Role Conjunctions in Expressive Description Logics

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Abstract. We show that adding role conjunctions to the prominent DLs $\mathcal{SHIF}$ and $\mathcal{SHOIN}$ causes a jump in the computational complexity of the standard reasoning tasks from ExpTime to 2ExpTime already for $\mathcal{SHI}$ and from NExpTime to N2ExpTime for $\mathcal{SHOIF}$. We further show that this increase in complexity is due to a subtle interaction between inverse roles, role hierarchies, and role transitivity in the presence of role conjunctions and that for the DL $\mathcal{SHQ}$ a jump in the computational complexity cannot be observed.

1 Introduction

Description Logics (DLs) are a family of logic based knowledge representation formalisms [1]. Most DLs are fragments of First-Order Logic restricted to unary and binary predicates, which are called concepts and roles in DLs. The constructors for building complex expressions are usually chosen such that the key inference problems, such as concept satisfiability, are decidable. The Description Logics $\mathcal{SHIF}$ and $\mathcal{SHOIN}$ provide a logical underpinning for the W3C standards OWL Lite and OWL DL [2] and highly optimized implementations for the standard reasoning tasks are available, e.g., FaCT++ [3], KAON2[1], Pellet [4], and RacerPro[2]. These systems are used in a wide range of applications, e.g., medicine [5–8, 6], bio informatics [9–11], life sciences [12, 13], or information integration [14–16].

The DLs $\mathcal{SHIF}$ and $\mathcal{SHOIN}$ provide quite a rich set of constructors for concepts (unary predicates). Current standardization efforts go, however, into the direction of also supporting a richer set of constructors for roles (binary predicates), but it was recently shown that role compositions in the proposed OWL2 (previously known as OWL 1.1)[3] cause an exponential blowup [17].

We show that an exponential blowup also occurs if we allow for conjunctions over roles, which naturally appear, for example, when conjunctive queries over knowledge bases are reduced to standard reasoning tasks. Using role conjunctions, the query $\langle x \rangle \leftarrow r(x, y) \wedge s(x, y) \wedge A(y)$ can, for example, be answered by retrieving all instances of the concept $\exists(r \sqcap s).A$ for A a concept name, r, s roles, and x, y variables. We show, by a reduction to the word problem for exponential space bounded alternating Turing machines, that the computational

[1] `http://kaon2.semanticweb.org`
[2] `http://www.racer-systems.com`
[3] `http://www.w3.org/TR/owl2-syntax/`

complexity of the standard reasoning tasks jumps from ExpTime to 2ExpTime already for $\mathcal{SHI}^{\sqcap}$ (without number restrictions). We further show that the standard reasoning tasks become N2ExpTime-hard in $\mathcal{SHOIF}^{\sqcap}$, although they are NExpTime-complete for $\mathcal{SHOIF}$. We show this by using an instance of the tiling problem. In both cases, the increase in complexity is due to a subtle interaction between inverse roles, role transitivity, and role hierarchies in the presence of role conjunctions. We demonstrate this by proving that for the DL $\mathcal{SHQ}^{\sqcap}$ that does not allow for inverse roles, the standard reasoning tasks remain in ExpTime.

A similar effect is known from propositional dynamic logics (PDL), where the intersection operator also causes a jump from ExpTime to 2ExpTime [18]. The logic PDL is very similar to the DL $\mathcal{ALC}$ extended with regular expressions over roles. It was further known that full Boolean role operators cause a jump from ExpTime to NExpTime for the basic DL $\mathcal{ALC}$ plus nominals [19] and this result can further be sharpened to only $\mathcal{ALC}$ extended with role conjunctions and role negation [20]. When placing a restriction on the use of role negations (to so called safe role expressions) as in $\mathcal{ALCQI}b$, the standard reasoning tasks remain in ExpTime [21].

In the following section, we give some basic definitions and notations used throughout the paper. In Section 3, we discuss the relationship between role conjunctions and conjunctive queries and give a polynomial reduction from the problem of knowledge base satisfiability in any DL between $\mathcal{ALCH}^{\sqcap}$ and $\mathcal{SHIO}^{\sqcap}$ to the problem of conjunctive query entailment in the respective DL without role conjunctions. In Section 4, we show that in $\mathcal{SHQ}$, i.e., without nominals and inverses, the standard reasoning tasks remain in ExpTime. In Section 5, we present the results for $\mathcal{SHI}^{\sqcap}$, followed by our grid construction technique for $\mathcal{SHOIF}^{\sqcap}$ in Section 6. Finally, we conclude and discuss some remaining open questions.

2 Preliminaries

Let N_C, N_R, and N_I be countably infinite sets of *concept names*, *role names*, and *individual names*. We assume that the set of role names contains a subset $N_{tR} \subseteq N_R$ of *transitive role names*. A *role* R is an element of $N_R \cup \{r^- \mid r \in N_R\}$, where roles of the form r^- are called *inverse roles*. A *role conjunction* is an expression of the form $\rho = (R_1 \sqcap \cdots \sqcap R_n)$. A *role inclusion axiom* (RIA) is an axiom of the form $R \sqsubseteq S$ where R and S are roles. A *role hierarchy* $\mathcal{R}$ is a finite set of role inclusion axioms.

An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty set $\Delta^{\mathcal{I}}$, the *domain* of $\mathcal{I}$, and a function $\cdot^{\mathcal{I}}$, which maps every concept name A to a subset $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, every role name $r \in N_R$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, every role name $r \in N_{tR}$ to a transitive binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and every individual name a to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The interpretation of an inverse role r^- is $\{\langle d, d' \rangle \mid \langle d', d \rangle \in r^{\mathcal{I}}\}$. The interpretation of a role conjunction $R_1 \sqcap \cdots \sqcap R_n$ is ${R_1}^{\mathcal{I}} \cap \cdots \cap {R_n}^{\mathcal{I}}$. An interpretation $\mathcal{I}$ satisfies a RIA $R \sqsubseteq S$ if $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$, and a role hierarchy $\mathcal{R}$ if $\mathcal{I}$ satisfies all RIAs in $\mathcal{R}$.

For a role hierarchy $\mathcal{R}$, we introduce the following standard DL notations:

1. We define the function Inv over roles as $\mathsf{Inv}(R) := R^-$ if $R \in N_R$ and $\mathsf{Inv}(R) := s$ if $R = s^-$ for a role name s.
2. We define $\sqsubseteq_{\mathcal{R}}$ as the smallest transitive reflexive relation on roles such that $R \sqsubseteq S \in \mathcal{R}$ implies $\mathcal{R} \sqsubseteq_{\mathcal{R}} S$ and $\mathsf{Inv}(R) \sqsubseteq_{\mathcal{R}} \mathsf{Inv}(S)$. We write $R \equiv_{\mathcal{R}} S$ if $R \sqsubseteq_{\mathcal{R}} S$ and $S \sqsubseteq_{\mathcal{R}} R$.
3. A role R is called *transitive w.r.t.* $\mathcal{R}$ (notation $R^+ \sqsubseteq_{\mathcal{R}} R$) if $R \equiv_{\mathcal{R}} S$ for some role S such that $S \in N_{tR}$ or $\mathsf{Inv}(S) \in N_{tR}$.
4. A role S is called *simple w.r.t.* $\mathcal{R}$ if there is no role R such that R is transitive w.r.t. $\mathcal{R}$ and $R \sqsubseteq_{\mathcal{R}} S$. A role conjunction $R_1 \sqcap \cdots \sqcap R_n$ is *simple w.r.t.* $\mathcal{R}$ if each conjunct is simple w.r.t. $\mathcal{R}$.

The set of $\mathcal{SHOIQ}^{\sqcap}$*-concepts* (or concepts for short) is the smallest set built inductively from N_C, N_R, and N_I using the following grammar, where $A \in N_C$, $o \in N_I$, n is a non-negative integer, ρ is a role conjunction and δ is a simple role conjunction:

$$C ::= A \mid \{o\} \mid \neg C \mid C_1 \sqcap C_2 \mid \forall \rho.C \mid \geqslant n\, \delta.C.$$

We use the following standard abbreviations: $\top \equiv A \sqcup \neg A$, $\bot \equiv A \sqcap \neg A$, $C_1 \sqcup C_2 \equiv \neg(\neg C_1 \sqcap \neg C_2)$, $\exists \rho.C \equiv \neg(\forall \rho.(\neg C))$, and $\leqslant n\, \delta.C \equiv \neg(\geqslant (n+1)\, \delta.C)$.

Given an interpretation $\mathcal{I}$, the semantics of $\mathcal{SHOIQ}^{\sqcap}$-concepts is defined as follows:

Given an interpretation $\mathcal{I}$, the semantics is defined as follows:

$$\begin{aligned} \{o\}^{\mathcal{I}} &= \{o^{\mathcal{I}}\},\ (C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}},\ (\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}, \\ (\forall \rho.C)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \text{ if } \langle d, d' \rangle \in \rho^{\mathcal{I}}, \text{ then } d' \in C^{\mathcal{I}}\}, \\ (\geqslant n\, \delta.C)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \sharp s^{\mathcal{I}}(d, C) \geq n\} \end{aligned}$$

where $\sharp M$ denotes the cardinality of the set M and $s^{\mathcal{I}}(d, C)$ is defined as $\{d' \in \Delta^{\mathcal{I}} \mid \langle d, d' \rangle \in s^{\mathcal{I}} \text{ and } d' \in C^{\mathcal{I}}\}$. Concepts of the form $\{o\}$ are called *nominals*.

A *general concept inclusion* (GCI) is an expression $C \sqsubseteq D$, where both C and D are concepts. A finite set of GCIs is called a *TBox*. An interpretation $\mathcal{I}$ *satisfies a GCI* $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and a TBox $\mathcal{T}$ if it satisfies every GCI in $\mathcal{T}$.

An (ABox) *assertion* is an expression of the form $C(a)$, $r(a, b)$, where C is a concept, r is a role, $a, b \in N_I$. An *ABox* is a finite set of assertions. We use $N_I(\mathcal{A})$ to denote the set of individual names occurring in $\mathcal{A}$. An interpretation $\mathcal{I}$ *satisfies an assertion* $C(a)$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, $r(a, b)$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in r^{\mathcal{I}}$. An interpretation $\mathcal{I}$ *satisfies an ABox* $\mathcal{A}$ if it satisfies each assertion in $\mathcal{A}$, which we denote with $\mathcal{I} \models \mathcal{A}$.

A *knowledge base* (KB) is a triple $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ with $\mathcal{R}$ a role hierarchy, $\mathcal{T}$ a TBox, and $\mathcal{A}$ an ABox. We use $N_I(\mathcal{K})$, $N_C(\mathcal{K})$, $N_R(\mathcal{K})$, $N_{tR}(\mathcal{K})$ to denote the sets of individual names, concept names, and (transitive) role names occurring in $\mathcal{K}$. We say that an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *satisfies* $\mathcal{K}$ if $\mathcal{I}$ satisfies $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{A}$. In this case, we say that $\mathcal{I}$ is a *model* of $\mathcal{K}$ and write $\mathcal{I} \models \mathcal{K}$. We say that $\mathcal{K}$ *is satisfiable* if $\mathcal{K}$ has a model. A concept D *subsumes* a concept C w.r.t. $\mathcal{K}$, denoted as $\mathcal{K} \models (C \sqsubseteq D)$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{K}$. A concept C is *satisfiable* w.r.t. $\mathcal{K}$ if there is a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^{\mathcal{I}} \neq \emptyset$.

It should be noted that the standard reasoning tasks such as knowledge base satisfiability, concept subsumption, or concept satisfiability are mutually reducible in polynomial time. For example, concept subsumption can be reduced to concept (un)satisfiability as follows: a concept D subsumes a concept C w.r.t. $\mathcal{K}$ iff the concept $C \sqcap \neg D$ is unsatisfiable w.r.t. $\mathcal{K}$.

The DLs $\mathcal{SHIQ}^{\sqcap}$ and $\mathcal{SHOQ}^{\sqcap}$ are obtained from $\mathcal{SHOIQ}^{\sqcap}$ by disallowing nominals and inverse roles respectively. Further disallowing number restrictions gives the DLs $\mathcal{SHI}^{\sqcap}$ and $\mathcal{SHO}^{\sqcap}$ respectively. Finally, the DL $\mathcal{SHOIQ}^{\sqcap}$ minus both nominals and inverse roles, results in the DL $\mathcal{SHQ}^{\sqcap}$. If we restrict number restrictions to the form $\leqslant n\, \delta.\top$ and $\geqslant n\, \delta.\top$, we denote this by the letter $\mathcal{N}$ instead of $\mathcal{Q}$. If we allow instead of number restrictions only the declaration of roles as functional, we write $\mathcal{F}$ instead of $\mathcal{N}$ or $\mathcal{Q}$.

3 Conjunctive Queries and Role Conjunctions

There is a close relationship between role conjunctions and conjunctive queries and often the complexity results for knowledge base satisfiability in a DL $\mathcal{L}^{\sqcap}$ and the query entailment problem for the DL $\mathcal{L}$ agree. In this section we show that the standard reasoning problems for DLs with role conjunctions and without counting can reduced to the problem of answering unions of conjunctive queries in the respective DL without role conjunctions. The opposite direction is, in general, not possible, i.e., conjunctive query entailment cannot be polynomially reduced to knowledge base satisfiability in the DL with role conjunctions. This is a straightforward consequence of the fact that knowledge base satisfiability for $\mathcal{ALCI}^{\sqcap}$ is ExpTime-complete [21], while conjunctive query entailment in $\mathcal{ALCI}$ is 2ExpTime-complete [22].

Let N_V be a countably infinite set of variables, and (N_C, N_R, N_I) a signature. An atom at is an expression of the form $A(v)$ or $r(v, v')$ where $v, v' \in N_V$, $A \in N_C$, and $r \in N_R$. A *Boolean conjunctive query* q is a conjunction of atoms. We use $N_V(q)$ to denote the set of variables occurring in q. A *union of Boolean conjunctive queries* Q is an expression $q_1 \vee \ldots \vee q_\ell$, where each q_i is a Boolean conjunctive query.

Let q be a Boolean conjunctive query and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an interpretation. For a total function $\pi\colon N_V(q) \to \Delta^{\mathcal{I}}$, we write $\mathcal{I} \models^{\pi} A(v)$ if $\pi(v) \in A^{\mathcal{I}}$ and $\mathcal{I} \models^{\pi} r(v, v')$ if $\langle \pi(v), \pi(v') \rangle \in r^{\mathcal{I}}$. If $\mathcal{I} \models^{\pi} at$ for all atoms at in q, we write $\mathcal{I} \models^{\pi} q$. We say that $\mathcal{I}$ *satisfies* q and write $\mathcal{I} \models q$ if there exists a π such that $\mathcal{I} \models^{\pi} q$. Let $\mathcal{K}$ be a knowledge base and q a conjunctive query. If $\mathcal{I} \models q$ for every mode $\mathcal{I}$ of $\mathcal{K}$, we say that $\mathcal{K}$ *entails* q and write $\mathcal{K} \models q$. $\mathcal{K}$ *entails* a union of conjunctive queries $Q = q_1 \vee \ldots \vee q_\ell$, written as $\mathcal{K} \models Q$, if, for every model $\mathcal{I}$ of $\mathcal{K}$, there exists some i with $1 \leq i \leq \ell$ such that $\mathcal{I} \models q_i$.

Please note that the omission of constants and answer variables in the definition of conjunctive queries and the restriction to concept and role names is for the sake of complexity without loss of generality [23, 22].

We show now how the problem of knowledge base satisfiability for any DL between $\mathcal{ALCH}^{\sqcap}$ and $\mathcal{SHIO}^{\sqcap}$ can polynomially be reduced to entailment of unions of conjunctive queries in the respective DL without role conjunctions.

Let $\mathcal{K} = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ be a knowledge base in a DL between $\mathcal{ALCH}^{\sqcap}$ and $\mathcal{SHIO}^{\sqcap}$. It is always possible to transform $\mathcal{K}$ preserving satisfiability such that all GCIs in $\mathcal{T}$ have one of the following *simplified forms*:

$$A \sqsubseteq \forall \rho.B \ \mid \ A \sqsubseteq \exists r.B \ \mid \ \bigsqcap A_i \sqsubseteq \bigsqcup B_j \mid \ A \equiv \{a\}, \tag{1}$$

where $A_{(i)}$ and $B_{(j)}$ are atomic concepts, a is an individual name, r is a role name, and ρ is a conjunction of roles. Furthermore, concept assertions in $\mathcal{A}$ are limited to the form $A(a)$ for a concept name A. If $i = 0$, we interpret $\bigsqcap A_i$ as $\top$ and if $j = 0$, we interpret $\bigsqcup B_j$ as $\bot$. Every knowledge base, which is not in this form, can be transformed in polynomial time into the desired form by using the standard structural transformation, which iteratively introduces definitions for compound sub-concepts and sub-roles (see, e.g., [24]). GCIs of the form $A \sqsubseteq \exists(R_1 \sqcap \ldots \sqcap R_n).B$ can be transformed into those without role conjunctions, by replacing $R_1 \sqcap \ldots \sqcap R_n$ with a fresh role name r and adding the role axioms $r \sqsubseteq R_i$ for each i with $1 \leq i \leq n$.

From $\mathcal{K}$ we obtain a knowledge base $\mathcal{K}'$ and a query q as follows: $\mathcal{K}' = (\mathcal{T}', \mathcal{R}, \mathcal{A})$ and $\mathcal{T}'$ is obtained from $\mathcal{T}$ by dropping each GCI of the form $A \sqsubseteq \forall \rho.B$ with $\rho = R_1 \sqcap \ldots \sqcap R_n$ and by adding, for each such GCI, an axiom $\neg B \sqsubseteq \bar{B}$ where $\bar{B}$ is a fresh concept. The query Q is a union of Boolean conjunctive queries such that, for each GCI of the form $A \sqsubseteq \forall \rho.B$ in $\mathcal{T}$, Q contains a conjunct $A(x) \wedge R_1(x, y) \wedge \ldots \wedge R_n(x, y) \wedge \bar{B}(y)$.

Please note that the concept $\bar{B}$ is only introduced since we allow only for concept names in a concept atom of the query and not for complex concepts.

