

Classical knowledge for quantum security (extended abstract)

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Abstract

We propose a decision procedure for analysing security of quantum cryptographic protocols, combining a classical algebraic rewrite system for knowledge with an operational semantics for quantum distributed computing. As a test case, we use our procedure to reason about security properties of a recently developed quantum secret sharing protocol that uses graph states. We analyze three different scenarios based on the safety assumptions of the classical and quantum channels and discover the path of an attack in the presence of an adversary. The epistemic analysis that leads to this and similar types of attacks is purely based on our classical notion of knowledge.

Keywords: Quantum cryptography, distributed measurement calculus, algebraic information update.

1 Introduction

Quantum communication is an inseparable part of quantum computing: it offers solutions to the risks caused by the exponential speed-up in the power of adversaries, which is in turn caused by quantum algorithms. While some advances have been made in the area of formal verification of quantum communication protocols [11], no applicable formal framework has yet been suggested for their automatic cryptanalysis. This is contrary to the fact that, similar to the situation in classical security, attacks have been discovered on proven-to-be-safe quantum protocols.

In this paper, we present a decision procedure to verify whether a protocol satisfies an epistemic security property. Our procedure derives knowledge properties of agents from the set of dynamic and epistemic traces of the protocol. The *dynamic traces* are generated from the protocol specification by operational rules of

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distributed measurement calculus (DMC) [5]. These are then expanded to the *epistemic traces* using *appearances* of agents about the actions of the protocol. The appearances are derived from the safety assumptions of the communication channels according to a set of rules. Our notions of knowledge and time are classical and have been used in the formal analysis of classical protocols, for example in the Halpern style models of [14,7] and in the algebraic Epistemic Systems of [2,17].

Both the DMC model and the algebra have been previously used to analyze the security of quantum key distribution (QKD) and its attacks [7,8,16]. The setting of this paper has advantages over both these attempts. First, we rely on the already existing rules of the semantics of DMC, as opposed to adding axioms for quantum mechanics to the algebra as pursued in [16]. Second, we use the algebraic axiomatics of dynamic and epistemic adjunctions to derive knowledge properties of the protocol, as opposed to model-checking them by traversing the tree of the protocol as done in [7,8]. Third, we set the actions of the adversary in a compositional way using the appearance maps of the algebra, as opposed to ad-hocly adding them to the specification of the protocol as suggested in [7,8].

We prove that our decision procedure is sound and terminating with regard to the pair of a DMC model and the algebraic axiomatics of Epistemic Systems. We apply our decision procedure to a new quantum secret sharing (QSS) protocol, which is based on graph states and has been proposed recently in [12]. For this protocol, we develop epistemic properties and prove them for three kinds of assumptions on the quantum channels: safe, unsafe with non-suspicious agents, and unsafe with suspicious agents. We show that in the second case, the protocol does not satisfy its desired epistemic property and is thus not secure, moreover, we discover the path of an intercept-exchange attack that caused this insecurity. A full analysis of the safety assumptions of all the channels and their impact on the security properties needs automation, which constitutes on going work. Also, we have only been working on a one-round basis and indeed, for a full analysis of protocols one needs to run the protocol in many runs and then use probabilities, for instance on the knowledge modalities. This would be a natural and exciting extension of the currently proposed framework.

In a nut shell, our framework is obtained by merging the model checking approach of [8,7] and the algebraic axiomatics of [16]. The former is based on a distributed extension [5] for an assembly language [6] that universally models computations of the one way model. Its knowledge operator is defined over Kripke structures in the style of Fagin et al [10] by using equivalence relations on the states. Reasoning about properties of a protocol is done on the state space of this structure using a logic with temporal and epistemic operators. The latter is based on the Stone-like duals of these relational systems and moreover, following [4], a quantale structure is assumed on the actions. This setting consists of a pair of a quantale of classical and quantum actions and its right module of bits and qubits involved in a protocol. The pair is endowed with a family of join-preserving maps, one for each agent involved in the protocol. The right adjoints to these endomorphisms give rise to a very useful notion of knowledge, both on propositions of module and actions of quantale.

