', \leftarrow', \rightarrow')$ $\quad B \to B + A \quad \phi : (B \to B) * (A \to A)$

iff $\phi = (\phi_B + \phi_A)$, $\cancel{\phi_B : B \to B' \quad \phi_A : A \to A'}$ $\quad \cancel{\phi = \phi_B + \phi_A}$

$\phi_B \overleftarrow{f} = \overleftarrow{\phi_A f}' \qquad \phi_B \overrightarrow{f} = \overrightarrow{\phi_A f}$



surjective

injective

GRF is category of graphs & their morphisms.

~~Categories~~ Functors

Functor. $\phi : (B, A, \leftarrow, \rightarrow, ;, id) \longrightarrow (\qquad)'$

if $\phi : (B, A, \leftarrow, \rightarrow) \rightarrow$ is a graph morphism

$\phi_A(id_x) = id_{(\phi_B x)}$

$\phi_A(f;g) = (\phi f);(\phi g)$

Examples ~~next week lift: $Set \rightarrow PFn$ lift f = f~~ injective.

map : Hask $\rightarrow$ Hask* injective

length : Hask* $\rightarrow (Nat, +, 0)$ surjective

every

Cat is category of small categories and functors

now do

Natural Transformations.

Monoids.

$\eta : \phi \to \psi$

N, M; Mon $\eta \in M$

$\phi, \psi : N \to M$,

$\eta ; \psi(f) = \phi(f) ; \eta$

$$\begin{tikzcd} \bullet \arrow[r, "\psi f"] & \bullet \\ \bullet \arrow[u, "\eta"] \arrow[r, "\phi f"'] & \bullet \arrow[u, "\eta"'] \end{tikzcd}$$

number

abs ψg

conc ϕg

$N = R^*$ $\quad \phi = R^* \to R+$

$\psi \quad R^* \to R\times$

$\eta =$

f, g numbers

$\psi (R^+ \times, 1 \to R, +, 1)$

$n = \log$

Ordering

$\eta : \phi \to \psi$

$(B, \leq), \beta' \subseteq$: Pre $\quad \eta : B \to \leq'$

ϕ, ψ monotonic

$\eta_x = (\phi x, \psi x)$ i.e proves domination

$\eta x ; \psi(x, y) = \phi(x, y) ; \eta_y$

$\phi x \leq \psi x$ all x

$$\begin{tikzcd} \psi x \arrow[r, "\psi(x,y)"] & \psi y \\ \phi x \arrow[u, "\eta_x"] \arrow[r, "\phi(x,y)"'] & \phi y \arrow[u, "\eta_y"'] \end{tikzcd}$$

isomorphism of categories

<u>after natural transformations</u>

$f : C \to C' \qquad f' : C' \to C$

$f ; f' = \mathrm{Id}_C \qquad f' ; f = \mathrm{Id}_C$

equivalence of categories

$\cancel{f ; f' \equiv}$

f fa

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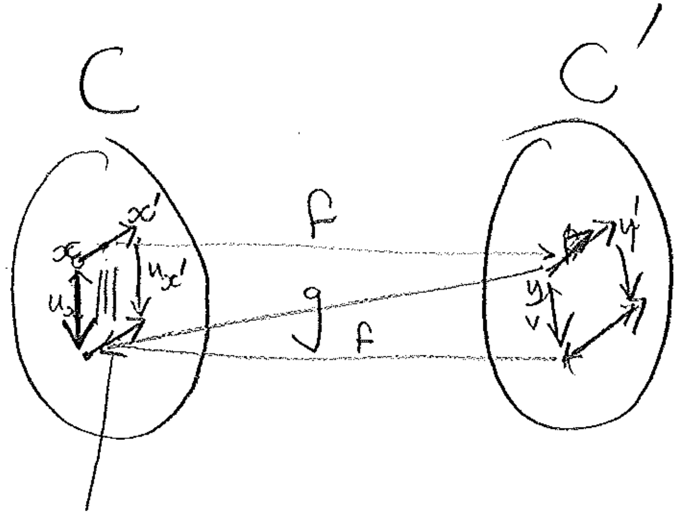


$u^{-1} ; g f(\to) u$

object t is terminal means

$$\forall x : B\ \exists 1 f.\ a : x \to t$$

could do Functors first. but I won't

written $o \cdots\to o^{t}$

$\boxed{\top = \text{max in an order.}}$

~~object n is initial means~~

$$\cancel{\forall x : B\ \exists 1 f : A.\ f : x \to t}$$

The dual of $C = (B, A, \to, \leftarrow, ;, id)$ is $\overset{\leftarrow}{C} = (B, A, \overset{\to}{,}\ \overset{\leftarrow}{,}\ , \circ, id)$

initial in C means terminal in C^{op} $\{\}$ in Grf $\{\}$ in Set in Mon, Cat?

in Set Arrows from $\{42\}$ to $\cancel{x}$ correspond to members of $\cancel{x}$

in Set. $\forall \xi$ in x $\Longrightarrow$ $(K\xi) : \{42\} \to S$

$$S \neq \{\} \Rightarrow \forall f : \{42\} \to S \Longrightarrow f = K\xi \text{ for exactly one } \xi \text{ in } S$$

in Set $f : x \to \{\} \Rightarrow x = \{\}$

Isomorphism

$$
\begin{tikzcd}
& \circ \arrow[r, "id"] \arrow[dr, "g"] & \circ \\
\cancel{A}\,x \arrow[ur, "f"] \arrow[r, "id"'] & \cancel{A}\,x \arrow[ur, "f"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \circ \arrow[dr, "f"] & \\
y\,B \arrow[ur, "g"] \arrow[rr, "id"'] & & \cancel{B}\,y
\end{tikzcd}
$$

means f, g are iso

~~A~~ and ~~B~~ isomorphic.

Theorem. all terminal objects are isomorphic

Proof. let $t1$ and $t2$ be terminal

$$
\begin{tikzcd}
t1 \arrow[loop left, dashed, "id"] \arrow[r, dashed, shift left] & t2 \arrow[l, dashed, shift left]
\end{tikzcd}
$$

commuting diagram
$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$$
commutes means $\longrightarrow = f;g$

f epimorphism
$$\begin{tikzcd} & \cdot \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr] & & \cdot \end{tikzcd}$$
for at most one $\dashrightarrow$

(1)

f is monomorphism
$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ T \arrow[ur, dashed, "g"] \arrow[ur, dashed, "h"'] \arrow[rr] & & B \end{tikzcd}$$
for at most one

$g;f = h;f \Rightarrow g = h$ monic all g, h

in Set monic $\equiv$ injective

$f\,x = f\,y \Rightarrow K_x;f = K_y;f \overset{\text{monic}}{\Rightarrow} K_x = K_y \Rightarrow x = y$

e.g if $S \subseteq T$ epic $\equiv$ surjective

$\lambda x{:}S.\, x$ is monic.

injex

injective $\Rightarrow$ monic

$\forall x\, f(g\,x) = f(h\,x)$

$\underset{\text{injective}}{\Longrightarrow} \forall x\; g x = h x$

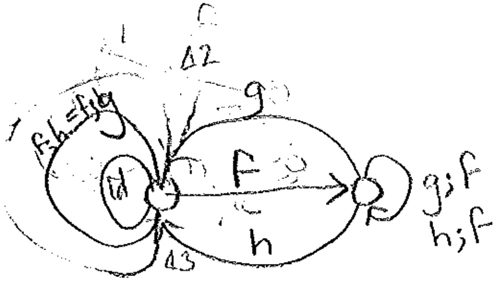
x is a subobject of y is an monic $f: x \to y$

a subc

all $f: \{42\} \to S$ is monic $g;f = h;f \Rightarrow g =$

all $f: S \to \{42\}$ are epic   no $f;g = f;h$

is even initial $a\{\} \to S$ monic yes in set



Slice Category

let $b \in B \in \mathcal{C}$. $\mathcal{C}/b = \{ f : x \to b \mid x \in \mathcal{C} \}, \vec{tri},$

has objects
$$\begin{tikzcd} x \arrow[d, "j"] \\ b \end{tikzcd} \qquad x \in \mathcal{C}$$

has arrows
$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "j"'] & y \arrow[d, "k"] \\ & b \end{tikzcd} : \begin{tikzcd} x \arrow[d, "g"] \\ b \end{tikzcd} \longrightarrow \begin{tikzcd} y \arrow[d, "h"] \\ b \end{tikzcd}$$

has composition
$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[dr, "j"'] & \cdot \arrow[r, "g"] \arrow[d, "k"] & \cdot \arrow[dl, "l"] \\ & b & \end{tikzcd} = \begin{tikzcd} \cdot \arrow[rr, "f;g"] \arrow[dr, "j"'] & & \cdot \arrow[dl, "l"] \\ & b & \end{tikzcd}$$

$\mathcal{C}^{\to} = P$

what is the arrow category

(it commutes because $j = f;k$ and $k = g;l$

what is terminal object? $\therefore\ j = (f;g);l$

2.5

Coproduct of ~~two categories~~

$(B\times B',\ A\times A',\ \leftarrow \times \leftarrow',\ \rightarrow \times \rightarrow',\ ;\times ;',\ (id, id'))$

$(B+B',\ A+A',\ +,\ +,\ +,\ id+id')$

sum.