Lemma 1. *Let $\mathcal{K}$ be a simplified knowledge base in a DL between $\mathcal{ALCH}^{\sqcap}$ and $\mathcal{SHIO}^{\sqcap}$ and let $\mathcal{K}'$ and Q be obtained from $\mathcal{K}$ as described above. Then $\mathcal{K}$ is satisfiable iff $\mathcal{K}' \not\models Q$.*

Proof. For the if direction: By assumption there exists a model $\mathcal{I}$ of $\mathcal{K}'$ such that $\mathcal{I} \not\models Q$. We claim that $\mathcal{I} \models \mathcal{K}$. In contrary of what is to be shown, assume that there exists a GCI $A \sqsubseteq \forall \rho.B$ with $\rho = R_1 \sqcap \ldots \sqcap R_n$ in $\mathcal{K}$ and elements $d, d' \in \Delta^{\mathcal{I}}$ such that $d \in A^{\mathcal{I}}, \langle d, d' \rangle \in {R_1}^{\mathcal{I}} \cap \ldots \cap {R_n}^{\mathcal{I}}$, and $d' \in (\neg B)^{\mathcal{I}}$. By definition, $\mathcal{K}'$ contains the axiom $\neg B \sqsubseteq \bar{B}$ and, since $\mathcal{I}$ satisfies $\mathcal{K}'$, we have that $d' \in \bar{B}^{\mathcal{I}}$. Also by definition, Q contains a disjunct $q_i = A(x) \wedge r_1(x, y) \wedge \ldots \wedge R_n(x, y) \wedge \bar{B}(y)$ and it is not hard to check that $\mathcal{I} \models^{\pi} q_i$ for $\pi : x \mapsto d, y \mapsto d'$. Thus $\mathcal{I} \models q_i$ and, by definition of the semantics of unions of conjunctive queries, $\mathcal{I} \models Q$, which is a contradiction.

For the only if direction: By assumption there exists a model $\mathcal{I}$ of $\mathcal{K}$. We extend $\mathcal{I}$ to a model of $\mathcal{K}'$ by interpreting the new concepts in $\mathcal{K}'$ as follows: for each GCI of the form $A \sqsubseteq \forall \rho.B$ with $\rho = R_1 \sqcap \ldots \sqcap R_n$, we set $\bar{B}^{\mathcal{I}} = (\neg B)^{\mathcal{I}}$. By definition of the semantics, $\mathcal{I}$ satisfies each of the new axioms $\neg B \sqsubseteq \bar{B}$ and, thus, $\mathcal{I} \models \mathcal{K}'$. We have to show that $\mathcal{I} \not\models Q$. Assume to the contrary that $\mathcal{I} \models^{\pi} q_i$ for a disjunct $q_i = A(x) \wedge r_1(x, y) \wedge \ldots \wedge R_n(x, y) \wedge \bar{B}(y)$ of Q. Then $\pi(y) \in \bar{B}^{\mathcal{I}} = \neg B^{\mathcal{I}}$ and, thus, $\pi(x) \notin \forall (R_1 \sqcap \ldots \sqcap R_n).B^{\mathcal{I}}$, which is a contradiction. $\square$

Please note that the above reduction produces only conjunctive queries with two variables. This fact, together with our 2ExpTime-hardness result for $\mathcal{SHI}^{\sqcap}$, implies that conjunctive query entailment for $\mathcal{SHI}$ is 2ExpTime-hard already for a bounded number of variables in the query. The previously known 2ExpTime-hardness result for conjunctive query entailment in $\mathcal{ALCI}$ [22] holds only if the number of variables in the queries is not bounded [23].

4 $\mathcal{SHQ}^{\sqcap}$ is ExpTime-complete

In this section, we show that adding role conjunctions to the DL $\mathcal{SHQ}$ does not increase the computational complexity of the standard reasoning problems. For this purpose, we devise a polynomial encoding of a given $\mathcal{SHQ}^{\sqcap}$ knowledge base to an equisatisfiable $\mathcal{ALCHQ}^{\sqcap}$ (i.e., $\mathcal{SHQ}^{\sqcap}$ minus role transitivity) knowledge base. Since it is known that the standard reasoning tasks for $\mathcal{ALCHQ}^{\sqcap}$ are in ExpTime [21, 23], this gives us the desired ExpTime upper bound. A corresponding lower bound straightforwardly follows from the ExpTime-hardness for $\mathcal{ALC}$ concept satisfiability checking w.r.t. general TBoxes [25], where $\mathcal{ALC}$ is the DL that restricts $\mathcal{ALCHQ}^{\sqcap}$ further by disallowing role hierarchies, number restrictions, and role conjunctions.

Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be an $\mathcal{SHQ}^{\sqcap}$ knowledge base. We say that $\mathcal{K}$ is *simplified* if $\mathcal{T}$ contains only axioms of the form

$$A \sqsubseteq \forall \rho.B \;\mid\; A \sqsubseteq \exists \rho.B \;\mid\; A \sqsubseteq \bowtie n\, \delta.B \;\mid\; \bigsqcap A_i \sqsubseteq \bigsqcup B_j, \tag{2}$$

where $A_{(i)}$ and $B_{(j)}$ are atomic concepts, ρ (δ) is a (simple) conjunction of roles, and $\bowtie$ stands for $\leqslant$ or $\geqslant$. Furthermore, concept assertions in $\mathcal{A}$ are limited to the form $A(a)$ for a concept name A. If $i = 0$, we interpret $\bigsqcap A_i$ as $\top$ and if $j = 0$, we interpret $\bigsqcup B_j$ as $\bot$. Every $\mathcal{SHQ}^{\sqcap}$ knowledge base, which is not in this form, can in polynomial time be transformed into the desired form by using the standard structural transformation, which iteratively introduces definitions for compound sub-concepts and sub-roles (see, e.g., [24]).

Encoding transitivity is often used, and polynomial encodings are known for many DLs such as $\mathcal{SHIQ}$ or $\mathcal{SHOIQ}$ [24, 26]. Intuitively, these encodings work by adding axioms that propagate the concepts that occur under universal quantifiers over paths of transitive roles. This ensures that even if we treat the transitive roles as non-transitive, we can obtain a model of the original knowledge base by transitively closing the relations of the originally transitive roles. For example, for a simplified $\mathcal{SHQ}$ knowledge base (without role conjunctions) such an encoding produces a knowledge base in which all transitive roles are regarded as non-transitive and that contains, for each axiom $A \sqsubseteq \forall r.B$ and $t \in N_{tR}$ such that $t \sqsubseteq_{\mathcal{R}} r$, additionally the axioms

$$A \sqsubseteq \forall t.A^t, \quad A^t \sqsubseteq \forall t.A^t, \quad A^t \sqsubseteq B, \tag{3}$$

for a fresh concept name $A^t \in N_C$. If we adapt this encoding in a naive way to $\mathcal{SHQ}^{\sqcap}$ (cf. also [23]), we would add, for each axiom $A \sqsubseteq \forall \rho.B$ with $\rho = r_1 \sqcap \ldots \sqcap r_n$ and $\tau = t_1 \sqcap \ldots \sqcap t_n$ such that $t_i \in N_{tR}$ and $t_i \sqsubseteq_{\mathcal{R}} r_i$ for each i with $1 \leq i \leq n$, the axioms

$$A \sqsubseteq \forall \tau.A^\tau, \quad A^\tau \sqsubseteq \forall \tau.A^\tau, \quad A^\tau \sqsubseteq B, \tag{4}$$

for A^τ a fresh concept name. This encoding is no longer polynomial since for an input knowledge base of size m, we can only use m as an upper bound for the number of transitive sub-roles for each T_i, which leaves us with an upper bound of m^n for the number of additional axioms. Furthermore, n can also only be bounded by m.

For our encoding, the tree or forest model property of $\mathcal{SHQ}^{\sqcap}$ is quite important and, therefore, we define more precisely, what we mean with forest models.

In the following, we assume without loss of generality, that the ABox contains at least one individual name, i.e., $N_I(\mathcal{A})$ is non-empty. Otherwise, one can always add an assertion $A(a)$ for a fresh A and a to $\mathcal{A}$.

Definition 1. *Let $\mathbb{N}$ denote the set of non-negative integers and $\mathbb{N}^+$ the set of all (finite) non-empty words over the alphabet $\mathbb{N}$. A non-empty set $F \subseteq \mathbb{N}^+$ is a* forest *if, for each $w \in \mathbb{N}^+$ and $c \in \mathbb{N}$, $w \cdot c \in F$ implies $w \in F$, where "$\cdot$" denotes concatenation. For $w' = w \cdot c$, we call w' a* successor *of w.*

Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be an $\mathcal{SHQ}^{\sqcap}$ knowledge base with $N_I(\mathcal{A}) = \{a_1, \ldots, a_m\}$. A forest base for $\mathcal{K}$ *is an interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ that interprets transitive roles in an unrestricted (i.e., not necessarily transitive) way and, additionally, satisfies the following conditions:*

- F1 *$\Delta^{\mathcal{J}}$ is a forest;*
- F2 *there is a total and bijective mapping f from $N_I(\mathcal{A})$ to $\Delta^{\mathcal{J}} \cap \mathbb{N}$ such that ${a_i}^{\mathcal{J}} = i$;*
- F3 *if $\langle w, w' \rangle \in r^{\mathcal{J}}$, then either $w, w' \in \mathbb{N}$ or w' is a successor of w.*

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is canonical for $\mathcal{K}$ *if there exists a forest base $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ for $\mathcal{K}$ such that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$, $a^{\mathcal{I}} = a^{\mathcal{J}}$ for $a \in N_I(\mathcal{K})$, $A^{\mathcal{I}} = A^{\mathcal{J}}$ for $A \in N_C(\mathcal{K})$, and $r^{\mathcal{I}} = r^{\mathcal{J}} \cup \bigcup_{t \sqsubseteq_{\mathcal{R}} r,\, t \in N_{tR}} (t^{\mathcal{J}})^+$, where the superscript $^+$ denotes the transitive closure. In this case, we say that $\mathcal{J}$ is a* forest base *for $\mathcal{I}$ and if $\mathcal{I} \models \mathcal{K}$ we say that $\mathcal{I}$ is a* canonical model for $\mathcal{K}$.

For a forest base (see Figure 1), we require in particular that *all* relationships between elements within a tree that can be inferred by transitively closing a role are omitted (cf. F3).

Please note that the above definition implicitly relies on the unique name assumption (UNA) (cf. F2). This is w.l.o.g. as we can guess an appropriate partition among the individual names and replace the individual names in each partition with one representative individual name from that partition. Furthermore, for a logic that is ExpTime-hard, we can do this without increasing the theoretical complexity of the standard reasoning problems.

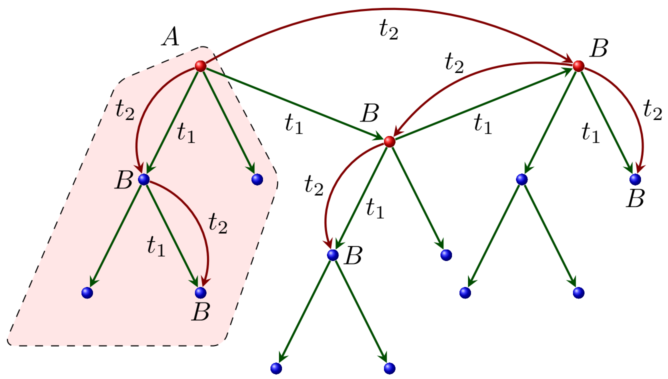

Fig. 1. A forest base for a forest model satisfying axiom $A \sqsubseteq \forall(t_2 \sqcap t_2).B$ where t_1 and t_2 are transitive roles

Lemma 2. *Let $\mathcal{K}$ be an $\mathcal{SHQ}^\sqcap$ knowledge base, then $\mathcal{K}$ is satisfiable iff $\mathcal{K}$ has a canonical model.*

Proof (Sketch). The if direction is trivial. For the only if direction, we can use any model $\mathcal{I}$ of $\mathcal{K}$, which exists by assumption, and unravel this model into a canonical model (see, e.g., [23]). $\square$

Our aim is now to transform $\mathcal{K}$ into an equisatisfiable $\mathcal{ALCHQ}^\sqcap$ knowledge base. We will use the canonical models to show that the obtained knowledge base is equisatisfiable with the original knowledge base.

Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be a simplified $\mathcal{SHQ}^\sqcap$ knowledge base. We construct $\mathcal{K}' = (\mathcal{R}', \mathcal{T}', \mathcal{A}')$ as an extension of $\mathcal{K}$ with new concepts and axioms. The signature of $\mathcal{K}'$ is defined by $N_I(\mathcal{K}') := N_I(\mathcal{K})$, $N_R(\mathcal{K}') := N_R(\mathcal{K})$, $N_{tR}(\mathcal{K}') := \emptyset$, $N_C(\mathcal{K}') := N_C(\mathcal{K}) \cup \{A_a,\, A_a^r \mid A \in N_C(\mathcal{K}),\, a \in N_I(\mathcal{K}),\, r \in N_R(\mathcal{K})\}$. Recall, that w.l.o.g., $N_I(\mathcal{K})$ is non empty, therefore there exists at least one A_a for every $A \in N_C(\mathcal{K})$. We obtain $\mathcal{K}'$ from $\mathcal{K}$ by extending $\mathcal{K}$ with the following axioms:

$$A \sqsubseteq \textstyle\bigsqcup_{a \in N_I(\mathcal{A})} A_a \qquad A \in N_C(\mathcal{K}) \tag{5}$$
$$A_a \sqsubseteq \forall r.A_a^r \qquad A \in N_C(\mathcal{K}),\, a \in N_I(\mathcal{A}),\, r \in N_R(\mathcal{K}) \tag{6}$$
$$A_a^t \sqsubseteq \forall t.A_a^t \qquad A \in N_C(\mathcal{K}),\, a \in N_I(\mathcal{A}),\, t \in N_{tR}(\mathcal{K}) \tag{7}$$
$$A_a^t \sqsubseteq A_a^r \qquad A \in N_C(\mathcal{K}),\, a \in N_I(\mathcal{A}),\, t \in N_{tR}(\mathcal{K}),\, r \in N_R(\mathcal{K}),\, t \sqsubseteq_{\mathcal{R}} r \tag{8}$$
$$A_a^{r_1} \sqcap \cdots \sqcap A_a^{r_n} \sqsubseteq B \qquad a \in N_I(\mathcal{A}),\, (A \sqsubseteq \forall\rho.B) \in \mathcal{T},\, \rho = r_1 \sqcap \cdots \sqcap r_n \tag{9}$$

Theorem 1. *Let $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ be a simplified $\mathcal{SHQ}^\sqcap$ knowledge base and $\mathcal{K}' = (\mathcal{R}', \mathcal{T}', \mathcal{A}')$ an $\mathcal{ALCHQ}^\sqcap$ knowledge base obtained from $\mathcal{K}$ as described above. Then (i) $\mathcal{K}'$ is obtained from $\mathcal{K}$ in polynomial time and (ii) $\mathcal{K}$ is satisfiable iff $\mathcal{K}'$ is satisfiable.*

Proof. For (i): Let k be the size of $\mathcal{K}$. It is easy to see that the number of axioms of the form (5) is bounded by k, of the form (9) by k^2, of the form (6) and (7) by k^3, and of the form (8) by k^4. Since the size of every axiom (5)–(9) is bounded by k, we obtain that the size of $\mathcal{K}'$ is polynomial in k and thus can be computed in polynomial time in the size of $\mathcal{K}$.

For the if direction of (ii): Let $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be a model of $\mathcal{K}'$. We define an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ as follows:

1. $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$;
2. $a^{\mathcal{I}} = a^{\mathcal{J}}$, $a \in N_I$;
3. $A^{\mathcal{I}} = A^{\mathcal{J}}$, $A \in N_C$;
4. $r^{\mathcal{I}} = r^{\mathcal{J}} \cup \bigcup_{t \sqsubseteq_{\mathcal{R}} r,\, t \in N_{tR}} (t^{\mathcal{J}})^+$, $r \in N_R$

According to case 4 of the definition for $\mathcal{J}$ it is easy to see that $\mathcal{I}$ interprets all transitive roles in $\mathcal{K}$ as transitive relations.

First we demonstrate that $\mathcal{I}$ satisfies all RIAs $(r \sqsubseteq s) \in \mathcal{R}$. By case 4 of the definition for $\mathcal{I}$ we have $r^{\mathcal{I}} = r^{\mathcal{J}} \cup \bigcup_{t \sqsubseteq_{\mathcal{R}} r,\, t \in N_{tR}} (t^{\mathcal{J}})^+$ and $s^{\mathcal{I}} = s^{\mathcal{J}} \cup \bigcup_{t \sqsubseteq_{\mathcal{R}} s,\, t \in N_{tR}} (t^{\mathcal{J}})^+$. Since $\mathcal{J}$ satisfies $\mathcal{R}$, we have $r^{\mathcal{J}} \subseteq s^{\mathcal{J}}$. Since $(r \sqsubseteq s) \in \mathcal{R}$, we have $\{t \sqsubseteq_{\mathcal{R}} r \mid t \in N_{tR}\} \subseteq \{t \sqsubseteq_{\mathcal{R}} s \mid t \in N_{tR}\}$. Therefore $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$.

Next, we demonstrate that $\mathcal{I}$ satisfies all GCIs $C \sqsubseteq D \in \mathcal{T}$. Since $\mathcal{K}$ is simplified, GCIs in $\mathcal{K}$ occur in four different forms (see (2)):

1. Let $C \sqsubseteq D$ be of the form $\bigsqcap A_i \sqsubseteq \bigsqcup B_j$: since ${A_i}^{\mathcal{I}} = {A_i}^{\mathcal{J}}$ and ${B_j}^{\mathcal{I}} = {B_j}^{\mathcal{J}}$ according to case 3 of the definition for $\mathcal{I}$, we have that $\mathcal{I}$ satisfies the GCI.
2. Let $C \sqsubseteq D$ be of the form $A \sqsubseteq \bowtie n\, \delta.B$: since $A^{\mathcal{I}} = A^{\mathcal{J}}$ and $B^{\mathcal{I}} = B^{\mathcal{J}}$ according to case 3 of the definition for $\mathcal{I}$, and, since δ is a conjunction of simple roles, we have $\delta^{\mathcal{I}} = \delta^{\mathcal{J}}$ according to case 4 of the definition for $\mathcal{I}$. Thus $\mathcal{I}$ satisfies the GCI.
3. Let $C \sqsubseteq D$ be of the form $A \sqsubseteq \exists\rho.B$: according to case 3 of the definition for $\mathcal{I}$, $A^{\mathcal{I}} = A^{\mathcal{J}}$ and $B^{\mathcal{I}} = B^{\mathcal{J}}$. Since further $\rho^{\mathcal{J}} \subseteq \rho^{\mathcal{I}}$ according to case 4 of the definition for $\mathcal{I}$, we have that $\mathcal{I}$ satisfies the GCI.
4. In order to show that $\mathcal{I}$ satisfies every GCI of the form $(A \sqsubseteq \forall\rho.B) \in \mathcal{T}$ with $\rho = r_1 \sqcap \cdots \sqcap r_n$, $n \geq 1$, let $c, d \in \Delta^{\mathcal{I}}$ be such that $c \in A^{\mathcal{I}}$ and $\langle c, d\rangle \in \rho^{\mathcal{I}}$. We need to demonstrate that $d \in B^{\mathcal{I}}$. By definition of the semantics we have that $\rho^{\mathcal{I}} \subseteq {r_j}^{\mathcal{I}}$ for every j with $1 \leq j \leq n$.

