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A Cut-Free Sequent Calculus for Algebraic Dynamic Epistemic Logic

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Abstract

We develop a cut-free sequent calculus for a Dynamic Epistemic Logic. The calculus is nested and represents a sub-structural action logic which acts on a propositional logic via a *dynamic modality* and its left adjoint *update*. Both logics are positive and have agent-indexed adjoint pairs of epistemic modalities. We prove admissibility (where appropriate) of *Weakening* and *Contraction* and *Cut*, as well as soundness and completeness theorems with regard to the algebraic semantics. To model epistemic protocols, we add assumption rules, prove that the admissibility results are preserved, and derive properties of a toy protocol that has honest and dishonest public and private announcements.

Keywords: Dynamic Epistemic Logic, Algebraic Modal Logic, Cut-Admissibility, Epistemic Protocols.

1 Introduction

The phrase "Dynamic Epistemic Logic" (DEL) refers to a family of logics, developed to reason about information acquisition as a result of communication actions that take place among agents in multi-agent protocols. An example of these is the logic of public and private announcements of [2], which extends the public announcement logic of, e.g., [5]. The DEL logical systems are usually presented by a Hilbert-style proof system and a relational semantics, whose central notion is an update product between the state and action Kripke models. There has been a lot of activity in the field, extending the domain and applicability of the logics, e.g. to belief revision, and developing semantic automated tools; for references and a comprehensive presentation of the literature see [11]. The field has, however, enjoyed lesser activity on the proof-theoretic side. This paper aims to take some steps towards filling this gap.

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An algebraic version of DEL has been developed in [1,7], where the update operator is the action of a quantale of communication actions on a module of facts and epistemic propositions, both of which are endowed with endo-maps for epistemic modalities. To illustrate the power of the setting in reasoning about dynamic epistemic properties, the algebra only deals with the positive fragment of the logic and modalities of modal logic K. Absence of negation is made up for with adjunction. The uniform technique of unfolding adjunctions simplify, to a great extent, proofs of epistemic properties, e.g. in the muddy children puzzle.

A sound and complete sequent calculus was included in [1,7], but with the big flaw that its cut rules were not eliminable. The richness of the logic, which consists of propositional and action connectives, the interaction between these, as well as with the epistemic modalities, made the problem of cut-alimination a challenge. In previous work [8,10] we developed a cut-free calculus for the propositional fragment of the logic. In this paper we build on that and develop a cut-free calculus for the full logic. Our calculus is a nested one, in the style of [4] (and see also p. 122 of [6]), and has two parts: an action part for linear operations on actions (sequential and parallel composition and non-deterministic choice) and a propositional part for operations on propositions (conjunction and disjunction). Moreover, the action logic acts on the propositional logic via the update operator whose right adjoint is the dynamic modality (weakest precondition of program logics such as Hoare). Both logics have adjoint epistemic modalities that interact with the update operator, in the style of the *action-knowledge* axiom of DEL. We prove that three kinds of cut rules are admissible: an action cut in the action logic, a propositional cut in the propositional logic, as well as a mixed action-propositional cut in the propositional logic.

To be able to use the calculus to prove properties of epistemic protocols, we need to encode the assumptions thereof. These include possibilities of agents regarding the propositions and actions, stability of atomic propositions (i.e. "facts") under updates, and applicability of actions (i.e. their preconditions). We show that adding these rules preserves our admissibility results, then encode and prove properties of a coin-toss protocol with honest and dishonest public and private announcements.

2 Sequent Calculus for Actions

2.1 Sequent Calculus

We refer to this logic as *action logic*. The set Q of *terms* q of the logic is generated over a set A of *agents* A and a set B of *basic actions* σ by the following grammar:

$$q ::= \bot \mid \top \mid 1 \mid \sigma \mid q \wedge q \mid q \vee q \mid q \bullet q \mid \Box_A\, q \mid \blacklozenge_A q$$

The binary connectives $\wedge$ and $\vee$ are lattice operations of meet and join and $\top$ and $\bot$ are their units; $\bullet$ is a monoid multiplication and 1 is its unit, the modalities $\Box_A$ and $\blacklozenge_A$ are endo-operators on the lattice monoid.

Action items Q and *action contexts* Θ are generated by the following syntax:

$$Q ::= q \mid \Theta^A \qquad \Theta ::= Q\ list$$

where Θ^A will be interpreted as $\blacklozenge_A(\bigodot \Theta)$, for $\bigodot \Theta$ the composition of the interpretations of elements in Θ.

Thus, *action contexts* are finite lists of action items, where *action items* are either terms or agent-annotated action contexts. The use of lists rather than sets or multisets reflects the non-commutativity (and non-idempotence) of the composition operation on actions. Lists may be empty. The concatenation of two lists is indicated by a comma, as in Θ, Θ' or (treating an action item Q as a one element list) as in Θ, Q or Q, Θ. Thus, Θ, Q, Θ' indicates a typical list of which Q is a member.

If one of the items inside a context is replaced by a "hole" $[\,]$, we have a *context-with-a-hole*. More precisely, we have the notions of *context-with-a-hole* Σ and *item-with-a-hole* R, defined using mutual recursion as follows:

$$\Sigma ::= \Theta, R, \Theta' \qquad\qquad R ::= [\,] \mid \Sigma^A$$

and so a context-with-a-hole is a context (i.e. a list of items) together with an *item-with-a-hole*, i.e. either a hole or an agent-annotated context-with-a-hole. To emphasise that a context-with-a-hole is not a context, we use Σ for the former and Θ for the latter; similarly for items-with-a-hole R and items Q.

Given a context-with-a-hole Σ and a context Θ, the result $\Sigma[\Theta]$ of applying the first to the second, i.e. replacing the hole $[\,]$ in Σ by Θ, is a context, defined recursively (together with the application of an item-with-a-hole to a context, to form a context) as follows:

$$(\Theta', R, \Theta'')[\Theta] = \Theta', R[\Theta], \Theta'' \qquad ([\,])[\Theta] = \Theta \qquad (\Sigma^A)[\Theta] = (\Sigma[\Theta])^A$$

The last of these looks more like an item; but that just forms a one element context.

Given contexts-with-a-hole Σ', Σ, and an item-with-a-hole R, the *combinations* $\Sigma' \bullet \Sigma$ and $R \bullet \Sigma$ are defined to be contexts with holes, as follows, by mutual recursion on the structures of Σ' and R:

$$(\Theta, R, \Theta') \bullet \Sigma = \Theta, (R \bullet \Sigma), \Theta' \qquad ([\,]) \bullet \Sigma = \Sigma \qquad (\Sigma''^A) \bullet \Sigma = (\Sigma'' \bullet \Sigma)^A$$

The last of these looks more like an item-with-a-hole; but that just forms a one element context-with-a-hole.

Lemma 2.1 *Given contexts-with-a-hole Σ', Σ, an item-with-a-hole R and a context Θ, the following hold:*

$$(\Sigma' \bullet \Sigma)[\Theta] = \Sigma'[\Sigma[\Theta]] \qquad (R \bullet \Sigma)[\Theta] = R[\Sigma[\Theta]]$$

Sequents consist of a context Θ (on the left), a turnstile and a term q (on the right). On the left, it is convenient to omit the list constructors, e.g. we write $1, \sigma, \bot \vdash \sigma'$ rather than $\langle 1, \sigma, \bot \rangle \vdash \sigma'$. The empty list is written $\langle\,\rangle$ or even omitted..

We have the following initial sequents (in which σ is restricted to being an atom):

$$\frac{}{\vdash 1}\ 1R \qquad \frac{}{\sigma \vdash \sigma}\ Id \qquad \frac{}{\Sigma[\bot] \vdash q}\ \bot L \qquad \frac{}{\Theta \vdash \top}\ \top R$$

The rules for the lattice operations, composition and the modalities are:

$$\frac{\Sigma[\;] \vdash q}{\Sigma[1] \vdash q}\ 1L$$

$$\frac{\Sigma[q_i] \vdash q}{\Sigma[q_1 \wedge q_2] \vdash q}\ \wedge L_i \qquad\qquad \frac{\Theta \vdash q_1 \quad \Theta \vdash q_2}{\Theta \vdash q_1 \wedge q_2}\ \wedge R$$

$$\frac{\Sigma[q_1] \vdash q \quad \Sigma[q_2] \vdash q}{\Sigma[q_1 \vee q_2] \vdash q}\ \vee L \qquad \frac{\Theta \vdash q_1}{\Theta \vdash q_1 \vee q_2}\ \vee R_1 \qquad \frac{\Theta \vdash q_2}{\Theta \vdash q_1 \vee q_2}\ \vee R_2$$

$$\frac{\Sigma[q_1, q_2] \vdash q}{\Sigma[q_1 \bullet q_2] \vdash q}\ \bullet L \qquad\qquad \frac{\Theta_1 \vdash q_1 \quad \Theta_2 \vdash q_2}{\Theta_1, \Theta_2 \vdash q_1 \bullet q_2}\ \bullet R$$

$$\frac{\Sigma[q^A] \vdash q'}{\Sigma[\blacklozenge_A q] \vdash q'}\ \blacklozenge_A L \qquad\qquad \frac{\Theta \vdash q}{\Theta^A \vdash \blacklozenge_A q}\ \blacklozenge_A R$$

$$\frac{\Sigma[q] \vdash q'}{\Sigma[(\Box_A q)^A] \vdash q'}\ \Box_A L \qquad\qquad \frac{\Theta^A \vdash q}{\Theta \vdash \Box_A\, q}\ \Box_A R$$

$$\frac{\Sigma[\Theta^A, \Theta'^A] \vdash q}{\Sigma[(\Theta, \Theta')^A] \vdash q}\ \mathit{Dist} \qquad\qquad \frac{\Sigma[\langle\;\rangle] \vdash q}{\Sigma[\langle\;\rangle^A] \vdash q}\ \mathit{Unit}$$

Various notational abbreviations are in use here, such as $\Sigma[\;]$ meaning $\Sigma[\langle\;\rangle]$, $\Sigma[q]$ meaning $\Sigma[\langle q\rangle]$, $\Sigma[q_1, q_2]$ meaning $\Sigma[\langle q_1, q_2\rangle]$ and q^A meaning $\langle q\rangle^A$. Where the empty list is an antecedent, it is omitted.

The two indicated occurrences of σ in the *Id* rule are *principal.* Each right rule has its conclusion's succedent as its *principal formula*; in addition, the $\blacklozenge_A R$ rule has Θ^A as a *principal item.* Each left rule has a *principal item*; these are as usual.

As an example of a derivation, we show that a sequence of $\blacklozenge_A$s preserves composition and conjunction in the following direction

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,\mathit{Id}}{q^B \vdash \blacklozenge_B q}\,\blacklozenge_B R}{(q^B)^A \vdash \blacklozenge_A\blacklozenge_B q}\,\blacklozenge_A R \quad \dfrac{\dfrac{\dfrac{}{q' \vdash q'}\,\mathit{Id}}{q'^B \vdash \blacklozenge_B q'}\,\blacklozenge_B R}{(q'^B)^A \vdash \blacklozenge_A\blacklozenge_B q'}\,\blacklozenge_A R}{(q^B)^A, (q'^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,\bullet R}{(q^B, q'^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,\mathit{Dist}}{((q,q')^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,\mathit{Dist}}{((q \bullet q')^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,\bullet L}{(((q \bullet q') \wedge q'')^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,\wedge L \quad \dfrac{\dfrac{\dfrac{\dfrac{}{q'' \vdash q''}\,\mathit{Id}}{q''^B \vdash \blacklozenge_B q''}\,\blacklozenge_B R}{(q''^B)^A \vdash \blacklozenge_A\blacklozenge_B q''}\,\blacklozenge_A R}{(((q \bullet q') \wedge q'')^B)^A \vdash \blacklozenge_A\blacklozenge_B q''}\,\wedge L}{(((q \bullet q') \wedge q'')^B)^A \vdash (\blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q') \wedge \blacklozenge_A\blacklozenge_B q''}\,\wedge R}{\blacklozenge_B((q \bullet q') \wedge q'')^A \vdash (\blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q') \wedge \blacklozenge_A\blacklozenge_B q''}\,\blacklozenge_B L}{\blacklozenge_A\blacklozenge_B((q \bullet q') \wedge q'') \vdash (\blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q') \wedge \blacklozenge_A\blacklozenge_B q''}\,\blacklozenge_A L$$

As a standard check on the rules, we show the following:

Lemma 2.2 *For every term q, the sequent $q \vdash q$ is derivable.*

Proof. By induction on the size of q. In case q is an atom, or $\bot$, or $\top$, the sequent $q \vdash q$ is already initial. For $q = 1$, the sequent $1 \vdash 1$ follows from the initial sequent $\vdash 1$ by one step of $1L$. For compound q, consider the cases. Meet, join and composition are routine. Suppose q is $\blacklozenge_A q'$; by inductive hypothesis, we can derive $q' \vdash q'$, and by $\blacklozenge_A R$ we can derive $q'^A \vdash \blacklozenge_A q'$, whence $\blacklozenge_A q' \vdash \blacklozenge_A q'$ by $\blacklozenge_A L$.