product of Preorders $=$ $(x, x') \leq_\times (y, y')$ iff $x \leq y$ and $x' \leq' y'$

product of Monoids $=$ $(f, f');(g, g') = (f;g, f';g')$

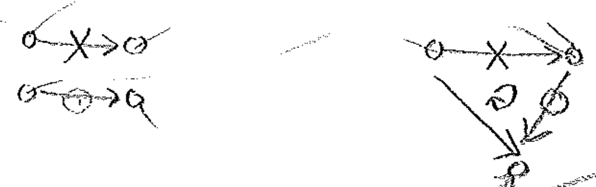
Functors 3.1

Projection: $P_1 : C \times C' \to C$ is a functor

arrow cat $C^{\to}$

$P_1((f, f');(g, g')) = $ ~~$B\times B$~~

$2+2$ $\quad$ 3



A, ~~$A\times A$~~, $\{(x,(f,g),y) \mid x;f = g;y\}$, first, third,

$$\begin{tikzcd} \cdot \arrow[r, "f"] & \cdot \arrow[r, "f'"] & \cdot \\ \cdot \arrow[u, "x"] \arrow[r, "g"'] & \cdot \arrow[u, "y"] \arrow[r, "g'"'] & \cdot \arrow[u, "z"'] \end{tikzcd} = x\uparrow$$

image is not a category

its ~~target~~ certainly is

Forgetful: $U : Mon \to Set$

$U(A, ;, id) = A$

$U(\phi : A \to B) = \phi : A \to B$

$U(C/b) = \{\overleftarrow{x} \mid x \to b \in C\}$

$; \to y \mid x \xrightarrow{f} y \to b = x \to b$

arrows $\{f : x \to y \mid \exists g : y \to b\}$

Functors are   need products & sums

2+2

$$\circ \xrightarrow{x} \circ$$

$$\circ \xrightarrow{\theta} \circ$$

$$\begin{tikzcd} \circ \arrow[r, "x"] \arrow[dr] & \circ \arrow[d, "\phi"] \\ & \circ \end{tikzcd}$$

image not a category

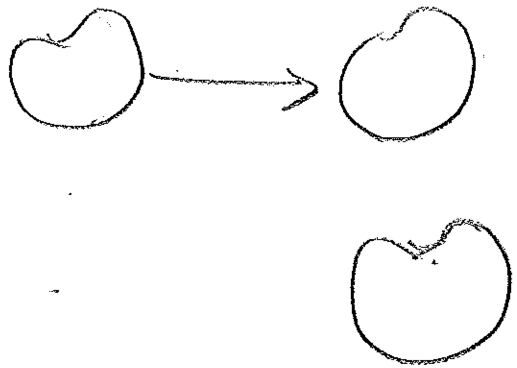
$$\mathrm{Proj}_1 : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

$$\mathrm{Proj}_1 (f, g) = f$$

$$P_1\big(((f,f');(g,g'))\big) = P_1(f;g,\ f';g') = f;g = P_1(f,f');P_1(g,g')$$

$$\mathrm{Inj}_1 : \mathcal{C} \to \mathcal{C} + \mathcal{C}$$

maps $\mathcal{C}$ into its first copy



Powersets

$\mathcal{P} : Set \to Set$

$\mathcal{P}_0 S = \{x \mid x \subseteq S\}$

$\mathcal{P}_1(f : S \to T) = \lambda x : \mathcal{P}S . \{f\xi \mid \xi \in x\}$ direct image

or inverse image

$\mathcal{P} : Set^{op} \to Set$ contravariant functor

$\mathcal{P}(\underbrace{f : S \to T}_{\text{in } Set^{op} = f : T \to S \text{ in } Set}) = \lambda x : \mathcal{P}S \{\xi : T \mid f\xi \in x\}$ note f a function from T to S

$\mathcal{P}f$

$wp : Rel^{op} \to Set$

$wp\,(S) = \mathcal{P}S$

$wp\,(\underbrace{R : \cancel{S \to T}}_{R \subseteq T \times S \text{ in } Rel}) \stackrel{T \to S \text{ in } Rel^{op}}{} =_{df} \lambda x : \mathcal{P}T . \{\xi : S \mid \forall \psi : T \ R(\psi, \xi) \Rightarrow \psi \in x\}$

Free $F: Set \to Mon$

$$F s = s^*$$

$$F(f: s \to t) = map\ f : s^* \to t^*$$

$$F(f;g) = map(f;g) = map\ f ; map\ t$$

$F: Grf \to Cat.$

$F(B, A, \leftarrow, \rightarrow) =$ path graph with "$s\hat{}\langle x \rangle ; \langle x \rangle \hat{} t = s \hat{} t$"
and $id\ x = \langle x, x \rangle$

F

Forgetful $U: Mon \to Set$

$$U(A, ;, id) = A$$

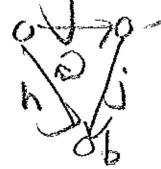
$$U(\phi: A \to B) = \phi: A \to B$$

$U: Cat \to Grf$

$S \xrightarrow{h}$ $\quad S^* \xrightarrow{\hat{h}}$

$U: C/b \to C.$

objects: $\{ x \mid x \to b \in C \}$

arrows $\{ f: x \to y \mid$ ~~are objects~~ $\exists h j.$  $\}$

one

homofmer: post: $(A, ;, id) \longrightarrow (\{f : A \to A\}, ;, id)$ a Monoid to a Set action 3.4

post $f\ h =_{df} \lambda h\ h;f$ called an <u>action</u>

$$\text{post } id = \lambda h\ h;id = id$$

$$(\text{post } f);(\text{post } g) = \lambda h\ (\text{pre } g)(\underbrace{(\text{post } f)\ h}_{h;f})$$

$$= \lambda h\ h;f;g$$

$$= \text{post}(f;g)$$

These are examples of homomorphism. They lead up to the Yoneda lemma (use for Hom too I think)

$\text{pre} : M^{op} \to \text{Set}$

$$\text{pre } f = \lambda h\ f;h$$

$$(\text{pre } f);(\text{pre } g) = \lambda h\ \text{pre } g\ (f;h)$$

$$= \lambda h\ \text{pre } g;f;h$$

$$= \ |$$

$\downarrow (B, \leq) \to (\mathcal{P}B, \subseteq)$ A   downset

$\downarrow x = \{z \mid z \subseteq x\}$

$\downarrow (x, y) = \downarrow x \to \downarrow y = \downarrow x \hookrightarrow \downarrow y$

$(\downarrow (x, y)) ; (\downarrow (y, z)) = (\downarrow x \hookrightarrow \downarrow y) ; (\downarrow y \to \downarrow z) = \downarrow x \hookrightarrow \downarrow z$

similarly $\uparrow (B, \subseteq)^{op} \to Set$ upset

Natural transform compare functors

like functors compare categories
morphisms compare objects

In monoids:

$\eta : \phi \to \psi$	$\phi, \psi : M \to M'$

$$\begin{tikzcd} \circ \arrow[r, "\psi f"] & \circ \\ \circ \arrow[u, "\eta"] \arrow[r, "\phi f"'] & \circ \arrow[u, "\eta"'] \end{tikzcd}$$

$\eta ; \psi f = \phi f ; \eta$ all f.

no example

Language for

$\cancel{\phi \eta \psi} : (S^*, \hat{}, \langle\rangle) \xrightarrow{\eta} ((N \to N), ;, id)$ ϕ is more accurate

$\eta : T \to T$ has image $\subsetneq T$ $S = \{+7, \times 12, -1, \ldots\}$

$\eta ; \psi f = \phi f ; \eta$ eg. ψ"$+n$" $= +n \mod 2^{16}$ $\psi\, n = +n \mod 2^{24}$

$\eta =$ truncate to single length

$\phi, \psi : (S^*, \hat{}, \langle\rangle) \to (PT, \cap, true)$

$\eta \subseteq T$

$\eta \cap \phi s = \psi s \cap \eta$ $\eta \cap \phi = \eta \cap \psi$

in preorders a natural transformation is a triple

$$\frac{\eta : \phi \to \psi}{} \quad \Bigg| \quad \frac{\phi, \psi : (B, \leq) \to (B', \leq') \text{ monotonic} \qquad \eta : B \to \leq' \qquad \forall x : B.\ \phi x \leq' \psi x}{}$$

$$\begin{tikzcd}
\psi x \arrow[r] & \psi y \\
\phi x \arrow[u, "\eta_x"] \arrow[r] & \phi y \arrow[u, "\eta_y"']
\end{tikzcd}$$

where $\eta_x =_{df} (\phi x, \psi x)$

is the witness that $\phi x \leq' \psi x$

Composition 4.4

let $\eta : F \to G$ and $\eta' : G \to H$ for $F\,G\,H : C \to C'$

define $(\eta ; \eta')_b = \eta_b ; \eta'_b : F \to H$

it is a natural transformation

$$
\begin{tikzcd}
F(B) \arrow[r, "\eta_B"] & Gb \arrow[r, "\eta'_B"] & H_b \\
F(A) \arrow[u, "Ff"] \arrow[r, "\eta_A"'] & G(A) \arrow[u, "Gf"] \arrow[r] & HA \arrow[u, "Hf"]
\end{tikzcd}
$$

$id_F = id : F \to F$

$$
\begin{tikzcd}
FA \arrow[r, "\eta_a"] & Fa \\
Fa \arrow[u, "id"] \arrow[r, "\eta_a"] & Fa \arrow[u, "id"']
\end{tikzcd}
$$

So Functors$(C \to C')$, Nat, ... ~~are~~ a category F...