   First we demonstrate that there exists $a \in N_I(\mathcal{A})$ such that $d \in (A_a^{r_j})^{\mathcal{J}}$ for every j with $0 \leq j \leq n$. Indeed, since $\mathcal{J}$ is a model of (5), and $c \in A^{\mathcal{I}} = A^{\mathcal{J}}$, there exists some $a \in N_I(\mathcal{A})$ such that $c \in {A_a}^{\mathcal{J}}$. Now, let us fix any j with $1 \leq j \leq n$. Since $\langle c, d\rangle \in {r_j}^{\mathcal{I}}$, by case 4 of the definition for $\mathcal{I}$, we have either *(i)* $\langle c, d\rangle \in {r_j}^{\mathcal{J}}$, or *(ii)* $\langle c, d\rangle \in (t^{\mathcal{J}})^+$ for some $t \sqsubseteq_{\mathcal{R}} r_j$, $t \in N_{tR}$. In case *(i)*, since $\mathcal{J}$ is a model of (6) for $r = r_j$, and $c \in {A_a}^{\mathcal{J}}$, we have $d \in (A_a^{r_j})^{\mathcal{J}}$ what was required to show. In case *(ii)*, there exist elements $c = d_1, d_2, \ldots, d_p = d$ with $p \geq 2$, such that $\langle d_{\ell-1}, d_\ell\rangle \in t^{\mathcal{J}}$ for every ℓ with $2 \leq \ell \leq p$. Since $c \in {A_a}^{\mathcal{J}}$, and $\mathcal{J}$ is a model of (6) for $r = t$, and of (7), by induction on ℓ with $2 \leq \ell \leq p$, it is easy to show that $d_\ell \in (A_a^t)^{\mathcal{J}}$, and, in particular,

$d \in (A_a^t)^{\mathcal{J}}$. Since $\mathcal{J}$ is a model of (8) for $r = r_j$, we have $d \in (A_a^{r_j})^{\mathcal{J}}$ what was required to show.

Now, since $d \in (A_a^{r_j})^{\mathcal{J}}$ for all j with $1 \leq j \leq n$, and $\mathcal{J}$ is a model of (9), we have $d \in B^{\mathcal{J}} = B^{\mathcal{I}}$, what was required to show.

It remains to demonstrate that $\mathcal{I}$ satisfies all assertions from $\mathcal{A}$. Since $\mathcal{K}$ is simplified, all concept assertions in $\mathcal{A}$ are of the form $A(a)$ for $A \in N_C$ and $\mathcal{I}$ satisfies every $A(a) \in \mathcal{A}$, since $a^{\mathcal{I}} = a^{\mathcal{J}}$ and $A^{\mathcal{I}} = A^{\mathcal{J}}$ according to the cases 1 and 2 of the definition for $\mathcal{I}$. Furthermore, $\mathcal{I}$ satisfies every role assertion $r(a,b) \in \mathcal{A}$, since $a^{\mathcal{I}} = a^{\mathcal{J}}$, $b^{\mathcal{I}} = b^{\mathcal{J}}$, and $r^{\mathcal{J}} \subseteq r^{\mathcal{I}}$ according to the cases 1 and 2 of the definition for $\mathcal{I}$.

Note that we did not use the canonical model property for $\mathcal{SHQ}^{\sqcap}$ for proving the if direction of (ii). We are going to use this property for proving the only if direction of (ii).

For the only if direction of (ii), let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a canonical model of $\mathcal{K}$ and $\mathcal{I}' = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}'})$ a forest base for $\mathcal{I}$ supplied with a bijection function f between $N_I(\mathcal{A})$ and $\Delta^{\mathcal{I}} \cap \mathbb{N}$. Such a model exists by Lemma 2 since $\mathcal{K}$ is satisfiable by assumption. Let $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be obtained from $\mathcal{I}$ as follows:

1. $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$;
2. $a^{\mathcal{J}} = a^{\mathcal{I}}$ for $a \in N_I(\mathcal{A})$, $A^{\mathcal{J}} = A^{\mathcal{I}}$ for $A \in N_C(\mathcal{K})$, $r^{\mathcal{J}} = r^{\mathcal{I}}$ for $r \in N_R(\mathcal{K})$
3. $(A_a)^{\mathcal{J}} = A^{\mathcal{I}} \cap \{d \in \Delta^{\mathcal{I}} \mid d = f(a) \cdot w \text{ and } w \in \mathbb{N}^*\}$, $A \in N_C(\mathcal{K})$, $a \in N_I(\mathcal{A})$;
4. $(A_a^r)^{\mathcal{J}} = \{d \in \Delta^{\mathcal{I}} \mid \exists c \in {A_a}^{\mathcal{J}} : \langle c, d\rangle \in r^{\mathcal{I}}\}$, $A \in N_C(\mathcal{K})$, $a \in N_I(\mathcal{A})$, $r \in N_R(\mathcal{K})$.

Since according to the cases 1–2 of the definition for $\mathcal{J}$, the interpretation of the symbols in $\mathcal{K}$ remains unchanged, $\mathcal{J}$ is a model of all GCIs in $\mathcal{K}'$ that are also in $\mathcal{K}$. It remains to demonstrate that $\mathcal{J}$ is a model of all GCIs that are new in $\mathcal{K}'$. These GCIs are of form (5)–(9).

1. In order to prove that $\mathcal{J}$ satisfies every axiom $A \sqsubseteq \sqcup_{a \in N_I(\mathcal{A})} A_a$ of form (5), take any $d \in A^{\mathcal{J}}$. We need to demonstrate that $d \in (A_a)^{\mathcal{J}}$ for some $a \in N_I(\mathcal{A})$. By Definition 1 of the canonical models for $\mathcal{SHQ}^{\sqcap}$, $d = f(a) \cdot w \in \Delta^{\mathcal{I}}$ for some $a \in N_I(\mathcal{A})$ and $w \in \mathbb{N}^*$. Hence $d \in {A_a}^{\mathcal{J}}$ according to case 3 of the definition for $\mathcal{J}$ what is required to show.
2. In order to prove that $\mathcal{J}$ satisfies every axiom $A_a \sqsubseteq \forall r.A_a^r$ of the form (6), take any $c, d \in \Delta^{\mathcal{J}}$ such that $c \in {A_a}^{\mathcal{J}}$ and $\langle c, d\rangle \in r^{\mathcal{J}}$. By definition of $(A_a^r)^{\mathcal{J}}$ (case 4), we have $d \in (A_a^r)^{\mathcal{J}}$.
3. In order to prove that $\mathcal{J}$ satisfies every axiom $A_a^t \sqsubseteq \forall t.A_a^t$ of the form (7), take any $c, d \in \Delta^{\mathcal{J}}$ such that $c \in (A_a^t)^{\mathcal{J}}$ and $\langle c, d\rangle \in t^{\mathcal{J}}$. We need to demonstrate that $d \in (A_a^t)^{\mathcal{J}}$. By definition of $(A_a^t)^{\mathcal{J}}$ (case 4), there exists $c' \in (A_a)^{\mathcal{J}}$ such that $\langle c', c\rangle \in t^{\mathcal{I}}$. Since $\langle c, d\rangle \in t^{\mathcal{J}} = t^{\mathcal{I}}$ and $t \in N_{tR}(\mathcal{K})$, we have $\langle c', d\rangle \in t^{\mathcal{I}}$, and, therefore, $d \in (A_a^t)^{\mathcal{J}}$, what is required to show.
4. In order to prove that $\mathcal{J}$ satisfies every axiom $A_a^t \sqsubseteq A_a^r$ of the form (8), take any $d \in (A_a^t)^{\mathcal{J}}$. We need to demonstrate that $d \in (A_a^r)^{\mathcal{J}}$. By definition of $(A_a^t)^{\mathcal{J}}$ (case 4), there exists $c \in {A_a}^{\mathcal{J}}$ such that $\langle c, d\rangle \in t^{\mathcal{I}}$. Since $t \sqsubseteq_{\mathcal{R}} r$, we have $\langle c, d\rangle \in t^{\mathcal{I}} \subseteq r^{\mathcal{I}}$, and so $d \in (A_a^r)^{\mathcal{J}}$, what is required to show.

5. Finally, in order to prove that $\mathcal{J}$ satisfies every axiom $A_a^{r_1} \sqcap \cdots \sqcap A_a^{r_n} \sqsubseteq B$ of form (9), take any $d \in (A_a^{r_1})^{\mathcal{J}} \cap \cdots \cap (A_a^{r_n})^{\mathcal{J}}$. We need to prove that $d \in B^{\mathcal{J}}$. By definition of $(A_a^r)^{\mathcal{J}}$ (case 4), there exist $c_i \in (A_a)^{\mathcal{J}}$ such that $\langle c_i, d\rangle \in r_i{}^{\mathcal{I}}$, $1 \le i \le n$. By definition of $(A_a)^{\mathcal{J}}$ (case 4), $c_i = f(a) \cdot w_i$, $1 \le i \le n$. We prove that there exists a $c \in (A_a)^{\mathcal{J}}$ such that $\langle c, d\rangle \in r_i{}^{\mathcal{I}}$ for every i with $1 \le i \le n$.
Since $\mathcal{I}'$ is a forest base for $\mathcal{I}$ and $\langle c_i, d\rangle = \langle f(a) \cdot w_i, d\rangle \in r_i{}^{\mathcal{I}} = r_i{}^{\mathcal{I}'} \cup \bigcup_{t \sqsubseteq_{\mathcal{R}} r_i,\, t \in N_{tR}} (t^{\mathcal{I}'})^+$, $1 \le i \le n$, there are two cases possible: either (i) $d = f(a) \cdot w$ for some $w \in \mathbb{N}^+$ and, for each i with $1 \le i \le n$, w_i is a proper prefix of w, or (ii) $d = f(b) \cdot w$ for some $w \in \mathbb{N}^*, b \neq a$, and $c_i = f(a)$, for every i with $1 \le i \le n$. In case (ii) we have found the required $c = f(a)$. In case (i), let w_j be the longest prefix of w among all w_i with $1 \le i \le n$ and define $c = c_j$. Note that $c_j \neq d$, since $\langle d, d\rangle \notin r_i{}^{\mathcal{I}}$ by the definition of the forest model. Since $r_i{}^{\mathcal{I}} = r_i{}^{\mathcal{I}'} \cup \bigcup_{t \sqsubseteq_{\mathcal{R}} r_i,\, t \in N_{tR}} (t^{\mathcal{I}'})^+$, there are two possible cases for each i with $1 \le i \le n$: either (1) $\langle c_i, d\rangle \in r_i{}^{\mathcal{I}'}$, d is a successor of c_i, and thus $c = c_i$, or (2) $\langle c_i, d\rangle \in (t^{\mathcal{I}'})^+$ and $\langle c, d\rangle \in r_i{}^{\mathcal{I}}$ since $(t^{\mathcal{I}'})^+ \subseteq r_i{}^{\mathcal{I}}$ and c_i is a prefix of $c = c_j$. In both cases we have demonstrated that $\langle c, d\rangle \in r_i{}^{\mathcal{I}}$ for each i with $1 \le i \le n$.
Since $c \in A_a{}^{\mathcal{J}} \subseteq A^{\mathcal{I}}$, $\langle c, d\rangle \in r_i{}^{\mathcal{I}}$ for every i with $1 \le i \le n$, and $\mathcal{I}$ is a model of the axiom $(A \sqsubseteq \forall\rho.B) \in \mathcal{T}$ with $\rho = r_1 \sqcap \cdots \sqcap r_n$, we have $d \in B^{\mathcal{I}} = B^{\mathcal{J}}$ what was required to show.

□

With the above theorem, we immediately get the following result.

Corollary 1. *The problem of satisfiability for $\mathcal{SHQ}^{\sqcap}$ knowledge bases is complete for* ExpTime *(and so are all the standard reasoning problems).*

Proof. Given an $\mathcal{SHQ}^{\sqcap}$ knowledge base $\mathcal{K}$, by Theorem 1, it is possible to construct in polynomial time an equisatisfiable $\mathcal{ALCHQ}^{\sqcap}$ knowledge base $\mathcal{K}'$. Since the problem of satisfiability for $\mathcal{ALCHQ}^{\sqcap}$ is in ExpTime [21, 23], this implies that the problem of satisfiability of $\mathcal{SHQ}^{\sqcap}$ knowledge bases is in ExpTime. Furthermore, the problem is ExpTime-hard since $\mathcal{ALCHQ}^{\sqcap}$ contains $\mathcal{ALC}$ for which all standard reasoning problems are ExpTime-hard. Since all standard reasoning problems like knowledge base satisfiability, concept satisfiability, concept non-subsumption and instance checking are inter-reducible in polynomial time to each other, all these problems are also ExpTime-complete for $\mathcal{SHQ}^{\sqcap}$. □

5 $\mathcal{SHI}^{\sqcap}$ and $\mathcal{SHIQ}^{\sqcap}$ are 2ExpTime-complete

In this section, we show that extending $\mathcal{SHI}$ with role conjunctions causes an exponential blow-up in the computational complexity of the standard reasoning tasks. We show this by a reduction from the word problem of an exponential space alternating Turing machine.

An *alternating Turning machine* (ATM) is a tuple $M = (\Gamma, \Sigma, Q, q_0, \delta_1, \delta_2)$, where Γ is a finite *working alphabet* containing a *blank symbol* $\boxdot$, $\Sigma \subseteq \Gamma \setminus \{\boxdot\}$ is the *input alphabet*; $Q = Q_\exists \uplus Q_\forall \uplus \{q_a\} \uplus \{q_r\}$ is a finite set of states partitioned into *existential states* $Q_\exists$, *universal states* $Q_\forall$, an *accepting state* q_a, and a *rejecting state* q_r; $q_0 \in Q_\exists$ is the *starting state*, and $\delta_1, \delta_2 \colon (Q_\exists \cup Q_\forall) \times \Gamma \to Q \times \Gamma \times \{L, R\}$ are *transition functions.* A *configuration* of M is a word $c = w_1 q w_2$ where $w_1, w_2 \in \Gamma^*$ and $q \in Q$. An *initial configuration* is $c^0 = q_0 w_0$ where $w_0 \in \Sigma^*$. The *size* $|c|$ of a configuration c is the number of symbols in c. The *successor configurations* $\delta_1(c)$ and $\delta_2(c)$ of a configuration $c = w_1 q w_2$ with $q \neq q_a, q_r$ over the transition functions δ_1 and δ_2 are defined as for deterministic Turing machines (see, e.g., [27]). The sets $\mathsf{C}_{\mathsf{acc}}(M)$ of *accepting configurations* and $\mathsf{C}_{\mathsf{rej}}(M)$ of *rejecting configurations* of M are the smallest sets such that (i) $c = w_1 q w_2 \in \mathsf{C}_{\mathsf{acc}}(M)$ if either $q = q_a$, or $q \in Q_\forall$ and $\delta_1(c), \delta_2(c) \in \mathsf{C}_{\mathsf{acc}}(M)$, or $q \in Q_\exists$ and $\delta_1(c) \in \mathsf{C}_{\mathsf{acc}}(M)$ or $\delta_2(c) \in \mathsf{C}_{\mathsf{acc}}(M)$, and (ii) $c = w_1 q w_2 \in \mathsf{C}_{\mathsf{rej}}(M)$ if either $q = q_r$, or $q \in Q_\exists$ and $\delta_1(c), \delta_2(c) \in \mathsf{C}_{\mathsf{rej}}(M)$, or $q \in Q_\forall$ and $\delta_1(c) \in \mathsf{C}_{\mathsf{rej}}(M)$ or $\delta_2(c) \in \mathsf{C}_{\mathsf{rej}}(M)$. The set of *reachable* configurations from an initial configuration c^0 in M is the smallest set $M(c^0)$ such that $c^0 \in M(c^0)$ and $\delta_1(c), \delta_2(c) \in M(c^0)$ for every $c \in M(c^0)$. A *word problem* for an ATM M is to decide given an initial configuration c^0 whether $c^0 \in \mathsf{C}_{\mathsf{acc}}(M)$. M is $g(n)$ *space bounded* if for every initial configuration c^0 we have: (i) $c^0 \in \mathsf{C}_{\mathsf{acc}}(M) \cup \mathsf{C}_{\mathsf{rej}}(M)$, and (ii) $|c| \leq g(|c^0|)$ for every $c \in M(c^0)$. A classical result $\mathsf{AExpSpace} = \mathsf{2ExpTime}$ [28] implies that there exists a 2^n space bounded ATM M for which the following decision problem is $\mathsf{2ExpTime}$-complete: given an initial configuration c^0 decide whether $c^0 \in \mathsf{C}_{\mathsf{acc}}(M)$.

In order to reduce the word problem of M to reasoning problems in $\mathcal{SHI}^\sqcap$, we introduce an auxiliary notion of a computation of an ATM that is more convenient to deal with when determining accepting computations. Let us denote by $\{0,1\}^*$ the set of all finite words over the letters 0 and 1, by ϵ the empty word, and, for every $b \in \{0,1\}^*$, by $b \cdot 0$ and $b \cdot 1$ a word obtained by appending 0 and 1 to b. A *computation of an ATM M from c^0* is a pair $P = (B, \pi)$, where $B \subseteq \{0,1\}^*$ is a forest, and $\pi : B \to M(c^0)$ a mapping from words to configurations reachable from c^0, such that (i) $\epsilon \in B$ and $\pi(\epsilon) = c^0$, and for every $b \in B$ with $\pi(b) = c = w_1 q w_2$ we have (ii) $q \neq q_r$, (iii) $q \in Q_\forall$ implies $\{b \cdot 0, b \cdot 1\} \subseteq B$, (iv) $q \in Q_\exists$ implies $b \cdot 0 \in B$ or $b \cdot 0 \in B$, (v) $b \cdot 0 \in B$ implies $\pi(b \cdot 0) = \delta_1(c)$, and (vi) $b \cdot 1 \in B$ implies $\pi(b \cdot 1) = \delta_2(c)$. A computation is *finite* if B is finite. It is easy to see that for any $g(n)$ space bounded ATM M, we have $c^0 \in \mathsf{C}_{\mathsf{acc}}(M)$ iff there exists a finite computation of M from c^0.

