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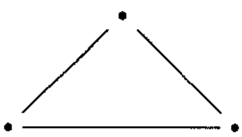
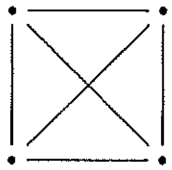
A fragment of graph theory (C.A.R. Hoare)

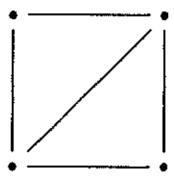
We define a graph as a quadruple (E, V, source, target), where E is the set of its edges, V is the set of its vertices, and source and target are total functions from E to V which identify the ends connected by each edge. An n-graph is one with n vertices: it is *complete* if it contains at least one edge between each of the $n\ (n-1)/2$ distinct pairs of vertices. A complete 3-graph is called a triangle, and a complete 4-graph is a tetrahedron.





One graph is a subgraph of another if it consists of a subset of the edges and vertices, with the corresponding restrictions of the source and target functions. A subgraph is called a *clique* if it is a subgraph of a complete subgraph; i.e. the parent graph has enough edges to connect every pair of its vertices. So every subgraph of a clique is a clique; and a subgraph with the same vertices as a clique is a clique.

A *nearly* complete n-graph is one in which all but one of the pairs of distinct vertices are connected by an edge; an example is the nearly complete tetrahedron:



A graph is said to be *nearly linear* if every subgraph which is a nearly complete tetrahedron is a clique; in other words, the missing diagonal can be found in the parent graph. In the remainder of this note, we shall deal with a fixed but unnamed nearly linear graph; all other graphs mentioned will be assumed to be subgraphs of it. The fundamental theorem of nearly linear graphs shows that they decompose into complete subgraphs that share no edges.

Following the example of category theory, an edge added to a diagram is dotted; but we do not require it to be unique.

Note that the definition of a clique does not require that the connecting edges be members of the subgraph; so

Theorem 1 *If A and B are cliques and $A \cap B$ contains an edge then $A \cup B$ is also a clique.*

Proof If the vertices of one of A and B are a subset of the vertices of the other, then the union has the same vertices as one of the operands, and is therefore a clique. Otherwise we need to demonstrate existence of an edge between an arbitrary vertex selected from $(A - B)$, and one selected from $(B - A)$. Because A and B are cliques, both the selected vertices are connected by four edges to both ends of the shared edge. The five mentioned edges from a near-tetrahedron. Near linearity provides the sixth edge connecting the arbitrary vertex of $(A - B)$ with the arbitrary vertex of $(B - A)$. □

insert figure

A~~n~~ path of length n is defined in the usual way as a set of $n + 1$ vertices organised as a sequence, with an edge between each adjacent pair. Two paths may be concatenated (by ^) if the last vertex of the first path is the same as the first vertex of the other. Let e be an edge, and let p be a path whose ends are the source and target of e. If $p \cup \{e\}$ is a clique, then p is said to be a *chain* on e. The path $\langle e \rangle$ is defined to consist of e together with its two end vertices, and it is always a chain on e. The path containing any two sides of a triangle is a chain on the third side. Chains can be arbitrarily shortened. If p^q^r is a chain on e, then the clique property ensures the existence of an edge f such that p^$\langle f \rangle$^r is also a chain on e. In a nearly linear graph, chains can be combined in various interesting ways, as described by the theorems that follow.

any hamiltonian path in a clique is a chain on ~~the omitted~~ any edge omitted from it.

Theorem 2. **(Chain insertion)** *If p is a chain on e and q^$\langle e \rangle$^r is a chain on f, then q^p^r is also a chain on f.*