HOM $\mathcal{C} \to Set$

$\mathcal{C}(x,y) =_{df} \{ g \mid \overleftarrow{g} = x \;\&\; \overrightarrow{g} = y \}$ is a set

Let x be an object of $\mathcal{C}$

different post for each x

co

$Hom(x,-) : C \to Set$

$Hom(x,-)(g : y \to z) = post : C(x,y) \to C(x,z)$

$$\begin{tikzcd} y \arrow[r, "g"] & z \\ x \arrow[u, "h"] \arrow[ur] & \end{tikzcd}$$

$f : D \to C$

$h : C \to A$

$$\begin{tikzcd} Hom\,D\,A \arrow[r, "{Hom(D,k)}"] & Hom(D,B) \\ Hom(C\,A) \arrow[u, "{Hom(f,A)}"] \arrow[r, "{Hom(C,k)}"] & Hom(C\,B) \arrow[u, "{Hom(f,B)}"'] \end{tikzcd}$$

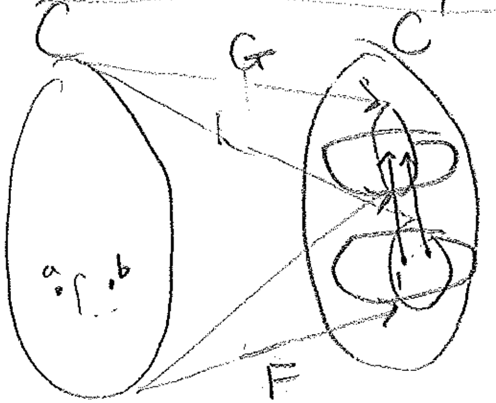
$Hom(-,y) : C^{op} \to Set \quad = \; pre$

$Hom(-,-) : C^{op} \times \mathcal{C} \to Set$

$Hom(-,-)(f,g) = \lambda h \;\; f;h;g$

in Categories

$\eta : F \to G$



$G, F : C \to C'$

$\eta : \mathrm{obj}\, C \to \mathrm{arr}\, C'$

$$\begin{tikzcd} \circ \arrow[r, "Gf"] & \circ \\ \circ \arrow[u, "\eta_x"] \arrow[r, "Ff"'] & \circ \arrow[u, "\eta_y"'] \end{tikzcd}$$

commutes in C'

η "shifts" the image of F to that of G.

e.g. let $F, G : \left(\begin{tikzcd} \overset{A}{\circ} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \overset{N}{\circ} \end{tikzcd} \right) \longrightarrow \mathrm{Set}$

images of F and G are (different) graphs (models of the diagram)

Let (ϕ_A, ϕ_N): image $F \to$ image G be a graph homomorphism

$$\begin{tikzcd} \circ \arrow[r, "G(\mathrm{source})"] & \circ\, GN \\ F(A) \arrow[u, "\phi_A"] \arrow[r, "F(\mathrm{source})"'] & \circ\, F(N) \arrow[u, "\phi_N"'] \end{tikzcd}$$

and similarly for target

it's a natural transformation!

Data refinement.

An Data Type is a functor into Set.

A data abstraction is a natural transformation

$$\begin{tikzcd} \circ \arrow[r, "G(succ)"] & \circ \\ \circ \arrow[u, "abs"] \arrow[r, "F(succ)"'] & \circ \arrow[u, "abs"'] \end{tikzcd}$$

similarly for pred, ...

a data diagram
$$\begin{tikzcd} \underset{int}{\circ} \arrow[r] & \overset{set}{\circ} \arrow[l, bend right] \arrow[dl] \\ \underset{bag}{\circ} & \end{tikzcd}$$
needs a different abstraction for each node.

~~The arrow category of C~~ $\cong$ ~~Func((o → o), C)~~

$$\begin{tikzcd} \circ \arrow[r, "G(!)"] & \circ \\ \circ \arrow[u, "\eta_0"] \arrow[r, "F(!)"'] & \circ \arrow[u, "\eta_1"'] \end{tikzcd}$$

Natural isomorphism

Equivalence of cat but they are not isomorphic

Mon = Cat$_{\#}$ with only one object ??

~~$F;G = \mathrm{Id}$ in CAT~~ MON

~~$\forall F\,G(f) = f$ in Cat$_1$~~

~~$\forall\, G\,F(g) = g$ in~~ no

$F : \mathrm{Mon} \to \mathrm{Cat}_1$ $\quad U : \mathrm{Cat}_1 \to \mathrm{Mon}$

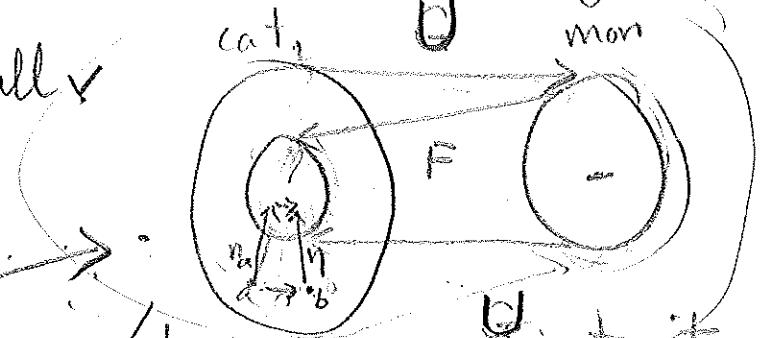
$F(A; id) = (\{0\}, A, !, !, ;, K id)$

$F(f : M \to M') = (\{(0,0)\}, f) : FM \to FM'$ in Cat$_1$

draw this first.

$U(\{x\}, A, !, !, ;, id) = (A, ;, id)$ all v

$U((\phi_0, \phi_A) : Gx \to Gy) = f_A$



$UF(f : M \to M') = f$ (ok)

$FU(\{v\}, \ldots) = (0, \ldots) \neq$

but isomorphic to it.

$F; U = \mathrm{Id}$ in Mon

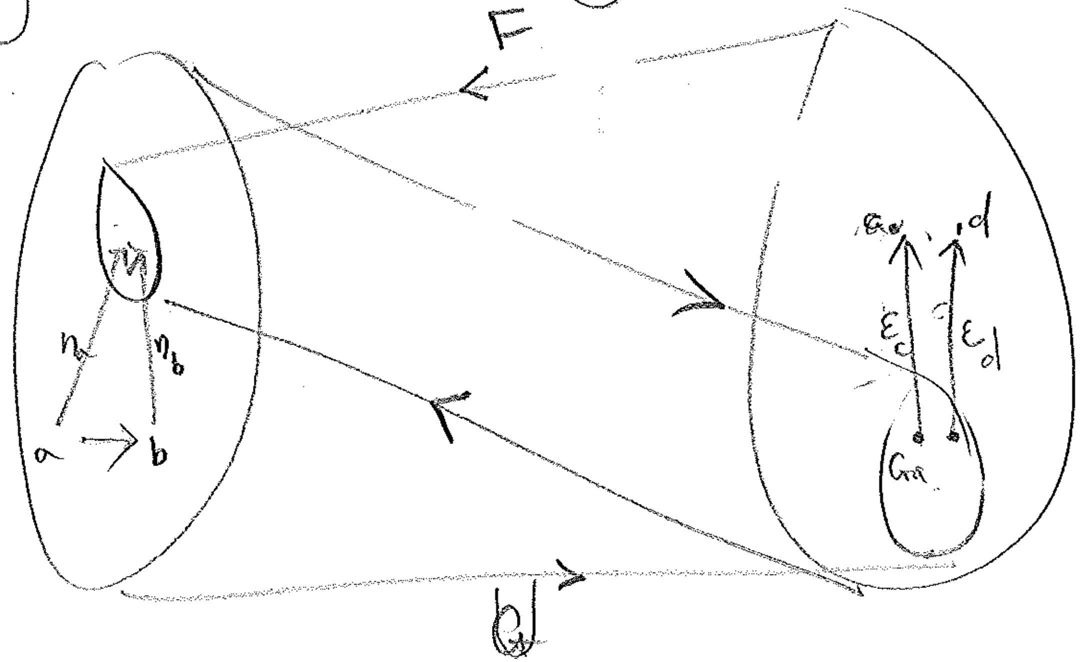
$$\begin{tikzcd} F(G_c) \arrow[r, "G;F"] & FG(c) \\ G_c \arrow[u, "\eta_{G_c}"] \arrow[r, "\mathrm{Id}"] & F(c) \arrow[u, "\eta_c"'] \end{tikzcd}$$ in Cat$_1$

$\eta_x(x, A, \ldots) =_{df} (0, A, \ldots)$

is an isomorphism

$\eta : \mathrm{Id}_{\mathrm{Mon}} \to U; F$ in Cat$_1$

In general ~~two~~ categories are equivalent if (F, U) is an equivalence of



p 261

$\varepsilon : F ; U \to \mathrm{Id}$

$\eta : \mathrm{Id} \to U ; F$

~~They~~ are both natural isomorphisms

left An adjunction is a right

$(F, U, \varepsilon, \eta)$ st $\varepsilon : F ; G \to \mathrm{Id}$ $\eta : \mathrm{Id} \to G ; F$ are nts natural transformations.

$f = \eta_a ; Ug$ iff $g = Ff ; \varepsilon_b$ for $f : a \to Ub$ & $g : Fa \to b$.

$C(a, Ub) \sim C(Fa, b)$

Products and Sums. pp137 – 164

a product of A and B is $(C, proj_1, proj_2)$

$proj_1 : C \to A$, $proj_2 : C \to B$.

use p_1 p_2

C is a product $A \times B$

$\forall D.\ \forall q_1 : D \to A,\ \forall q_2\ D \to B$

$\exists 1 q\ (: D \to C)\ q ; proj_1 = q_1$ and $q ; proj_2 = q_2$

call it $\langle q_1, q_2 \rangle$

$$\begin{tikzcd} & D \arrow[dl, "q_1"'] \arrow[d, dashed, "q \langle q_1, q_2 \rangle"] \arrow[dr, "q_2"] & \\ A & C \arrow[l, "proj_1"] \arrow[r, "proj_2"'] & B \end{tikzcd}$$

draw it the other way up.

in **Set** a product is $(A \times B, fst, snd)$ because

For $q : D \to A \times B$

$q = \lambda z (q_1 z, q_2 z)$

iff $q_1 = q ; proj_1$ and $q_2 = q ; proj_2$

in a Preorder $\leq$ it is glb

$(a \wedge b, (a \wedge b, a), (a \wedge b, b))$

$$\begin{tikzcd} a & & b \\ & a \wedge b \arrow[ul] \arrow[ur] & \\ & D \arrow[u, dashed] \arrow[uul] \arrow[uur] & \end{tikzcd}$$

in Mon, and Pre all arrows are morphisms.