Now suppose q is $\Box_A q'$. By inductive hypothesis, we can derive $q' \vdash q'$, and by $\Box_A L$ we get $(\Box_A q')^A \vdash q'$; from this we obtain $q \vdash q$ by $\Box_A R$. $\Box$

The *size of a term* is just the (weighted) number of operator occurrences, counting each operator $\blacklozenge_A$ and $\Box_A$ as having weight 2; the *size of an item* Θ^A is the size of Θ plus 1, and the *size of a context* is the sum of the sizes of its items. The *size of a sequent* $\Theta \vdash q$ is just the sum of the sizes of Θ and q. Note that each premiss of a rule instance has lower size than the conclusion, except for the rule D, whose presence leads to non-termination of a naive implementation of the calculus.

Lemma 2.3 *The $\blacklozenge_A L$ and $\Box_A R$ rules are invertible, i.e. the following are admissible:*

$$\frac{\Sigma[\blacklozenge_A q] \vdash q'}{\Sigma[q^A] \vdash q'}\ \blacklozenge_A Inv \qquad\qquad \frac{\Theta \vdash \Box_A q}{\Theta^A \vdash q}\ \Box_A Inv$$

Proof. Induction on the height of the derivation of the premiss. $\Box$

Lemma 2.4 *The $\bullet L$, $\vee L$ and $\wedge R$ rules are invertible.*

Proof. Induction on the height of the derivation of the premiss. $\Box$

Lemma 2.5 *The rule $\top L^-$ is admissible:*

$$\frac{\Sigma[\top] \vdash q}{\Sigma[\Theta] \vdash q}\ \top L^-$$

Proof. Induction on the depth of the derivation of the premiss and case analysis. $\Box$

Theorem 2.6 *The* Cut *rule is admissible*

$$\frac{\Theta \vdash q \quad \Sigma'[q] \vdash q'}{\Sigma'[\Theta] \vdash q'}\ Cut$$

Proof. Strong induction on the rank of the cut, where the *rank* is given by the pair (size of cut formula q, sum of heights of derivations of premisses).

(i) The first premiss is an instance of Id.

$$\frac{\overline{\sigma \vdash \sigma}\ Id \quad \Sigma'[\sigma] \vdash q'}{\Sigma'[\sigma] \vdash q'}\ Cut$$

transforms to

$$\Sigma'[\sigma] \vdash q'$$

(ii) The first premiss is an instance of $\bot L$.

$$\dfrac{\dfrac{}{\Sigma[\bot] \vdash q}\ \bot L \qquad \Sigma'[q] \vdash q'}{\Sigma'[\Sigma[\bot]] \vdash q'}\ Cut$$

transforms to (using Lemma 2.1 to identify $\Sigma'[\Sigma[\bot]]$ and $(\Sigma' \bullet \Sigma)[\bot]$) to

$$\dfrac{}{\Sigma'[\Sigma[\bot]] \vdash q'}\ \bot L$$

(iii) The first premiss is an instance of $\top R$.

$$\dfrac{\dfrac{}{\Theta \vdash \top}\ \top R \qquad \Sigma'[\top] \vdash q'}{\Sigma'[\Theta] \vdash q'}\ Cut$$

transforms to the following using Lemma 2.5

$$\dfrac{\Sigma'[\top] \vdash q'}{\Sigma'[\Theta] \vdash q'}\ \top L^{-}$$

(iv) The first premiss is an instance of $1L$. Straightforward

(v) The first premiss is an instance of $\wedge L$. Straightforward

(vi) The first premiss is an instance of $\vee L$. Straightforward

(vii) The first premiss is an instance of $\bullet L$. Straightforward

(viii) The first premiss is an instance of $\blacklozenge_A L$.

$$\dfrac{\dfrac{\Sigma[q^A] \vdash q'}{\Sigma[\blacklozenge_A q] \vdash q'}\ \blacklozenge_A L \qquad \Sigma'[q'] \vdash q''}{\Sigma'[\Sigma[\blacklozenge_A q]] \vdash q''}\ Cut$$

transforms (using Lemma 2.1 to identify $\Sigma'[\Sigma[\blacklozenge_A q]]]$ and $(\Sigma' \bullet \Sigma)[\blacklozenge_A q]]$) to

$$\dfrac{\dfrac{\Sigma[q^A] \vdash q' \qquad \Sigma'[q'] \vdash q''}{\Sigma'[\Sigma[q^A]] \vdash q''}\ Cut}{\Sigma'[\Sigma[\blacklozenge_A q]] \vdash q''}\ \blacklozenge_A L$$

(ix) The first premiss is an instance of $\Box_A L$.

$$\dfrac{\dfrac{\Sigma[q] \vdash q'}{\Sigma[(\Box_A q)^A] \vdash q'}\ \Box_A L \qquad \Sigma'[q'] \vdash q''}{\Sigma'[\Sigma[(\Box_A q)^A]] \vdash q''}\ Cut$$

transforms to

$$\dfrac{\dfrac{\Sigma[q] \vdash q' \quad \Sigma'[q'] \vdash q''}{\Sigma'[\Sigma[q] \vdash q''}\ Cut}{\Sigma'[\Sigma[(\Box_A q)^A]] \vdash q''}\ \Box_A L$$

(x) The first premiss is an instance of $\wedge R$.

$$\dfrac{\dfrac{\Theta \vdash q_1 \qquad \Theta \vdash q_2}{\Theta \vdash q_1 \wedge q_2}\ \wedge R \qquad \Sigma[q_1 \wedge q_2] \vdash q'}{\Sigma[\Theta] \vdash q'}\ Cut$$

is dealt with in two ways, according as whether or not the cut formula is principal in the second premiss. Details are routine.

(xi) The first premiss is an instance of $\vee R$.

$$\dfrac{\dfrac{\Theta \vdash q_i}{\Theta \vdash q_1 \vee q_2}\ \vee R_i \qquad \Sigma[q_1 \vee q_2] \vdash q'}{\Sigma[\Theta] \vdash q'}\ Cut$$

is dealt with in two ways, according as whether or not the cut formula is principal in the second premiss. Details are routine.

(xii) The first premiss is an instance of $\bullet R$.

$$\dfrac{\dfrac{\Theta \vdash q \quad \Theta' \vdash q'}{\Theta, \Theta' \vdash q \bullet q'}\ \bullet R \qquad \Sigma[q \bullet q'] \vdash q''}{\Sigma[\Theta, \Theta'] \vdash q''}\ Cut$$

transforms to

$$\dfrac{\Theta' \vdash q' \qquad \dfrac{\Theta \vdash q \qquad \dfrac{\Sigma[q \bullet q'] \vdash q''}{\Sigma[q, q'] \vdash q''}\ Inv \bullet L}{\Sigma[\Theta, q'] \vdash q''}\ Cut}{\Sigma[\Theta, \Theta'] \vdash q''}\ Cut$$

(xiii) The first premiss is an instance of $Dist$.

$$\dfrac{\dfrac{\Sigma[\Theta^A, \Theta'^A] \vdash q}{\Sigma[(\Theta, \Theta')^A] \vdash q}\ Dist \qquad \Sigma'[q] \vdash q'}{\Sigma'[\Sigma[(\Theta, \Theta')^A]] \vdash q'}\ Cut$$

transforms to

$$\dfrac{\dfrac{\Sigma[\Theta^A, \Theta'^A] \vdash q \quad \Sigma'[q] \vdash q'}{\Sigma'[\Sigma[\Theta^A, \Theta'^A]] \vdash q'}\ Cut}{\Sigma'[\Sigma[(\Theta, \Theta')^A]] \vdash q'}\ Dist$$

(xiv) The first premiss is an instance of $Unit$.

$$\dfrac{\dfrac{\Sigma[\ ] \vdash q}{\Sigma[\langle\ \rangle^A] \vdash q}\ Unit \qquad \Sigma'[q] \vdash q'}{\Sigma'[\Sigma[\langle\ \rangle^A]] \vdash q'}\ Cut$$

transforms to

$$\dfrac{\dfrac{\Sigma[\ ] \vdash q \qquad \Sigma'[q] \vdash q'}{\Sigma'[\Sigma[\ ]] \vdash q'}\ Cut}{\Sigma'[\Sigma[\langle\ \rangle^A]] \vdash q'}\ Unit$$

(xv) The first premiss is an instance of $\blacklozenge_A R$.

$$\dfrac{\dfrac{\Theta \vdash q}{\Theta^A \vdash \blacklozenge_A q}\ \blacklozenge_A R \qquad \Sigma'[\blacklozenge_A q] \vdash q'}{\Sigma'[\Theta^A] \vdash q'}\ Cut$$

transforms to

$$\dfrac{\Theta \vdash q \qquad \dfrac{\Sigma'[\blacklozenge_A q] \vdash q'}{\Sigma'[q^A] \vdash q'}\ Inv\blacklozenge_A L}{\Sigma'[\Theta^A] \vdash q'}\ Cut$$

(xvi) The first premiss is an instance of $\Box_A R$. This now depends on the form of the second premiss.

(a) Id (This case cannot occur, since the principal term of Id is always an atom.)

(b) $\bot L$, non-principal.

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\ \Box_A R \qquad \dfrac{}{\Sigma[\Box_A q][\bot] \vdash q'}\ \bot L}{\Sigma[\Theta][\bot] \vdash q'}\ Cut$$

transforms to

$$\dfrac{}{\Sigma[\Theta][\bot] \vdash q'}\ \bot L$$

(c) $\top R$, principal.

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\ \Box_A R \qquad \dfrac{}{\Sigma[\Box_A q] \vdash \top}\ \top R}{\Sigma[\Theta] \vdash \top}\ Cut$$

transforms to

$$\dfrac{}{\Sigma[\Theta] \vdash \top}\ \top R$$

(d) $1L$, non-principal

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\ \Box_A R \qquad \dfrac{\Sigma[\Box_A q][\ ] \vdash q'}{\Sigma[\Box_A q][1] \vdash q'}\ 1l}{\Sigma[\Theta][1] \vdash q'}\ Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Box_A q][\ ] \vdash q'}{\Sigma[\Theta][\ ] \vdash q'}\ Cut}{\Sigma[\Theta][1] \vdash q'}\ 1L$$

(e) $\wedge L$, non-principal

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\ \Box_A R \qquad \dfrac{\Sigma[\Box_A m][q_i] \vdash q'}{\Sigma[\Box_A m][q_1 \wedge q_2] \vdash q'}\ \wedge L_i}{\Sigma[\Theta][q_1 \wedge q_2] \vdash q'}\ Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Box_A q][q_i] \vdash q'}{\Sigma[\Theta][q_1, q_2] \vdash q'}\; Cut}{\Sigma[\Theta][q_1 \wedge q_2] \vdash q'}\; \wedge L$$

(f) $\vee L$, non-principal

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\; \Box_A R \qquad \dfrac{\Sigma[\Box_A q][q_1] \vdash q' \quad \Sigma[\Box_A q][q_2] \vdash q'}{\Sigma[\Box_A q][q_1 \vee q_2] \vdash q'}\; \vee L}{\Sigma[\Theta][q_1 \vee q_2] \vdash q'}\; Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Box_A q][q_1] \vdash q'}{\Sigma[\Theta][q_1] \vdash q'}\; Cut \qquad \dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Box_A q][q_2] \vdash q'}{\Sigma[\Theta][q_2] \vdash q'}\; Cut}{\Sigma[\Theta][q_1 \vee q_2] \vdash q'}\; \vee L$$

(g) $\bullet L$, non-principal

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\; \Box_A R \qquad \dfrac{\Sigma[\Box_A q][q_1, q_2] \vdash q'}{\Sigma[\Box_A q][q_1 \bullet q_2] \vdash q'}\; \bullet L}{\Sigma[\Theta][q_1 \bullet q_2] \vdash q'}\; Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \quad \Sigma[\Box_A q][q_1, q_2] \vdash q'}{\Sigma[\Theta][q_1, q_2] \vdash q'}\; Cut}{\Sigma[\Theta][q_1 \bullet q_2] \vdash q'}\; \bullet L$$

(h) $\blacklozenge_B L$, non-principal

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\; \Box_A R \qquad \dfrac{\Sigma[\Box_A q][q''^B] \vdash q'}{\Sigma[\Box_A q][\blacklozenge_B q''] \vdash q'}\; \blacklozenge_B L}{\Sigma[\Theta][\blacklozenge_B q''] \vdash q'}\; Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \quad \Sigma[\Box_A q][q''^B] \vdash q'}{\Sigma[\Theta][q''^B] \vdash q'}\; Cut}{\Sigma[\Theta][\blacklozenge_B q''] \vdash q'}\; \blacklozenge_B L$$

(i) $\Box_B L$, non-principal

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\; \Box_A R \qquad \dfrac{\Sigma[\Box_A q][q''] \vdash q'}{\Sigma[\Box_A q][(\Box_B q'')^B] \vdash q'}\; \Box_B L}{\Sigma[\Theta][(\Box_B q'')^B] \vdash q'}\; Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \quad \Sigma[\Box_A q][q''] \vdash q'}{\Sigma[\Theta][q''] \vdash q'}\; Cut}{\Sigma[\Theta][(\Box_B q'')^B] \vdash q'}\; \Box_B L$$

(j) $\Box_A L$, principal

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\,\Box_A R \qquad \dfrac{\Sigma'[q] \vdash q'}{\Sigma'[(\Box_A q)^A] \vdash q'}\,\Box_A L}{\Sigma'[\Theta^A] \vdash q'}\,Cut$$

transforms to

$$\dfrac{\Theta^A \vdash q \quad \Sigma'[q] \vdash q'}{\Sigma'[\Theta^A] \vdash q'}\,Cut$$

(k) $\wedge R$, principal

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\,\Box_A R \qquad \dfrac{\Sigma[\Box_A q] \vdash q_1 \qquad \Sigma[\Box_A q] \vdash q_2}{\Sigma[\Box_A q] \vdash q_1 \wedge q_2}\,\wedge R}{\Sigma[\Theta] \vdash q_1 \wedge q_2}\,Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Box_A q] \vdash q_1}{\Sigma[\Theta] \vdash q_1}\,Cut \qquad \dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Box_A q] \vdash q_2}{\Sigma[\Theta] \vdash q_2}\,Cut}{\Sigma[\Theta] \vdash q_1 \wedge q_2}\,\wedge R$$

(l) $\vee R$ Similar.