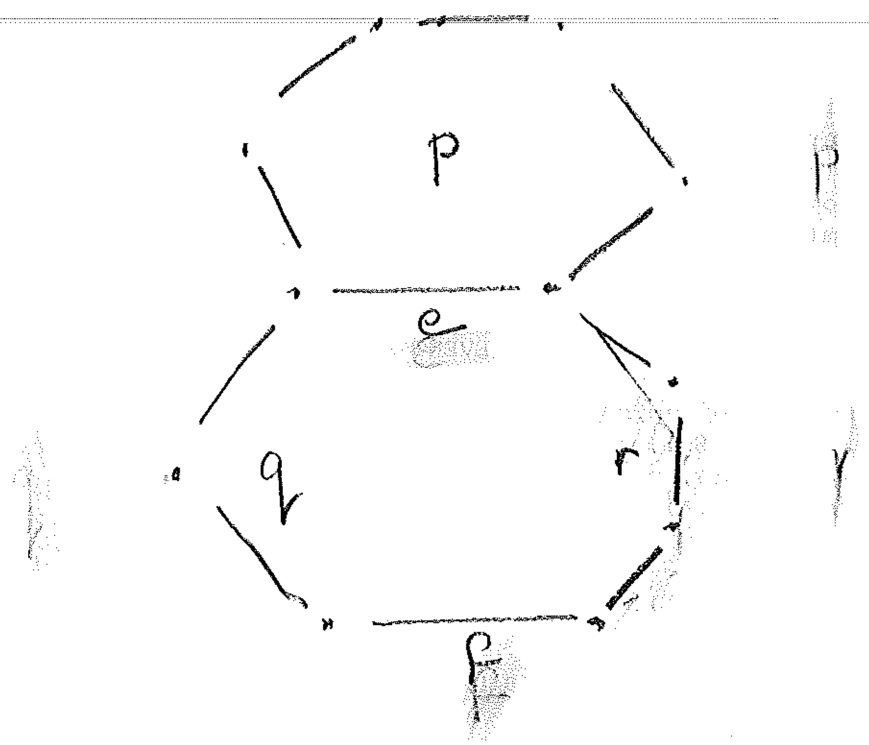
Proof By construction, p shares with q^$\langle e \rangle$^r the edge e. By Theorem 1, the union of all the vertices of p, q, r form a clique, containing all the vertices of q^$\langle e \rangle$^r. So q^p^r is a clique. It has the same endpoints as f, and so satisfies the definition. □

a chain on e, together with e, is ~~drawn~~ drawn as the perimeter of a polygon, whose interior can be filled by all the remaining edges of the clique.

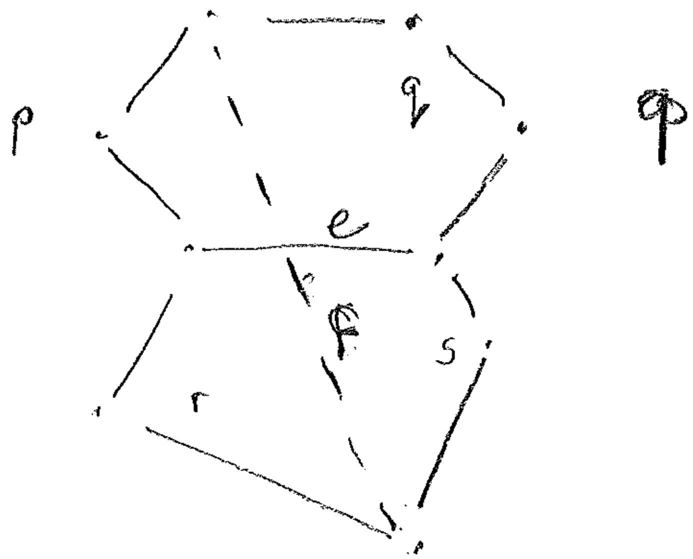
Theorem 3. **(Chain crossover)** *If p^q and r^s are both chains on e, then so is p^$\langle f \rangle$^s for some edge f.*

Proof By definition of a chain, the two cliques containing the paths p^q and r^s share the edge e. Their union is therefore a clique, and existence of the required edge f is assured. □

A chain is pictured as a circuit enclosing an area. The two theorems show how such areas can be combined and split again in all possible ways. In fact, there are enough edges to fill the whole area enclosed by a chain, and any circuit of the vertices defines a chain.



Theorem 2



Theorem 3

Directed Graphs

All definitions and theorems given so far apply equally to directed and undirected graphs. Considerations of direction permit finer distinctions to be made; in particular we can find graphs for which certain kinds of near tetrahedra are cliques, but others are not. We will concentrate attention primarily on acyclically directed subgraphs. There is essentially only one complete acyclic n-graph, as shown by the following theorem. (E.W.Dijkstra, private communication)

Theorem 4. *In a complete acyclic n-graph, there is a vertex with indegree i, for each i between 0 and* $n-1$*. Its outdegree is* $(n-i-1)$.

Proof Consider the relation L, where xLy means that there exists an edge directed from x to y. By the acyclic condition

$$\neg(xLy \;\&\; yLx) \qquad\qquad \text{(strictness)}$$

By the definition of completeness

$$xLy \;\vee\; yLx \;\vee\; x = y \qquad\qquad \text{(totality)}$$

If there is a directed path from x to y, the acyclic condition rules out two of the three possibilities listed above

$$xLw \;\&\; wLy \Rightarrow xLy \qquad\qquad \text{(transitivity)}$$

L is therefore a finite strict total ordering. The i^{th} vertex in the ordering is the target for $i-1$ edges from vertices below it, and the source of an edge to the $n-i-1$ vertices above it. □

A *nearly* complete n-graph is obtained by removing just a single edge from a complete n-graph. In the undirected case, all ways of doing this lead to the same result. In the acyclically directed case, removal of each of the $n(n-1)/2$ edges leads to a different graph. This is because the source and target vertices of the missing edge can be identified from their degrees, as shown by the following theorem.