A category is <u>cartesian</u> if all pairs of objects have a product.

Spans

A span on A, B is
$$\begin{tikzcd} & C \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ A & & B \end{tikzcd}$$
i.e (C, q_1, q_2)

A span^{A,B} morphism is
$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr] & C \arrow[dl] \arrow[d] \\ A & B \end{tikzcd}$$

composition
$$\begin{tikzcd} \circ \arrow[r] \arrow[d] \arrow[drr] & \circ \arrow[r] \arrow[dl] \arrow[dr] & \circ \arrow[dll] \arrow[d] \\ \circ & & \circ \end{tikzcd}$$

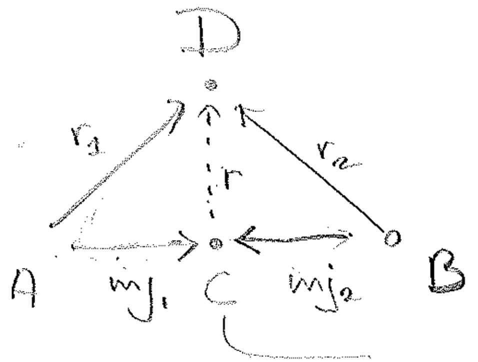
A product of A, B is the terminal object in the category of spans over A, B

Thm. it is unique up to unique isomorphism (if it exists)

Sums.

A sum of A and B is

(C, inj_1, inj_2)



call it $\langle r_1 | r_2 \rangle$

initial object in category of cospans

In set disjoint union.

$A+B = (\{0\} \times A \cup \{1\} \times B, \lambda x.(0,x), \lambda y.(1,y))$

all $r : A+B \to D$

$r\,z = \text{if } p_1 z = 0 \text{ then } r_1(p_2 z) \text{ else } r_2(p_2 z)$

for unique r_1 r_2

in a preorder it is lub

in pre it is +

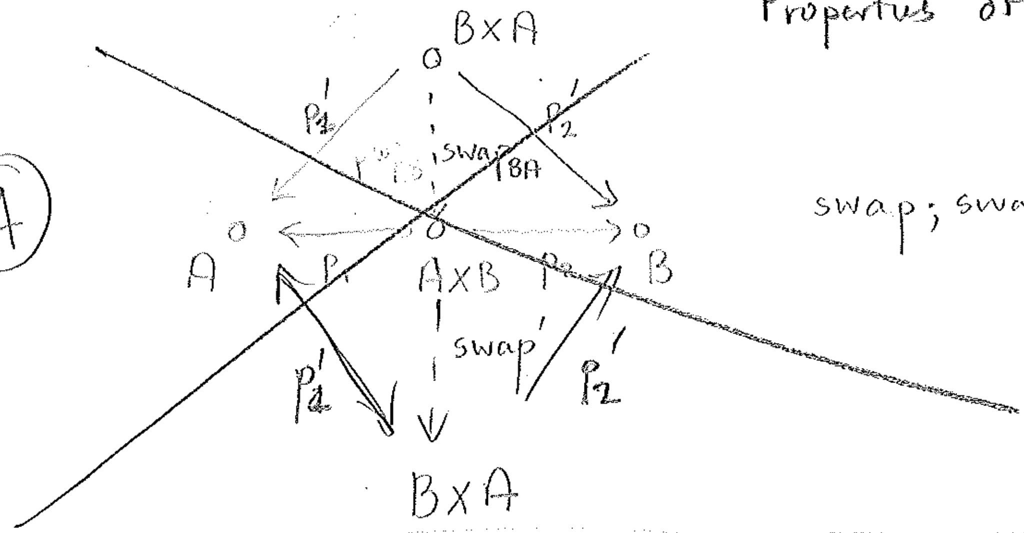
in mon???

VI

VII

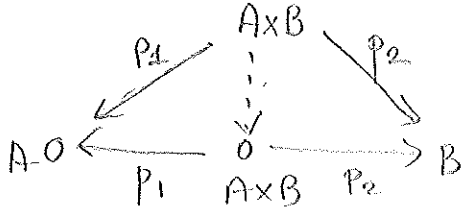
Properties of × in cartesian ~~closed~~ category

(4)



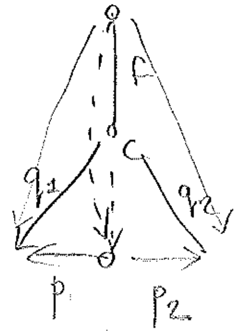
swap; swap = id

(1)



$\langle p_1, p_2\rangle = id_{A\times B}$ — the only such arrow

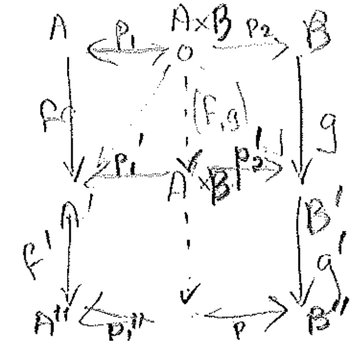
(2)



$f; \langle q_1, q_2\rangle = \langle fq_1, fq_2\rangle$

(3)

(4)

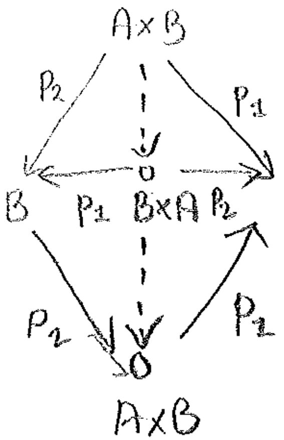


$f\times g = \langle p_1;f, p_2;g\rangle =$

$(f\times g);(f'\times g') = (f;f')\times(g\times g')$

(5)

(6)

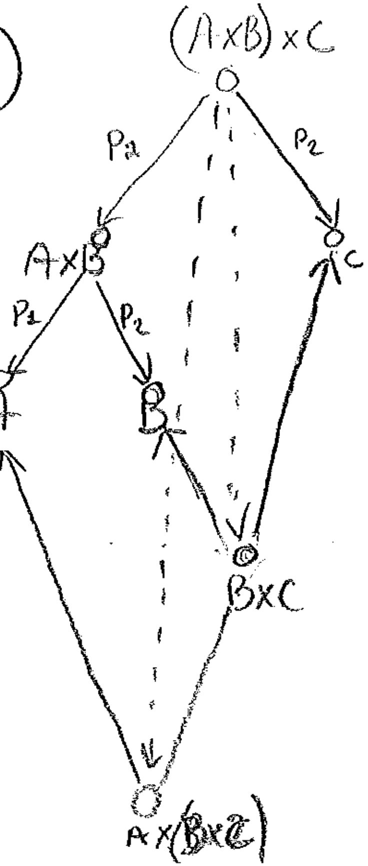


$$swap_{AB}: A \times B \to B \times A$$

$$= \langle p_2, p_1 \rangle$$

$$swap ; swap = id$$

(7)



$$\alpha : (A \times B) \times C \to A \times (B \times C)$$

$$\alpha = \langle p_{1\,(AB)C} ; p_{1\,AB}, \langle p_{1\,A} ; p_2, p_2 \rangle \rangle$$

Duality $\langle inj_1 \mid inj_2 \rangle = id_{A+B}$

$$\langle r_1 \mid r_2 \rangle ; f = \langle (r_1 ; f) \mid (r_2 ; f) \rangle$$

etc

Limits and Colimits

1

Equaliser of $f, g : A \to B$ in **Set**. equationally defined subset

$E(f,g) = \{x : A \mid fx = gx\} \hookrightarrow A$

$= \{(x,y) \mid fx = gy\}$

$\mathbb{R} \times \mathbb{R} \to \mathbb{R}$

eg $E(\lambda x, y\ x^2 + y^2 \quad 1)$ = unit circle

$E(\text{source}, \text{target}) = \{O \mid o \in N\}$ endo morphisms

thm. $E(f,g) ; f = E(f,g) ; g$ domainrestriction

if $e' ; f = e' ; g \Rightarrow e' \overset{?}{\subseteq} E(f,g)$ ✗

$\Rightarrow \exists i \quad i ; E(f,g) = e'$ e' factors through $E(f,g)$

category of ~~weak~~ equalisers

$$\begin{tikzcd} \circ \arrow[r] \arrow[dr, "e'"'] & \circ \arrow[d, "e"] \\ & A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ & B \end{tikzcd}$$

objects $= \{e' : E \to A \mid e ; f = e ; g\}$

arrows $= \{i : e' \to e \mid i ; e = e'\}$

subcat of A/A

equalisers are monic

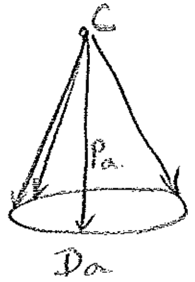
the equaliser is the terminal object in this category

coequaliser $CE(f,g)$ = equaliser in opposite category

$f ; e = g ; e$ $\quad c = \lambda y : B . \{[y]_{R^\#}\} \mid R^\#$ is smallest equiv including $R = \{(f(s), g(s)) \mid s \in A\}$

Cones and Cocones

Let $D : \mathcal{G} \to \mathcal{C}$ ($\mathcal{G}$ a graph, $\mathcal{C}$ a category) (i.e. a diagram in $\mathcal{C}$)

