Shifting the identity of predicate transformers.

In the space* of predicate transformers, let J be a preorder, i.e.

$$J \sqsubseteq \mathbb{I} \qquad \text{reflexive}$$

$$J \sqsubseteq J;J \qquad \text{transitive}$$

$$J;\sqcap S = \sqcap\{J;P \mid P \in S\} \qquad \text{conjunctive.}$$

It follows that J is idempotent

$$J;J \sqsubseteq \mathbb{I};J = J$$

We define a subset J° of predicate transformers

$$J^{\circ} \triangleq \{P \mid J;P;J = P\}.$$

We prove that J° is a subspace of predicate transformers, except that it has J as its identity instead of $\mathbb{I}$. ~~and $J;\perp$ instead of $\perp$~~

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*a complete distributive lattice

II

1. $J \in J^o$

$$J;J;J \equiv J \qquad \text{by idempotence}$$

2. $\top \in J^o$

$$J;\top;J = J;\top \qquad \top \text{ is a condition.}$$

$$= \top \qquad \text{by conjunctivity}$$

3. $\bot \in J^o$ ~~similarly~~ not a proof $J;\bot$ [illegible]

4. If $P, Q \in J^o$, so is $P;Q$.

$$J;P;Q;J \equiv J;(J;P;J);(J;Q;J);J$$

$$\equiv J;P;J;J;Q;J \qquad \text{by idempotence}$$

$$\equiv P;Q \qquad \text{assumption}$$

5. If $S \subseteq J^o$, the $\sqcap S \in J^o$

$$J;(\sqcap S);J \equiv J;\sqcap\{P;J \mid P \in S\} \qquad \text{property of space}$$

$$\equiv \sqcap\{J;P;J \mid P \in S\} \qquad J \text{ conjunctive}$$

$$\equiv \sqcap\{P \mid P \in S\} \qquad \text{since } S \subseteq J^o$$

2

6 If $S \subseteq J^\circ$, $\bigsqcup S \in J^\circ$

($\sqsubseteq$) $J;\bigsqcup S;J \sqsubseteq \mathbb{I};\bigsqcup S;\mathbb{I} = \bigsqcup S$. (since $J \sqsubseteq \mathbb{I}$)

($\sqsupseteq$) $J;P;J = P$ for all $P \in S$

$P \sqsubseteq \bigsqcup S$ for all $P \in S$

$\therefore J;\bigsqcup S;J \sqsupseteq P$ by monotonicity;

$\therefore J;\bigsqcup S;J \sqsupseteq \bigsqcup S$ lub

are there any?

7. If b is a condition ~~in J° then $\bar{b} \in J^\circ$~~ and its negation is $J;\bar{b};J$

then $J;b;J$ is a condition

$\top = J;\top;J = J;(b \sqcup \bar{b});J$ by (2)

$= (J;b;J) \sqcup (J;\bar{b};J)$ by (6)

$\cancel{= b \sqcup J;\bar{b};J}$ by assumption

$\bot = J;\bot;J = J;(b \sqcap \bar{b});J$

$= (J;b;J) \sqcap (J;\bar{b};J)$

$\cancel{= b \sqcap J;\bar{b};J}$

By uniqueness of negation, $\overline{J;b;J} = J;\bar{b};J$, which is clearly in J°

$\top \sqsubseteq H;\top$

~~$\mathbb{I}$~~ $\sqsubseteq H$

$K;L \sqsupseteq K$

$H;H \sqsubseteq H$

$L;K \sqsubseteq L$

$\bot \sqsubseteq H;\bot;H$ by disjunctivity

true $\to$ true, false $\quad p \cup q$

8. In defining fixed points, we confine attention to elements of J^o

$$\mu X. FX \triangleq \sqcap\{X \mid FX \sqsubseteq X \ \& \ X \in J^o\}$$

Now

$$J; \mu X. FX; J = \sqcap\{J;X;J \mid FX \sqsubseteq X \ \& \ X \in J^o\}$$

$$= \mu X. FX.$$

But we still need to prove this is a fixed point of F. This depends on the assumption

$$F : J^o \to J^o$$

i.e $\forall X.\ X = J;X;J \Rightarrow FX = J;FX;J.$

According to (1) - (7) above, this will be guaranteed if F is an expression in our algebra.

Proof: presumably the same as the standard proof, applied in the domain J^o

Shifting the top of predicate transformers.

every relation is universally conjunctive

In the space of predicate transformers,

It is universally ~~nonempty~~ conjunctive ?

let K be idempotent and conjunctive.