We encode a computation of the ATM M in a binary tree (see Figure 2) whereby the configurations of M are encoded on exponentially long chains that grow from the nodes of the tree—the i^{th} element of a chain represents the i^{th} element of the configuration. In our construction, we distinguish odd and even configurations in the computation using concept names Odd and Even. Every odd configuration has two even successor configurations reachable by roles r_e^1 and r_e^2 respectively; likewise, every even configuration has two odd successor configurations reachable by inverses of r_o^1 and r_o^2. We further alternate between the concepts P_0, P_1, and P_2 within the levels of the binary tree. This allows us to distinguish the predecessor and the successor configuration represented by

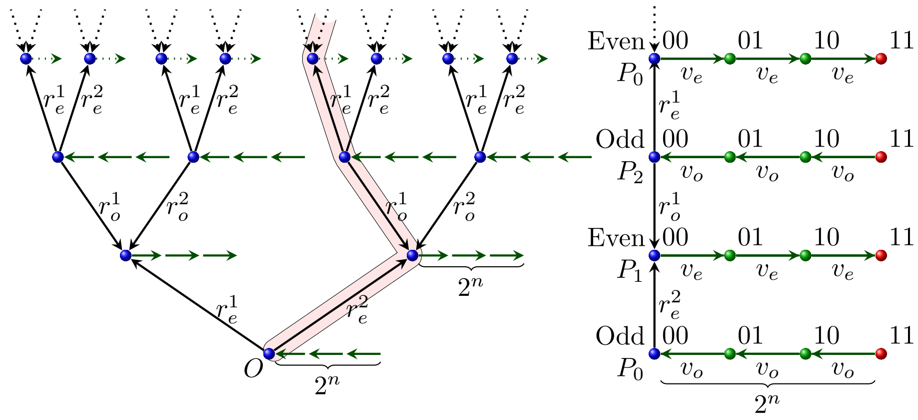

Fig. 2. The alternating binary tree structure for simulating a computation of the ATM (left) and a detailed picture for the highlighted path (right)

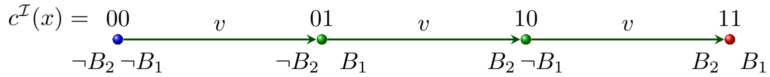

Fig. 3. Expressing exponentially long chains using a counter and binary encoding

the exponentially long chains. We enforce these chains (see Figure 3) by using the well know "integer counting" technique [29]. A counter $c^{\mathcal{I}}(x)$ is an integer between 0 and $2^n - 1$ that is assigned to an element x of the interpretation $\mathcal{I}$ using n atomic concepts $B_1, \ldots, B_n$ such that the i^{th} bit of $c^{\mathcal{I}}(x)$ is equal to 1 iff $x \in {B_i}^{\mathcal{I}}$. We first define the concept Z that can be used to initialize the counter to zero, and the concept E to detect whether the counter has reached the final value $2^n - 1$ and, thus, the end of the chain is reached:

$$Z \equiv \neg B_1 \sqcap \ldots \sqcap \neg B_n \tag{10}$$

$$E \equiv \quad B_1 \sqcap \ldots \sqcap B_n \tag{11}$$

Every element that is not the end of the chain has a v-successor:

$$\neg E \sqsubseteq \exists v.\top \tag{12}$$

The lowest bit of the counter is always flipped over v, while any other bit of the counter is flipped over v if and only if the previous bit is flipped from 1 to 0:

$$\top \equiv (B_1 \sqcap \forall v.\neg B_1) \sqcup (\neg B_1 \sqcap \forall v.B_1) \tag{13}$$

$$B_{k-1} \sqcap \forall v.\neg B_{k-1} \equiv (B_k \sqcap \forall v.\neg B_k) \sqcup (\neg B_k \sqcap \forall v.B_k) \qquad 1 < k \leq n \tag{14}$$

For convenience, let us denote by $j[i]_2$ the i^{th} bit of j in binary coding (the lowest bit of j is $j[1]_2$).

Lemma 3. *Let $\mathcal{K}$ be a knowledge base containing axioms* (13) *and* (14)*. Then, for every model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{K}$ and $x, y \in \Delta^{\mathcal{I}}$ with $\langle x, y\rangle \in v^{\mathcal{I}}$, we have $c^{\mathcal{I}}(y) = c^{\mathcal{I}}(x) + 1$.*

Proof. Consider the set $Y := \{y \mid \langle x, y\rangle \in v^{\mathcal{I}}\}$. We prove that, for each $y \in Y, c^{\mathcal{I}}(y) = c^{\mathcal{I}}(x) + 1$. Please note that we do not enforce that x has only a single successor y, i.e., our domain is not restricted such that we have a sequence of elements with increasing counter values. We only require that if y is a v-successor of some x, then the counter value is incremented by one.

By induction on k with $1 \leq k \leq n$, we prove that, for each $y \in Y, c^{\mathcal{I}}(x)[k]_2 \neq c^{\mathcal{I}}(y)[k]_2$ if and only if either $k = 1$ or, otherwise, $c^{\mathcal{I}}(x)[k-1]_2 = 1$ and $c^{\mathcal{I}}(y)[k-1]_2 = 0$. Note that, in particular, the induction hypothesis implies that the values of $c^{\mathcal{I}}(y)[k]_2$ are the same for all $y \in Y$.

The base case $k = 1$ of the induction holds since $\mathcal{I}$ is a model of (13), and, therefore, for each $y \in Y, c^{\mathcal{I}}(x)[1]_2 \neq c^{\mathcal{I}}(y)[1]_2$. The induction step holds because $\mathcal{I}$ is a model of (14) which implies that $c^{\mathcal{I}}(x)[k-1]_2 = 1$ and, for each $y \in Y, c^{\mathcal{I}}(y)[k-1]_2 = 1$ if and only if $c^{\mathcal{I}}(y)[k]_2 \neq c^{\mathcal{I}}(x)[k]_2$. $\square$

The tree-like structure in Figure 2 is induced by the following formulas. First, we initialize the origin O of the tree by saying that it belongs to an odd row labeled with P_0 and, with the concept Z, we initialize an exponential chain:

$$O \sqsubseteq \mathrm{Odd} \sqcap P_0 \sqcap Z \tag{15}$$

Every initial element of an exponential chain has two successors alternating between odd and even values:

$$Z \sqcap \mathrm{Odd} \sqsubseteq \exists r_e^1.\mathrm{Even} \sqcap \exists r_e^2.\mathrm{Even} \tag{16}$$

$$Z \sqcap \mathrm{Even} \sqsubseteq \exists {r_o^1}^-.\mathrm{Odd} \sqcap \exists {r_o^2}^-.\mathrm{Odd} \tag{17}$$

For convenience, we introduce super-roles r^1, r^2 and r of the created roles to keep track of the relations between the nodes and their successors:

$$r_e^1 \sqsubseteq r^1 \qquad r_o^1 \sqsubseteq {r^1}^- \qquad r_e^2 \sqsubseteq r^2 \qquad r_o^2 \sqsubseteq {r^2}^- \qquad r^1 \sqsubseteq r \qquad r^2 \sqsubseteq r \tag{18}$$

The new roles are used to initialize the value Z for the successors and increment P_j over r modulo 3 (we denote $j + 1 \mod 3$ as $[j + 1]_3$):

$$Z \sqsubseteq \forall r.Z \qquad P_j \sqsubseteq \forall r.P_{[j+1]_3} \qquad 0 \leq j \leq 2 \tag{19}$$

In order to have the roles on the exponential chain correspond to the odd and even rows, we replace axiom (12) with the following axioms:

$$\neg E \sqcap \mathrm{Even} \sqsubseteq \exists v_e.\top \qquad \neg E \sqcap \mathrm{Odd} \sqsubseteq \exists v_o^-.\top \tag{20}$$

$$v_o \sqsubseteq v^- \qquad v_e \sqsubseteq v \tag{21}$$

$$\mathrm{Odd} \sqsubseteq \forall v.\mathrm{Odd} \qquad \mathrm{Even} \sqsubseteq \forall v.\mathrm{Even} \tag{22}$$

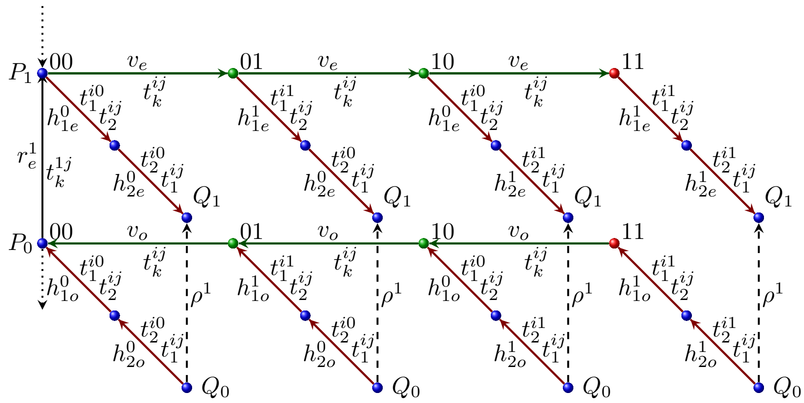

Fig. 4. A zoom-in and extension of Figure 2, which illustrates the use of the auxiliary side chains to connect the elements of the exponentially long chains with the corresponding elements in the successor chains

The values of P_j are copied across the elements of the same row:

$$P_j \sqsubseteq \forall v.P_j \qquad \neg P_j \sqsubseteq \forall v.\neg P_j \qquad 0 \leq j \leq 2 \tag{23}$$

If we take a look at Figure 2 we notice that the roles r_o^i, r_e^i, v_o and v_e are directed in such a way that, from every element of an exponential chain, only elements of the neighboring chains are reachable by a sequence of roles. In other words, if we introduce a common transitive super-role t of these roles, then every element of the chain will be connected via t to exactly all elements of the parent chain and all elements of the successor chains. Unfortunately, this is not sufficient to simulate a computation of the Turing machine, as we need to connect exactly the corresponding elements of a chain and its two successor chains to compute the successor configurations. In order to achieve this goal, we will add auxiliary chains to the exponential chain that, using transitive super-roles and role conjunctions, will allow us to restrict the reachability relation only to the corresponding elements.

The detailed construction for the side chains of two successive configurations is shown in Figure 4. Every element of the exponential v-chain has n additional "side" successors reachable by roles h_{ke}^j and h_{ko}^j with $j \in \{0, 1\}$ and $1 \leq k \leq n$. Intuitively, k corresponds to the counting concepts and j to the counter value. We will also count the level in the h-chains using concepts H_k, $0 \leq k \leq n$—all elements of the v-chain belong to H_0, and every h-successor of an element in H_{k-1} belongs to H_k. The following axioms initialize the side chains according to this description:

$$O \sqsubseteq H_0 \qquad H_0 \sqsubseteq \forall r.H_0 \qquad H_0 \sqsubseteq \forall v.H_0 \tag{24}$$

$$H_{k-1} \sqcap \neg B_k \sqsubseteq (\neg\text{Even} \sqcup \exists h^0_{ke}.H_k) \sqcap (\neg\text{Odd} \sqcup \exists {h^0_{ko}}^-.H_k) \qquad 1 \le k \le n \tag{25}$$

$$H_{k-1} \sqcap B_k \sqsubseteq (\neg\text{Even} \sqcup \exists h^1_{ke}.H_k) \sqcap (\neg\text{Odd} \sqcup \exists {h^1_{ko}}^-.H_k) \qquad 1 \le k \le n \tag{26}$$

$$h^j_{ke} \sqsubseteq h \qquad h^j_{ko} \sqsubseteq h^- \qquad j \in \{0,1\},\ 1 \le k \le n \tag{27}$$

$$\text{Even} \sqsubseteq \forall h.\text{Even} \qquad \text{Odd} \sqsubseteq \forall h.\text{Odd} \tag{28}$$

We use these roles to express that the elements within an h-chain have the same values for B_k and P_j:

$$B_k \sqsubseteq \forall h.B_k \qquad \neg B_k \sqsubseteq \forall h.\neg B_k \qquad 0 \le k \le n \tag{29}$$

$$P_j \sqsubseteq \forall h.P_j \qquad \neg P_j \sqsubseteq \forall h.\neg P_j \qquad 0 \le j \le 2 \tag{30}$$

For the final elements of the h-chains, we introduce the special concepts Q_i that correlate with the concepts P_j:

$$H_n \sqsubseteq (P_j \sqcap Q_j) \sqcup (\neg P_j \sqcap \neg Q_j) \qquad 0 \le j \le 2 \tag{31}$$

These concepts will be used to connect the last elements of the h-chains with the corresponding elements in the chains for the two successor configurations using role conjunctions ρ^1 and ρ^2 introduced later on (see Figure 4). In order to connect these elements, we introduce transitive super-roles t^{ij}_k with $i \in \{1,2\}$, $j \in \{0,1\}$, and $1 \le k \le n$:

$$r^i_o \sqsubseteq t^{ij}_k \qquad r^i_e \sqsubseteq t^{ij}_k \tag{32}$$

$$v_o \sqsubseteq t^{ij}_k \qquad v_e \sqsubseteq t^{ij}_k \tag{33}$$

$$h^j_{ko} \sqsubseteq t^{ij}_k \qquad h^j_{ke} \sqsubseteq t^{ij}_k \tag{34}$$

$$h^j_{ko} \sqsubseteq t^{ij'}_{k'} \qquad h^j_{ke} \sqsubseteq t^{ij'}_{k'} \qquad j' \in \{0,1\},\ 1 \le k' \le n,\ k' \ne k \tag{35}$$

Intuitively, the index i in t^{ij}_k is inherited from the roles r^i_o and r^j_o (32)—all role implications hold for both values of i. Likewise, the index j is inherited from h^j_{ko} and h^j_{ke}, but only when the values of the index k match (34)—otherwise the role implications hold for both values of j (35). Roles v_o and v_e do not filter any indexes and imply all roles t^{ij}_k (33). Axioms (32)–(35) make sure that the first and the last elements of every h-chain are connected with t^{i0}_k (t^{i1}_k) iff the k^{th} bit of the counter is 0 (1). Thus, only the corresponding last elements of the h-chains in the successor configurations are connected with t^{ij}_k for all k with $1 \le k \le n$ and some i and j, because they have the same values for the counter. To make use of this property we introduce roles s^i_k that are obtained from t^{ij}_k by abstracting from j and forgetting the direction:

$$t^{ij}_k \sqsubseteq s^i_k \qquad {t^{ij}_k}^- \sqsubseteq s^i_k \qquad i \in \{1,2\},\ j \in \{0,1\},\ 1 \le k \le n \tag{36}$$

Now define the role conjunctions $\rho^1 = s_1^1 \sqcap \cdots \sqcap s_n^1$ and $\rho^2 = s_1^2 \sqcap \cdots \sqcap s_n^2$ that connect the last elements of the h-chains iff they are the corresponding elements for the r^1 and r^2 successors in our binary tree on Figure 2. Note that ρ^1 and ρ^2 are not simple.

We now specify how the created tree structure relates to an alternating Turing machine. Let c^0 be an initial configuration of an ATM $M = (\Gamma, \Sigma, Q, q_0, \delta_1, \delta_2)$ and $n = |c^0|$ (w.l.o.g., we assume that $n > 2$). In order to decide whether $c^0 \in \mathsf{C}_{\mathsf{acc}}(M)$, we try to build all the required accepting successor configurations of c^0 for M. We encode the configurations of M on the 2^n-long v-chains. A chain corresponding to a configuration c is connected via the roles r^1 and r^2 to two chains that correspond to $\delta_1(c)$ and $\delta_2(c)$ respectively. We use an atomic concept A_a for every symbol a that can occur in configurations and we make sure that all elements of the same h-chain are assigned to the same symbol:

$$A_a \sqsubseteq \forall h.A_a \qquad\qquad \neg A_a \sqsubseteq \forall h.\neg A_a \tag{37}$$

It is a well-known property of the transition functions of Turing machines that the symbols c_i^1 and c_i^2 at the position i of $\delta_1(c)$ and $\delta_2(c)$ are uniquely determined by the symbols c_{i-1}, c_i, c_{i+1}, and c_{i+2} of c at the positions $i-1, i, i+1$, and $i+2$.[4] We assume that this correspondence is given by the (partial) functions λ_1 and λ_2 such that $\lambda_1(c_{i-1}, c_i, c_{i+1}, c_{i+2}) = c_i^1$ and $\lambda_2(c_{i-1}, c_i, c_{i+1}, c_{i+2}) = c_i^2$. We use this property in our encoding as follows: for every quadruple of symbols $a_1, a_2, a_3, a_4 \in Q \cup \Gamma$, we introduce a concept name $S_{a_1a_2a_3a_4}$ which expresses that the current element of the v-chain is assigned with the symbol a_2, its v-predecessor with a_1 and its next two v-successors with respectively a_3 and a_4 (a_1, a_3, and a_4 are $\boxdot$ if there are no such elements):

$$\begin{aligned}
Z \sqcap A_{a_2} \sqcap \exists v.(A_{a_3} \sqcap \exists v.A_{a_4}) &\sqsubseteq S_{\boxdot a_2a_3a_4} & a_2, a_3, a_4 \in Q \cup \Gamma \quad (38)\\
A_{a_1} \sqcap \exists v.(A_{a_2} \sqcap \exists v.(A_{a_3} \sqcap \exists v.A_{a_4})) &\sqsubseteq \forall v.S_{a_1a_2a_3a_4} & a_1, a_2, a_3, a_4 \in Q \cup \Gamma \quad (39)\\
A_{a_1} \sqcap \exists v.(A_{a_2} \sqcap \exists v.(A_{a_3} \sqcap E)) &\sqsubseteq \forall v.S_{a_1a_2a_3\boxdot} & a_1, a_2, a_3 \in Q \cup \Gamma \quad (40)\\
A_{a_1} \sqcap \exists v.(A_{a_2} \sqcap E) &\sqsubseteq \forall v.S_{a_1a_2\boxdot\boxdot} & a_1, a_2 \in Q \cup \Gamma \quad (41)
\end{aligned}$$

Furthermore, all elements of the same h-chain have the same values of $S_{a_1a_2a_3a_4}$:

$$S_{a_1a_2a_3a_4} \sqsubseteq \forall h.S_{a_1a_2a_3a_4} \qquad\qquad \neg S_{a_1a_2a_3a_4} \sqsubseteq \forall h.\neg S_{a_1a_2a_3a_4} \tag{42}$$

Finally, the properties of the transition functions are expressed using the following axioms, where, as previously defined $\rho^1 = s_1^1 \sqcap \cdots \sqcap s_n^1$ and $\rho^2 = s_1^2 \sqcap \cdots \sqcap s_n^2$:

$$\begin{aligned}
S_{a_1a_2a_3a_4} \sqcap Q_j &\sqsubseteq \forall \rho^1.[\neg Q_{[j+1]_3} \sqcup A_{\lambda_1(a_1,a_2,a_3,a_4)}] & 0 \le i \le 2 \quad (43)\\
S_{a_1a_2a_3a_4} \sqcap Q_j &\sqsubseteq \forall \rho^2.[\neg Q_{[j+1]_3} \sqcup A_{\lambda_2(a_1,a_2,a_3,a_4)}] & 0 \le i \le 2 \quad (44)
\end{aligned}$$

Intuitively, these axioms say that whenever $S_{a_1a_2a_3a_4}$ holds at the end of an h-chain where Q_j holds, then $A_{\lambda_1(a_1,a_2,a_3,a_4)}$ should hold for every ρ^1 (ρ^2) successor

[4] If any of the indexes $i-1, i+1$, or $i+2$ are out of range for the configuration c, we assume that the corresponding symbols c_{i-1}, c_{i+1}, and c_{i+2} are the blank symbol $\boxdot$.

for which $Q_{[j+1]_3}$ holds. As noted before, only the corresponding last elements of the h-chains can be connected by ρ^1 and ρ^2. The concepts Q_j and $Q_{[j+1]_3}$ restrict the attention to the last elements of the h-chains and make sure that the information is propagated to the successor configuration and not to the predecessor configuration.