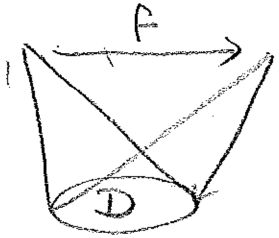


a cone over D is an object C (its apex) and and a family p : nodes$(\mathcal{G}) \to$ arrows$(\mathcal{C})$ s.t

$$\begin{tikzcd} & C \arrow[dl, "p_a"'] \arrow[dr, "p_b"] & \\ Da \arrow[rr, "Ds"'] \arrow[urr, phantom, "\mathcal{G}"] & & Db \end{tikzcd}$$

all a, b, s

a morphism of cones: $f : C \to C'$ =



s.t

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p_a"'] & \mathcal{G} & C' \arrow[dl, "p'_a"] \\ & Da & \end{tikzcd}$$

all a

limit (D) =$_{df}$ terminal cone

cones $D : (\overset{a}{o}) \to \mathcal{C}$ = $\overset{o}{\downarrow} Da$ - cone slice category = $\mathcal{C}/Da$

cones $(D : (\overset{a}{o} \; \overset{b}{o}) \to \mathcal{C}')$ = spans over Da D_b

terminal cone $(D : \text{Set} \to \mathcal{C}')$ = product

$$\begin{tikzcd} & C \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Da & & Db \end{tikzcd}$$

complete means has all limits

cones $(D : o \rightrightarrows o \to \mathcal{C})$

$$\begin{tikzcd} & E \arrow[dl, "e"'] \arrow[d, "h"] \\ Da \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D_b \end{tikzcd}$$

= equalisers $e;f = e;g = h$

Pull-backs

let $D : \begin{tikzcd} & S \arrow[d, "f"] \\ U \arrow[r, "g"'] & T \end{tikzcd} \longrightarrow \mathcal{C}$ pull-back = limit D

$$\begin{tikzcd}
\cdot \arrow[dr, dashed] \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "k"'] & & \\
& \cdot \arrow[r, "p_1"] \arrow[d, "p_2"'] & S \arrow[d, "f"] \\
& U \arrow[r, "g"'] & T
\end{tikzcd}$$

pullback diagram

= product in $\mathcal{C}/T$ (draw)

in set $\text{pullback}(f,g) = \left(\underbrace{\{ (s,u) \mid fs = gu \}}_{\subseteq S \times T}, p_1, p_2 \right)$

if $g = U \hookrightarrow T$ $= (s,u) \mid fs = u \;\&\; u \in U$

$= \{ (s, fs) \mid fs \in U \}$

$\equiv f^{-1}(U)$

$$\begin{tikzcd}
f^{-1}(U) \arrow[r, hook, "p_1"] \arrow[d, "f|U"'] & S \arrow[d, "f"] \\
U \arrow[r, hook, "\iota"'] & T
\end{tikzcd}$$

domain restriction.

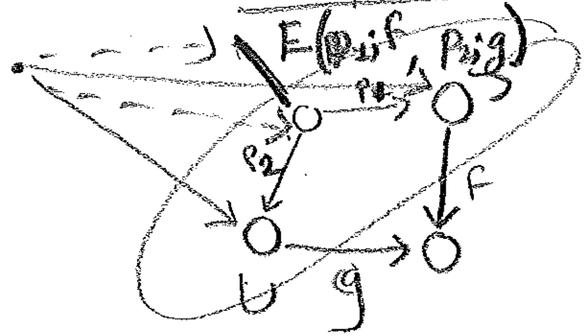
$$\begin{tikzcd}
\cdot \arrow[dr] \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "k"'] & & \\
& \cdot \arrow[r, "id"] \arrow[d, "id"'] & \cdot \arrow[d, "f"] \\
& \cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

f monic means

$$\begin{tikzcd}
\cdot \arrow[r, "g"] \arrow[d, "g"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

$\Rightarrow$ g monic

pullback from product and equaliser



$= (x, y) \quad fx =$

Natural numbers in $\mathcal{C}$ with terminal $*$

$$\mathcal{C}' = \left\{ D : \begin{tikzcd} * \arrow[d, "z"] \\ n \arrow[loop below, "s"] \end{tikzcd} \longrightarrow \mathcal{C} \right\} \Bigg| \left\{ f \;\middle|\; \begin{tikzcd} & * \arrow[dl, "z"'] \arrow[dr, "z'"] & \\ n \arrow[rr, "f"] \arrow[d, "s"] & & n' \arrow[d, "s'"] \\ n \arrow[rr, "f"] & & n' \end{tikzcd} \text{ commutes} \right\}$$

a natural numbers object $* \xrightarrow{\text{zero}} \mathbb{N} \xrightarrow{\text{succ}} \mathbb{N}$ is initial in this category

$$\begin{tikzcd} & \mathbb{N} \arrow[r, "\text{succ}"] \arrow[d, dashed, "!f"] & \mathbb{N} \arrow[d, dashed, "!f"] \\ 0 \arrow[ur, "\text{zero}"] \arrow[r, "Ka"'] & A \arrow[r, "t"] & A \end{tikzcd}$$

defines f by primitive recurs

$f\,0 = a$

$f;t = \text{succ};f$ i.e. $f(n+1) = t(f\,n)$

Pushout

$$
\begin{tikzcd}
0 \arrow[r, "f"] \arrow[d, "g"'] & S \arrow[d, "i_1"] & \\
U \arrow[r, "i_2"'] & S+T \arrow[dr, "CE(\ ,\ )"] & \\
 & & S+T/\equiv
\end{tikzcd}
$$

where $\equiv$ is equivalence specified by

$$= \cancel{S} \qquad x \equiv y \text{ iff } fx = gy$$

pushout = coproduct

- in coslice category

(g, f), in $\mathcal{C}/S+T$

remember to ask about tutorials.

Adjunctions.

1.5

between preorders (called Galois connections) 1.5 and 1.6.

$(f,g): S \to T$	$f: S \to T$ (left adjoint), $g: T \to S$ (right adjoint)
examples	$fc \le d$ in T iff $c \le gd$ in S, all $c \in S$, $d \in T$.
$(I, I): S \to S$	$Ic \le d$ iff $c \le Id$
$(\times 10, \div 10): \mathbb{N} \to \mathbb{N}$	$c \times 10 \le d$ iff $c \le d \div 10$
$(K_\perp, K_*): \{*\} \to T_{(\perp)}$	$K_\perp \overset{*}{c} \le d$ iff $c \le K_* d$ $= \text{true}$ all c in S
$(K_*, K_\top): T_{(\top)} \to \{*\}$	similar
$(\Delta_S, \wedge): S_{(\wedge)} \to S \times S$	$\Delta c =_{df} (c,c) \le (d,e)$ in $S \times S$ iff $c \le d \wedge e$ in S
$(\vee, \Delta_S): S \times S \to S$	similar.
$(K, \Lambda): S \to (T \to S)$	$Kc =_{df} \lambda x.c \le f$ iff $c \le \Lambda f$ $c \in S$, $f \in T \to S$ $\Lambda = \lim\inf\{fx \mid x \in T\}$
(V, K)	$V = \lim\sup$

Galois connections

equivalent definitions

$(f,g): S \to T$	$f: S \to T \quad g: T \to S$
	$fc \leq d$ in T iff $c \leq gd$ in S let $c = gd$
	$\hom_T(fc, d)$ is isomorphic to $\hom_S(c, gd)$
	$f(gd) \leq d$ in T and $c \leq g(fc)$
	$g;f \leq Id_T$ and $Id_S \leq g;f$ *
	$gd = \bigvee\{c \in S \mid fc \leq d\}$ f and g determine each other *
	$fc = \bigwedge\{d \in T \mid \quad\}$ *
	$f(\bigwedge h) = \bigwedge(f \circ h)$ all $h: R \to S$ (R complete) *
	i.e f continuous.
	g cocontinuous. *
	later
	adjunctions $S \to T$ form a category

~~9.3~~ 10.1

Equivalent definition.

$$fc \le d \quad \text{iff} \quad c \le gd$$

$$\hom(fc, d) \equiv \hom(c, gd)$$

because $f(gd) \le d$ in T and $c \le gfc$

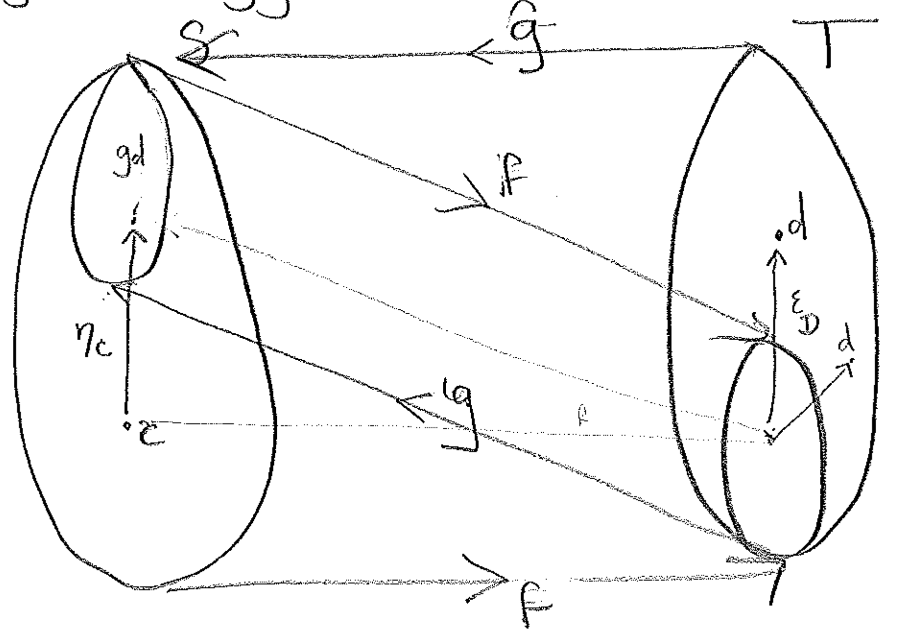
define $\varepsilon_d =_{df} (f(gd), d)$ in T $\eta_c = (c, gfc)$ in S

so $\varepsilon : g;f \to 1_d$ $\eta : 1_d \to f;g$.

$$f(c, gd); \varepsilon_d = (fc, d) \quad \text{iff} \quad (c, gd) = \eta_c; g(fc, d)$$

$f x ; \varepsilon_d = (fc, d)$ has (unique) solution for x in (c, gd)

$(c, gd) = \eta_c; gy$ — y



$K_\perp(c^{id}) =$ $\qquad c = id_*;$ $\qquad c : x \to *$

$I; T \to y = T$ $\qquad x = *$

c

Adjunctions in categories

10.2

$(F, G, \varepsilon, \eta) : S \to T$ | $F : S \to T$, $G : T \to S$ functors.