Theorem 5. *A nearly complete acyclic n-graph has* $n-2$ *vertices with degree* $n-1$*, and two with degree* $n-2$*. A complete graph can be obtained by incrementing the higher of the anomalous outdegrees, and the indegree of the other anomalous vertex.*

Proof Removal of an edge decrements the outdegree of its source (the lower vertex) and the indegree of its target (the higher vertex). In the original complete graph, all vertices had degree $n - 1$, so now two of them have degree $n - 2$. □

On removal from a complete acyclic graph of an edge between two vertices adjacent in the ordering, these two anomalous vertices acquire the same indegree and the same outdegree. In this case, the missing edge can be restored in either of the two directions, maybe opposite to the original. In all other cases, acyclicity determines the direction.

Figure 1 displays and names the six varieties of directed near-tetrahedron. Their essential distinctness can be checked by counting indegrees and outdegrees, as shown. A graph is said to have property B2 if any of its subgraphs matching diagram B2 is a clique, i.e. the graph contains the missing diagonal. (In this case it may be in either of the two directions, without violation of acyclicity). Similar definitions can be given for five other properties, based on the other five diagrams. All six properties are mutually independent, as shown by the following theorem.

Theorem 6. *For each of the six properties defined by Fig 1, there exists a graph which violates it, but satisfies all the other five properties.*

Proof Consider the graph containing just the five edges of just one of the diagrams. Absence of the sixth edge causes violation of the corresponding property. All the other properties are satisfied vacuously, because no diagram matches any other. □

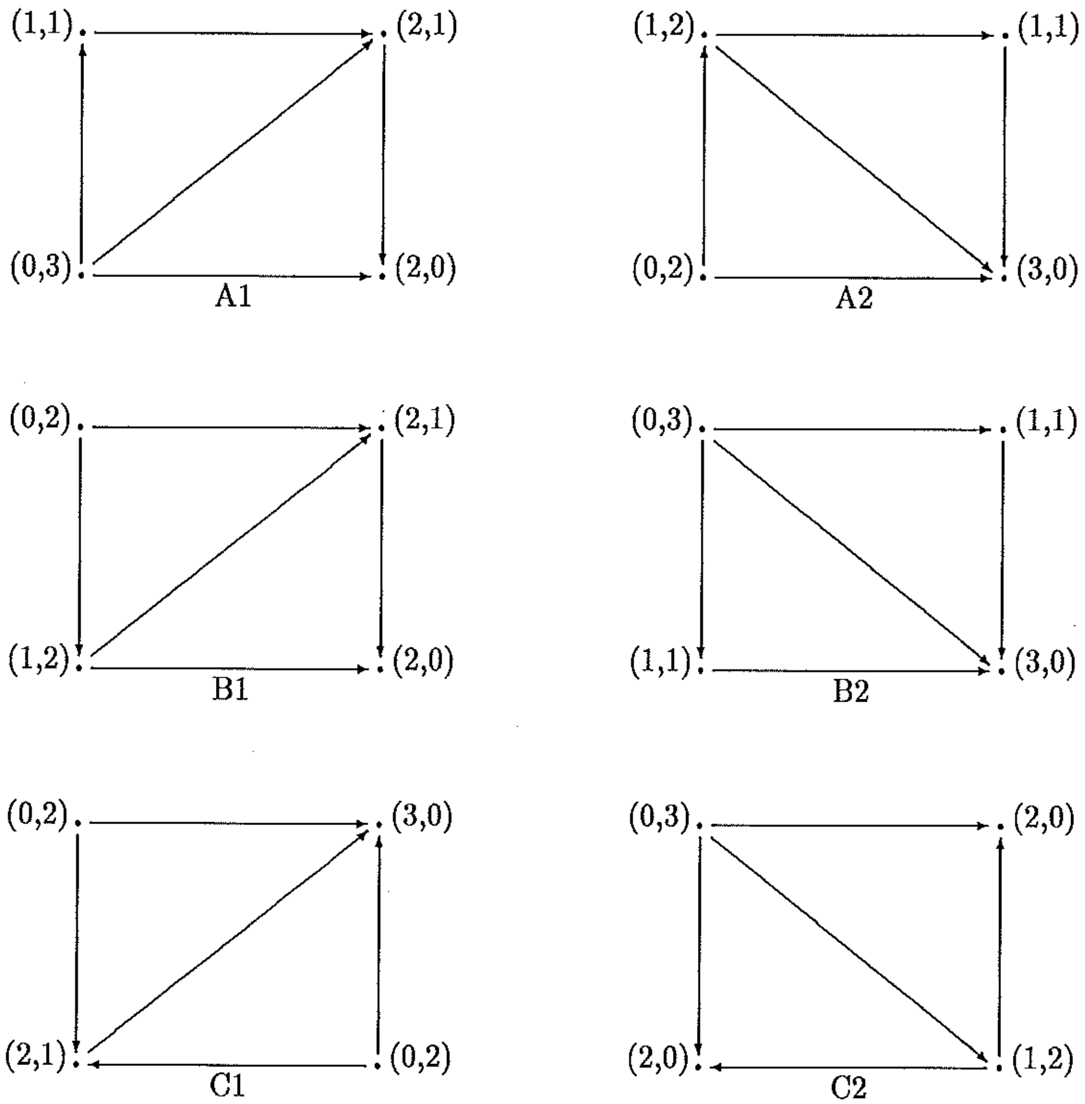
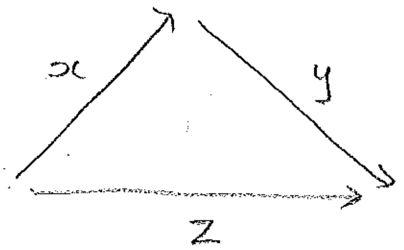