We now make sure that the elements in the root chain of our tree correspond to the initial configuration c^0:

$$O \sqsubseteq A_{c_1^0} \sqcap \forall v.(A_{c_2^0} \sqcap \cdots \forall v.(A_{c_n^0} \sqcap \forall v.O_{\boxdot}) \cdots) \tag{45}$$

$$O_{\boxdot} \sqsubseteq A_{\boxdot} \sqcap \forall v.O_{\boxdot} \tag{46}$$

In order to distinguish between the configurations with existential and universal states, we introduce two concepts $S_\forall$ and $S_\exists$, which are implied by the corresponding states and propagated to the first elements of the configuration:

$$A_q \sqsubseteq S_\exists \qquad q \in Q_\exists \qquad\qquad A_q \sqsubseteq S_\forall \qquad q \in Q_\forall \tag{47}$$

$$\exists v.S_\exists \sqsubseteq S_\exists \qquad\qquad \exists v.S_\forall \sqsubseteq S_\forall \tag{48}$$

Now instead of creating always two successor configurations, we create only configurations that are required for acceptance. Thus, we replace axioms (16) and (17) with the axioms (49)–(51) below:

$$Z \sqcap \mathrm{Odd} \sqcap S_\forall \sqsubseteq \exists r_e^1.\top \sqcap \exists r_e^2.\top \qquad Z \sqcap \mathrm{Even} \sqcap S_\forall \sqsubseteq \exists {r_o^1}^-.\top \sqcap \exists {r_o^2}^-.\top \tag{49}$$

$$Z \sqcap \mathrm{Odd} \sqcap S_\exists \sqsubseteq \exists r_e^1.\top \sqcup \exists r_e^2.\top \qquad Z \sqcap \mathrm{Even} \sqcap S_\exists \sqsubseteq \exists {r_o^1}^-.\top \sqcup \exists {r_o^2}^-.\top \tag{50}$$

$$\mathrm{Odd} \sqsubseteq \forall r.\mathrm{Even} \qquad\qquad \mathrm{Even} \sqsubseteq \forall r.\mathrm{Odd} \tag{51}$$

Finally we forbid configurations with rejecting states in our model:

$$A_{q_r} \sqsubseteq \bot \tag{52}$$

To summarize, our construction proves the following theorem:

Theorem 2. *Let c^0 be an initial configuration for the ATM M and $\mathcal{K}$ a knowledge base consisting of the axioms* (10)–(15) *and* (18)–(52)*. Then $c^0 \in \mathsf{C}_{\mathsf{acc}}(M)$ if and only if O is (finitely) satisfiable in $\mathcal{K}$.*

Proof. By c_i we denote the i^{th} symbol in the configuration c when $1 \leq i \leq |c|$ and the blank symbol $\boxdot$ otherwise.

($\Rightarrow$) Assume that $c^0 \in \mathsf{C}_{\mathsf{acc}}(M)$. Since M is 2^n space bounded, there exists a finite computation $P = (B, \pi)$ of M from c^0 such that $|\pi(b)| \leq 2^n$ for every $b \in B$. We will use this computation in order to guide the construction of a finite model $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ for $\mathcal{K}$ that satisfies O.

We define $\Delta^\mathcal{I} := \{x_{b,i,k} \mid b \in B,\ 0 \leq i < 2^n,\ 0 \leq k \leq n\}$. The interpretation of the concepts B_j, Z, E, O, Odd, Even, P_j, Q_j, H_k, A_a, $S_{a_1a_2a_3a_4}$, $O_{\boxdot}$, $S_\exists$, and $S_\forall$ is defined by:

– ${B_j}^\mathcal{I} = \{x_{b,i,k} \mid b \in B,\ i[j]_2 = 1, 0 \leq k \leq n\}, 1 \leq j \leq n$;

- $Z^{\mathcal{I}} = \{x_{b,0,k} \mid b \in B,\, 0 \le k \le n\}$, $E^{\mathcal{I}} = \{x_{b,2^n-1,k} \mid b \in B,\, 0 \le k \le n\}$;
- $O^{\mathcal{I}} = \{x_{\epsilon,0,0}\}$;
- $\mathrm{Odd}^{\mathcal{I}} = \{x_{b,i,k} \mid b \in B,\, |b| \text{ is odd},\, 0 \le k \le n\}$,
  $\mathrm{Even}^{\mathcal{I}} = \{x_{b,i,k} \mid b \in B,\, |b| \text{ is even},\, 0 \le k \le n\}$;
- ${P_j}^{\mathcal{I}} = \{x_{b,i,k} \mid b \in B,\, [|b|]_3 = j,\, 0 \le i < 2^n,\, 0 \le k \le n\}$, $0 \le j \le 2$;
- ${Q_j}^{\mathcal{I}} = \{x_{b,i,n} \mid b \in B,\, [|b|]_3 = j,\, 0 \le i < 2^n\}$, $0 \le j \le 2$;
- ${H_k}^{\mathcal{I}} = \{x_{b,i,k} \mid b \in B,\, 0 \le i < 2^n\}$, $0 \le k \le n$;
- ${A_a}^{\mathcal{I}} = \{x_{b,i,k} \mid b \in B,\, 0 \le i < 2^n,\, 0 \le k \le n,\, \pi(b)_{i+1} = a\}$, $a \in Q \cup \Gamma$;
- ${S_{a_1a_2a_3a_4}}^{\mathcal{I}} = \{x_{b,i,k} \mid b \in B,\, 0 \le i < 2^n,\, 0 \le k \le n,\, \pi(b)_i = a_1,\, \pi(b)_{i+1} = a_2,\, \pi(b)_{i+2} = a_3,\, \pi(b)_{i+3} = a_4\}$, $a_1, a_2, a_3, a_4 \in Q \cup \Gamma$;
- ${O_{\boxdot}}^{\mathcal{I}} = \{x_{\epsilon,i,k} \mid n \le i < 2^n,\, 0 \le k \le n\}$;
- ${S_\exists}^{\mathcal{I}} = \{x_{b,i,0} \mid b \in B,\, 0 \le i < 2^n,\, \exists q \in Q_\exists :\ \pi(b) = w_1 q w_2\}$,
  ${S_\forall}^{\mathcal{I}} = \{x_{b,i,0} \mid b \in B,\, 0 \le i < 2^n,\, \exists q \in Q_\forall :\ \pi(b) = w_1 q w_2\}$.

The roles r_o^i, r_e^i, r^i, r, v_0, v_e, v, h_{ko}^j, h_{ke}^j, h, t_k^{ij}, and s_k^i are interpreted as follows:

- $(r_e^1)^{\mathcal{I}} = \{\langle x_{b,0,0}, x_{b\cdot 0,0,0}\rangle \mid b \cdot 0 \in B,\, |b| \text{ is even}\}$,
  $(r_e^2)^{\mathcal{I}} = \{\langle x_{b,0,0}, x_{b\cdot 1,0,0}\rangle \mid b \cdot 1 \in B,\, |b| \text{ is even}\}$,
  $(r_o^1)^{\mathcal{I}} = \{\langle x_{b\cdot 0,0,0}, x_{b,0,0}\rangle \mid b \cdot 0 \in B,\, |b| \text{ is odd}\}$,
  $(r_o^2)^{\mathcal{I}} = \{\langle x_{b\cdot 0,0,0}, x_{b,0,0}\rangle \mid b \cdot 1 \in B,\, |b| \text{ is odd}\}$,
  $(r^1)^{\mathcal{I}} = (r_e^1)^{\mathcal{I}} \cup ((r_o^1)^-)^{\mathcal{I}}$,
  $(r^2)^{\mathcal{I}} = (r_e^2)^{\mathcal{I}} \cup ((r_o^2)^-)^{\mathcal{I}}$,
  $r^{\mathcal{I}} = (r^1)^{\mathcal{I}} \cup (r^2)^{\mathcal{I}}$;
- $(v_e)^{\mathcal{I}} = \{\langle x_{b,i-1,0}, x_{b,i,0}\rangle \mid b \in B,\, |b| \text{ is odd},\, 1 \le i < 2^n\}$,
  $(v_o)^{\mathcal{I}} = \{\langle x_{b,i,0}, x_{b,i-1,0}\rangle \mid b \in B,\, |b| \text{ is even},\, 1 \le i < 2^n\}$,
  $v^{\mathcal{I}} = (v_e)^{\mathcal{I}} \cup ((v_o)^-)^{\mathcal{I}}$;
- $(h_{ke}^j)^{\mathcal{I}} = \{\langle x_{b,i,k-1}, x_{b,i,k}\rangle \mid b \in B,\, |b| \text{ is odd},\, 0 \le i < 2^n,\, i[k]_2 = j\}$, $1 \le k \le n$, $j \in \{0,1\}$,
  $(h_{ko}^j)^{\mathcal{I}} = \{\langle x_{b,i,k}, x_{b,i,k-1}\rangle \mid b \in B,\, |b| \text{ is even},\, 0 \le i < 2^n,\, i[k]_2 = j\}$, $1 \le k \le n$, $j \in \{0,1\}$,
  $h^{\mathcal{I}} = \{\langle x_{b,i,k-1}, x_{b,i,k}\rangle \mid b \in B,\, 0 \le i < 2^n,\, 1 \le k \le n\}$;
- $(t_k^{ij})^{\mathcal{I}} = ((r_o^i)^{\mathcal{I}} \cup (r_e^i)^{\mathcal{I}} \cup {v_0}^{\mathcal{I}} \cup {v_e}^{\mathcal{I}} \cup (h_{ko}^j)^{\mathcal{I}} \cup (h_{ke}^j)^{\mathcal{I}} \cup \bigcup_{1 \le k' \le n,\, k' \ne k}^{j' \in \{0,1\}} [(h_{k'o}^{j'})^{\mathcal{I}} \cup (h_{k'o}^{j'})^{\mathcal{I}}])^+$, $1 \le k \le n$, $i \in \{1,2\}$, $j \in \{0,1\}$,
- $(s_k^i)^{\mathcal{I}} = \bigcup_{j \in \{0,1\}} [(t_k^{ij})^{\mathcal{I}} \cup ((t_k^{ij})^-)^{\mathcal{I}}]$, $1 \le k \le n$, $i \in \{1,2\}$.

Clearly $\mathcal{I}$ satisfies the concept O and interprets t_k^{ij} as transitive relations. It is straightforward to check using the properties of the computation of an ATM that $\mathcal{I}$ satisfies all axioms (10)–(15) and (18)–(52) in $\mathcal{K}$. In particular, $\mathcal{I}$ satisfies axiom (43) for $\rho^1 = s_1^1 \sqcap \cdots \sqcap s_n^1$ since $\{\langle x, y\rangle \in (\rho^1)^{\mathcal{I}} \mid x \in {Q_j}^{\mathcal{I}},\, y \in {Q_{[j+1]_3}}^{\mathcal{I}}\} \subseteq \{x_{b,i,n}, x_{b\cdot 0,i,n} \mid b \in B,\, 0 \le i < 2^n\}$ and $x_{b,i,n} \in (S_{a_1a_2a_3a_4})^{\mathcal{I}}$ implies $x_{b\cdot 0,i,n} \in (A_{\lambda_1(a_1,a_2,a_3,a_4)})^{\mathcal{I}}$ by the definition of $\mathcal{I}$, since $\pi(b \cdot 0) = \delta^1(\pi(b))$

by the case (v) from the definition of a computation of an ATM, and since $\lambda_1(c_{i-1}, c_i, c_{i+1}, c_{i+1}) = c_i^1$ whenever $c^1 = \delta^1(c)$.

($\Leftarrow$) Assume that $\mathcal{I}$ is a model of $\mathcal{K}$. We build a computation $P = (B, \pi)$ of M from c^0 witnessed by $\mathcal{I}$. The elements $b \in B$ and the values $\pi(b)$ are built inductively on $|b|$ together with elements $x_{b,i,k} \in \Delta^{\mathcal{I}}$ with $0 \leq i < 2^n$ and $0 \leq k \leq n$. We demonstrate by induction that (i) when $|b|$ is even, we have $x_{b,i,k} \in \mathrm{Odd}^{\mathcal{I}}$, $\langle x_{b,i,0}, x_{b,i-1,0}\rangle \in {v_o}^{\mathcal{I}}$ when $i \geq 1$, and $\langle x_{b,i,k}, x_{b,i,k-1}\rangle \in {(h^j_{ko})}^{\mathcal{I}}$ when $k \geq 1$ and $i[k]_2 = j$, (ii) when $|b|$ is odd, we have $x_{b,i,k} \in \mathrm{Even}^{\mathcal{I}}$, $\langle x_{b,i-1,0}, x_{b,i,0}\rangle \in {v_e}^{\mathcal{I}}$ when $i \geq 1$, and $\langle x_{b,i,k-1}, x_{b,i,k}\rangle \in {(h^j_{ke})}^{\mathcal{I}}$ when $k \geq 1$ and $i[k]_2 = j$, (iii) $x_{b,i,n} \in {Q_j}^{\mathcal{I}}$ iff $x_{b,i,k} \in {P_j}^{\mathcal{I}}$ for all k with $0 \leq k \leq n$ if $[|b|]_3 = j$, $0 \leq j \leq 2$, and (iv) $\pi(b)_{i+1} = a$ implies $x_{b,i,k} \in {A_a}^{\mathcal{I}}$ for every i and k with $0 \leq i < 2^n$ and $0 \leq k \leq n$.

For the base case $b = \epsilon$, we define $x_{\epsilon,0,0} := x$ for some $x \in O^{\mathcal{I}}$ and $\pi(\epsilon) := c^0$. Since $\mathcal{I}$ is a model of (15), we have $x_{\epsilon,0,0} \in \mathrm{Odd}^{\mathcal{I}}$, $x_{\epsilon,0,0} \in {P_0}^{\mathcal{I}}$, and $x_{\epsilon,0,0} \in Z^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of (10), (11), (13), (14), (20), and (21), it is easy to show using Lemma 3, that there exist elements $x_{\epsilon,i,0} \in \Delta^{\mathcal{I}}$ with $1 \leq i < 2^n$ such that $x_{\epsilon,i,0} \in \mathrm{Odd}^{\mathcal{I}}$, $\langle x_{\epsilon,i,0}, x_{\epsilon,i-1,0}\rangle \in {v_o}^{\mathcal{I}}$, and $c^{\mathcal{I}}(x_{\epsilon,i,0}) = i$. Furthermore, since $\mathcal{I}$ is a model of (24)–(29), there exist elements $x_{\epsilon,i,k} \in \Delta^{\mathcal{I}}$ with $1 \leq k \leq n$ such that $x_{\epsilon,i,k} \in {H_k}^{\mathcal{I}}$, $x_{\epsilon,i,k} \in \mathrm{Odd}^{\mathcal{I}}$, $c^{\mathcal{I}}(x_{\epsilon,i,k}) = i$, and $\langle x_{\epsilon,i,k}, x_{\epsilon,i,k-1}\rangle \in {(h^j_{ko})}^{\mathcal{I}}$ when $i[k]_2 = j$. Thus, we have demonstrated property (i) for $b = \epsilon$. Property (ii) for $b = \epsilon$ holds vacuously since $|\epsilon|$ is even. Property (iii) holds since $x_{\epsilon,0,0} \in {P_0}^{\mathcal{I}}$ and $\mathcal{I}$ is a model of (23), (30), and (31). Property (iv) for $b = \epsilon$ holds since $\mathcal{I}$ is a model of (45), (46), and (37).

Now assume that we have constructed some $b \in B$, all elements $x_{b,j,k} \in \Delta^{\mathcal{I}}$ with $1 \leq i < 2^n$, $0 \leq k \leq n$, and the value of $\pi(b)$. Let $\pi(b)_j = q \in Q$ be the state of the configuration $\pi(b)$ occurring at the position j. By the induction hypothesis (iv), we have $x_{b,j,0} \in {A_q}^{\mathcal{I}}$. If $q \in Q_\exists$, then since $\mathcal{I}$ is a model of (47) and (48), we have $x_{b,0,0} \in {S_\exists}^{\mathcal{I}}$. Since $x_{b,0,0} \in Z^{\mathcal{I}}$, and $x_{b,0,0} \in \mathrm{Odd}^{\mathcal{I}}$ ($x_{b,0,0} \in \mathrm{Even}^{\mathcal{I}}$), and $\mathcal{I}$ is a model of (49)–(51), there exists either $x_{b\cdot 0,0,0} \in \mathrm{Even}^{\mathcal{I}}$ ($x_{b\cdot 0,0,0} \in \mathrm{Odd}^{\mathcal{I}}$) such that $\langle x_{b,0,0}, x_{b\cdot 0,0,0}\rangle \in {(r^1_e)}^{\mathcal{I}}$ ($\langle x_{b\cdot 0,0,0}, x_{b,0,0}\rangle \in {(r^1_o)}^{\mathcal{I}}$), or $x_{b\cdot 1,0,0} \in \mathrm{Even}^{\mathcal{I}}$ ($x_{b\cdot 1,0,0} \in \mathrm{Odd}^{\mathcal{I}}$) such that $\langle x_{b,0,0}, x_{b\cdot 1,0,0}\rangle \in {(r^2_e)}^{\mathcal{I}}$ ($\langle x_{b\cdot 1,0,0}, x_{b,0,0}\rangle \in {(r^2_o)}^{\mathcal{I}}$). In either case we add the respective elements $b \cdot 0$ or $b \cdot 1$ to B. If $q \in Q_\forall$ then similarly, since $\mathcal{I}$ is a model of (47), (48), and (49)–(51), we have $x_{b,0,0} \in {S_\forall}^{\mathcal{I}}$, and there exist $x_{b\cdot 0,0,0}, x_{b\cdot 1,0,0} \in \mathrm{Even}^{\mathcal{I}}$ ($x_{b\cdot 0,0,0}, x_{b\cdot 1,0,0} \in \mathrm{Odd}^{\mathcal{I}}$) such that $\langle x_{b,0,0}, x_{b\cdot 0,0,0}\rangle \in {(r^1_e)}^{\mathcal{I}}$ and $\langle x_{b,0,0}, x_{b\cdot 1,0,0}\rangle \in {(r^2_e)}^{\mathcal{I}}$ ($\langle x_{b\cdot 0,0,0}, x_{b,0,0}\rangle \in {(r^1_o)}^{\mathcal{I}}$ and $\langle x_{b\cdot 1,0,0}, x_{b,0,0}\rangle \in {(r^2_o)}^{\mathcal{I}}$). In this case, we add both elements $b \cdot 0$ and $b \cdot 1$ to B. Note that it is not possible that $q = q_r$ since $\mathcal{I}$ is a model of (52). Since $\mathcal{I}$ is a model of (18) and (19), we have $\langle x_{b,0,0}, x_{b\cdot 0,0,0}\rangle \in r^{\mathcal{I}}$, $x_{b\cdot 0,0,0} \in Z^{\mathcal{I}}$, and $x_{b\cdot 0,0,0} \in {P_j}^{\mathcal{I}}$ for $j = [|b| + 1]_3$ when $b \cdot 0 \in B$. Likewise, when $b \cdot 1 \in B$, we have $\langle x_{b,0,0}, x_{b\cdot 1,0,0}\rangle \in r^{\mathcal{I}}$, $x_{b\cdot 1,0,0} \in Z^{\mathcal{I}}$, and $x_{b\cdot 1,0,0} \in {P_j}^{\mathcal{I}}$ for $j = [|b| + 1]_3$.