(m) $\bullet R$, non-principal. The cut formula $\Box_A q$ can occur in the first part Θ' or second part Θ'' of the list Θ', Θ''. Without loss of generality assume it occurs in the first part, then

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\,\Box_A R \qquad \dfrac{\Sigma[\Box_A q] \vdash q_1 \qquad \Theta'' \vdash q_2}{\Sigma[\Box_A q], \Theta'' \vdash q_1 \bullet q_2}\,\bullet R}{\Sigma[\Theta], \Theta'' \vdash q_1 \bullet q_2}\,Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Box_A q] \vdash q_1}{\Sigma[\Theta] \vdash q_1}\,Cut \qquad \Theta'' \vdash q_2}{\Sigma[\Theta], \Theta'' \vdash q_1 \bullet q_2}\,\bullet R$$

(n) $\blacklozenge_B R$, principal.

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\,\Box_A R \qquad \dfrac{\Sigma[\Box_A q] \vdash q'}{\Sigma[\Box_A q]^B \vdash \blacklozenge_B q'}\,\blacklozenge_B R}{\Sigma[\Theta]^B \vdash \blacklozenge_B q'}\,Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \quad \Sigma[\Box_A q] \vdash q'}{\Sigma[\Theta] \vdash q'}\,Cut}{\Sigma[\Theta]^B \vdash \blacklozenge_B q'}\,\blacklozenge_B R$$

(o) $\Box_B R$, principal.

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\ \Box_A R \quad \dfrac{\Sigma[\Box_A q]^B \vdash q'}{\Sigma[\Box_A q] \vdash \Box_B q'}\ \Box_B R}{\Sigma[\Theta] \vdash \Box_B q'}\ Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \quad \Sigma[\Box_A q]^B \vdash q'}{\Sigma[\Theta]^B \vdash q'}\ Cut}{\Sigma[\Theta] \vdash \Box_B q'}\ \Box_B q'$$

(p) *Dist*, non-principal. The cut formula $\Box_A q$ can occur in the context-with-a-hole $\Sigma[\ ]$, or either in first part Θ' or the second part Θ'' of the list $(\Theta', \Theta'')^A$. If it occurs in $\Sigma[\ ]$, then

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\ \Box_A R \quad \dfrac{\Sigma[\Box_A q][\Theta'^B, \Theta''^B] \vdash q'}{\Sigma[\Box_A q][(\Theta', \Theta'')^B] \vdash q'}\ Dist}{\Sigma[\Theta][(\Theta', \Theta'')^B] \vdash q'}\ Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Box_A q][\Theta'^B, \Theta''^B] \vdash q'}{\Sigma[\Theta][\Theta'^B, \Theta''^B] \vdash q'}\ Cut}{\Sigma[\Theta][(\Theta', \Theta'')^B] \vdash q'}\ Dist$$

If it occurs in Θ' then

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\ \Box_A R \quad \dfrac{\Sigma[\Sigma'[\Box_A q]^B, \Theta''^B] \vdash q'}{\Sigma[(\Sigma'[\Box_A q], \Theta'')^B] \vdash q'}\ Dist}{\Sigma[(\Sigma'[\Theta], \Theta'')^B] \vdash q'}\ Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Sigma'[\Box_A q]^B, \Theta''^B] \vdash q'}{\Sigma[\Sigma'[\Theta]^B, \Theta''^B] \vdash q'}\ Cut}{\Sigma[(\Sigma'[\Theta], \Theta'')^B] \vdash q'}\ Dist$$

The case in which the cut formula occurs in Θ'' is similar.

(q) *Unit*, non-principal.

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\ \Box_A R \quad \dfrac{\Sigma[\Box_A q][\ ] \vdash A}{\Sigma[\Box_A q][\langle\ \rangle^A] \vdash A}\ Unit}{\Sigma[\Theta][\langle\ \rangle^A] \vdash q'}\ Cut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \qquad \Sigma[\Box_A q][\ ] \vdash q'}{\Sigma[\Theta][\ ] \vdash q'}\ Cut}{\Sigma[\Theta][\langle\ \rangle^A] \vdash q'}\ Unit$$

$\Box$

3 Sequent Calculus for Propositions

Given sets A of agents A and B of basic actions σ, we have as above an action logic with a set Q of terms q. Now let At be a set of (propositional) *atoms* p; the set M of *formulae* m of our propositional logic is generated by the following grammar:

$$m ::= \bot \mid \top \mid p \mid m \wedge m \mid m \vee m \mid \Box_A m \mid \blacklozenge_A m \mid m \cdot q \mid [q]m$$

Here the last two binary connectives are mixed action-proposition connectives: the operator $[q]\_$ is the *dynamic modality* operator and $\_ \cdot q$ is (as we shall see) its left adjoint, called *update*, just as $\blacklozenge_A$ is the left adjoint of $\Box_A$. We refer to this logic as *algebraic dynamic epistemic logic* (*AlgDEL*).

As in the action logic, we have *propositional items* I and *propositional contexts* Γ, generated by the following grammar:

$$I ::= m \mid \Gamma^A \mid \Gamma^\Theta \qquad\qquad \Gamma ::= I \ \textit{multiset}$$

where Γ^A will be interpreted as $\blacklozenge_A(\bigwedge\Gamma)$, for $\bigwedge\Gamma$ the conjunction of the interpretations of elements in Γ, and Γ^Θ as $(\bigwedge\Gamma)\cdot\bigodot\Theta$, for $\bigodot\Theta$ the composition of the interpretations of elements in Θ.

Note that, in contrast to the syntax for action contexts, the *propositional contexts* are (finite) *multi-sets* of items, making the role of the *Contraction* rule explicit. The union of two multi-sets is indicated by a comma, as in Γ, Γ' or (treating an item I as a one element multiset) as in Γ, I. A *propositional item* can be either a formula or an agent-annotated contexts, as in [8,9,10]; but it can also be a propositional context Γ annotated by an action context Θ.

To express the rules correctly, we need, as in Section 2, some notion of propositional context (or item) with a hole. There are now two kinds of hole, one for propositions and one for actions, both represented by [ ]; we use the notations Δ for a *propositional context-with-a-p-hole*, J for a *propositional item-with-a-p-hole*, Λ for a *propositional context-with-an-a-hole* and K for a *propositional item-with-an a-hole*, defined, using mutual recursion, as follows:

$$\Delta ::= \Gamma, J \qquad J ::= [\,] \mid \Delta^A \mid \Delta^\Theta \qquad \Lambda ::= \Gamma, K \qquad K ::= \Gamma^\Sigma \mid \Lambda^\Theta$$

in which we recall from Section 2 that Σ indicates an action context-with-an-a-hole.

We can now define various applications of something with an appropriate hole to a p-context Γ or an a-context Θ, constructing p-contexts:

$$(\Gamma', J)[\Gamma] = \Gamma', J[\Gamma] \qquad ([\,])[\Gamma] = \Gamma \qquad (\Delta^A)[\Gamma] = \Delta[\Gamma]^A$$

$$(\Gamma', K)[\Theta] = \Gamma', K[\Theta] \qquad (\Gamma'^\Sigma)[\Theta] = \Gamma'^{\Sigma[\Theta]} \qquad (\Lambda^{\Theta'})[\Theta] = \Lambda[\Theta]^{\Theta'}$$

Given p-contexts-with-a-p-hole Δ', Δ, and a p-item-with-a-p-hole J, the *combinations* $\Delta' \bullet \Delta$ and $J \bullet \Delta$ are defined as follows by mutual recursion on Δ' and J, giving in each case a p-context-with-a-p-hole:

$$(\Gamma, J)\bullet\Delta = \Gamma, (J\bullet\Delta) \quad ([\,])\bullet\Delta = \Delta \quad (\Delta''^A)\bullet\Delta = (\Delta''\bullet\Delta)^A \quad (\Delta''^\Theta)\bullet\Delta = (\Delta''\bullet\Delta)^\Theta$$

and, likewise, given a p-context-with-an-a-hole Λ, a p-item-with-an-a-hole K, and an a-context-with-an-a-hole Σ, the *combinations* $\Lambda \bullet \Sigma$ and $K \bullet \Sigma$ are defined by mutual recursion on Λ and K, giving in each case a p-context-with-an-a-hole:

$$(\Gamma, K) \bullet \Sigma = \Gamma, (K \bullet \Sigma) \quad (\Gamma^{\Sigma'}) \bullet \Sigma = \Gamma^{\Sigma' \bullet \Sigma} \quad (\Lambda^{\Theta}) \bullet \Sigma = (\Lambda \bullet \Sigma)^{\Theta}$$

Lemma 3.1 *Given propositional contexts-with-a-p-hole Δ', Δ, a propositional item-with-a-p-hole J and a propositional context Γ, the following hold:*

$$(\Delta' \bullet \Delta)[\Gamma] = \Delta'[\Delta[\Gamma]] \qquad (J \bullet \Delta)[\Gamma] = J[\Delta[\Gamma]]$$

Lemma 3.2 *Given a propositional context-with-an-a-hole Λ, an action context-with-a-hole Σ, a propositional item-with-an-a-hole K and an action context Θ, the following hold:*

$$(\Lambda \bullet \Sigma)[\Theta] = \Lambda[\Sigma[\Theta]] \qquad (K \bullet \Sigma)[\Theta] = K[\Sigma[\Theta]]$$

We have the following initial sequents (in which p is restricted to being an atom):

$$\frac{}{\Gamma, p \vdash p}\ Id \qquad \frac{}{\Delta[\bot] \vdash m}\ \bot L \qquad \frac{}{\Gamma \vdash \top}\ \top R$$

The rules for the lattice operations and the modal operators are:

$$\frac{\Delta[m_1, m_2] \vdash m}{\Delta[m_1 \wedge m_2] \vdash m}\ \wedge L \qquad \frac{\Gamma \vdash m_1 \quad \Gamma \vdash m_2}{\Gamma \vdash m_1 \wedge m_2}\ \wedge R$$

$$\frac{\Delta[m_1] \vdash m \quad \Delta[m_2] \vdash m}{\Delta[m_1 \vee m_2] \vdash m}\ \vee L \qquad \frac{\Gamma \vdash m_1}{\Gamma \vdash m_1 \vee m_2}\ \vee R1 \qquad \frac{\Gamma \vdash m_2}{\Gamma \vdash m_1 \vee m_2}\ \vee R2$$

$$\frac{\Delta[m^A] \vdash m'}{\Delta[\blacklozenge_A(m)] \vdash m'}\ \blacklozenge_A L \qquad \frac{\Gamma \vdash m}{\Gamma', \Gamma^A \vdash \blacklozenge_A(m)}\ \blacklozenge_A R$$

$$\frac{\Delta[(\Box_A m, \Gamma)^A, m] \vdash m'}{\Delta[(\Box_A m, \Gamma)^A] \vdash m'}\ \Box_A L \qquad \frac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R$$

The rules for the dynamic operations are:

$$\frac{\Delta[m^q] \vdash m'}{\Delta[m \cdot q] \vdash m'}\ \cdot L \qquad\qquad \frac{\Gamma \vdash m' \qquad \Theta \vdash q'}{\Gamma^{\Theta} \vdash m' \cdot q'}\ \cdot R$$

$$\frac{\Delta[([q]m, \Gamma)^q, m] \vdash m'}{\Delta[([q]m, \Gamma)^q] \vdash m'}\ DyL \qquad\qquad \frac{\Gamma^q \vdash m}{\Gamma', \Gamma \vdash [q]m}\ DyR$$

$$\frac{\Delta[((\Gamma^{\Theta})^{As})^{\Theta'^{As}}] \vdash m}{\Delta[(\Gamma^{\Theta,\Theta'})^{As}] \vdash m}\ DyDist \qquad\qquad \frac{\Delta[\Gamma^{\Theta,\Theta'}] \vdash m}{\Delta[(\Gamma^{\Theta})^{\Theta'}] \vdash m}\ ReArr$$

As in the action logic, the two indicated occurrences of p in the *Id* rule are *principal* and each right rule has its conclusion's succedent as its *principal formula*. But in addition, $\blacklozenge_A R$ (similarly DyR) rule has Γ^A as a *principal item* and Γ' (which is there to ensure admissibility of *Weakening*) as its *parameter*. Each left rule has a *principal item*; these are as usual, except that the $\Box_A L$ (similarly DyL) rule has the formula $\Box_A m$ *principal* as well as the principal item $(\Box_A m, \Gamma)^A$. Also, note that the $\Box_A L$ (similarly DyL) rule duplicates the principal item in the conclusion into the premiss (which is to make *Contraction* admissible); in examples, we may omit this duplicated item for simplicity. The As in the $DyDist$ rule denotes a list of agents. The parentheses are to clarify the scope of the annotations and will be dropped when there is no ambiguity.