(1,1)
(2,1)
(0,3)
(2,0)
A1
(1,2)
(1,1)
(0,2)
(3,0)
A2
(0,2)
(2,1)
(1,2)
(2,0)
B1
(0,3)
(1,1)
(1,1)
(3,0)
B2
(0,2)
(3,0)
(2,1)
(0,2)
C1
(0,3)
(2,0)
(2,0)
(1,2)
C2

Figure 1:

2. Applications

Start with any set S, and any ternary predicate $T(x, y, z)$, where x, y, z range over members of S. We will represent the predicate by writing its arguments round the edges of an acyclically directed triangle, in the order shown:



The conjunction of several such propositions will be drawn as a collection of such triangles, sharing edges with the same direction and label. An example is obtained by annotating the edges of Fig 1 (B 1).

$$
\begin{tikzcd}
\cdot \arrow[r, "b"] \arrow[d, "a"'] & \cdot \arrow[d, "c"] \\
\cdot \arrow[r, "d"'] \arrow[ur, "x"] & \cdot
\end{tikzcd}
$$

$$T(a, x, b) \ \& \ T(x, c, d). \quad \ldots\ldots \quad (1)$$

Existentially quantified variables may have their names omitted; for example (cf. B2)

$$
\begin{tikzcd}
\cdot \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr] & \cdot \arrow[d, "c"] \\
\cdot \arrow[r, "d"'] & \cdot
\end{tikzcd}
$$

$$\exists y.\ T(a, d, y) \ \& \ T(b, c, y) \quad \ldots\ldots \quad (2)$$

Now suppose that predicate T has the property that

$$\left(\exists x.(1)\right) \Rightarrow (2) \quad \text{for all } a, b, c, d.$$

This means that every occurrence of diagram B1 ensures the existence of the additional arrow that makes the diagram complete. Such a predicate is said to have property B1. A similar definition is given for the other five properties defined by the diagrams of Figure 1.

Our main examples will be drawn from Computing Science. S will be the set of observations that can be made of some class of complex system and its components; and the predicate $T(x, y, z)$ will means that z is a possible observation of small assembly whose two components can give rise to observations x and y respectively.

A2. S is the set of machine states, represented as functions from variables to values. Two programs may operate in parallel on disjoint states, the result being simply the union of the results produced by the components

$$T(x, y, z) \triangleq \mathrm{dom}.x \cap \mathrm{dom}.y = \emptyset \;\&\; z = x \cup y$$

Disjointness ensures that z is also a function

B1. S is A^* for some set of events A.
Parallel composition is often implemented by arbitrary interleaving of the sequences of events in which the two components engage

$T(x, y, z) = z$ is an interleaving of x and y

This is the trace definition of $|||$ in CSP.

C1. S is A^* for some alphabet A. Composition is defined as concatenation of strings

$$T(x, y, z) = (z = x^\frown y)$$

This is the primitive composition operation in regular expressions and other grammars.