$\hom(Fa, b) \simeq \hom(a, Gb)$ (naturally)

$\varepsilon : (G;F) \to \mathbb{I}_T$ $\eta : (F;G) \to \mathbb{I}d_S$

$Fc ; \varepsilon_d = d \in T$ iff $c = \eta_c ; Gd \in S$ all $d : Fx \to y$ $c : x \to Gy$ $d \in S$ $g \in T$

$(I_S, I_S, \mathrm{Id}, \mathrm{Id}) : S \to S$ | $\mathbb{I}_S c ; id_d = d$ iff $c = id_c ; \mathbb{I}c$

$(K_T, K_*, \; , \;) : \{*\} \to T$ if $\perp$ is initial in T | $(K_T c) ; \top \to y = d$ in T iff $c = id_* ; K_* d$ in $\{*\}$ if $\perp$ is initial x in T $d : K_T(x) \to y$ in T, $c : x \to K_* y$ in $*$

a product functor

$(\Delta_S, \otimes, (p_1 \times p_2), \;) : S \to S \times S$ if S is cartesian

$(x \otimes x \to x)$ $(\varepsilon_1, \varepsilon_2)$ $(c\overset{\varepsilon}{x}, c\overset{\varepsilon}{y})$ $(p_1(c\overset{\varepsilon}{x}), p_2(c\overset{\varepsilon}{x}))$

$\Delta c ; p_1 \times p_2 = (d_1 \times d_2)$ iff $c = \langle id_x, id_x \rangle ; d_1 \otimes d_2$ $d : (x \times x) \to (y \times z)$ in $S \times S$

in $S \times S$ in S $c : x \to y \times z$ in S

$x \otimes x$ in S, $\xi \otimes \xi$, mediator

unique

$p_1(c\,x) = d_1 x$

$p(c\,x) = d_2 x$

$c\,x = d_1 x \times d_2 x$

$(K, \Lambda, \; , \;) : S \to (T \to S)$ $Kc ; ? = f$ iff $c = ? ; \Lambda f$

functors diagrams in S of shape T

Cartesian closure in Preorders

$Bool = \begin{array}{c}\uparrow true \\ false\end{array}$ $\leq$

Boolean algebra

$a \wedge b \leq c$ iff $a \leq (b \Rightarrow c)$

eg $\Rightarrow$ implication (intuitionistic) in set

false $\to$ true $\leq$ is $\vdash$ total order with $\top$

$b \Rightarrow c = \top$ if $b \leq c$ else c

$\mathrm{Hom}(a \wedge b, c) \simeq \mathrm{Hom}(a, b \Rightarrow c)$

$\mathrm{Hom}(a \times b, c) \underset{\text{curry } \theta}{\simeq} \mathrm{Hom}(a, \mathrm{Hom}(b, c))$ w $a, b, c \in$ Sets

$\xleftarrow{\text{uncurry } \check{\theta}}$ does it preserve structure — set of functions

$\_ \times b$ (id_B) is a (endo) functor
$b \to \_$ is a (endo) functor
} it is adjoint to $\_ \times b$

$$\begin{tikzcd}
b \Rightarrow c & (b \Rightarrow c) \times b \arrow[r, "eval^b_c"] & c \\
a \arrow[u, dashed, "g \,=\, \text{curry } f"] & a \times b \arrow[u, "(g \times id_b)"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$g = \theta f$

$\boxed{\begin{array}{l} f : a \times b \to c \qquad\qquad g = \text{curry } f \\ f = \underset{a \to (b \to c)}{g} \times id_b \,;\, eval^b_{bc} \text{ for unique } g \end{array}}$

$eval^b : (b \to \_) ; \_ \times b \to id$

$$\begin{tikzcd}
a \times b \arrow[r, "id\,f"] & c \\
(b \to a \times b) \times b \arrow[u, "\varepsilon^b_{a \times b}"] \arrow[r, "(b \to f) \times b"] & (b \to c) \times b \arrow[u, "\varepsilon^b_c"']
\end{tikzcd}$$

$g(\beta)$ $\quad$ $f(g\,\beta)$

(g, β) $\quad$ $g;$ $\quad$ $(g;f, \beta)$

Exponentials

$\times b$

$$\overset{F}{\hom}(x \times b, y) \cong \overset{G}{\hom}(x, b \to y)$$

$\xrightarrow{\theta}$ curry

$\xleftarrow[df]{\breve\theta}$ uncurry

$\theta f = g \qquad f = \breve\theta g$

$\theta f = \lambda\xi.(\lambda\beta\, f(\xi,\beta)) \quad \xi \in x,\ \beta \in b, \quad f : x \times b \to y$ curry

$\breve\theta g = \lambda(\xi,\beta).(g\xi)\beta \quad$ " " $, \quad g : x \to (b \to y)$ uncurry

$\varepsilon_y =_{df} \breve\theta(id_{(b\to y)})$

$= \lambda(\xi,\beta)(\xi\beta)$ apply.

$\eta_x =_{df} \theta(id_{x \times b})$

$= \lambda\xi.(\lambda\beta(\xi,\beta))$ = curried cons

$\therefore \theta(\varepsilon_y) = id_{b\to y}$

$\breve\theta\, \eta_x = id_{x\times b}$

$(b \Rightarrow g); \varepsilon = \breve\theta g$

$l \Rightarrow r =_{df} \theta(id \times l; \varepsilon; r) = \lambda\xi.(\lambda\beta\ r(\varepsilon(id \times l)(\xi,\beta))$

internal hom

$\uparrow$ contra

$\varepsilon(\xi, l\beta)$

$r(\varepsilon(l\beta)$

$\lambda\beta$

$\lambda\xi \qquad l;\varepsilon;r$

page 297 II.3
10.4

$$(g \times b);\varepsilon = \lambda \varepsilon, \beta \; g \times b(\varepsilon, \beta) \qquad \eta;(b \Rightarrow f) = \theta f$$

$$= (g\varepsilon)(\beta)$$

$$= \check{\theta}_g$$

$$(g \times b);\varepsilon = f \quad \text{iff} \quad g = \eta;(b \to f)$$

$$(b \Rightarrow f) \times b;\varepsilon = \lambda(\phi, \beta)\; f(\phi, \beta)$$

$$= \lambda(\phi, \beta)\; f\, \varepsilon(\phi, \beta)$$

$$= f;\varepsilon$$

$$\cancel{\eta;g = b \Rightarrow g}$$

$$\eta;(b \Rightarrow (p \times q)) = p;\eta;b \Rightarrow q$$

$$\eta;(b \Rightarrow (g \times b)) = g;\eta$$

$$\varepsilon : (b \Rightarrow \_) \times b \to \mathbb{I} \qquad \eta : \mathbb{I} \to b \Rightarrow (\_ \times b)$$

$$\theta((p \times q);f;r) \stackrel{\text{def}}{=} \lambda \varepsilon(\lambda \beta\; r(f(p \times q(\varepsilon, \beta)))) \qquad p;\theta f;(q \Rightarrow r)$$

use notes

$$\lambda \varepsilon (q \Rightarrow r)((\theta f)(p\varepsilon))$$

$$(p \times q \Rightarrow r);\theta^b = \theta^b;(p \Rightarrow (q \Rightarrow r)) \qquad r(f(p\xi, q\beta))$$

$$\lambda \varepsilon \quad q;\theta f(p\varepsilon);r$$

$$\lambda \varepsilon, \lambda \beta.\; r(((\theta f)(p\varepsilon))(q\beta)) = \lambda \varepsilon \lambda \beta\; r(\theta f(p\varepsilon)(q\beta))$$

h. natural transformation

$$\lambda \varepsilon \lambda \beta.\; r(\quad = \quad r(f(p \times q)(\varepsilon, \beta)))$$

$\mathrm{map} : \mathrm{Set} \to \mathrm{Mon}$ $\qquad$ $U(A, ;, id) \overset{=}{_N} A : \mathrm{Mon} \to \mathrm{Set}$

$\mathrm{map}(c) = (c^*, \hat{}, \langle\rangle)$ $\qquad$ $U(f : (A ; id) \to (A' ; id')) = f : A \to A'$

$\mathrm{map}(f : c \to d) = \lambda\langle x_0, x_1, \ldots\rangle : \mathrm{map}(c). \langle f x_0, f x_1 \ldots\rangle : \mathrm{map}(d)$

x is a set $\qquad$ m is a monoid

$\eta_x = (\lambda\xi . \langle\xi\rangle : x \to U(\mathrm{map}\, x))$ $\qquad$ $\varepsilon_m = \lambda\langle a_0, a_1, \ldots\rangle : a_0 ;_M a_1 ;_M \ldots$

$x \to U(\mathrm{map}\, x)$ $\quad$ x^* $\qquad$ $: \mathrm{map}(Um) \to m$

$= \mathrm{unitseq}_x$ $\qquad$ $= \mathrm{Fold}_m$

$(\mathrm{map}, U, \varepsilon, \eta)$ is an adjunction.