We also include all the four initial sequents and all the fifteen rules of the action logic, as well as the variants of the L rules (including $\bot L$, *Dist* and *Unit*) of the action logic obtained by replacing any Σ by Λ and the succedent action q by a formula m. Thus, for example, $\blacklozenge_A L$ is included in the form

$$\frac{\Sigma[q^A] \vdash m}{\Sigma[\blacklozenge_A q] \vdash m}\ \blacklozenge_A L$$

and we leave it to the context to disambiguate whether such a rule is from the logic for propositions or that for actions.

As an example of a derivation we show that a sequence of $\blacklozenge_A$s preserves an information update by a composition of actions as follows (in which we use a superfix BA to indicate first an annotation by B and then by A):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id \quad \dfrac{}{q \vdash q}\,Id}{m^q \vdash m \cdot q}\,\cdot R}{(m^q)^B \vdash \blacklozenge_B(m \cdot q)}\,\blacklozenge_B R}{(m^q)^{BA} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q)}\,\blacklozenge_A R \quad \dfrac{\dfrac{\dfrac{}{q' \vdash q'}\,Id}{q'^B \vdash \blacklozenge_B q'}\,\blacklozenge_B R}{q'^{BA} \vdash \blacklozenge_A\blacklozenge_B q'}\,\blacklozenge_A R}{((m^q)^{BA})^{q'^{BA}} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,\cdot R}{(m^{q,q'})^{BA} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,DyDist}{(m^{q \bullet q'})^{BA} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,\bullet L}{(m \cdot (q \bullet q'))^{BA} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,\cdot L}{\blacklozenge_B(m \cdot (q \bullet q'))^A \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,\blacklozenge_B L}{\blacklozenge_A\blacklozenge_B(m \cdot (q \bullet q')) \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,\blacklozenge_A L$$

But we can also have the following (also sound) form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id}{m^B \vdash \blacklozenge_B m}\,\blacklozenge_B R}{m^{BA} \vdash \blacklozenge_A \blacklozenge_B m}\,\blacklozenge_A R \quad \dfrac{\dfrac{\dfrac{}{q \bullet q' \vdash q \bullet q'}\,Id}{(q \bullet q')^B \vdash \blacklozenge_B (q \bullet q')}\,\blacklozenge_B R}{(q \bullet q')^{BA} \vdash \blacklozenge_A \blacklozenge_B (q \bullet q')}\,\blacklozenge_A R}{((m^{BA})^{(q \bullet q')^{BA}} \vdash \blacklozenge_A \blacklozenge_B m \cdot \blacklozenge_A \blacklozenge_B (q \bullet q')}\,\cdot R}{(m^{(q \bullet q')})^{BA} \vdash \blacklozenge_A \blacklozenge_B m \cdot \blacklozenge_A \blacklozenge_B (q \bullet q')}\,DyDist}{(m \cdot (q \bullet q'))^{BA} \vdash \blacklozenge_A \blacklozenge_B m \cdot \blacklozenge_A \blacklozenge_B (q \bullet q')}\,\cdot L}{\blacklozenge_B (m \cdot (q \bullet q'))^{A} \vdash \blacklozenge_A \blacklozenge_B m \cdot \blacklozenge_A \blacklozenge_B (q \bullet q')}\,\blacklozenge_B L}{\blacklozenge_A \blacklozenge_B (m \cdot (q \bullet q')) \vdash \blacklozenge_A \blacklozenge_B m \cdot \blacklozenge_A \blacklozenge_B (q \bullet q')}\,\blacklozenge_A L$$

Here the rule $DyDist$ is applied to the list $\langle q \bullet q' \rangle$, treated as the concatenation of the empty list $\langle\ \rangle$ and the list $\langle q \bullet q' \rangle$.

Lemma 3.3 *For every formula m and every context Γ, the sequent $\Gamma, m \vdash m$ is derivable.*

Lemma 3.4 *The following* Weakening *and* Contraction *rules are admissible:*

$$\dfrac{\Delta[\Gamma] \vdash m}{\Delta[\Gamma, \Gamma'] \vdash m}\,Wk \qquad \dfrac{\Delta[\top] \vdash m}{\Delta[\Gamma] \vdash m}\,\top L^- \qquad \dfrac{\Delta[\Gamma, \Gamma] \vdash m}{\Delta[\Gamma] \vdash m}\,Contr$$

Lemma 3.5 *The $\wedge L$, $\vee L$, $\wedge R$, $\cdot L$, DyR, $\blacklozenge_A L$, and $\Box_A R$ rules are invertible.*

Theorem 3.6 *The following* Cut *rules are admissible:*

$$\dfrac{\Gamma \vdash m \quad \Delta[m] \vdash m'}{\Delta[\Gamma] \vdash m'}\,PrCut \qquad \dfrac{\Theta \vdash q \quad \Lambda[q] \vdash m}{\Lambda[\Theta] \vdash m}\,DyCut$$

Proof. Strong induction on the rank of the cut, where the *rank* is given by the pair: (size of cut formula m, sum of heights of derivations of premisses). This will need some changes to replace Deltas by Lambdas

We classify the cases into two major groups: the first one for the dynamic cut $DyCut$ and the second one for the propositional cut $PrCut$.

(I) Reductions for admissibility of $DyCut$.

(i) Cuts where the second premiss is an instance of the rules of $AlgDEL$. This case breaks down to two groups, when the cut formula is principal and when it is not.

(a) The cuts in the principal cases can only be done with dynamic rules in the second premiss. All of these propagate up by cutting with the assumption of the rule.

- $\cdot R$

$$\dfrac{\Theta \vdash q \quad \dfrac{\Gamma \vdash m' \quad q \vdash q'}{\Gamma^q \vdash m' \cdot q'}\,\cdot R}{\Gamma^\Theta \vdash m' \cdot q'}\,DyCut$$

which transforms to the following, using the admissible Cut of action logic:

$$\dfrac{\Gamma \vdash m' \quad \dfrac{\Theta \vdash q \quad q \vdash q'}{\Theta \vdash q'}\,Cut}{\Gamma^\Theta \vdash m' \cdot q'}\,\cdot R$$

- *DyL*

$$\dfrac{\Theta \vdash q \qquad \dfrac{\Lambda([q]m,\Gamma)^q, m] \vdash m'}{\Lambda[([q]m,\Gamma)^q] \vdash m'}\, DyL}{\Lambda[([\bullet\Theta]m,\Gamma)^{\Theta}] \vdash m'}\, DyCut$$

  Here we have to replace the instance of Θ in the dynamic modality with $\bullet\Theta$, which is the composition of all the items in Θ. This cut transforms to

$$\dfrac{\dfrac{\Theta \vdash q \qquad \Lambda[([q]m,\Gamma)^q, m] \vdash m'}{\Lambda[([\bullet\Theta]m,\Gamma)^{\Theta}, m] \vdash m'}\, DyCut}{\Lambda[([\bullet\Theta]m,\Gamma)^{\Theta}] \vdash m'}\, DyL$$

- *DyDist*, there are two cases e.g. the cut formula can be the first context Θ or the second one Θ'. Consider the first one

$$\dfrac{\Theta \vdash q \qquad \dfrac{\Lambda[(\Gamma^{q^{As}})^{\Theta'^{As}}] \vdash m}{\Lambda[(\Gamma^{q,\Theta'})^{As}] \vdash m}\, DyDist}{\Lambda[(\Gamma^{\Theta,\Theta'})^{As}] \vdash m}\, DyCut$$

  which transforms to

$$\dfrac{\dfrac{\Theta \vdash q \quad \Lambda[(\Gamma^{q^{As}})^{\Theta'^{As}}] \vdash m}{\Lambda[(\Gamma^{\Theta^{As}})^{\Theta'^{As}}] \vdash m}\, DyCut}{\Lambda[(\Gamma^{\Theta,\Theta'})^{As}] \vdash m}\, DyDist$$

  The second case is reduced identically.
- *ReArr* is similar to *DyDist*.

(b) The cuts in the non-principal cases propagate up by cutting with the assumption of the rule. These are all routine. Here is an example when the second premiss is an instance of a propositional rule

$$\dfrac{\Theta \vdash q \qquad \dfrac{\Lambda[q][(\Box_A m,\Gamma)^A, m] \vdash m'}{\Lambda[q][(\Box_A m,\Gamma)^A] \vdash m'}\, \Box_A R}{\Lambda[\Theta][(\Box_A m,\Gamma)^A] \vdash m'}\, DyCut$$

which transforms to

$$\dfrac{\dfrac{\Theta \vdash q \qquad \Lambda[q][(\Box_A m,\Gamma)^A, m] \vdash m'}{\Lambda[\Theta][(\Box_A m,\Gamma)^A, m] \vdash m'}\, DyCut}{\Lambda[\Theta][(\Box_A m,\Gamma)^A] \vdash m'}\, \Box_A R$$

Here is an example where the second premiss is an instance of a dynamic rule

$$\dfrac{\Theta \vdash q \quad \dfrac{\Lambda[q][m'^{q'}] \vdash m''}{\Lambda[q][m' \cdot q'] \vdash m''}\, \cdot L}{\Lambda[\Theta][m' \cdot q'] \vdash m''}\, DyCut$$

which transforms to

$$\dfrac{\dfrac{\Theta \vdash q \quad \Lambda[q][m'^{q'}] \vdash m''}{\Lambda[\Theta][m'^{q'}] \vdash m''}\ DyCut}{\Lambda[\Theta][m' \cdot q'] \vdash m''}\ \cdot L$$

(ii) Cuts where the first premiss is an instance of rules of action logic. These break down to four groups:

(a) Principal cuts where the first premiss is an instance of $1L$, *Unit* and *Dist*; in these cases the cut propagates up by cutting with the assumptions of these rules, as follows:

$1L$

$$\dfrac{\dfrac{\Sigma[\ ] \vdash q}{\Sigma[1] \vdash q}\ 1L \quad \Lambda[q] \vdash m}{\Lambda[\Sigma[1]] \vdash m}\ DyCut$$

transforms to

$$\dfrac{\dfrac{\Sigma[\ ] \vdash q \quad \Lambda[q] \vdash m}{\Lambda[\Sigma[\ ]] \vdash m}\ DyCut}{\Lambda[\Sigma[1]] \vdash m}\ 1L$$

Dist

$$\dfrac{\dfrac{\Sigma[\Theta^A, \Theta'^A] \vdash q}{\Sigma[(\Theta, \Theta')^A] \vdash q}\ Dist \quad \Lambda[q] \vdash m}{\Lambda[\Sigma[(\Theta, \Theta')^A]] \vdash m}\ DyCut$$

transforms to

$$\dfrac{\dfrac{\Sigma[\Theta^A, \Theta'^A] \vdash q \quad \Lambda[q] \vdash m}{\Lambda[\Sigma[\Theta^A, \Theta'^A]] \vdash m}\ DyCut}{\Lambda[\Sigma[(\Theta, \Theta')^A]] \vdash m}\ Dist$$

Unit

$$\dfrac{\dfrac{\Sigma[\langle\ \rangle] \vdash q}{\Sigma[\langle\ \rangle^A] \vdash q}\ Unit \quad \Lambda[q] \vdash m}{\Lambda[\Sigma[\langle\ \rangle^A]] \vdash m}\ DyCut$$

transforms to

$$\dfrac{\dfrac{\Sigma[\langle\ \rangle] \vdash q \quad \Delta[q] \vdash m}{\Lambda[\Sigma[\langle\ \rangle]] \vdash m}\ DyCut}{\Lambda[\Sigma[\langle\ \rangle^A]] \vdash m}\ Unit$$

(b) Principal cuts where the first premiss is an instance of a left rule; these propagate up by cutting with the assumption of the rules, as follows:

- $\vee L$, and $\wedge L$ are routine.
- $\bullet L$

$$\dfrac{\dfrac{\Sigma[q_1, q_2] \vdash q}{\Sigma[q_1 \bullet q_2] \vdash q}\ \bullet L \quad \Lambda[q] \vdash m}{\Lambda[\Sigma[q_1 \bullet q_2]] \vdash m}\ DyCut$$

transforms to

$$\dfrac{\dfrac{\Sigma[q_1, q_2] \vdash q \quad \Delta[q] \vdash m}{\Lambda[\Sigma[q_1, q_2]] \vdash m}\ DyCut}{\Lambda[\Sigma[q_1 \bullet q_2]] \vdash m}\ \bullet L$$

- $\blacklozenge_A L$

$$\dfrac{\dfrac{\Sigma[q^A] \vdash q'}{\Sigma[\blacklozenge_A q] \vdash q'}\ \blacklozenge_A L \qquad \Lambda[q'] \vdash m}{\Lambda[\Sigma[\blacklozenge_A q]] \vdash m}\ DyCut$$

transforms to

$$\dfrac{\dfrac{\Sigma[q^A] \vdash q' \quad \Lambda[q'] \vdash m}{\Lambda[\Sigma[q^A]] \vdash m}\ DyCut}{\Lambda[\Sigma[\blacklozenge_A q]] \vdash m}\ \blacklozenge_A L$$

- $\Box_A L$

$$\dfrac{\dfrac{\Sigma[q] \vdash q'}{\Sigma[(\Box_A q)^A] \vdash q}\ \Box_A L \qquad \Lambda[q'] \vdash m}{\Lambda[\Sigma[(\Box_A q)^A]] \vdash q'}\ DyCut$$

transforms to

$$\dfrac{\dfrac{\Sigma[q] \vdash q' \quad \Lambda[q'] \vdash m}{\Lambda[\Sigma[q] \vdash q'}\ DyCut}{\Lambda[\Sigma[(\Box_A q)^A]] \vdash q'}\ \Box_A L$$

(c) Non-principal cuts with right rules whose left rule is invertible, i.e. $\blacklozenge_A R, \bullet R, \vee R$; here the size of the cut formula decreases, as follows:

- $\vee R$ is routine.
- $\bullet R$

$$\dfrac{\dfrac{\Theta_1 \vdash q_1 \qquad \Theta_2 \vdash q_2}{\Theta_1 \bullet \Theta_2 \vdash q_1 \bullet q_2}\ \bullet R \qquad \Lambda[q_1 \bullet q_2] \vdash m}{\Lambda[\Theta_1 \bullet \Theta_2] \vdash m}\ DyCut$$

transforms to

$$\dfrac{\dfrac{\Theta_2 \vdash q_2 \qquad \dfrac{\Theta_1 \vdash q_1 \quad \dfrac{\Delta[q_1 \bullet q_2] \vdash m}{\Lambda[q_1, q_2] \vdash m}\ Inv \bullet L}{\Lambda[\Theta_1, q_2] \vdash m}\ DyCut}{\Lambda[\Theta_1, \Theta_2] \vdash m}\ DyCut}{\Lambda[\Theta_1 \bullet \Theta_2] \vdash m}\ \bullet L$$

- $\blacklozenge_A R$

$$\dfrac{\dfrac{\Theta \vdash q}{\Theta^A \vdash \blacklozenge_A q}\ \blacklozenge_A R \qquad \Lambda[\blacklozenge_A q] \vdash m}{\Lambda[\Theta^A] \vdash m}\ DyCut$$

transforms to

$$\dfrac{\Theta \vdash q \quad \dfrac{\Lambda[\blacklozenge_A q] \vdash m}{\Delta[q^A] \vdash m}\ Inv\blacklozenge_A}{\Lambda[\Theta^A] \vdash m}\ DyCut$$

(d) First premiss is an instance of $\Box_A R$ and $\wedge R$, whose left rules are not invertible; these need case analysis on the form of the second premiss. Cases for $\wedge R$ are routine, so we deal with cases for $\Box_A R$. These form two groups: when the cut formula is principal and when it is not. In both cases the cuts propagate up by cutting with the assumption of the rule. The principal cuts are more interesting, so we only we present these here.

- $\cdot R$

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\,\Box_A R \quad \dfrac{\Gamma \vdash m' \qquad \Box_A q \vdash q'}{\Gamma^{\Box_A q} \vdash m' \cdot q'}\,\cdot R}{\Gamma^{\Theta} \vdash m' \cdot q'}\,\mathit{DyCut}$$

transforms to

$$\dfrac{\Gamma \vdash m' \quad \dfrac{\Theta \vdash \Box_A q \qquad \Box_A q \vdash q'}{\Theta \vdash q'}\,\mathit{Cut}}{\Gamma^{\Theta} \vdash m' \cdot q'}\,\cdot R$$

- DyL

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\,\Box_A R \quad \dfrac{\Lambda[(([\Box_A q]m, \Gamma)^{\Box_A q}, m] \vdash m'}{\Lambda[(([\Box_A q]m, \Gamma)^{\Box_A q}] \vdash m'}\,\mathit{DyL}}{\Lambda[(([\bullet\Theta]m, \Gamma)^{\Theta}] \vdash m'}\,\mathit{DyCut}$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \quad \Lambda[(([\Box_A q]m, \Gamma)^{\Box_A q}, m] \vdash m'}{\Lambda[(([\bullet\Theta]m, \Gamma)^{\Theta}, m] \vdash m'}\,\mathit{DyCut}}{\Lambda[(([\bullet\Theta]m, \Gamma)^{\Theta}] \vdash m'}\,\mathit{DyL}$$

- DyDist, there are two cases: either Θ' contains the cut term, or Θ'' does (where Θ', Θ'' is the sequence of action items mentioned in the principal item of the rule). Consider the first case:

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\,\Box_A R \quad \dfrac{\Lambda[((\Gamma^{\Sigma[\Box_A q]^{As}})^{\Theta''})^{As}] \vdash m}{\Lambda[(\Gamma^{\Sigma[\Box_A q], \Theta''})^{As}] \vdash m}\,\mathit{DyDist}}{\Lambda[(\Gamma^{\Sigma[\Theta], \Theta''})^{As}] \vdash m}\,\mathit{DyCut}$$

which transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \quad \Lambda[(\Gamma^{\Box_A q^{As}})^{\Theta''^{As}}] \vdash m}{\Lambda[(\Gamma^{\Theta^{As}})^{\Theta''^{As}}] \vdash m}\,\mathit{DyCut}}{\Lambda[(\Gamma^{\Theta, \Theta''})^{As}] \vdash m}\,\mathit{DyDist}$$

The second case is similar.

- ReArr, there two cases: either Θ is the cut formula or Θ' is. Consider the first case:

$$\dfrac{\dfrac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\,\Box_A R \quad \dfrac{\Lambda[\Gamma^{\Box_A q, \Theta'}] \vdash m}{\Lambda[(\Gamma^{\Box_A q})^{\Theta'}] \vdash m}\,\mathit{ReArr}}{\Lambda[(\Gamma^{\Theta})^{\Theta'}] \vdash m}\,\mathit{DyCut}$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash \Box_A q \quad \Lambda[\Gamma^{\Box_A q, \Theta'}] \vdash m}{\Lambda[\Gamma^{\Theta, \Theta'}] \vdash m}\,\mathit{DyCut}}{\Lambda[(\Gamma^{\Theta})^{\Theta'}] \vdash m}\,\mathit{ReArr}$$

(II) Reductions for admissibility of $PrCut$. The $AlgDEL$ calculus is obtained by adding dynamic rules to the proposition-only calculus of [9]; so the cases here are those of [9] together with the following:

(i) Principal cuts where the first premiss is an instance of dynamic rules.

(a) $\cdot R$

$$\dfrac{\dfrac{\Gamma \vdash m' \qquad \Theta \vdash q'}{\Gamma^{\Theta} \vdash m' \cdot q'}\ \cdot R \qquad \Sigma[m' \cdot q'] \vdash m''}{\Sigma[\Gamma^{\Theta}] \vdash m''}\ PrCut$$

transforms to

$$\dfrac{\Theta \vdash q' \qquad \dfrac{\Gamma \vdash m' \qquad \dfrac{\Sigma[m' \cdot q'] \vdash m''}{\Sigma[m'^{q}] \vdash m''}\ Inv \cdot L}{\Sigma[\Gamma^{q}] \vdash m''}\ PrCut}{\Sigma[\Gamma^{\Theta}] \vdash m''}\ DyCut$$

(b) $\cdot L$

$$\dfrac{\dfrac{\Delta[m^{q}] \vdash m'}{\Delta[m \cdot q] \vdash m'}\ \cdot L \qquad \Delta'[m'] \vdash m''}{\Delta'[\Delta[m \cdot q]] \vdash m''}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\Delta[m^{q}] \vdash m' \qquad \Delta'[m'] \vdash m''}{\Delta'[\Delta[m^{q}]] \vdash m''}\ PrCut}{\Delta'[\Delta[m \cdot q]] \vdash m''}\ \cdot L$$

(c) $DyDist$

$$\dfrac{\dfrac{\Delta[(\Gamma^{\Theta^{As}})^{\Theta'^{As}}] \vdash m}{\Delta[(\Gamma^{\Theta,\Theta'})^{As}] \vdash m}\ DyDist \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[\Gamma^{(\Theta,\Theta')^{As}}]] \vdash m'}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\Delta[(\Gamma^{\Theta^{As}})^{\Theta'^{As}}] \vdash m \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[(\Gamma^{\Theta^{As}})^{\Theta'^{As}}]] \vdash m}\ PrCut}{\Delta'[\Delta[(\Gamma^{\Theta,\Theta'})^{As}]] \vdash m'}\ DyDist$$

(d) $ReArr$

$$\dfrac{\dfrac{\Delta[\Gamma^{\Theta,\Theta'}] \vdash m}{\Delta[\Gamma^{\Theta^{\Theta'}}] \vdash m}\ ReArr \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[(\Gamma^{\Theta})^{\Theta'}]] \vdash m'}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\Delta[\Gamma^{\Theta,\Theta'}] \vdash m \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[\Gamma^{\Theta,\Theta'}]] \vdash m'}\ PrCut}{\Delta'[\Delta[\Gamma^{\Theta^{\Theta'}}]] \vdash m'}\ ReArr$$

(ii) The first premiss is an instance of $\Box_A R$ rules and second premiss an instance of one of the dynamic rules.

(a) $\cdot R$

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \dfrac{\Delta[\Box_A m] \vdash m' \qquad \Theta \vdash q}{\Delta[\Box_A m]^\Theta \vdash m' \cdot q'}\ \cdot R}{\Delta[\Gamma]^\Theta \vdash m' \cdot q'}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m] \vdash m'}{\Delta[\Gamma] \vdash m'}\ PrCut \qquad \Theta \vdash q'}{\Delta[\Gamma]^\Theta \vdash m' \cdot q'}\ \cdot R$$

(b) DyR

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \dfrac{\Delta[\Box_A m]^{q'} \vdash m'}{\Delta[\Box_A m] \vdash [q']m'}\ DyR}{\Delta[\Gamma] \vdash [q']m'}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m]^{q'} \vdash m'}{\Delta[\Gamma]^{q'} \vdash m'}\ PrCut}{\Delta[\Gamma] \vdash [q']m'}\ DyR$$

(c) DyL

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][(([q']m', \Gamma')^{q'}, m'] \vdash m''}{\Delta[\Gamma][(([q']m', \Gamma')^{q'}, m'] \vdash m''}\ PrCut}{\Delta[\Gamma][(([q']m', \Gamma')^{q'}] \vdash m''}\ DyL$$

(d) $DyDist$, there are two cases, when the cut formula $\Box_A m$ is in Δ and when it is in Γ', the reduction for the first case is as follows, the reduction for the second case is similar.

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \dfrac{\Delta[\Box_A m][(\Gamma'^{\Theta^{As}})^{\Theta'^{As}}] \vdash m'}{\Delta[\Box_A m][(\Gamma'^{\Theta,\Theta'})^{As}] \vdash m'}\ DyDist}{\Delta[\Gamma][(\Gamma'^{\Theta,\Theta'})^{As}] \vdash m'}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][(\Gamma'^{\Theta^{As}})^{\Theta'^{As}} \vdash m'}{\Delta[\Gamma][(\Gamma'^{\Theta^{As}})^{\Theta'^{As}} \vdash m'}\ PrCut}{\Delta[\Gamma][(\Gamma'^{\Theta,\Theta'})^{As}] \vdash m'}\ DyDist$$

(e) $ReArr$, there are two cases, when the cut formula $\Box_A m$ is in Δ and when it is in Γ', the reduction for the first case is as follows, the reduction for the second case is similar.