$K;\top = \top$ (not $\bot$)

$K = \lambda x\ \text{true} \lhd x = \text{true} \rhd \Pi$ as a predicate transformer ~~but $K;\bot \neq \bot$~~ $K;\top = \top$

$K;\bot \neq \bot$

$\top \lhd b \leq x \rhd b$

$$K \equiv K;K \qquad K;(\mathbb{I} \sqcap K) = K^{\langle b \rangle} \sqsubseteq K;b \sqcap c$$

$$K;\sqcap S = \sqcap\{K;P \mid P \in S\}. \quad \text{~~provided } S \neq \{\}\text{~~}$$

Define $J = \mathbb{I} \sqcap K$

Now $J \sqsubseteq \mathbb{I}$ (property of $\sqcap$)

$J \sqsubseteq J;J$

$$
\begin{aligned}
(\,J;J &= (\mathbb{I} \sqcap K);(\mathbb{I} \sqcap K) \\
&= (\mathbb{I} \sqcap K) \sqcap (K;(\mathbb{I} \sqcap K)) && \checkmark \\
&= \mathbb{I} \sqcap K \sqcap K \sqcap (K;K) && \text{K conjunctive} \\
&\sqsupseteq \mathbb{I} \sqcap K && \text{K idempotent}
\end{aligned}
$$

Furthermore J is conjunctive, because $\mathbb{I}$ ~~and K are~~.

Define $K{\downarrow} = J^{\circ} \cap \{P \mid P \sqsubseteq K\}$.

~~$(\mathbb{I} \sqcap K);\sqcap S = \sqcap\{\mathbb{I};x \sqcap K;x \mid x \in S\} = \sqcap((\mathbb{I} \sqcap K);x \mid x \in S)$~~

~~$\mathbb{I} \sqcap K;\sqcap\{\} = \mathbb{I} \sqcap K;\top$~~

Now we show that $K{\downarrow}$ is an almost complete subspace of predicate transformers, but with J as its identity and K as its top.

(1) $J \in K{\downarrow}$

$J \in J^\circ$ and $J = \mathrm{I\!I} \sqcap K \sqsubseteq K$.

(2) $\bot \in K{\downarrow}$

because $\bot \sqsubseteq K$ and $\bot \in J^\circ$

(3) $JKJ \in K{\downarrow}$. We prove $K \in J^\circ$

$$J;K;J = (\mathrm{I\!I} \sqcap K);K;(\mathrm{I\!I} \sqcap K)$$

use conjunctive here

$$= K \sqcap (K;K) \sqcap (K;K;K)$$

$$= K; \cancel{(\mathrm{I\!I} \sqcap K)} \sqsubseteq K \qquad \text{idempotence } (\text{~~transitivity~~})$$

$$\sqsubseteq K$$

(4) If $P, Q \in K{\downarrow}$, so is $(P;Q)$

$$(P;Q) \sqsubseteq K;K \qquad \text{by monotonicity}$$

$$= K \qquad \text{by idempotence}$$

(5) If $S \subseteq K\downarrow$ then $\sqcap S \in K\downarrow$ provided S is nonempty

Let $P \in S$, $P \sqsubseteq K$, because $S \subseteq K\downarrow$

$\therefore \sqcap S \sqsubseteq P \sqsubseteq K$

(6) If $S \subseteq K\downarrow$ then $\sqcup S \in K\downarrow$

for all $P \in S$, $P \sqsubseteq K$

$\therefore \sqcup P \sqsubseteq K$. (lub)

b is a condition in $\mathbb{J}$ ($\overline{\mathbb{J}};\overline{B};\mathbb{J}) \sqcap K$ is a condition in)

(7) If $b \in K\downarrow$ then its complement in $K\downarrow$ is $(\overline{b} \sqcap K)$

$\top = \mathbb{J};\top;\mathbb{K}$.

$K = \top \sqcap K = \mathbb{J};(b \sqcup \overline{b});\mathbb{J} \sqcap K$

$= (\mathbb{J}b\mathbb{J} \sqcap K) \sqcup (\mathbb{J}\overline{b}\mathbb{J} \sqcap K)$

$= b \sqcup (\overline{b} \sqcap K)$

$K;\ K;P \sqcup K;Q$

$\sqsubseteq$

$\bot = \bot \sqcap K = (b \sqcap \overline{b}) \sqcap K$

$= b \sqcap (\overline{b} \sqcap K)$

$\therefore (\overline{b} \sqcap K)$ is the complement of b in $K\downarrow$

(8) Assume $F : K\downarrow \to K\downarrow$. It follows that $F(K) \in K\downarrow$ and so $F(K) \sqsubseteq K$. The meet in the following definition is

$\mu X\ FX \triangleq \sqcap \{X \mid FX \sqsubseteq X \ \&\ X \in \mathbb{J}^{o} \ \&\ X \sqsubseteq K\}$

therefore nonempty, and so belongs to $K\downarrow$

copy to Oege
JiFeng

Dear Hilary,

Thanks for your interested and interesting reply. I think our choice is Heyting algebras, because we find galois connections so useful. Also, I think it is easier for us to work with morphisms distributing through arbitrary meets (or joins). I should very much like to know more about duality, yours and/or Stone's.

But we are very hooked on <u>monotone</u> functions as morphisms, so that our Homsets are themselves complete distributive lattices. This means that the normal product and coproduct constructions in the category have to be weakened to local adjunctions or worse. Nevertheless, we would like to find some kind of cartesian closedness.

The best introduction to the motivation is probably Dijkstra's original paper on Guarded Commands.