If we add an element $b \cdot 0$ to B then we define $\pi(b \cdot 0) := \delta_1(\pi(b))$. Since $x_{b\cdot 0,0,0} \in Z^{\mathcal{I}}$ and $\mathcal{I}$ is a model of (10), (11), (13), (14), (20)–(31), one can construct elements $x_{b\cdot 0,i,k} \in \Delta^{\mathcal{I}}$ with $0 \leq i < 2^n$ and $0 \leq k \leq n$ similarly as in the base case, such that the induction properties (*i*)–(*iii*) are satisfied. Similarly, if we add $b \cdot 1$ to B then we define $\pi(b \cdot 1) := \delta_2(\pi(b))$ and construct elements $x_{b\cdot 1,i,k} \in \Delta^{\mathcal{I}}$ with $0 \leq i < 2^n$ and $0 \leq k \leq n$. It remains thus to prove property (*iv*) for the new elements in B.

Assume that $b \cdot 0 \in B$. If $|b|$ is even then, since we have demonstrated that $\langle x_{b,0,0}, x_{b\cdot 0,0,0} \rangle \in (r_e^1)^{\mathcal{I}}$, property (*i*) for b and property (*ii*) for $b \cdot 0$, it is easy to show that since $\mathcal{I}$ is a model of (32)–(36), and since $\mathcal{I}$ interprets t_k^{ij} as transitive relations, we have $\langle x_{b,i,n}, x_{b\cdot 0,i,n} \rangle \in (\rho^1)^{\mathcal{I}}$ and $\langle x_{b\cdot 0,i,n}, x_{b,i,n} \rangle \in (\rho^1)^{\mathcal{I}}$ for every i with $0 \leq i < 2^n$, where $\rho^1 = s_1^1 \sqcap \cdots \sqcap s_n^1$. Likewise, if $|b|$ is odd, then $\langle x_{b\cdot 0,0,0}, x_{b,0,0} \rangle \in (r_o^1)^{\mathcal{I}}$ and using property (*ii*) for b and property (*i*) for $b \cdot 0$ one can also show that $\langle x_{b,i,n}, x_{b\cdot 0,i,n} \rangle \in (\rho^1)^{\mathcal{I}}$ and $\langle x_{b\cdot 0,i,n}, x_{b,i,n} \rangle \in (\rho^1)^{\mathcal{I}}$ for every i with $0 \leq i < 2^n$. Since $\mathcal{I}$ is a model of axioms (21), (27), and (38)–(42), using property (*iv*) for b it is easy to show that, for every i with $0 \leq i < 2^n$ such that $\pi(b)_i = a_1$, $\pi(b)_{i+1} = a_2$, $\pi(b)_{i+2} = a_3$, and $\pi(b)_{i+3} = a_4$, we have $x_{b,i,k} \in (S_{a_1,a_2,a_3,a_4})^{\mathcal{I}}$ for every k with $0 \leq k \leq n$. Now, using property (*iii*) for b and $b \cdot 0$ and the fact that $\mathcal{I}$ is a model of (43)–(44) and (37), and λ_1 corresponds to the transition function δ_1 of M, we obtain property (*iv*) for $b \cdot 0$. Analogously, one can show that if $b \cdot 1 \in B$ then $\langle x_{b,i,n}, x_{b\cdot 1,i,n} \rangle \in (\rho^2)^{\mathcal{I}}$ and $\langle x_{b\cdot 1,i,n}, x_{b,i,n} \rangle \in (\rho^2)^{\mathcal{I}}$ for $\rho^2 = s_1^2 \sqcap \cdots \sqcap s_n^2$, and, consequently, that property (*iv*) holds for $b \cdot 1$. □

When analyzing the number of introduced axioms and their size, we see that their number is polynomial in n and their size is linear in n, where n is the size of the initial configuration. Hence, we get the following result.

Corollary 2. *The problem of (finite) concept satisfiability in the DL $\mathcal{SHI}^{\sqcap}$ is* 2ExpTime*-hard (and so are all the standard reasoning problems).*

The corresponding upper bound from [23] gives us the following result.

Corollary 3. *The problem of concept satisfiability in $\mathcal{SHI}^{\sqcap}$ and $\mathcal{SHIQ}^{\sqcap}$ is* 2ExpTime*-complete (and so are all the standard reasoning problems).*

Corollary 4. *The problem of entailment for unions of conjunctive queries in $\mathcal{SHI}$ is* 2ExpTime*-complete already for queries with at most two variables.*

Proof. By Lemma 1 the problem of knowledge base satiafiability in $\mathcal{SHI}^{\sqcap}$ can be reduced in polynomial time to the problem of non-entailment for a union of conjunctive queries containing at most two variables. The matching 2ExpTime upper bound follows from the results in [23]. □

6 $\mathcal{SHOIF}^{\sqcap}$ is N2ExpTime-hard

For proving the lower bound of reasoning in $\mathcal{SHOIF}^{\sqcap}$, we use a reduction from the double exponential domino tiling problem. We demonstrate how, by using $\mathcal{SHOIF}^{\sqcap}$ formulas, one can encode a $2^{2^n} \times 2^{2^n}$ grid-like structure illustrated

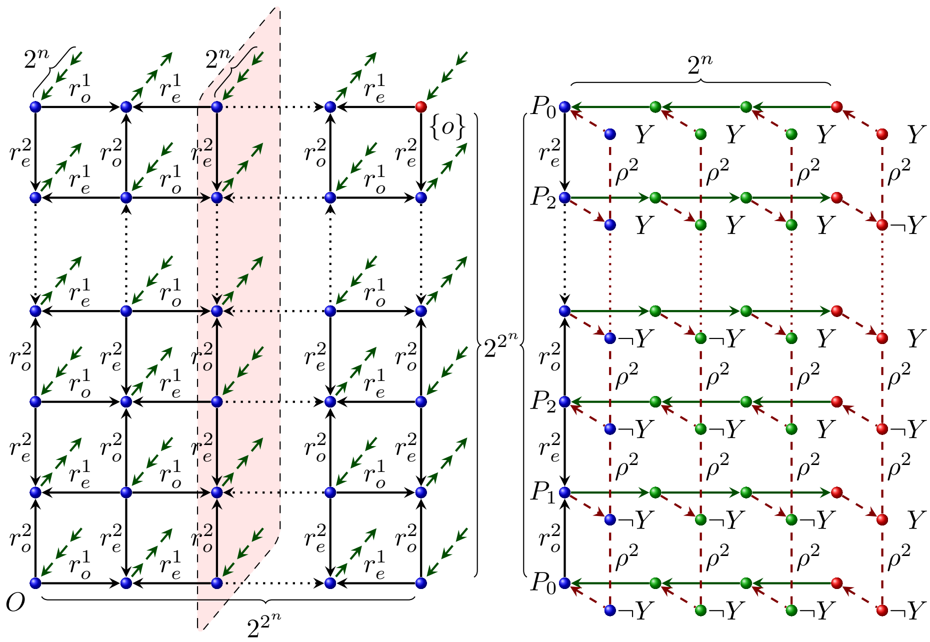

Fig. 5. A doubly exponential grid structure (left) and a detailed picture corresponding to the selected vertical slice in the grid (right)

in Figure 5. As in our tree-like structure in Figure 2 we will use four roles r_o^1, r_e^1, r_o^2, and r_e^2 with alternating directions to create the grid. Roles r_o^1 and r_e^1 induce horizontal edges and roles r_o^2 and r_e^2 induce vertical edges. The nodes of the grid are also partitioned on even and odd in a similar way as before: the odd nodes have only outgoing r-edges and the even nodes have only incoming r-edges. In fact our grid structure in Figure 5 is obtained from the tree structure in Figure 2 by merging the nodes that are reachable with the same number of horizontal and vertical edges up to a certain level, that is the nodes having the same "coordinates". The key idea of our construction is that in $\mathcal{SHOIF}^\sqcap$ it is possible to express doubly exponential counters for encoding the coordinates—a similar technique has been recently used in [17] for proving N2ExpTime-hardness of $\mathcal{SROIQ}$. We use a pair of counters to encode the coordinates of the grid: the counters are initialized in the origin O of the grid; the first counter is incremented across horizontal edges and the second counter is incremented across the vertical edges. We use nominals and inverse functional roles as in the hardness prove for $\mathcal{SHOIQ}$ [21] to enforce the uniqueness of the nodes with the same coordinates.

To store the values of the counters we will use exponentially long v-chains that grow from the nodes of the grid. The i^{th} element of the chain encodes the i^{th} bit of the horizontal counter using concept X and the i^{th} bit of the vertical counter using concept Y (see the right part of Figure 2). We will use auxiliary side h-chains like in our construction for $\mathcal{SHI}^{\sqcap}$ to connect the corresponding elements of the v-chains, which allows a proper incrementation of the counters.

In order to express the grid-like structure in Figure 5, we reuse all axioms (10)–(36) that define r-, v-, and h-chains, and add axioms to deal with the new counters and to merge the nodes with equal coordinates. First, we initialize both counters for the origin of our grid using auxiliary concepts Z^1 and Z^2:

$$O \sqsubseteq Z^1 \sqcap Z^2 \qquad Z^1 \sqsubseteq \neg X \sqcap \forall v.Z^1 \qquad Z^2 \sqsubseteq \neg Y \sqcap \forall v.Z^2 \tag{53}$$

Next, we introduce two concepts X^f and Y^f which express that the corresponding bit of the counter needs to be flipped in the successor value. Thus, the ending bit of the counter should always be flipped, while any other bit of the counter should be flipped if and only if the lower bit of the counter (accessible via v) is flipped from 1 to 0:

$$E \sqsubseteq X^f \sqcap Y^f \tag{54}$$

$$\exists v.(X \sqcap X^f) \sqsubseteq X^f \qquad \exists v.\neg(X \sqcap X^f) \sqsubseteq \neg X^f \tag{55}$$

$$\exists v.(Y \sqcap Y^f) \sqsubseteq Y^f \qquad \exists v.\neg(Y \sqcap Y^f) \sqsubseteq \neg Y^f \tag{56}$$

Additionally, we express that the values of X, Y, X^f, and Y^f agree across all elements of the same h-chain:

$$X \sqsubseteq \forall h.X \qquad \neg X \sqsubseteq \forall h.\neg X \qquad Y \sqsubseteq \forall h.Y \qquad \neg Y \sqsubseteq \forall h.\neg Y \tag{57}$$

$$X^f \sqsubseteq \forall h.X^f \qquad \neg X^f \sqsubseteq \forall h.\neg X^f \qquad Y^f \sqsubseteq \forall h.Y^f \qquad \neg Y^f \sqsubseteq \forall h.\neg Y^f \tag{58}$$

Finally, we express when the bits are flipped and when they are not flipped for the successor configurations using the property that the end elements of h-chains are related to exactly the corresponding elements of the successor chains via the roles ρ^1 and ρ^2. The axioms are analogous to axioms (43) and (44) that propagate the information to the successor configurations:

$$Q_j \sqcap X^f \sqsubseteq (X \sqcap \forall \rho^1.[\neg Q_{[j+1]_3} \sqcup \neg X]) \sqcup (\neg X \sqcap \forall \rho^1.[\neg Q_{[i+1]_3} \sqcup X]) \tag{59}$$

$$Q_j \sqcap \neg X^f \sqsubseteq (X \sqcap \forall \rho^1.[\neg Q_{[j+1]_3} \sqcup X]) \sqcup (\neg X \sqcap \forall \rho^1.[\neg Q_{[j+1]_3} \sqcup \neg X]) \tag{60}$$

$$Q_j \sqcap Y^f \sqsubseteq (Y \sqcap \forall \rho^2.[\neg Q_{[j+1]_3} \sqcup \neg Y]) \sqcup (\neg Y \sqcap \forall \rho^2.[\neg Q_{[j+1]_3} \sqcup Y]) \tag{61}$$

$$Q_j \sqcap \neg Y^f \sqsubseteq (Y \sqcap \forall \rho^2.[\neg Q_{[j+1]_3} \sqcup Y]) \sqcup (\neg Y \sqcap \forall \rho^2.[\neg Q_{[j+1]_3} \sqcup \neg Y]) \tag{62}$$

The following formulas express that the counters are copied for other directions:

$$Q_j \sqsubseteq (X \sqcap \forall \rho^2.[\neg Q_{[j+1]_3} \sqcup X]) \sqcap (\neg X \sqcap \forall \rho^2.[\neg Q_{[i+1]_3} \sqcup \neg X]) \tag{63}$$

$$Q_j \sqsubseteq (Y \sqcap \forall \rho^1.[\neg Q_{[j+1]_3} \sqcup Y]) \sqcap (\neg Y \sqcap \forall \rho^1.[\neg Q_{[i+1]_3} \sqcup \neg Y]) \tag{64}$$

The following is an analog of Lemma 3 for doubly exponential counters:

Lemma 4. *Let $\mathcal{K}$ be a knowledge base containing axioms* (10)–(36)*,* (51)*, and* (54)–(64)*, and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ a model of $\mathcal{K}$. Let $x_{i,k} \in \Delta^{\mathcal{I}}$ with $0 \leq i < 2^n$ and $0 \leq k \leq n$, and $y \in \Delta^{\mathcal{I}}$ be such that*

– $(x.1)$ *$x_{i,k} \in P_j{}^{\mathcal{I}}$ for some $j \in \{0,1,2\}$, $x_{i,k} \in H_k{}^{\mathcal{I}}$, $c^{\mathcal{I}}(x_{i,k}) = i$,*
– $(x.2)$ *there exist integers p_1 and q_1 with $0 \le p_1, q_1 < 2^{2^n}$ such that $x_{i,k} \in X^{\mathcal{I}}$ iff $p_1[2^n - i]_2 = 1$ and $x_{i,k} \in Y^{\mathcal{I}}$ iff $q_1[2^n - i]_2 = 1$,*
– *and either:*
  - $(x.3o)$ *$x_{i,k} \in Odd^{\mathcal{I}}$, $\langle x_{i,0}, x_{i-1,0}\rangle \in v_o{}^{\mathcal{I}}$ when $i \ge 1$, $\langle x_{i,k}, x_{i,k-1}\rangle \in (h^{\ell}_{ko})^{\mathcal{I}}$ when $i[k]_2 = \ell$ and $k \ge 1$, $\ell \in 0,1$, and either:*
    $(x.3o.1)$ *$\langle x_{0,0}, y\rangle \in (r^1_e)^{\mathcal{I}}$, or* $(x.3o.2)$ *$\langle x_{0,0}, y\rangle \in (r^2_e)^{\mathcal{I}}$, or*
  - $(x.3e)$ *$x_{i,k} \in Even^{\mathcal{I}}$, $\langle x_{i-1,0}, x_{i,0}\rangle \in v_e{}^{\mathcal{I}}$ when $i \ge 1$, $\langle x_{i,k-1}, x_{i,k}\rangle \in (h^{\ell}_{ke})^{\mathcal{I}}$ when $i[k]_2 = \ell$ and $k \ge 1$, $\ell \in 0,1$, and either:*
    $(x.3e.1)$ *$\langle y, x_{0,0}\rangle \in (r^1_o)^{\mathcal{I}}$, or* $(x.3e.2)$ *$\langle y, x_{0,0}\rangle \in (r^2_o)^{\mathcal{I}}$*

Then there exist elements $y_{i,k} \in \Delta^{\mathcal{I}}$ with $0 \le i < 2^n$ and $0 \le k \le n$, such that $y_{0,0} = y$, and respectively:

– $(y.1)$ *$y_{i,k} \in P_{[j+1]_3}{}^{\mathcal{I}}$, $x_{i,k} \in H_k{}^{\mathcal{I}}$, $c^{\mathcal{I}}(y_{i,k}) = i$,*
– $(y.2)$ *there exist integers p_2 and q_2 with $0 \le p_2, q_2 < 2^{2^n}$ such that $y_{i,k} \in X^{\mathcal{I}}$ iff $p_2[2^n - i]_2 = 1$ and $y_{i,k} \in Y^{\mathcal{I}}$ iff $q_2[2^n - i]_2 = 1$,*
– $(y.3e)$ *$y_{i,k} \in Even^{\mathcal{I}}$, $\langle y_{i-1,0}, y_{i,0}\rangle \in v_e{}^{\mathcal{I}}$ when $i \ge 1$, $\langle y_{i,k-1}, y_{i,k}\rangle \in (h^{\ell}_{ke})^{\mathcal{I}}$ when $i[k]_2 = \ell$, $k \ge 1$, $\ell \in \{0,1\}$, and $(x.3o)$ holds, or*
– $(y.3o)$ *$y_{i,k} \in Odd^{\mathcal{I}}$, $\langle y_{i,0}, y_{i-1,0}\rangle \in v_o{}^{\mathcal{I}}$ when $i \ge 1$, $\langle y_{i,k}, y_{i,k-1}\rangle \in (h^{\ell}_{ko})^{\mathcal{I}}$ when $i[k]_2 = \ell$, $k \ge 1$, $\ell \in \{0,1\}$, and $(x.3e)$ holds,*
– $(y.4.1)$ *$p_2 = p_1 + 1 \mod 2^{2^n}$ and $q_2 = q_1$ when $(x.3o.1)$ or $(x.3e.1)$ holds, and*
– $(y.4.2)$ *$p_2 = p_1$ and $q_2 = q_1 + 1 \mod 2^{2^n}$ when $(x.3o.2)$ or $(x.3e.2)$ holds.*

Proof. We prove the lemma only for the case when conditions $(x.3o)$ and $(x.3o.1)$ hold. All other cases are proved analogously.