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \dfrac{\Delta[\Box_A m][\Gamma'^{\Theta,\Theta'}] \vdash m}{\Delta[\Box_A m][\Gamma'^{\Theta^{\Theta'}}] \vdash m}\ ReArr}{\Delta[\Gamma][\Gamma'^{\Theta^{\Theta'}}] \vdash m'}\ PrCut$$

transforms to

$$\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][\Gamma'^{\Theta,\Theta'}] \vdash m'}{\dfrac{\Delta[\Gamma][\Gamma'^{\Theta,\Theta'}] \vdash m'}{\Delta[\Gamma][\Gamma'^{\Theta^{\Theta'}}] \vdash m'}\; ReArr}\; PrCut$$

(iii) The first premiss is an instance of DyR rules and second premiss is an instance of all the rule. The cases here are almost identical to the cases in the previous item when $\Box_A R$ was the first premiss. We present two of them here.

(a) The second premiss is an instance of DyL

$$\dfrac{\dfrac{\Gamma^q \vdash m}{\Gamma \vdash [q]m}\; DyR \qquad \dfrac{\Delta'[(([q]m, \Gamma')^q, m] \vdash m'}{\Delta'[(([q]m, \Gamma')^q] \vdash m'}\; DyL}{\Delta'[(\Gamma, \Gamma')^q] \vdash m'}\; PrCut$$

transforms to

$$\dfrac{\dfrac{\Gamma^q \vdash m \qquad \dfrac{\Gamma \vdash [q]m \qquad \Delta'[(([q]m, \Gamma')^q, m] \vdash m'}{\Delta'[(\Gamma, \Gamma')^q, m] \vdash m'}\; PrCut}{\dfrac{\Delta'[(\Gamma, \Gamma')^q, \Gamma^q] \vdash m'}{\Delta'[(\Gamma, \Gamma')^q, (\Gamma, \Gamma')^q] \vdash m'}\; Wk}\; PrCut}{\Delta'[(\Gamma, \Gamma')^q] \vdash m}\; Contr$$

(b) The second premiss is an instance of $DyDist$. There are two cases, we consider the one in which the cut formula $[q]m$ is in Δ.

$$\dfrac{\dfrac{\Gamma^q \vdash m}{\Gamma \vdash [q]m}\; DyR \qquad \dfrac{\Delta[[q]m][(\Gamma'^{\Theta^{As}})^{\Theta'^{As}}] \vdash m'}{\Delta[[q]m][(\Gamma'^{\Theta,\Theta'})^{As}] \vdash m'}\; DyDist}{\Delta[\Gamma][(\Gamma'^{\Theta,\Theta'})^{As}] \vdash m'}\; PrCut$$

transforms to

$$\dfrac{\Gamma \vdash [q]m \qquad \Delta[[q]m][(\Gamma'^{\Theta^{As}})^{\Theta'^{As}}] \vdash m'}{\dfrac{\Delta[\Gamma][(\Gamma'^{\Theta^{As}})^{\Theta'^{As}}] \vdash m'}{\Delta[\Gamma][(\Gamma'^{\Theta,\Theta'})^{As}] \vdash m'}\; DyDist}\; PrCut$$

We finish by giving two of the reductions for the proposition-only case from [9], to illustrate the argument. Note the use of $\blacklozenge_A Inv$ in the first. The second is one of many cases where the first premiss is an instance of $\Box_A R$.

(i) The first premiss is an instance of $\blacklozenge_A R$.

$$\dfrac{\dfrac{\Gamma \vdash m}{\Gamma', \Gamma^A \vdash \blacklozenge_A(m)}\; \blacklozenge_A R \qquad \Delta'[\blacklozenge_A(m)] \vdash m'}{\Delta'[\Gamma', \Gamma^A] \vdash m'}\; Cut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash m \qquad \dfrac{\Delta'[\blacklozenge_A(m)] \vdash m'}{\Delta'[m^A] \vdash m'}\,\blacklozenge_A Inv}{\Delta'[\Gamma^A] \vdash m'}\,Cut}{\Delta'[\Gamma', \Gamma^A] \vdash m'}\,Wk$$

(ii) The first premiss is an instance of $\Box_A R$ and the second an instance of $\Box_A L$, principal

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta'[(\Box_A m, \Gamma')^A, m] \vdash m'}{\Delta'[(\Box_A m, \Gamma')^A] \vdash m'}\,\Box_A L}{\Delta'[(\Gamma, \Gamma')^A] \vdash m'}\,Cut$$

transforms to

$$\dfrac{\dfrac{\dfrac{\Gamma^A \vdash m \qquad \dfrac{\Gamma \vdash \Box_A m \qquad \Delta'[(\Box_A m, \Gamma')^A, m] \vdash m'}{\Delta'[(\Gamma, \Gamma')^A, m] \vdash m'}\,Cut}{\Delta'[(\Gamma, \Gamma')^A, \Gamma^A] \vdash m'}\,Cut}{\Delta'[(\Gamma, \Gamma')^A, (\Gamma, \Gamma')^A] \vdash m'}\,Wk}{\Delta'[(\Gamma, \Gamma')^A] \vdash m}\,Contr$$

□

4 Algebraic Semantics

4.1 Actions

Definition 4.1 Let A be a set, with elements called *agents*, as before. A *lattice monoid with adjoint modalities* `LMAM` over A is both a bounded lattice $(Q, \vee, \wedge, \top, \bot)$ and a unital monoid $(Q, 1, \bullet, \leq)$ where $\bullet$ preserves joins, with two $\mathcal{A}$-indexed families $\{\blacklozenge_A\}_{A \in \mathsf{A}} \colon Q \to Q$ and $\{\Box_A\}_{A \in \mathsf{A}} \colon Q \to Q$ of order-preserving maps, each $\blacklozenge_A$ being left adjoint to $\Box_A$. Thus, apart from the lattice axioms, the following hold, for all $q, q', q'' \in Q$:

$$q \bullet (q' \vee q'') = (q \bullet q') \vee (q \bullet q'') \quad \text{and} \quad (q' \vee q'') \bullet q = (q' \bullet q) \vee (q'' \bullet q) \tag{1}$$

$$q \bullet 1 = q \quad \text{and} \quad 1 \bullet q = q \tag{2}$$

$$q \leq q' \text{ implies } \blacklozenge_A q \leq \blacklozenge_A q' \tag{3}$$

$$q \leq q' \text{ implies } \Box_A q \leq \Box_A q' \tag{4}$$

$$\blacklozenge_A q \leq q' \quad \text{iff} \quad q \leq \Box_A q' \tag{5}$$

Proposition 4.2 *In any* `LMAM` *the following hold, for all* $q, q' \in Q$*:*

$$\blacklozenge_A(q \vee q') = \blacklozenge_A q \vee \blacklozenge_A q' \tag{6}$$
$$\Box_A(q \wedge q') = \Box_A q \wedge \Box_A q' \tag{7}$$
$$\blacklozenge_A(q \wedge q') \leq \blacklozenge_A q \wedge \blacklozenge_A q' \tag{8}$$
$$\Box_A q \vee \Box_A q' \leq \Box_A(q \vee q') \tag{9}$$
$$\blacklozenge_A \bot = \bot \qquad \Box_A \top = \top \tag{10}$$
$$q \bullet (q' \wedge q'') \leq (q \bullet q') \wedge (q \bullet q'') \tag{11}$$
$$(q' \wedge q'') \bullet q \leq (q' \bullet q) \wedge (q'' \bullet q) \tag{12}$$
$$\blacklozenge_A \Box_A q \leq q \tag{13}$$
$$q \leq \Box_A \blacklozenge_A q \tag{14}$$

Definition 4.3 An LMAM Q is *multiplicative* whenever $\blacklozenge_A$ satisfies the following, for all $q, q' \in Q$:

$$\blacklozenge_A(q \bullet q') \leq \blacklozenge_A q \bullet \blacklozenge_A q' \tag{15}$$
$$\blacklozenge_A 1 \leq 1 \tag{16}$$

Proposition 4.4 *In any multiplicative* LMAM Q *the following hold, for all* $q, q' \in Q$*:*

$$\Box_A q \bullet \Box_A q' \leq \Box_A(q \bullet q') \tag{17}$$
$$1 \leq \Box_A 1 \tag{18}$$

Let Q be a multiplicative LMAM over A. An *interpretation* of the action logic (over A, and with set B of basic actions) *in* Q is a map $[\![-]\!] \colon \mathsf{B} \to Q$. The meaning of action terms is obtained by induction on the structure of the terms:

$$[\![q_1 \vee q_2]\!] = [\![q_1]\!] \vee [\![q_2]\!], \quad [\![q_1 \wedge q_2]\!] = [\![q_1]\!] \wedge [\![q_2]\!], \quad [\![q_1 \bullet q_2]\!] = [\![q_1]\!] \bullet [\![q_2]\!],$$
$$[\![\blacklozenge_A q]\!] = \blacklozenge_A [\![q]\!], \quad [\![\Box_A q]\!] = \Box_A [\![q]\!],$$
$$[\![\top]\!] = \top, \quad [\![\bot]\!] = \bot, \quad [\![1]\!] = 1\,.$$

The meanings of items and of contexts are obtained by mutual induction on their structure:

$$[\![q]\!] = \text{as above}$$
$$[\![\Theta^A]\!] = \blacklozenge_A [\![\Theta]\!]$$
$$[\![I_1, \cdots, I_n]\!] = [\![I_1]\!] \bullet \cdots \bullet [\![I_n]\!]$$
$$[\![\langle\ \rangle]\!] = 1$$

Note that, since $\bullet$ is associative (but not necessarily commutative), the meaning of a context Θ depends on its presentation as a list of items in a particular order.

A sequent $\Theta \vdash q$ is *true* in an interpretation $[\![-]\!]$ in Q iff $[\![\Theta]\!] \leq [\![q]\!]$; it is *true* in Q iff true in all interpretations in Q, and it is *valid* iff true in every multiplicative LMAM.

Lemma 4.5 *Let* Θ, Θ' *be contexts with* $[\![\Theta]\!] \leq [\![\Theta']\!]$ *and* Σ *a context-with-a-hole. Then*

$$[\![\Sigma[\Theta]]\!] \leq [\![\Sigma[\Theta']]\!].$$

Proof. Routine induction on the structure of Σ (using also a similar result for items-with-a-hole). $\Box$

Theorem 4.6 (Soundness) *Any derivable sequent is valid, i.e.* $\Theta \vdash q$ *implies* $[\![\Theta]\!] \leq [\![q]\!]$ *is* true *in any interpretation* $[\![-]\!]$ *of* B *in any multiplicative* `LMAM` *(over* A*).*

Proof. We show that the initial sequents of the sequent calculus are valid and that the rules are truth-preserving.

- Axioms. These are routine.
- The right rules.
  - $\wedge R$, $\vee R$ and $\bullet R$ are routine.
  - $\blacklozenge_A R$. We have to show

$$[\![\Theta]\!] \leq [\![q]\!] \quad \text{implies} \quad [\![\Theta^A]\!] \leq [\![\blacklozenge_A q]\!]$$

Assume $[\![\Theta]\!] \leq [\![q]\!]$, by monotonicity of $\blacklozenge_A$ it follows that $\blacklozenge_A[\![\Theta]\!] \leq \blacklozenge_A[\![q]\!]$, by definition of $[\![-]\!]$ this is equivalent to $[\![\Theta^A]\!] \leq [\![\blacklozenge_A q]\!]$.

  - $\Box_A R$. We have to show

$$[\![\Theta^A]\!] \leq [\![q]\!] \quad \text{implies} \quad [\![\Theta]\!] \leq [\![\Box_A q]\!]$$

Assume $[\![\Theta^A]\!] \leq [\![q]\!]$, by definition of $[\![-]\!]$ this is equivalent to $\blacklozenge_A[\![\Theta]\!] \leq [\![q]\!]$, by property (5) of definition 4.1 this is equivalent to $[\![\Theta]\!] \leq \Box_A[\![q]\!]$, equivalent to $[\![\Theta]\!] \leq [\![\Box_A q]\!]$ by definition of $[\![-]\!]$.

- The left rules. These are done by induction on the structure of Σ
  - $\wedge L$, $\vee L$ and $\bullet L$ are routine.
  - $\blacklozenge_A L$. It is enough to show

$$[\![\Sigma[\blacklozenge_A q]]\!] \leq [\![\Sigma[q^A]]\!]$$

By definition of $[\![-]\!]$, we have $[\![\blacklozenge_A q]\!] = [\![q^A]\!]$, by lemma 4.5 we obtain $[\![\Sigma[\blacklozenge_A q]]\!] \leq [\![\Sigma[q^A]]\!]$.

  - $\Box_A L$. It is enough to show

$$[\![\Sigma[(\Box_A q)^A]]\!] \leq [\![\Sigma[q]]\!]$$

By definition of $[\![-]\!]$, we have $[\![(\Box_A q)^A]\!] = [\![\blacklozenge_A \Box_A q]\!]$, from this and (13) in proposition 4.2 it follows that $[\![(\Box_A q)^A]\!] \leq [\![q]\!]$, hence by lemma 4.5 we obtain $[\![\Sigma[(\Box_A q)^A]]\!] \leq [\![\Sigma[q]]\!]$.