2 Decision Procedure

The procedure has three main steps. First, we write as program in the language of the distributed measurement calculus (DMC) to implement the specification of the quantum protocol and generate a set of *dynamic traces* for it. This is done by executing the rules of the operational semantics of DMC. Second, we write formulae with dynamic and epistemic modalities to express security properties of the protocol. This is done in the algebraic syntax of Epistemic Systems. Finally, we apply an algebraic rewrite system to decide whether the protocol satisfies the properties.

Step (1) Specify and produce dynamic traces in DMC.

Programs of DMC are implemented as *networks of agents*. A network of agents is denoted by \mathcal{N} and is defined as follows

$$\mathcal{N} = |\psi\rangle \parallel \mathbf{A}(Q).\mathcal{E} \mid \mathbf{B}(Q').\mathcal{E}' \dots$$

It consists of a set of agents acting in parallel (denoted by \parallel) on a given entanglement resource $|\psi\rangle$. An *agent* $\mathbf{A}(Q).\mathcal{E}$ is specified by a name \mathbf{A} , a set Q of qubits it owns, and an event sequence \mathcal{E} . The event sequence can be a computation in the measurement calculus, a classical message reception $c?x$ and sending $c!y$, or a qubit reception $qc?q$ and sending $qc!q'$. Note that, contrary to the original definitions in [5] we now write specifications from left to right; also agents may have extra classical parameters a , written as $A(a, Q)$. As an example, here is one round of Ekert's implementation of QKD:

$$QKD = E_{12} \parallel \mathbf{A}(a, 1).[H_1^a; M_1; c!a; c?b] \mid \mathbf{B}(b, 2).[H_2^b; M_2; c?a; c!b]$$

The set of traces of a program are generated by following the rules of the small-step semantics as specified in [5], but moreover, we work with projections, annotate actions with agents that performed them, and name the preparation actions of the initial entanglement resource $|\psi\rangle$ and the distribution actions of qubits. For example, $P_i^{A,\alpha}$ stands for the spin α projection of qubit i done by agent A and N_i^C is the preparation of qubit i by agent C . The preparation actions are made explicit by juxtaposing them to the left most of the traces; for QKD the entanglement resource E_{12} is created by applying $N_1^C; N_2^C; E_{12}^C$ to a 2-qubit system $q_1 \otimes q_2$, where N is preparation in the $|+\rangle$ state and C is the agent who prepared the entanglement resource. Distributing these qubits to agents A and B is denoted by a quantum broadcast action $qc!_X^C q_i$, which stands for agent C sending qubit q_i to agent X and agent X receiving it from him. This is a shorthand for a quantum send $qc!_X^C q_i$ and a quantum receive $qc?_X^C q_i$. Similarly, we also shorthand a classical send $c!_X^C a$ and receive $c?_X^C a$ to a broadcast $c!_X^C a$.

According to these conventions two of the four possible traces for a successful run of QKD become as follows

$$\pi = N_1^C; N_2^C; E_{1,2}^C; qc!_A^C q_1; qc!_B^C q_2; P_1^{A,X}; P_2^{B,X}; c!_B^A a; c!_A^B b$$

$$\pi' = N_1^C; N_2^C; E_{1,2}^C; qc!_A^C q_1; qc!_B^C q_2; P_1^{A,Z}; P_2^{B,Z}; c!_B^A a; c!_A^B b$$

Step (2) Write security properties in Epistemic Systems.

The input to the rewrite system is an expression of the form

$$l \vdash r$$

where l is the initial state and r is an epistemic property that contains the disjunction of dynamic traces produced above. An example is the following expression

$$q_i \vdash [\pi] \Box_A \Box_A s_i^j$$

which is read as

After running the trace π of the protocol on qubit q_i , agent A knows that B knows that the value of bit i is j .