C2 S is $M \times M$, whose elements are pairs representing machine states before and after execution of a program. $T(x, y, z)$ represents sequential execution of components producing observations x and y. The initial state of z is the same as that of the first component x, the final state is that produced by y; and the initial state of y is the final state of x. In summary, T is defined

$$T((x, x'), (y, y'), (z, z')) = (z = x \;\&\; z' = y' \;\&\; y = x').$$

This is an element-wise definition of the familiar concept of relational composition.

Some of these predicates describe parallel execution and some sequential; some are symmetric in their first two arguments and some are not; some are functional in their third argument and some are non-deterministic. Let us ignore such important distinctions, to concentrate on properties of even greater interest.

Theorem(?) Each example enjoys the property with the same name, and all preceding properties.

3. Algebra.

A physical system or any of its components may be represented in mathematics by the set of all observations to which it may give rise, under any possible circumstances. The purpose of the mathematical calculations is to predict the possible observations that may be made of an assembly, given a corresponding knowledge of its its components. If there are only two components, the relationship between the three observations may be expressed by a ternary relation T, as illustrated in the previous section. The construction of an assembly from components A and B can be defined by lifting the predicate T to sets. The resulting algebraic operation will be denoted by semicolon:

$$A;B \triangleq \{c \mid \exists a,b : a \in A \ \& \ b \in B \ \& \ T(a,b,c)\}.$$

It is also interesting to enquire what would be the result of <u>removing</u> a component from an assembly: this might be either the first or second component; and the two effects are defined in terms of the representing sets:

$$C/B \triangleq \{a \mid \exists b,c:\ b \in B \ \&\ c \in C \ \&\ T(a,b,c)\}$$

$$A \backslash C \triangleq \{b \mid \exists a,c:\ a \in A \ \&\ c \in C \ \&\ T(a,b,c)\}.$$

In general, removal of components is not a reasonable, economic or even a feasible way of constructing a product; but it can be very useful in reasoning about its construction, especially if it is subject to powerful algebraic laws.

Already for an arbitrary predicate T there are a number of interesting laws

1. $;$ / and \ all distribute through arbitrary unions, finite, infinite, or empty.
2. The following are all equivalent

(a) $(A;B) \cap C \neq \emptyset$

(b) $(C/B) \cap A \neq \emptyset$

(c) $(A\backslash C) \cap A \neq \emptyset$

(d) $\exists a, b, c :: a \in A \ \& \ b \in B \ \& \ c \in C \ \& \ T(a,b,c)$

The second law is identical to the Schroeder equivalences (exchange laws) of the relational calculus, where $C/B = C;\breve{B}$ and $A\backslash C = \breve{A};C$. The converse operator is a source of great power in the relational calculus. We make do with the much weaker but more general relative converses / and \. Fortunately, much of the power can be restored in cases when T enjoys some or all of the properties described in the previous section.

Theorem 2 is in fact a nuisance: it means that there are at least three dissimilar methods of saying the same thing. A single notation would be much simpler; so we extend the same graphical notation used in the previous section, labelling the edges (of a triangle) with sets instead of single elements:

$$\begin{tikzcd} & {} \arrow[dr, "B"] & \\ {} \arrow[ur, "A"] \arrow[rr, "C"'] & & {} \end{tikzcd}$$

$$\exists a, b, c : a \in A \;\&\; b \in B \;\&\; c \in C \;\&\; T(a, b, c).$$

This asserts the existence of a member of the each of the sets labelling an edge of the diagram; and the diagram represents a true proposition when the set labels are replaced by suitably chosen elements from each of the sets.

This definition can be extended to diagrams with more than one triangle, for example:

$$\begin{tikzcd} \bullet \arrow[r, "B"] & \bullet \arrow[d, "C"] \\ \bullet \arrow[u, "A"] \arrow[ur, "X"] \arrow[r, "D"'] & \bullet \end{tikzcd}$$

$$\exists a, b, c, d, x : \left(a \in A \,\&\, b \in B \,\&\, c \in C \,\&\, d \in D \,\&\, x \in X \,\&\, T(a, b, x) \,\&\, T(x, c, d) \right).$$

Note that this is a <u>stronger</u> assertion than just the conjunction of the two component triangles, because the member chosen for the shared edge (labelled X) has to be the same on both sides; if the triangles ~~were~~

were separated the assertion would be weaker:

new line

$$\exists a, b, x \left(a \in A \,\&\, b \in B \,\&\, x \in X \,\&\, T(a,b,x) \right) \,\&\, \exists c, d, x \left(c \in C \,\&\, T(x,c,d) \right)$$

An edge may be labelled by a conjunction of sets, each of which is said to label it. An edge labelled by the universal set may be left unlabelled