$\mathrm{map}\, c ; \mathrm{Fold}_m = d$ (in mon) iff $c = \mathrm{unit}_x ; Ud$ (in Set) for $c : x \to Um$ in Set

for all $d : \mathrm{map}\, x \to m$ in Mon there is one $c : x \to Um$ solving this equation.

for all $c : x \to Um$ there is one d solving this equation

$d : \mathrm{map}\, x \to m$ in Mon

$\hom(\mathrm{map}\, x \to m) \simeq \hom(x \to Um)$ naturally in x and m

Algebras and Monads

R-algebra is (A, a) | $R : \mathcal{A} \to \mathcal{A}$ a functor
$a : RA \to A$ sum

e.g $\mathcal{A} = (\mathrm{pfn}(N \to N), \leq)$

$R : \mathrm{pfn}\,N \to \mathrm{pfn}\,N$ objects as preorder

$Rh =_{df} \lambda n.\ \text{if } n = 0 \text{ then } 1 \text{ else } n * h(n-1)$

(is monotonic in h) nb $h(\mathrm{fac}) = \mathrm{fac}$

(Fac, id_N) is an R-algebra because

id_N is iso

$id_N : R(\mathrm{Fac}) \to \mathrm{fac}$

$R : \mathrm{Set} \to \mathrm{Set}$ | $R(A) = \{*\} + A$

$(N, 0 + \mathrm{succ})$, is an R-algebra

$0 + \mathrm{succ} = \lambda x.\ \text{if } x = * \text{ then } 0 \text{ else } \mathrm{succ}(x)$

$(N, 23 + \mathrm{square})$ is another

a fixed point of R is (A, a) s.t. a is an isomorphism $Ra \simeq a$

denoted in Haskell by $N = \mathrm{zero} \mid \mathrm{succ}\,N$

algebra morphism

$f : (A, a) \to (B, b)$

①

$$
\begin{tikzcd}
RA \arrow[r, "a"] \arrow[d, "Rf"'] & A \arrow[d, "f"] \arrow[dd, dotted, bend left, "= f;g"] \\
RB \arrow[r, "b"] \arrow[d, "Rg"'] & B \arrow[d, "g"] \\
RC \arrow[r, "c"] & C
\end{tikzcd}
$$

= least fixed point of R

② The initial algebra on R (if it exists) is initial object in this category

eg. (fac, id) $(N,\ 0 + \mathrm{succ})$

$$\boxed{\begin{array}{c} \text{in ML} \\ N =_{df} \text{zero} \mid \text{succ } N \end{array}}$$

(eg)

$$
\begin{tikzcd}
1 + N \arrow[r, "0+\mathrm{succ}"] \arrow[d, "1+23^{2n}"'] & N \arrow[d, dashed] \\
1 + 1 \arrow[r, "23+\mathrm{square}"'] & N
\end{tikzcd}
$$

$f\ \lambda n.\ \text{if } n = 0 \text{ then } 23 \text{ else square}(f(n-1))$
$= 23^{2n}$

dev

initial algebra is isomorphism

$$
\begin{tikzcd}
RA \arrow[r, "a"] \arrow[d, "Rf"'] & A \arrow[d, dashed, "f"] \\
RRA \arrow[r, "Ra"] \arrow[d, "Ra"'] & RA \arrow[d, "a"] \\
RA \arrow[r, "a"] & A
\end{tikzcd}
$$

$f;a = id$

$\therefore a;f = Rf;Ra = R(f;a)$
$= R(id) = id.$

$R(A) = \{\bullet\} + \mathrm{Char} \times A$

$\mathrm{Char}^* = (\mathrm{Str},\ \mathrm{nil} + \mathrm{cons})$ where $\mathrm{nil} + \mathrm{cons} \equiv$ if $x = \bullet$ then <> else let $(x_0, x') = x$ in $\mathrm{cons}(x_0, x')$

is an isomorphism

All polynomial endofunctors have initial algebras

and final coalgebras $(b : B \to S(b))$

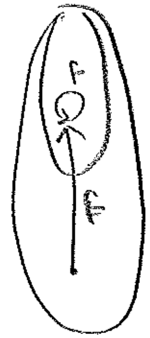
B which contain infinite lists.

Haskell's algebraic data types.

Closure in a preorder S

$f : S \to S$ monotonic | $x \le fx$ increasing

| $f(fx) \le fx$ idempotent.



$\eta_x = (x, fx)$ i.e. $\eta : id \to f$

$\mu_x = (ffx, fx)$ $\quad \mu : f^2 \to f$ $\Big\}$ are natural

e.g. $f : \mathcal{P}\mathbb{R} \to \mathcal{P}\mathbb{R}$ is topological closure of set

$$\eta_{fx} ; \mu_x = id_{fx}$$

$$(f \circ \eta)_x ; \mu_x = id_{fx}$$

$$\begin{tikzcd} f \arrow[r, "\eta f"] \arrow[dr, "="'] & f^2 \arrow[d, "\mu"] & f \arrow[l, "f\eta"'] \arrow[dl, "="] \\ & f & \end{tikzcd}$$

$$f \circ \mu_x ; \mu_x = \mu f_x ; \mu$$

$$f(\cancel{f}ffx, \cancel{f}fx);(ffx, fx) = (fffx, ffx);(ffx, fx)$$

$$\begin{tikzcd} f^3 \arrow[r, "f\mu"] \arrow[d, "\mu f"'] & f^2 \arrow[d, "\mu"] \\ f^2 \arrow[r, "\mu"'] & f \end{tikzcd}$$

horizontal composition of natural transformations

coclosure

Monads

(T, η, μ) — unit, multiplic | $T: \mathcal{A} \to \mathcal{A}$ a functor, $\eta: id \to T$ and $\mu: T^2 \to T$ satisfying the diagrams

$(\mathcal{P}, unit, union)$.

$(map, unit, concat)$

if $F \dashv U$

then $(F;U, \eta, U\epsilon F)$

① $unit: id \to \mathcal{P}$ $\qquad f: a \to b$

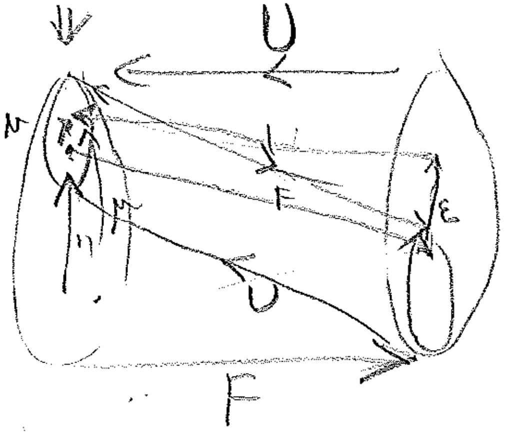
$id\, f ; unit_b = unit_a ; \mathcal{P}(f)$

i.e $unit(f x) = (map\, f)(unit\, x)$

② $\mu : map^2 \to map$

$map(map\, f) ; \mu = \mu ; map\, f$ $\quad xs : a^{**}$ $\quad f:$

$concat((map (map\, f))x = map\, f\, concat\, x$:



$\eta map ; concat = id map$

(3a) $\eta_{x^*} ; concat_x = id_{x^*}$

$concat(unit(xs)) = xs$

(3b) $map(\eta_x) ; concat = id map$

$concat((map\, unit)(xs)) = xs$

4 $map\, concat ; concat$

$= concat ; concat$

The Kleisli category of monad $\mathbb{T} = (T : \mathcal{C} \to \mathcal{C}, \eta, \mu)$

$$= \left(obj(\mathcal{C}), \{ f : A \to TB \mid f \in arr(\mathcal{C}) \}, \overleftarrow{f} = A, \overrightarrow{f} = B, \circledS, \eta_A : A \to TA \right)$$

$f : A \to B$ in K

$g : B \to C$ in K

A

$$f \circledS g = f ; Tg ; \mu_C = A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC$$

naturality $\eta ; \mathbb{I} \to T$ unit

$$\eta_A \circledS f = \eta_A ; Tf ; \mu_B = f ; \eta_B ; \mu_B = f$$

$$f \circledS \eta_A = f ; T\eta ; \mu$$

$$f \circledS (g \circledS h) = f ; T(g ; Th ; \mu) ; \mu = f ; Tg ; T^2h ; T\mu ; \mu$$

$$(f \circledS g) \circledS h = f ; Tg ; \mu ; Th ; \mu \qquad \mu ; Th ; \mu \quad \mu T ; \mu \quad \mu :$$

Prolog is the Kleisli Category of list monad.

(A ~~Preordered~~ , Monoid)

A two-monoid $(A, *, ;, id)$ 13,9

$(A, ;, id, \leq)$

one-cells

$(A, ;)$ a monoid $\quad A \; ; \; id$

$(A, \leq)$ a preorder

two-cells $(p \leq p') \wedge q \leq q' \Rightarrow p;q \leq p';q'$

define $(p, p') * (q, q') = (p, q')$ if $p' = q$ & $p \leq p'$ and $q \leq q'$

$(id, id) * (p, p') =$

$(p, p');(q, q') = (p;q, p';q')$

$p' = q$
$r' = s$

$((p, p') * (q, q'));((r, r') * (s, s')) = (p, q');(r, s') = (p;r, q';s')$

$((p, p');(r, r')) * ((q, q');(s, s')) = (p;r, p';r');(q;s, q';s')$

A two-category.