  - *Dist*. It is enough to show

$$[\![\Sigma[(\Theta, \Theta')^A]]\!] \leq [\![\Sigma[\Theta^A, \Theta'^A]]\!]$$

By definition of $[\![-]\!]$ and (15) in definition 4.3 we have

$$[\![(\Theta, \Theta')^A]\!] = \blacklozenge_A[\![\Theta, \Theta']\!] = \blacklozenge_A([\![\Theta]\!] \bullet [\![\Theta']\!]) \quad \leq \quad \blacklozenge_A[\![\Theta]\!] \bullet \blacklozenge_A[\![\Theta']\!]$$

By definition of $[\![-]\!]$, for the right hand side we have

$$\blacklozenge_A[\![\Theta]\!] \bullet \blacklozenge_A[\![\Theta']\!] = [\![\Theta^A]\!] \bullet [\![\Theta'^A]\!] = [\![\Theta^A, \Theta'^A]\!]$$

Hence, we obtain $[\![(\Theta, \Theta')^A]\!] \leq [\![\Theta^A, \Theta'^A]\!]$. From this by lemma 4.5 it follows that $[\![\Sigma[(\Theta, \Theta')^A]]\!] \leq [\![\Sigma[\Theta^A, \Theta'^A]]\!]$.

· *Unit*. It is enough to show

$$[\![\Sigma[\langle\ \rangle^A]]\!] \leq [\![\Sigma[\ ]]\!]$$

which is equivalent to the following by definition of $[\![\ ]\!]$

$$\blacklozenge_A 1 \leq 1$$

This is true by (16) of definition 4.3.

□

Theorem 4.7 (Completeness) *Any valid sequent is derivable, i.e. if* $[\![\Theta]\!] \leq [\![q]\!]$ *for every multiplicative* LMAM *and every interpretation* $[\![-]\!]$ *therein, then* $\Theta \vdash q$.

Proof. We follow the Lindenbaum-Tarski proof method of completeness (building the counter-model). We show the following

(i) The logical equivalence $\cong$ defined as $\vdash\dashv$ over the formulae of the logic is an equivalence relation, i.e. it is reflexive, transitive (by the admissibility of *Cut*), and symmetric.

(ii) The order relation $\leq$ defined as $\vdash$ on the above equivalence classes is a partial order, i.e. reflexive, transitive and anti-symmetric.

(iii) The operations $\wedge, \vee, \bullet, \blacklozenge_A$, and $\Box_A$ on the above equivalence classes (defined in a routine fashion) are well-defined. To avoid confusion with the brackets of the sequents, i.e. $\Sigma[\Theta']$, we occasionally drop the brackets of the equivalence classes and for example write $\blacklozenge_A q$ for $[\blacklozenge_A q]$.

(a) For $\blacklozenge_A[q] := [\blacklozenge_A q]$ we show

$$[q] \cong [q'] \implies [\blacklozenge_A q] \cong [\blacklozenge_A q']$$

The proof tree of one direction is as follows, the other direction is identical

$$\dfrac{\dfrac{q \vdash q'}{q^A \vdash \blacklozenge_A q'}\ \blacklozenge_A R}{\blacklozenge_A q \vdash \blacklozenge_A q'}\ \blacklozenge_A L$$

(b) For $\Box_A[q] := [\Box_A q]$ we show

$$[q] \cong [q'] \implies [\Box_A q] \cong [\Box_A q']$$

The proof tree of one direction is as follows, the other direction is identical

$$\dfrac{\dfrac{q \vdash q'}{(\Box_A q)^A \vdash q'}\ \Box_A L}{\Box_A q \vdash \Box_A q'}\ \Box_A R$$

(c) Similarly for $[q_1] \wedge [q_2] := [q_1 \wedge q_2]$, $[q_1] \vee [q_2] := [q_1 \vee q_2]$ and $[q_1] \bullet [q_2] := [q_1 \bullet q_2]$.

(iv) The equivalence classes and their corresponding operations form a multiplicative lattice monoid. To show this, we prove properties of definition 4.1 in our logic.

(a) Proof trees for properties of meet and joint are routine.

(b) The proof tree for one direction of the first half of (1) is as follows (instances of Id refer to lemma 3.3). Proof trees for the other direction and the other half are similar.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,Id \quad \dfrac{}{q' \vdash q'}\,Id}{q, q' \vdash q \bullet q'}\,\bullet R}{q, q' \vdash (q \bullet q') \vee (q \bullet q'')}\,\vee R \quad \dfrac{\dfrac{\dfrac{}{q \vdash q}\,Id \quad \dfrac{}{q'' \vdash q''}\,Id}{q, q'' \vdash q \bullet q''}\,\bullet R}{q, q'' \vdash (q \bullet q') \vee (q \bullet q'')}\,\vee R}{q, (q' \vee q'') \vdash (q \bullet q') \vee (q \bullet q'')}\,\vee L}{q \bullet (q' \vee q'') \vdash (q \bullet q') \vee (q \bullet q'')}\,\bullet L$$

(c) Proof trees for the first half of (2) are as follows, second half is proved similarly.

$$\dfrac{\dfrac{\dfrac{}{q \vdash q}\,Id}{q, 1 \vdash q}\,1L}{q \bullet 1 \vdash q}\,\bullet L \qquad\qquad \dfrac{\dfrac{}{\vdash 1}\,1R \quad \dfrac{}{q \vdash q}\,Id}{q \vdash q \bullet 1}\,\bullet R$$

(d) Proof trees for (3) and (4), i.e. order preservation of $\blacklozenge_A$ and $\Box_A$ are as follows

$$\dfrac{\dfrac{q \vdash q'}{q^A \vdash \blacklozenge_A q'}\,\blacklozenge_A R}{\blacklozenge_A q \vdash \blacklozenge_A q'}\,\blacklozenge_A L \qquad\qquad \dfrac{\dfrac{q \vdash q'}{(\Box_A q)^A \vdash q'}\,\Box_A L}{\Box_A q \vdash \Box_A q'}\,\Box_A R$$

(e) Proof trees for (5), i.e. the adjunction between $\blacklozenge_A$ and $\Box_A$ are as follows

$$\dfrac{\dfrac{q \vdash \Box_A q'}{q^A \vdash q'}\,\Box_A Inv}{\blacklozenge_A q \vdash q'}\,\blacklozenge_A L \qquad\qquad \dfrac{\dfrac{\blacklozenge_A q \vdash q'}{q^A \vdash q'}\,\blacklozenge_A Inv}{q \vdash \Box_A q'}\,\Box_A R$$

(f) The proof tree for (15) is as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,Id}{q^A \vdash \blacklozenge_A q}\,\blacklozenge_A R \quad \dfrac{\dfrac{}{q' \vdash q'}\,Id}{q'^A \vdash \blacklozenge_A q'}\,\blacklozenge_A R}{q^A, q'^A \vdash \blacklozenge_A q \bullet \blacklozenge_A q'}\,\bullet R}{(q, q')^A \vdash \blacklozenge_A q \bullet \blacklozenge_A q'}\,D}{(q \bullet q')^A \vdash \blacklozenge_A q \bullet \blacklozenge_A q'}\,\bullet L}{\blacklozenge_A (q \bullet q') \vdash \blacklozenge_A q \bullet \blacklozenge_A q'}\,\blacklozenge_A L$$

(g) The proof tree for (16) is as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,1R}{\langle\,\rangle^A \vdash 1}\,Unit}{1^A \vdash 1}\,1L}{\blacklozenge_A 1 \vdash 1}\,\blacklozenge_A L$$

$\square$

4.2 Propositions

Definition 4.8 Let $\mathcal{A}$ be a set, with elements called *agents*. A DLAM over $\mathcal{A}$ is a bounded distributive lattice $(L, \wedge, \vee, \top, \bot)$ with two $\mathcal{A}$-indexed families $\{\blacklozenge_A\}_{A\in\mathcal{A}} \colon L \to L$ and $\{\Box_A\}_{A\in\mathcal{A}} \colon L \to L$ of order-preserving maps, with each $\blacklozenge_A$ left adjoint to $\Box_A$, i.e. $\blacklozenge_A(l) \leq l'$ iff $l \leq \Box_A(l')$.

Definition 4.9 A multiplicative LMAM Q *acts on* a DLAM L (with the same sets of agents) whenever we have two pointwise order-preserving maps $\_ \cdot \_ \colon L \times Q \to L$ and $[\_]\_ \colon Q \times L \to L$, with $- \cdot q$ left adjoint to $[q]-$, i.e. $l \cdot q \leq l'$ iff $l \leq [q]l'$, and moreover the following hold for all $l \in L, q, q_1, q_2 \in Q$ and A in A:

$$l \cdot (q_1 \bullet q_2) = (l \cdot q_1) \cdot q_2 \tag{19}$$

$$l \cdot 1 = l \tag{20}$$

$$\blacklozenge_A(l \cdot q) \leq \blacklozenge_A l \cdot \blacklozenge_A q \tag{21}$$

Proposition 4.10 *Whenever a multiplicative* LMAM Q *acts on a* DLAM L*, the following hold, for all* l, l_i *in* L *and* q, q_i *in* Q*:*

$$(l_1 \vee l_2) \cdot q = (l_1 \cdot q) \vee (l_2 \cdot q) \tag{22}$$

$$[q]\,(l_1 \wedge l_2) = [q]\,l_1 \wedge [q]\,l_2 \tag{23}$$

$$\bot \cdot q = \bot \qquad [q]\top = \top \tag{24}$$

$$([q]l) \cdot q \leq l \tag{25}$$

$$l \leq [q](l \cdot q) \tag{26}$$

$$[q_1 \bullet q_2]\,l = [q_1]\,[q_2]\,l \tag{27}$$

$$[1]\,l = l \tag{28}$$

Let Q be an LMAM acting on a DLAM L and $[\![-]\!]^Q$ an interpretation of the set of terms of the action logic (over a set B of basic actions) in Q, as defined in the previous subsection. An *interpretation* of the set M of formulae of the propositional logic (over a set At of atoms) *in* L is given by a map $[\![-]\!] \colon At \to L$; extension to an interpretation of formulae is obtained by induction on the structure of the formulae:

$$\begin{aligned} [\![m_1 \vee m_2]\!] = [\![m_1]\!] \vee [\![m_2]\!], &\quad [\![m_1 \wedge m_2]\!] = [\![m_1]\!] \wedge [\![m_2]\!], \\ [\![\blacklozenge_A(m)]\!] = \blacklozenge_A([\![m]\!]), &\quad [\![\Box_A m]\!] = \Box_A[\![m]\!], \\ [\![\top]\!] = \top, &\quad [\![\bot]\!] = \bot, \\ [\![m \cdot q]\!] = [\![m]\!] \cdot [\![q]\!]^Q, &\quad [\![[q]m]\!] = [[\![q]\!]^Q][\![m]\!]\,. \end{aligned}$$

Given the meaning of action items and contexts as defined in the previous subsection, the meanings of propositional items and contexts are obtained as elements of L by mutual induction on their structure as follows:

$$\begin{aligned} [\![m]\!] &= \text{as above} \\ [\![\Gamma^A]\!] &= \blacklozenge_A([\![\Gamma]\!]) \\ [\![\Gamma^\Theta]\!] &= [\![\Gamma]\!] \cdot [\![\Theta]\!]^Q \\ [\![I_1, \cdots, I_n]\!] &= [\![I_1]\!] \wedge \cdots \wedge [\![I_n]\!] \\ [\![\emptyset]\!] &= \top \end{aligned}$$

Theorem 4.11 (Soundness) *Any derivable sequent is valid, i.e.* $\Gamma \vdash m$ *implies* $[\![\Gamma]\!] \leq [\![m]\!]$ *is* true *in any interpretation* $[\![-]\!]$ *in any* `DLAM` L *acted upon by any* `LMAM` Q *(for given sets A of agents, B of basic actions and At of atoms).*

Proof. The technique is the same as that used in the previous subsection; a summary of proofs of soundness of dynamic rules are as follows, full proofs of propositional-only rules has been presented in previous work [9].

- $\cdot L$ immediately follows from the definition of $[\![m^q]\!]$.
- $\cdot R$, follows from order-preservation of $\cdot$ and definition of $[\![\Gamma^\Theta]\!]$.
- DyL, follows from property 25 in proposition 4.10 and definition of meet.
- DyR, follows from definition of adjunction from definition 4.8 and that of meet.
- $DyDist$, follows from property 21 of definition 4.9.
- $ReArr$, follows from property 19 of definition 4.9.

□

Theorem 4.12 (Completeness) *Any valid sequent is derivable, i.e. if* $[\![\Gamma]\!] \leq [\![m]\!]$ *for every* `DLAM` L *acted upon by any* `LMAM` Q *and every interpretation* $[\![-]\!]$ *therein, then* $\Gamma \vdash m$.