First, we define $y_{0,0} := y$. Since $\mathcal{I}$ is a model of (18), (19), (10), and (51) from the conditions $(x.3o.1)$, $(x.1)$, and $(x.3o)$ we have $\langle x_{0,0}, y_{0,0}\rangle \in r^{\mathcal{I}}$, $y_{0,0} \in Z^{\mathcal{I}}$, $y_{0,0} \in P_{[j+1]_3}{}^{\mathcal{I}}$, and $y_{0,0} \in \mathrm{Even}^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of (10), (11), (13), (14), (20), and (21), it follows from Lemma 3 that there exist elements $y_{i,0} \in \Delta^{\mathcal{I}}$ with $1 \le i < 2^n$ such that $y_{i,0} \in \mathrm{Even}^{\mathcal{I}}$, $\langle y_{i-1,0}, y_{i,0}\rangle \in v_e{}^{\mathcal{I}}$, and $c^{\mathcal{I}}(y_{i,0}) = i$. Furthermore, since $\mathcal{I}$ is a model of (24)–(30), there exist elements $y_{i,k} \in \Delta^{\mathcal{I}}$ with $1 \le k \le n$ such that $y_{i,k} \in H_k{}^{\mathcal{I}}$, $y_{i,k} \in \mathrm{Even}^{\mathcal{I}}$, $c^{\mathcal{I}}(y_{i,k}) = i$, $y_{i,k} \in P_{[j+1]_3}{}^{\mathcal{I}}$, and $\langle y_{i,k-1}, y_{i,k}\rangle \in (h^{\ell}_{ke})^{\mathcal{I}}$ when $i[k]_2 = \ell$. Therefore we have proved the claims $(y.1)$ and $(y3.e)$. It remains thus to prove the claims $(y.2)$ and $(y.4.1)$.

Obviously, it is possible to find integers $p_2, q_2 < 2^{2^n}$ that satisfy claim $(y.2)$. We now prove that claim $(y.4.1)$ holds for these integers. Since $\mathcal{I}$ is a model of (31), it is easy to show using $(x.1)$ and $(y.1)$ that $x_{i,n} \in Q_j{}^{\mathcal{I}}$ and $y_{i,n} \in Q_{[j+1]_3}{}^{\mathcal{I}}$ for every i with $0 \le i < 2^n$. Now using axioms (32)–(36), and properties $(x.3o)$ and $(y.3e)$, it is easy to show that $\langle x_{i,n}, y_{i,n}\rangle \in (\rho^1)^{\mathcal{I}}$ for $\rho^1 = s^1_1 \sqcap \cdots \sqcap s^1_n$ and every i with $0 \le i < 2^n$. Since $\mathcal{I}$ is a model of (54) and $x_{2^n-1,n} \in E^{\mathcal{I}}$, we have $x_{2^n-1,n} \in X^{f\,\mathcal{I}}$. Furthermore, since $\mathcal{I}$ is a model of (55), for every i with $1 \le i < 2^n$, we have $x_{i-1,n} \in (X^f)^{\mathcal{I}}$ if and only if $x_{i,n} \in (X \sqcap X^f)^{\mathcal{I}}$. Since

$x_{i,n} \in {Q_j}^{\mathcal{I}}$, $y_{i,n} \in {Q_{[j+1]_3}}^{\mathcal{I}}$, and $\langle x_{i,n}, y_{i,n}\rangle \in (\rho^1)^{\mathcal{I}}$ for every i with $0 \le i < 2^n$, using axioms (59) and (60) it is easy to show that $p_2 = p_1 + 1 \mod 2^{2^n}$, and using axiom (64) it is easy to show that $q_2 = q_1$, what was required to prove in (y.4.1). □

In order to avoid creating r-successors after the maximal values of the counters are reached, we replace axioms (16) and (17) with (65) and (66), which express that the corresponding successor has to be created unless the highest bit flips from 1 to 0:

$$Z \sqcap \mathrm{Odd} \sqsubseteq ((X \sqcap X^f) \sqcup \exists r_e^1.\top) \sqcap ((Y \sqcap Y^f) \sqcup \exists r_e^2.\top) \tag{65}$$

$$Z \sqcap \mathrm{Even} \sqsubseteq ((X \sqcap X^f) \sqcup \exists {r_o^1}^-.\top) \sqcap ((Y \sqcap Y^f) \sqcup \exists {r_o^2}^-.\top) \tag{66}$$

In order to merge the elements with the same coordinates, we first merge the elements that have the maximal values for both counters:

$$Z \sqcap X \sqcap X^f \sqcap Y \sqcap Y^f \sqsubseteq \{o\} \tag{67}$$

The preceding elements with the same coordinates are then merged by asserting functionality of the roles r^1 and r^2 that are respective superroles of r_e^1, ${r_o^1}^-$, r_e^2, and ${r_o^2}^-$ according to (18):

$$\mathsf{Func}(r^1) \qquad\qquad \mathsf{Func}(r^2) \tag{68}$$

Lemma 5. *Let $\mathcal{K}$ be a knowledge base containing axioms* (10)–(15)*,* (18)–(36)*,* (51)*, and* (53)–(68)*. Then for every model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{K}$ and every $x \in O^{\mathcal{I}}$, there exist $x_{p,q} \in \Delta^{\mathcal{I}}$ with $0 \le p, q < 2^{2^n}$ such that (i) $x = x_{0,0}$, (ii) when $p \ge 1$, then $\langle x_{p-1,q}, x_{p,q}\rangle \in (r^1)^{\mathcal{I}}$, and (iii) when $q \ge 1$, then $\langle x_{p,q-1}, x_{p,q}\rangle \in (r^2)^{\mathcal{I}}$.*

Proof. By induction on $p+q$ with $0 \le p, q < 2^{2^n}$, we construct non-empty sets of elements $X_{p,q} \subseteq \Delta^{\mathcal{I}}$ and prove that (1) for every $x \in X_{p,q}$ there exist elements $x_{i,k}$ with $0 \le i < 2^n$ and $0 \le k \le n$ such that $x_{0,0} = x$ and conditions (x.1), (x.2), (x.3o) (when $p+q$ is even), and (x.3e) (when $p+q$ is odd) of Lemma 4 hold; (2) if $p \ge 1$ then for every $x \in X_{p-1,q}$ there exists $y \in X_{p,q}$ such that $\langle x, y\rangle \in (r^1)^{\mathcal{I}}$, and, if $q \ge 1$ then for every $x \in X_{p,q-1}$ there exists $y \in X_{p,q}$ such that $\langle x, y\rangle \in (r^2)^{\mathcal{I}}$. After that, we demonstrate that every set $X_{p,q}$ contains exactly 1 element, which we define by $x_{p,q}$. Then properties (*ii*) and (*iii*) of the lemma will be consequences of property (2).

For the base case $p = q = 0$, we set $X_{0,0} := \{x\}$ for some $x \in O^{\mathcal{I}}$ that is given by the condition of the lemma. Since $\mathcal{I}$ is a model of (15), we have $x \in \mathrm{Odd}^{\mathcal{I}}$, $x \in {P_0}^{\mathcal{I}}$, and $x \in Z^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of (10), (11), (13), (14), (20), and (21), it follows from Lemma 3 that there exist elements $x_{i,0} \in \Delta^{\mathcal{I}}$ with $0 \le i < 2^n$ such that $x_{0,0} = x$, $x_{i,0} \in \mathrm{Odd}^{\mathcal{I}}$, $\langle x_{i,0}, x_{i-1,0}\rangle \in {v_o}^{\mathcal{I}}$ when $i \ge 1$, and $c^{\mathcal{I}}(x_{i,0}) = i$. Furthermore, since $\mathcal{I}$ is a model of (24)–(29), there exist elements $x_{i,k} \in \Delta^{\mathcal{I}}$ with $1 \le k \le n$ such that $x_{i,k} \in {H_k}^{\mathcal{I}}$, $x_{i,k} \in \mathrm{Odd}^{\mathcal{I}}$, $c^{\mathcal{I}}(x_{i,k}) = i$, and $\langle x_{i,k}, x_{i,k-1}\rangle \in (h_{ko}^j)^{\mathcal{I}}$ when $i[k]_2 = j$. Therefore, condition (1) for the base case holds. Condition (2) for the base case hols vacuously, since $p = q = 0$.

For the induction step $p+q>0$, we construct the set $X_{p,q}$ provided we have constructed the sets $X_{p-1,q}$ if $p \geq 1$ and $X_{p,q-1}$ if $q \geq 1$. We first initialize $X_{p,q}$ to the empty set, and then add new elements as described below.

If $p \geq 1$, by the induction hypothesis (1), for every element $x \in X_{p-1,q}$ there exist elements $x_{i,k}$ with $0 \leq i < 2^n$ and $0 \leq k \leq n$ such that $x_{0,0} = x$ and the conditions (x.1), (x.2), (x.3o) (when $p-1+q$ is even), and (x.3e) (when $p-1+q$ is odd) of Lemma 4 hold. Since $p-1 < p < 2^{2^n}$, there exists i with $0 \leq i < 2^n$ such that $(p-1)[2^n-i]_2 = 0$, and therefore, by property (x.2), $x_{i,0} \notin X^{\mathcal{I}}$. If $i>0$ then since $\mathcal{I}$ is a model of (21) and (55) using the conditions (x.3o) (when $p-1+q$ is even), and (x.3e) (when $p-1+q$ is odd) it is easy to show that $x_{0,0} \notin (X^f)^{\mathcal{I}}$. Therefore $x = x_{0,0} \notin X^{\mathcal{I}} \cap (X^f)^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of (65) and (66), there exists $y \in \Delta^{\mathcal{I}}$ such that $\langle x,y\rangle \in (r^1_e)^{\mathcal{I}}$ (when $p-1+q$ is even) and $\langle y,x\rangle \in (r^1_o)^{\mathcal{I}}$ (when $p-1+q$ is odd). We add the constructed element y to the set $X_{p,q}$. By applying Lemma 4 to $x_{i,k}$ and y, we can show that condition (1) for the constructed element $y \in X_{p,q}$ is satisfied. Note also that $\langle x,y\rangle \in (r^1)^{\mathcal{I}}$ since $\mathcal{I}$ is a model of (18). Analogously, if $q \geq 1$, for every element $x \in X_{p,q-1}$, we construct an element $y \in X_{p,q}$ such that $\langle x,y\rangle \in (r^2)^{\mathcal{I}}$ and condition (1) is satisfied for y. After adding the respective elements y for all elements in $X_{p-1,q}$ (when $p \geq 1$) and $X_{p,q-1}$ (when $q \geq 1$), we have satisfied condition (2) for $X_{p,q}$. Note that since either $p \geq 1$ and $X_{p-1,q}$ is non-empty, or $q \geq 1$ and $X_{p,q-1}$ is non-empty, the constructed set $X_{p,q}$ is non-empty as well.

It remains therefore to prove that every set $X_{p,q}$ with $0 \leq p,q < 2^{2^n}$ contains exactly one element. First consider the set $X_{p',q'}$ for $p' = q' = 2^{2^n}-1$. By condition (1), for every element $x \in X_{p',q'}$, there exist elements $x_{i,k}$ with $0 \leq i < 2^n$ and $0 \leq k \leq n$ such that $x_{0,0} = x$ and conditions (x.1), (x.2), (x.3o) (when $p-1+q$ is even), and (x.3e) (when $p-1+q$ is odd) of Lemma 4 hold. Since $p' = q' = 2^{2^n}-1$, we have $p'[2^n-i]_2 = q'[2^n-i]_2 = 1$ for every i with $0 \leq i < 2^n$, and therefore, by property (x.2), we have $x_{i,0} \in X^{\mathcal{I}}$ and $y_{i,0} \in X^{\mathcal{I}}$ for every i with $0 \leq i < 2^n$. Since $\mathcal{I}$ is a model of (54)–(56), we have $x_{0,0} \in (X^f)^{\mathcal{I}}$ and $x_{0,0} \in (Y^f)^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of (10) we also have $x_{0,0} \in Z^{\mathcal{I}}$. Therefore, since $\mathcal{I}$ satisfies (67), we obtain that $x = x_{0,0} = o^{\mathcal{I}}$. We have demonstrated that for every $x \in X_{p',q'}$ we have $x = o^{\mathcal{I}}$, and, consequently, $X_{p',q'}$ contains at most one element. Furthermore, since $\mathcal{I}$ is a model of (68) and by condition (2) it follows that every set $X_{p,q}$ with $p+q < p'+q'$ also contains at most one element. Since every set $X_{p,q}$ with $0 \leq p,q < 2^{2^n}$ is non-empty, each of them contains exactly one element which we define by $x_{p,q}$. □

Our complexity result for $\mathcal{SHOIF}^{\sqcap}$ is now obtained by a reduction from the bounded domino tiling problem. A *domino system* is a triple $D = (T,H,V)$, where $T = \{1,\ldots,k\}$ is a finite set of *tiles* and $H,V \subseteq T \times T$ are *horizontal* and *vertical matching relations*. A *tiling* of $m \times m$ for a domino system D with *initial condition* $c^0 = \langle t^0_1,\ldots,t^0_n\rangle$, $t^0_i \in T, 1 \leq i \leq n$, is a mapping $t\colon \{1,\ldots,m\} \times \{1,\ldots,m\} \to T$ such that $\langle t(i-1,j),t(i,j)\rangle \in H, 1 < i \leq m, 1 \leq j \leq m$, $\langle t(i,j-1),t(i,j)\rangle \in V, 1 \leq i \leq m, 1 < j \leq m$, and $t(i,1) = t^0_i, 1 \leq i \leq n$. It is well known [30] that there exists a domino system D_0 that is N2ExpTime-complete for the following decision problem: given an initial condition c^0 of size n, check if D_0 admits the tiling of $2^{2^n} \times 2^{2^n}$ for c^0.

In order to encode the domino problem on our grid, we use new atomic concepts $T_1, \ldots, T_d$ for the tiles of the domino system D_0. The following axioms express that every element in our structure is assigned with a unique tile and that it is not possible to have horizontal and vertical successors that do not agree with the matching relations

$$\top \sqsubseteq T_1 \sqcup \cdots \sqcup T_k \tag{69}$$

$$T_i \sqcap T_j \sqsubseteq \bot \qquad 1 \leq i < j \leq d \tag{70}$$

$$T_i \sqcap \exists r^1.T_j \sqsubseteq \bot \qquad \langle i, j\rangle \notin H \tag{71}$$

$$T_i \sqcap \exists r^2.T_j \sqsubseteq \bot \qquad \langle i, j\rangle \notin V \tag{72}$$

Finally, we express the initial condition of the grid:

$$O \sqsubseteq T_{t_1^0} \sqcap \forall r^1.(T_{t_2^0} \sqcap \forall r^1.(T_{t_3^0} \sqcap \forall r^1.(T_{t_4^0} \sqcap \ldots \forall r^1.T_{t_n^0} \ldots))) \tag{73}$$

Note that the size and the number of formulas that we have constructed is polynomial in the size of c^0. Since D_0 is fixed, we obtain a polynomial reduction from the doubly exponential domino tiling problem to the problem of $\mathcal{SHOIF}^{\sqcap}$ knowledge base satisfiability.

Theorem 3. *Let c^0 be an initial condition of size n for the domino system D_0 and $\mathcal{K}$ a knowledge base consisting of axioms* (10)–(15), (18)–(36), (51), *and* (53)–(73). *Then D_0 admits the tiling of $2^{2^n} \times 2^{2^n}$ for c^0 if and only if O is (finitely) satisfiable in $\mathcal{K}$.*

Proof. ($\Rightarrow$) Let $t : 2^{2^n} \times 2^{2^n}$ be a tiling for the domino system $D_0 = (T, H, V)$ with the initial condition c^0. We use t to build a finite model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{K}$ that satisfies O.

We define $\Delta^{\mathcal{I}} := \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n}, 0 \leq i < 2^n, 0 \leq k \leq n\}$. The interpretation of the individual o is defined by $o^{\mathcal{I}} = x_{2^{2^n}-1, 2^{2^n}-1}$. The interpretation of the concepts B_j, Z, E, O, Odd, Even, P_j, Q_j, H_k, X, Y, X^f, Y^f, Z^1, Z^2, E^1, E^2, and T_ℓ are defined by:

- $B_j^{\mathcal{I}} = \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n}, i[j]_2 = 1, 0 \leq k \leq n\}, 1 \leq j \leq n$;
- $O^{\mathcal{I}} = \{x_{0,0,0,0}\}$,
  $Z^{\mathcal{I}} = \{x_{p,q,0,k} \mid 0 \leq p, q < 2^{2^n}, 0 \leq k \leq n\}$,
  $E^{\mathcal{I}} = \{x_{p,q,2^n-1,k} \mid 0 \leq p, q < 2^{2^n}, 0 \leq k \leq n\}$;
- $\mathrm{Odd}^{\mathcal{I}} = \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n}, p + q \text{ is even}, 0 \leq k \leq n\}$,
  $\mathrm{Even}^{\mathcal{I}} = \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n}, p + q \text{ is odd}, 0 \leq k \leq n\}$;
- $P_j^{\mathcal{I}} = \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n}, [p + q]_3 = j, 0 \leq i < 2^n, 0 \leq k \leq n\}$, $0 \leq j \leq 2$;
- $Q_j^{\mathcal{I}} = \{x_{p,q,i,n} \mid 0 \leq p, q < 2^{2^n}, [p + q]_3 = j, 0 \leq i < 2^n\}$, $0 \leq j \leq 2$;
- $H_k^{\mathcal{I}} = \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n}, 0 \leq i < 2^n\}$, $0 \leq k \leq n$;
- $X^{\mathcal{I}} = \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n}, p[2^n - i]_2 = 1, 0 \leq i < 2^n, 0 \leq k \leq n\}$;

- $Y^{\mathcal{I}} = \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n},\ q[2^n - i]_2 = 1,\ 0 \leq i < 2^n,\ 0 \leq k \leq n\}$;
- $(X^f)^{\mathcal{I}} = \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n},\ 0 \leq i < 2^n,\ \forall i' > i\ p[2^n - i']_2 = 1,\ 0 \leq k \leq n\}$;
- $(Y^f)^{\mathcal{I}} = \{x_{p,q,i,k} \mid 0 \leq p, q < 2^{2^n},\ 0 \leq i < 2^n,\ \forall i' > i\ q[2^n - i']_2 = 1,\ 0 \leq k \leq n\}$;
- $(Z^1)^{\mathcal{I}} = (Z^2)^{\mathcal{I}} = \{x_{0,0,i,0} \mid 0 \leq i < 2^n\}$;
- $T_\ell{}^{\mathcal{I}} = \{x_{p,q,0,0} \mid t(p+1, q+1) = \ell\}$.