$\mathcal{A}, \mathcal{B} \in C_0$
$A, B \in C_1$
$\alpha, \beta \in C_2$

objects, arrows, elements, fam

$(C_0, C_1, \leftarrow_2, \rightarrow, *, id)$ $\quad$ $(C_0, C_1, \leftarrow_1, \rightarrow_1, *, id)$ is a category base

for each $\mathcal{A}, \mathcal{B} \in C_0$ $\quad$ $(C_2, \downarrow, \uparrow, \circ, id)_{\mathcal{A}\mathcal{B}}$

small cat, Func, Nat

for each $\mathcal{A}, \mathcal{B}$ in C_0

if $A, B : \mathcal{A} \to \mathcal{B}$

$\left(\hom(A;B), C_{2AB}, \uparrow_{A,B}, \downarrow_{AB}, \circ, id^{V}\right)_{AB}$ is a category vertical

$= \alpha : A \to B : \mathcal{A} \to \mathcal{B}$

a 2-cell

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "A"] \arrow[r, bend right=50, "B"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$$

horizontal category $(C_0, C_2, \leftarrow, \rightarrow, *, id)$ is a category

extended to $C2$.

$$(\alpha \circ \gamma) * (\beta \circ \delta) = \begin{tikzcd} \mathcal{A} \arrow[r, bend left=60, "AF"] \arrow[r, "BG" description] \arrow[r, bend right=60, "C"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma"] & \mathcal{B} \arrow[r, bend left=60, "A'H"] \arrow[r, "B'K" description] \arrow[r, bend right=60, "C'"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{A} \arrow[r, bend left=60, "A;A'"] \arrow[r] \arrow[r, bend right=60, "2"'] \arrow[r, phantom, shift left=3, "\Downarrow(\alpha*\beta)"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma*\delta"] & \mathcal{C} \end{tikzcd} = (\alpha*\beta)\circ\gamma*\delta$$

interchange law.

insert $\beta : H \to K : \mathcal{A} \to$

$$\beta G_A = \beta_{FA} : G;H \to G;K$$

$$\begin{tikzcd}
KX \arrow[r] & KY \\
HX \arrow[u, "\beta_{GX}"] \arrow[r, "Hf"'] & HY \arrow[u, "\beta_{GY}"']
\end{tikzcd}$$

$Y := GY$

$X := GX$

$\alpha : F \to G$

$H\alpha : F;H \to G;H$

$$\begin{tikzcd}
GX \arrow[r] & GY \\
FX \arrow[u, "\alpha_X"] \arrow[r] & FY \arrow[u, "\alpha_Y"']
\end{tikzcd}$$

put H in front of everything.

Rel, (Sets, relations $\leftarrow, \rightarrow$, ;, id) not really needed. 13. 2

+ $(\mathcal{P}(A \times B), \subseteq)$ considered as a category

$\alpha \circ^{\text{vert}} \gamma = (A, B) \circ (B, C) = (A, C)$, | $\alpha * \beta = (A, B) * (A', B') = (A;A', B;B')$

$(\alpha \circ \gamma) * (\beta \circ \delta) = (A, C) * (A', C') = (A;A', C;C') = (A;A', B;B') \circ (B;B', C;C')$

$= (\alpha * \beta) \circ (\gamma * \delta)$

Func ($\underline{cat}$, $\underline{functs}$, $\leftarrow \rightarrow$;, id),

$(\mathrm{Func}(\mathcal{A}, \mathcal{B}), \mathrm{nat}(\mathcal{A}, \mathcal{B}), \uparrow, \downarrow, \circ, \mathrm{id})$

if $\alpha, \beta : F \to G$ then $\alpha \circ \beta : F \to G = \lambda x\, \alpha_x ;_{\mathcal{B}} \beta_x$ vertical

for x in $\mathcal{A}$, α_x is arrow in $\mathcal{B}$

$\alpha : F \to G : \mathcal{A} \to \mathcal{B}$ $* (\beta : H \to K : \mathcal{B} \to \mathcal{C}) = \gamma : F;H \to G;K : \mathcal{A} \to \mathcal{C}$

$F;H \to G;H \to GH \to$

$\gamma = H\alpha \circ \beta G = \beta F \circ K\alpha$, in $\mathcal{C}$

$H * \alpha \circ \beta * G$

$\overset{??}{=} (H \circ \beta) * (\alpha \circ G)$

$H\alpha = \mathrm{id}H * \alpha \quad \beta G = \beta * \mathrm{id}G$

with vertical composition

Conceptual definition $\boxed{\text{omit}}$

Konstantt 3.(3)

C_0, $C(A,B)$. $\overset{*\ \text{hor}}{\text{comp}}: C(A,B)\times C(B,C) \to C(A,C)$, $\leftarrow C(A,B) \to C(A,A)$

$\to C$

$$
\begin{tikzcd}
A\to B \times B\to C \times C\to D \arrow[r, "\text{comp}\times\text{id}"] \arrow[d, "\text{id}\times\text{comp}"'] & (A\to C)\times(C\to D) \arrow[d, "\text{comp}"] \\
{} \arrow[r, "\text{comp}"'] & (A\to D)
\end{tikzcd}
$$

assoc

$$
\begin{tikzcd}
 & A\to B \arrow[dl, "\leftarrow\times\text{id}"'] \arrow[d, "\text{id}"] \arrow[dr, "\text{id}\times\to"] & \\
(A\to A)\times A\to B \arrow[r, "\text{comp}"] & A\to B & A\to B \times B\to B \arrow[l, "\text{comp}"']
\end{tikzcd}
$$

A ~~monoidal two-monad~~ double two monoid

13.A 14.1

$\left(A ; id, \otimes, T\right)$ $(A, ;, id)$ a monoid

$(A, \otimes, T)$ a monoid

$$(f \otimes g);(h \otimes j) = (f;h) \otimes (g;j)$$

$\otimes$ is a functor $A \times A \to A$

$$F(f,g);(h,j) = F(f;h, g;j) = f;h \otimes g;j$$

$$= (f \otimes g);(h \otimes j) = F(f,g); F(h,j)$$

a monoidal ~~monad~~ monoid — $\otimes$ not quite assoc +

weakening $f \otimes (g \otimes h);a \simeq a;(f \otimes g) \otimes h$ like a cartesian monad.

strengthening $(f \otimes g);r = r;g$

$(f \otimes g);l = l;f$ [illegible]

$(f \otimes g);s = s;(g \otimes f)$ symmetric.

monoidal category

a sort of monoid

$$(\mathcal{C}, \otimes, \top)$$

$a, r, l,$ (s if symmetric) are all isomorphisms

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a functor, $\top \in \mathrm{obj}\ \mathcal{C}$

$\eta_A : FA \to GA$

$$\begin{tikzcd} GA \arrow[r, "f"] & GB \\ FA \arrow[u, "\eta"] \arrow[r, "f \times \top"'] & FB \arrow[u, "\eta"'] \end{tikzcd}$$

$r : \_ \otimes \top \to \mathrm{Id}$ $\qquad r_A : A \otimes \top \to A$

is $\qquad f \otimes \top ; r_B = r_A ; f \qquad f : A \to B$

$l : \top \otimes \_ \to \mathrm{Id}$

$a : F \to G$ where $F(f,g,h) = f \otimes (g \otimes h)$

$G(f,g,h) = (f \otimes g) \otimes h$

$$f \otimes (g \otimes h) ; a_{\vec{f}, \vec{g}, \vec{h}} = a_{\overleftarrow{f}\overleftarrow{g}\overleftarrow{h}} ; (f \otimes g) \otimes h .$$

$s : \otimes \to \lambda fg.\, g \otimes f$

$$f \otimes g ; s = s ; g \otimes f$$

$\left(\begin{matrix}\mathcal{C}, \text{product}, \text{terminal} \\ \mathcal{C}, \text{coproduct}, \text{initial}\end{matrix}\right)$ is the canonical example

Each equation of monoid
is an equality up to unique isomorphism

$$A \otimes (\top \times B) \overset{1}{\simeq} (A \otimes \top) \otimes B$$

assoc

$$\begin{tikzcd}
A \otimes (\top \times B) \arrow[rr, "a"] \arrow[dr, "id \otimes l"'] & & (A \otimes \top) \otimes B \arrow[dl, "r \otimes id", shift left] \\
& A \arrow[ur, shift left] &
\end{tikzcd}$$

$a; l \otimes id = id \times r;$

$s;s = id_{A \otimes B}$ $\quad s; l = r$ $\quad s; r = l$

$$\boxed{\begin{array}{c} id \times s; a; s \times id = a; s; a \\ A \times (B \times C) = (C \times A) \times B \end{array}}$$

$id \times a; a; a \times id = a; a$

$(A \times (B \times (C \times D)))$

$A \times ((B \times C) \times D)$ $\qquad (A \times B) \times (C \times D)$

$(A \times (B \times C)) \times D$ $\qquad ((A \times B) \times C) \times D$

$((A \times B) \times C) \times D$

Monoidal Closed.

$A \otimes \_$ has a right adjoint. $A \multimap \_$

apply : $A \otimes (A \multimap B) \to B$ natural transformation

define $c(A,B,C) : A \otimes (A \multimap B) \otimes (B \multimap C)$

$= (\mathrm{apply}_{AB} \otimes \mathrm{id}_{B \multimap C}) ; \mathrm{apply}_{BC}$ makes monoidal category into a two-category

In Boolean logic $B \to A = \neg A \Rightarrow \neg B$

Let $^* : C^{op} \to C$ functor is a <u>duality</u> if $\_ \multimap \_ \simeq (r;* \times l;*);s;\multimap$

$(f;g)^* = g^*;f^*$

$d : A \multimap B \xrightarrow{\sim} B^* \multimap A^*$

<u>and</u> $c;d = d \otimes d;s;c$ not iso

$(A \multimap B);(B \multimap C) \longrightarrow (C^* \multimap A^*)$

Rel is a monoidal (closed) preorder-enriched category!

$$\left(R : A \to B\right) \otimes \left(S : C \to D\right) = \left\{ \left((a,c),(b,d)\right) \mid a R b \wedge c S d \right\} : A \times C \to B \times D$$

models disjoint concurrency

$$R \otimes (S \otimes T) ; \alpha = \alpha ; (R \otimes S) \times T$$

$$s_{AC} = \left\{ \left((a,c),(c,a)\right) \mid a \in A, c \in C \right\}$$

$R \otimes S ; l_{BD} = l_{AC} ; R$ it is symmetric etc

$$(R \otimes S) ; s_{BD} \overset{\text{def}}{=} s_{AC} ; R \otimes S$$

its right adjoint is $\otimes$ itself. (something degenerate wrong here!)

$$\mathrm{Hom}(S \otimes T, U) \cong \mathrm{Hom}(T, S \otimes U)$$

what are θ, ε, η, etc?