Proof. We show that the Lindenbaum-Tarski (LT) algebra of the syntax of the *AlgDEL* logic forms a `DLAM` on which the LT algebra of the syntax of *Action* logic acts. That the LT algebra of propositional-only part of *AlgDEL* forms a `DLAM` has been shown in [9], it is routine to define well-defined dynamic operations of $m \cdot q$ and $[q]m$ on equivalence classes over the logical consequence relation. It remains to show that these satisfy the axioms of definition 4.9. The proof trees for meet and join are routine. Note that the rules for $\cdot$ and its right adjoint are exactly the same as those for $\blacklozenge_A$ and $\Box_A$, so order-preservation and adjunction follow identically. the proof trees for property 19 are:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id \quad \dfrac{}{q \vdash q}\,Id}{m^q \vdash m \cdot q}\,\cdot R \quad \dfrac{}{q' \vdash q'}\,Id}{((m^q))^{q'} \vdash (m \cdot q) \cdot q'}\,\cdot R}{(m^{q,q'}) \vdash (m \cdot q) \cdot q'}\,DyDist}{m^{q \bullet q'} \vdash (m \cdot q) \cdot q'}\,\bullet L}{m \cdot (q \bullet q') \vdash (m \cdot q) \cdot q'}\,\cdot L
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id \quad \dfrac{\dfrac{}{q \vdash q}\,Id \quad \dfrac{}{q' \vdash q'}\,Id}{q, q' \vdash q \bullet q'}\,\bullet R}{m^{q,q'} \vdash m \cdot (q \bullet q')}\,\cdot R}{(m^q)^{q'} \vdash m \cdot (q \bullet q')}\,ReArr}{(m \cdot q)^{q'} \vdash m \cdot (q \bullet q')}\,\cdot L}{(m \cdot q) \cdot q' \vdash m \cdot (q \bullet q')}\,\cdot L$$

The proof trees for property 20 are:

$$\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id}{m^1 \vdash m}\,1L}{m \cdot 1 \vdash m}\,\cdot L
\qquad
\dfrac{\dfrac{}{m \vdash m}\,Id \quad \dfrac{}{\vdash 1}\,1R}{m \vdash m \cdot 1}\,\cdot R$$

The proof tree for property 21 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id}{m^A \vdash \blacklozenge_A m}\,\blacklozenge_A R \qquad \dfrac{\dfrac{}{q \vdash q}\,Id}{q^A \vdash \blacklozenge_A q}\,\blacklozenge_A R}{(m^q)^A \vdash \blacklozenge_A m \cdot \blacklozenge_A q}\,DyDist}{(m \cdot q)^A \vdash \blacklozenge_A m \cdot \blacklozenge_A q}\,\cdot L}{\blacklozenge_A (m \cdot q) \vdash \blacklozenge_A m \cdot \blacklozenge_A q}\,\blacklozenge_A L$$

□

We conclude this section by relating to previous work [1,7]. The algebraic semantics developed there was referred to as an *Epistemic System*. The pair of an `LMAM` Q and the `DLAM` M on which it acts (M, Q) is a *pre-Epistemic system*. These are finite versions of Epistemic Systems, and, as spelled out in detail in [7], their completion yields an Epistemic System in which they faithfully embed.

5 Interpretation and Assumptions

The interpretation of propositional and action connectives are the usual ones: $\wedge, \vee$ for conjunction and disjunction on propositions and non-deterministic choice and parallel composition for actions, 1 stands for the skip action in which nothing happens. The dynamic connectives $[q]m$ and $m \cdot q$ stand for the weakest precondition and strongest postcondition, the former is often read as "after action q proposition m holds". As for the epistemic modalities, we follow previous work [1,7] and read $\blacklozenge_A$ as the appearance to agent A of a proposition (or an action), and $\Box_A$ as the belief of agent A about a proposition (or an action).

Each epistemic protocol has assumptions about atomic actions and facts (i.e. atomic propositions) involved in the protocol and the uncertainty of agents about these. For each atomic action σ, there is a *kernel* proposition k to which the action cannot apply, i.e. $k \cdot \sigma$ will lead to $\bot$ [2]. The atomic actions that we consider are *epistemic*, in that they do not change *facts of the world*. So an atomic action σ has no effect on a proposition p, i.e. if p is true before the action, it will stay true after it, in other words $p \cdot \sigma \vdash p$. Finally, each agent A has some uncertainty about each atomic proposition p (and action σ); these, following the approach of [2], are all the propositions (or actions) that appear to A as true (or as happening in case of actions) when in reality p is true (or σ is happening). So we have one or more assumptions of the form "appearance to agent A of fact p is proposition n" and "appearance to agent A of action σ is the term w". In the case of actions, these would for instance, enable us to encode honest and dishonest public and private announcements. To model these extra information, we add the following rules to our calculus

$$\dfrac{\Delta[\bot] \vdash m}{\Delta[k \cdot \sigma] \vdash m}\,Ker \qquad\qquad \dfrac{\Delta[p] \vdash m}{\Delta[p^\sigma] \vdash m}\,Fact \qquad\qquad \dfrac{\Delta[n^w] \vdash m}{\Delta[(p^A)^{\sigma^A}] \vdash m}\,App_A$$

[2] These model co-preconditions, where in DEL [2] a precondition is the proposition to which the action can apply.

Lemma 5.1 *Addition of the assumption rules preserves admissibility of* Contraction *and* Weakening *and the invertibility of rules.*

Theorem 5.2 *Addition of the assumption rules preserves admissibility of the propositional and dynamic* Cut *rules.*

Proof. For the propositional cut, we check three cases for each assumption rule, when the rule is principal and when it is one of the many cases of $\Box_A R$ and DyR rules. The second premiss of the dynamic cut can only be an instance of $Fact$ or App_A; the former cut permutes with $Fact$, and for the latter we do a case analysis on the action premiss.

(i) Ker principal

$$\dfrac{\dfrac{\Delta[\bot] \vdash m}{\Delta[k \cdot \sigma] \vdash m}\,Ker \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[k \cdot \sigma]] \vdash m'}\,PrCut$$

transforms to

$$\dfrac{\dfrac{\Delta[\bot] \vdash m \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[\bot]] \vdash m'}\,PrCut}{\Delta'[\Delta[k \cdot \sigma]] \vdash m'}\,Ker$$

(ii) $Fact$ principal

$$\dfrac{\dfrac{\Delta[p] \vdash m}{\Delta[p^\sigma] \vdash m}\,Fact \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[p^\sigma]] \vdash m'}\,PrCut$$

transforms to

$$\dfrac{\dfrac{\Delta[p] \vdash m \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[p]] \vdash m'}\,PrCut}{\Delta'[\Delta[p^\sigma]] \vdash m'}\,Fact$$

(iii) App_A is principal

$$\dfrac{\dfrac{\Delta[n^w] \vdash m}{\Delta[(p^A)^{\sigma^A}] \vdash m}\,App_A \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[(p^A)^{\sigma^A}]] \vdash m'}\,PrCut$$

transforms to

$$\dfrac{\dfrac{\Delta[n^w] \vdash m \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[n^w]] \vdash m'}\,PrCut}{\Delta'[\Delta[p^A]^{\sigma^A}] \vdash m'}\,App_A$$

(iv) First premiss is an instance of $\Box_A R$, second is one of Ker

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta[\Box_A m][\bot] \vdash m'}{\Delta[\Box_A m][k \cdot \sigma] \vdash m'}\,Ker}{\Delta[\Gamma][k \cdot \sigma] \vdash m'}\,PrCut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][\bot] \vdash m'}{\Delta[\Gamma][\bot] \vdash m'}\ PrCut}{\Delta[\Gamma][k \cdot \sigma] \vdash m'}\ Ker$$

(v) First premiss is an instance of $\Box_A R$, second is one of $Fact$

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \dfrac{\Delta[\Box_A m][p] \vdash m'}{\Delta[\Box_A m][p^\sigma] \vdash m'}\ Fact}{\Delta[\Gamma][p^\sigma] \vdash m'}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][p] \vdash m'}{\Delta[\Gamma][p] \vdash m'}\ PrCut}{\Delta[\Gamma][p^\sigma] \vdash m'}\ Fact$$

(vi)

(vii) First premiss is an instance of $\Box_A R$, second is one of App

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \dfrac{\Delta[\Box_A m][n^w] \vdash m'}{\Delta[\Box_A m][(p^A)^{\sigma^A}] \vdash m'}\ App_A}{\Delta[\Gamma][(p^A)^{\sigma^A}] \vdash m'}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][n^w] \vdash m'}{\Delta[\Gamma][n^w] \vdash m'}\ PrCut}{\Delta[\Gamma][(p^A)^{\sigma^A}] \vdash m'}\ App_A$$

(viii) Cases for when the first premiss is an instance of DyR and the second one is one of $Ker, Fact, App$ are identical.

Cases for the dynamic cut are left as an exercise. □

6 Applications

Consider a simple coin-toss protocol, when, in front of agents B and C, agent A tosses a coin and covers it in his palm, then takes a look and makes an honest public announcement that it is heads up. We have $\mathcal{A} = \{A, B, C\}$, $At = \{H, T\}$, for *Heads* and *Tails*, the kernel of the honest public announcement, denoted by $H!$, is T and it appears as it is to all the agents, whereas initially both H and T appear as $H \vee T$ to the agents. So, as assumption rules, for $p \in At, X \in \mathcal{A}$ we have

$$\dfrac{\Delta[\bot] \vdash m}{\Delta[T \cdot H!] \vdash m}\ Ker \qquad \dfrac{\Delta[p] \vdash m}{\Delta[p^{H!}] \vdash m}\ Fact \qquad \dfrac{\Delta[(H \vee T)^{H!}] \vdash m}{\Delta[(p^X)^{H!^X}] \vdash m}\ App_X$$

As an example of a derivation, we show that if the coin is heads up then after this announcement B believes that the coin is heads and also that C believes this too.

$$
\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{H\vdash H}\,Id}{H^{H!}\vdash H}\,Fact \quad \dfrac{\dfrac{}{\bot\vdash H}\,\bot L}{T^{H!}\vdash H}\,Ker}{(H\vee T)^{H!}\vdash H}\,\vee L}{(H^B)^{H!^B}\vdash H}\,App_B}{(H^{H!})^B\vdash H}\,DyDist
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{H\vdash H}\,Id}{H^{H!}\vdash H}\,Fact \quad \dfrac{\dfrac{}{\bot\vdash H}\,\bot L}{T^{H!}\vdash H}\,Ker}{(H\vee T)^{H!}\vdash H}\,\vee L}{(H^C)^{H!^C}\vdash H}\,App_C}{(H^{H!})^C\vdash H}\,DyDist}{H^{H!}\vdash \Box_C H}\,\Box_C R \quad \dfrac{\dfrac{}{\bot\vdash \Box_C H}\,\bot L}{T^{H!}\vdash \Box_C H}\,Ker}{(H\vee T)^{H!}\vdash \Box_A H}\,\vee L}{(H^B)^{H!^B}\vdash \Box_C H}\,App_B}{(H^{H!})^B\vdash \Box_C H}\,DyDist
}{(H^{H!})^B\vdash H\wedge\Box_C H}\,\wedge R}{H^{H!}\vdash \Box_B(H\wedge\Box_C H)}\,\Box_B R}{H\vdash [H!]\Box_B(H\wedge\Box_C H)}\,DyR
$$

If, instead of an honest public announcement, A had publicly lied that the coin was tails and B and C did not expect this and still thought that this was an honest announcement, then by the exact same derivation steps and a different set of appearance and kernel assumption rules, we could have proved the same thing. Denoting the public lying by $T\dagger$, the assumptions for this version of the protocol are as follows:

$$
\frac{\Delta[\bot]\vdash m}{\Delta[H\cdot T\dagger]\vdash m}\,Ker \qquad \frac{(H\vee T)^{T!}\vdash m}{\Delta[(p^X)^{T\dagger^X}]\vdash m}\,App_{X\in\{B,C\}} \qquad \frac{(H\vee T)^{T\dagger}\vdash m}{\Delta[(p^X)^{T\dagger^A}]\vdash m}\,App_A
$$

Similarly, if A had made an honest private announcement to B, an action denoted by $H!_B$, and it appeared to C that nothing, i.e. the unit action 1, had happened, we would have the following assumption rules for appearances:

$$
\frac{\Delta[(H\vee T)^{H!_B}]\vdash m}{\Delta[(H^B)^{(H!_B)^B}]\vdash m}\,App_B \qquad \frac{\Delta[(H\vee T)^{1}]\vdash m}{\Delta[(H^C)^{(H!_B)^C}]\vdash m}\,App_C
$$

If C suspected this announcement by thinking either that nothing happened or that A announced "heads" to B, the appearance assumption for C would change to

$$
\frac{\Delta[(H\vee T)^{1\vee H!_B}]\vdash m}{\Delta[(H^C)^{(H!_B)^C}]\vdash m}\,App_C
$$

7 Summary and future work

Thus, the algebraic ideas about dynamic epistemic logic from [1,7] may be represented in terms of a cut-free but complete sequent calculus, albeit one with a complex but powerful notation and rules admitting substantial non-determinism in root-first proof search. Nevertheless, and in contrast (we believe) to the algebraic approach of [1,7], this sequent calculus should be the basis of implementation allowing automated reasoning about suitable encodings of situations and protocols involving both epistemic operators and actions. Suitable refinements of the calculus, addressing issues such as termination and backtracking, have yet to be developed.

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