The roles r^i_o, r^i_e, r^i, r, v_0, v_e, v, h^j_{ko}, h^j_{ke}, h, t^{ij}_k, and s^i_k are interpreted as follows:

- $(r^1_e)^{\mathcal{I}} = \{\langle x_{p-1,q,0,0}, x_{p,q,0,0}\rangle \mid 1 \leq p < 2^{2^n},\ 0 \leq q < 2^{2^n},\ p+q \text{ is odd}\}$,
  $(r^1_o)^{\mathcal{I}} = \{\langle x_{p,q,0,0}, x_{p-1,q,0,0}\rangle \mid 1 \leq p < 2^{2^n},\ 0 \leq q < 2^{2^n},\ p+q \text{ is odd}\}$,
  $(r^2_e)^{\mathcal{I}} = \{\langle x_{p,q-1,0,0}, x_{p,q,0,0}\rangle \mid 0 \leq p < 2^{2^n},\ 1 \leq q < 2^{2^n},\ p+q \text{ is even}\}$,
  $(r^2_o)^{\mathcal{I}} = \{\langle x_{p,q,0,0}, x_{p,q-1,0,0}\rangle \mid 0 \leq p < 2^{2^n},\ 1 \leq q < 2^{2^n},\ p+q \text{ is even}\}$,
  $(r^1)^{\mathcal{I}} = (r^1_e)^{\mathcal{I}} \cup ((r^1_o)^-)^{\mathcal{I}}$,
  $(r^2)^{\mathcal{I}} = (r^2_e)^{\mathcal{I}} \cup ((r^2_o)^-)^{\mathcal{I}}$,
  $r^{\mathcal{I}} = (r^1)^{\mathcal{I}} \cup (r^2)^{\mathcal{I}}$;
- $(v_e)^{\mathcal{I}} = \{\langle x_{p,q,i-1,0}, x_{p,q,i,0}\rangle \mid 0 \leq p, q < 2^{2^n},\ p+q \text{ is odd},\ 1 \leq i < 2^n\}$,
  $(v_o)^{\mathcal{I}} = \{\langle x_{p,q,i,0}, x_{p,q,i-1,0}\rangle \mid 0 \leq p, q < 2^{2^n},\ p+q \text{ is even},\ 1 \leq i < 2^n\}$,
  $v^{\mathcal{I}} = (v_e)^{\mathcal{I}} \cup ((v_o)^-)^{\mathcal{I}}$;
- $(h^j_{ke})^{\mathcal{I}} = \{\langle x_{p,q,i,k-1}, x_{p,q,i,k}\rangle \mid 0 \leq p, q < 2^{2^n},\ p+q \text{ is odd},$
  $0 \leq i < 2^n,\ i[k]_2 = j\},\ 1 \leq k \leq n,\ j \in \{0,1\}$
  $(h^j_{ko})^{\mathcal{I}} = \{\langle x_{p,q,i,k}, x_{p,q,i,k-1}\rangle \mid 0 \leq p, q < 2^{2^n},\ p+q \text{ is even},$
  $0 \leq i < 2^n,\ i[k]_2 = j\},\ 1 \leq k \leq n,\ j \in \{0,1\}$
  $h^{\mathcal{I}} = \{\langle x_{p,q,i,k-1}, x_{p,q,i,k}\rangle \mid 0 \leq p, q < 2^{2^n},\ 0 \leq i < 2^n,\ 1 \leq k \leq n\}$;
- $(t^{ij}_k)^{\mathcal{I}} = ((r^i_o)^{\mathcal{I}} \cup (r^i_e)^{\mathcal{I}} \cup v_0{}^{\mathcal{I}} \cup v_e{}^{\mathcal{I}} \cup (h^j_{ko})^{\mathcal{I}} \cup (h^j_{ke})^{\mathcal{I}} \cup$
  $\bigcup^{j' \in \{0,1\}}_{1 \leq k' \leq n,\, k' \neq k} [(h^{j'}_{k'o})^{\mathcal{I}} \cup (h^{j'}_{k'o})^{\mathcal{I}}])^+, 1 \leq k \leq n,\ i \in \{1,2\},\ j \in \{0,1\}$,
- $(s^i_k)^{\mathcal{I}} = \bigcup_{j \in \{0,1\}} [(t^{ij}_k)^{\mathcal{I}} \cup ((t^{ij}_k)^-)^{\mathcal{I}}],\ 1 \leq k \leq n,\ i \in \{1,2\}$.

Clearly $\mathcal{I}$ satisfies the concept O and interprets t^{ij}_k as transitive relations. It is straightforward to check using the definition for the tiling problem that $\mathcal{I}$ satisfies all axioms (10)–(15), (18)–(36), (51), and (53)–(73). In particular, $\mathcal{I}$ satisfies axiom (59) for $\rho^1 = s^1_1 \sqcap \cdots \sqcap s^1_n$ and since $\{\langle x, y\rangle \in (\rho^1)^{\mathcal{I}} \mid x \in Q_j{}^{\mathcal{I}},\ y \in Q_{[j+1]_3}{}^{\mathcal{I}}\} \subseteq \{x_{p-1,q,i,n}, x_{p,q,i,n} \mid 1 \leq p < 2^{2^n},\ 0 \leq p, q < 2^{2^n},\ 0 \leq i < 2^n\}$ by the definition of $Q_j{}^{\mathcal{I}}$ and $(s^1_k)^{\mathcal{I}}$, and $x_{p-1,q,i,n} \in (X^f)^{\mathcal{I}}$ implies $x_{p-1,q,i,n} \in X^{\mathcal{I}}$ iff $x_{p,q,i,n} \notin X^{\mathcal{I}}$ by definition of $(X^f)^{\mathcal{I}}$ and $X^{\mathcal{I}}$, and by the properties of bit coded numbers. $\mathcal{I}$ satisfies (69)–(73) by the definition of $T_\ell{}^{\mathcal{I}}$, $(r^1)^{\mathcal{I}}$, and $(r^2)^{\mathcal{I}}$, and since t is a tiling for D_0 for the initial condition c^0.

($\Leftarrow$) Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model of $\mathcal{K}$ and $x \in O$. By Lemma 5, there exist $x_{p,q} \in \Delta^{\mathcal{I}}$ with $0 \leq p, q < 2^{2^n}$ that satisfy conditions (i)–(iii) of Lemma 5. Let us define a function $t : 2^{2^n} \times 2^{2^n} \to \{1, \ldots, d\}$ by setting $t(i,j) = \ell$ if and only if $x_{i-1,j-1} \in {T_\ell}^{\mathcal{I}}$ for every i and j with $1 \le i, j \le 2^{2^n}$. This function is defined correctly because $\mathcal{I}$ satisfies axioms (69) and (70). We demonstrate that t is a tiling for $D_0 = (T, H, V)$ with the initial condition c^0.

Since $\mathcal{I}$ is a model of (73), $x_{0,0} = x \in O^{\mathcal{I}}$ and $\langle x_{p-1,0}, x_{p,0} \rangle \in (r^1)^{\mathcal{I}}$ for $1 \le p < 2^n$ by Lemma 5, we have $x_{p,0} \in (T_{t^0_{p+1}})^{\mathcal{I}}$ for every p with $0 \le p < n$, and, therefore, $t(i, 1) = t^0_i$ for each i with $1 \le i \le n$ by definition of $t(i, j)$. Thus t satisfies the initial condition c^0.

In order to prove that t satisfies the matching conditions H and V of $\mathcal{D}_0$, assume that $t(i-1, j) = \ell_1$ and $t(i, j) = \ell_2$ for some i, j with $1 < i \le 2^{2^n}$ and $1 \le j \le 2^{2^n}$. By definition of $t(j, k)$, we have $x_{i-2,j-1} \in {T_{\ell_1}}^{\mathcal{I}}$ and $x_{i-1,j-1} \in {T_{\ell_2}}^{\mathcal{I}}$. Since by condition (*ii*) of Lemma 5, we have $\langle x_{i-2,j-1}, x_{i-1,j-1} \rangle \in (r^1)^{\mathcal{I}}$ and $\mathcal{I}$ is a model of (71), it is not possible that $\langle \ell_1, \ell_2 \rangle \notin H$. Therefore $\langle \ell_1, \ell_2 \rangle \in H$, which proves that t satisfies the horizontal matching condition. Analogously, using condition (*iii*) of Lemma 5 and axiom (72) we can show that t satisfies the vertical matching condition. $\square$

Corollary 5. *The problem of (finite) concept satisfiability in the DL $\mathcal{SHOIF}^{\sqcap}$ is* N2ExpTime*-hard (and so are all the standard reasoning problems).*

Proof. Since the $2^{2^n} \times 2^{2^n}$ tiling problem for D_0 with the initial condition is N2ExpTime-complete, and by Theorem 3 this problem is reducible in polynomial time to the problem of concept satisfiability in $\mathcal{SHOIF}^{\sqcap}$, the problem of concept satisfiability in $\mathcal{SHOIF}^{\sqcap}$ is N2ExpTime-hard. Since all standard reasoning problems like knowledge base satisfiability, concept satisfiability, concept non-subsumption and instance checking are inter-reducible in polynomial time to each other, all these problems are also N2ExpTime-hard for $\mathcal{SHOIF}^{\sqcap}$. $\square$

Corollary 6. *The problem of entailment for unions of conjunctive queries in $\mathcal{SHOIF}$ is* co-N2ExpTime*-hard already for queries with at most two variables.*

Proof. The proof of Lemma 1 can be easily extended to $\mathcal{SHOIF}$ knowledge bases where functional restrictions are not applied to role conjunctions. Since in our reduction from the domino tiling problem we did not use functional restrictions on role conjunctions, using the extended version of Lemma 1 and Theorem 3 it is easy to show that tiling problem for D_0 is reducible to the problem of non-entailment for unions of conjunctive queries in $\mathcal{SHOIF}$. Therefore, the letter problem is N2ExpTime-hard. $\square$

7 Conclusions

Our investigation of the computational complexity of DLs with role conjunctions is motivated by the facts that (i) role constructors recently gained attention since the upcoming OWL2 standard supports a much richer set of role constructors and (ii) conjunctive query answering in a DL $\mathcal{L}$ is often reducible to the knowledge base satisfiability problem for $\mathcal{L}$ with role conjunctions (e.g., for $\mathcal{SHIQ}$ and $\mathcal{SHOQ}$ this is the case). We have shown that role conjunctions cause an exponential blowup for the DLs $\mathcal{SHI}^\sqcap$ and $\mathcal{SHOIF}^\sqcap$. The main culprit for this are inverse roles, which we show by proving ExpTime-completeness of $\mathcal{SHQ}^\sqcap$. The obtained complexity results for knowledge base satisfiability in $\mathcal{SHQ}^\sqcap$ and $\mathcal{SHIQ}^\sqcap$ agree with the ones for conjunctive query entailment in $\mathcal{SHQ}$ and $\mathcal{SHIQ}$ (the ExpTime upper bound for conjunctive queries in $\mathcal{SHQ}$ has, to the best of our knowledge, only been shown for queries with simple roles [31]). It remains an open question whether $\mathcal{SHOIF}^\sqcap$ is N2ExpTime-complete. This is an interesting questions, since the decidability of conjunctive query entailment in $\mathcal{SHOIN}$ and, thus, OWL DL is a long-standing open problem.

References

1. Baader, F., Calvanese, D., McGuinness, D.L., Nardi, D., Patel-Schneider, P.F., eds.: The Description Logic Handbook. Cambridge University Press (2003)
2. Bechhofer, S., van Harmelen, F., Hendler, J., Horrocks, I., McGuinness, D.L., Patel-Schneider, P.F., Stein, L.A.: OWL web ontology language reference. Technical report, World Wide Web Consortium (2004) `http://www.w3.org/TR/2004/REC-owl-ref-20040210/`.
3. Tsarkov, D., Horrocks, I.: FaCT++ description logic reasoner: System description. In Furbach, U., Shankar, N., eds.: Proceedings of the International Joint Conference on Automated Reasoning (IJCAR 2006). Volume 4130 of Lecture Notes in Computer Science., Springer-Verlag (2006) 292 – 297
4. Sirin, E., Parsia, B., Cuenca Grau, B., Kalyanpur, A., Katz, Y.: Pellet: A practical OWL-DL reasoner. Accepted for the Journal of Web Semantics, Available online at `http://www.mindswap.org/papers/PelletJWS.pdf` (2006)
5. de Coronado, S., Haber, M.W., Sioutos, N., Tuttle, M.S., Wright, L.W.: NCI thesaurus: Using science-based terminology to integrate cancer research results. In: Proceedings of MEDINFO 2004, IOS Press (2004)
6. Schulz, S., Hahn, U.: Parts, locations, and holes - formal reasoning about anatomical structures. In: Proceedings of the 8th Conference on AI in Medicine in Europe (AIME 2001). Volume 2101 of Lecture Notes in Artificial Intelligence., Springer-Verlag (2001)
7. Spackman, K.A.: Managing clinical terminology hierarchies using algorithmic calculation of subsumption: Experience with SNOMED-RT. Journal of the American Medical Informatics Association (2000) Fall Symposium Special Issue.
8. Wroe, C., Cimino, J.J., Rector, A.L.: Integrating existing drug formulation terminologies into an HL7 standard classification using OpenGALEN. Journal of the American Medical Informatics Association (2001) Special Conference Issue.
9. Wroe, C., Goble, C.A., Greenwood, M., Lord, P., Miles, S., Papay, J., Payne, T., Moreau, L.: Automating experiments using semantic data on a bioinformatics grid. IEEE Intelligent Systems **19**(1) (2004) Special Issue on e-Science.
10. Wolstencroft, K., McEntire, R., Stevens, R., Tabernero, L., Brass, A.: Constructing Ontology-Driven Protein Family Databases. Bioinformatics **21**(8) (2005) 1685–1692
11. Wolstencroft, K., Brass, A., Horrocks, I., Lord, P., Sattler, U., Turi, D., Stevens, R.: A Little Semantic Web Goes a Long Way in Biology. In: Proceedings of the 5th International Semantic Web Conference (ISWC 2005). (2005)

12. Golbreich, C., Horridge, M., Horrocks, I., Motik, B., Shearer, R.: OBO and OWL: Leveraging semantic web technologies for the life sciences. In: Proceedings of the 7th International Semantic Web Conference (ISWC 2007). Lecture Notes in Computer Science, Springer-Verlag (2007)
13. Golbreich, C., Horrocks, I.: The OBO to OWL mapping, GO to OWL 1.1! In: Proceedings of the 3rd OWL Experiences and Directions Workshop (OWLED 2007). Number 258 in CEUR Workshop Proceedings, CEUR (`http://ceur-ws.org/`) (2007)
14. Peim, M., Franconi, E., Paton, N.W.: Estimating the quality of answers when querying over description logic ontologies. Data and Knowledge Engineering **47**(1) (2003) 105–129
15. Calvanese, D., De Giacomo, G., Lenzerini, M., Nardi, D., Rosati, R.: Description logic framework for information integration. In: Proceedings of the 6th International Conference on the Principles of Knowledge Representation and Reasoning (KR 1998). (1998)
16. Calvanese, D., De Giacomo, G., Lenzerini, M.: Description logics for information integration. In: Computational Logic: Logic Programming and Beyond. Volume 2408 of Lecture Notes in Computer Science., Springer-Verlag (2002) 41–60
17. Kazakov, Y.: $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ are harder than $\mathcal{SHOIQ}$. In: Proceedings of the 11th International Conference on the Principles of Knowledge Representation and Reasoning (KR 2008), AAAI Press/The MIT Press (2008)
18. Lange, M., Lutz, C.: 2-ExpTime lower bounds for Propositional Dynamic Logics with intersection. Journal of Symbolic Logic **70**(5) (2005) 1072–1086
19. Schmidt, R.A., Tishkovsky, D.: Deciding ALBO with tableau. In: Proceedings of the 2007 Description Logic Workshop (DL 2007). Volume 250., CEUR (`http://ceur-ws.org/`) (2007) 135–146
20. Lutz, C., Sattler, U., Wolter, F.: Description logics and the two-variable fragment. In: Proceedings of the 2001 Description Logic Workshop (DL 2001), CEUR (`http://ceur-ws.org/`) (2001) 66–75
21. Tobies, S.: Complexity Results and Practical Algorithms for Logics in Knowledge Representation. PhD thesis, RWTH Aachen (2001)
22. Lutz, C.: Inverse roles make conjunctive queries hard. In: Proceedings of the 2007 Description Logic Workshop (DL 2007). (2007)
23. Glimm, B., Horrocks, I., Lutz, C., Sattler, U.: Conjunctive query answering for the description logic $\mathcal{SHIQ}$. Journal of Artificial Intelligence Research **31** (2008) 151–198
24. Kazakov, Y., Motik, B.: A resolution-based decision procedure for $\mathcal{SHOIQ}$. Journal of Automated Reasoning **40**(2–3) (2008) 89–116
25. Schild, K.: A correspondence theory for terminological logics: preliminary report. In: Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI 1991). (1991) 466–471
26. Hustadt, U., Motik, B., Sattler, U.: Reducing $\mathcal{SHIQ}^-$ Description Logic to Disjunctive Datalog Programs. In: Proceedings of the 9th International Conference on the Principles of Knowledge Representation and Reasoning (KR 2004), AAAI Press/The MIT Press (2004) 152–162
27. Sipser, M.: Introduction to the Theory of Computation, Second Edition. Course Technology (February 2005)
28. Chandra, A.K., Kozen, D.C., Stockmeyer, L.J.: Alternation. Journal of the ACM **28**(1) (1981) 114–133

29. Tobies, S.: The complexity of reasoning with cardinality restrictions and nominals in expressive description logics. Journal of Artificial Intelligence Research **12** (2000) 199–217
30. Börger, E., Grädel, E., Gurevich, Y.: The classical decision problem. Journal of Logic, Language and Information **8**(4) (1999) 478–481
31. Lutz, C.: Two upper bounds for conjunctive query answering in $\mathcal{SHIQ}$. In: Proceedings of the 2008 Description Logic Workshop (DL 2008). (2008)