The l and r expressions are generated as follows:

- The initial state l is made of propositions m that are formed by closing atomic classical and quantum variables s_i^j and q_i under \neg, \wedge, \vee and logical constants \perp, \top . The variables are generated as follows

$$\kappa ::= s_i^j \mid q_l \mid q_l \otimes q_w$$

- The epistemic property r is generated as follows

$$r ::= m \mid [\pi]m \mid \Box_A(m)$$

where $\Box_A(m)$ is the epistemic modality and for π a dynamic trace $[\pi]m$ is the dynamic modality.

One such expression for Ekert's QKD is

$$q_1 \otimes q_2 \vdash \left[N_1^C; N_2^C; E_{1,2}^C; \mathbf{qc}!_A^C q_1; \mathbf{qc}!_B^C q_2; P_1^{A,X}; P_2^{B,X}; \mathbf{c}!_B^A a; \mathbf{c}!_A^B b \right] \Box_A \Box_B (s_1^0 \wedge s_2^0)$$

Proving this property together with a permutation of it for B , that is

$$q_1 \otimes q_2 \vdash \left[N_1^C; N_2^C; E_{1,2}^C; \mathbf{qc}!_A^C q_1; \mathbf{qc}!_B^C q_2; P_1^{A,X}; P_2^{B,X}; \mathbf{c}!_B^A a; \mathbf{c}!_A^B b \right] \Box_B \Box_A (s_1^0 \wedge s_2^0)$$

will imply that A and B *share* a piece of data, which is the results of each other's measurements, that is $(s_1^0 \wedge s_2^0)$. The sharing property is expressed by the nested knowledge property, that A knows that B knows it, and vice versa ³. That the data is *secret* is proved by showing that an adversary E does not know it, that is the following expression

$$q_1 \otimes q_2 \vdash \left[N_1^C; N_2^C; E_{1,2}^C; \mathbf{qc}!_A^C q_1; \mathbf{qc}!_B^C q_2; P_1^{A,X}; P_2^{B,X}; \mathbf{c}!_B^A a; \mathbf{c}!_A^B b \right] \neg \Box_E (s_1^0 \wedge s_2^0)$$

³ It is arguable whether one has to nest the knowledge modalities infinitely many times and thus use the common knowledge operator to express the sharing property, but for now we restrict ourselves to a two level nesting.

Step (3). Generate Epistemic traces and verify the property.

We proceed by analyzing uncertainty of agents about the states and actions of protocols. These are referred to as *appearance* maps and are denoted by f_A for an agent A . They encode all possible actions or propositions that appear possible to an agent, given the action that is happening or the proposition that is true in reality, we refer the reader to [2,17] for discussions and examples. Here, we treat these maps more practically and introduce a general set of rules to generate them. These rules are presented below.

- (i) The agents have no uncertainty about the steps of the protocol they are involved in.
- (ii) Qubits are encoded as black boxes and thus appear as they are, that is as identity to all agents. Classical bits appear as either 0 or 1 to agents.
- (iii) The owner of an action has no uncertainty about his actions, but is uncertain about other agents' actions. His appearances of these latter actions are generated by instantiating their variables.
- (iv) There is only one adversary present in each protocol. This adversary can intercept the unsafe channels, either quantum or classical, by stopping the messages, changing the content of the messages, creating new messages and sending them. On a quantum channel, the change of the content of the message is done by measuring the sent qubit and the creation of new messages by preparing fresh qubits. On the classical channel, the change is simply affected by reading and writing the values of the bits.
- (v) On the safe channels, the adversary can either be passive or not present at all. In the latter case, he cannot even see if messages are passing through and what is their content. In the former case, on a classical channel, he can see the value of the bits as well as the sender and receiver of each message, but cannot change anything. On a quantum channel, he can only see that a qubit is passing, but cannot see its state.
- (vi) Communication actions on a safe channel are either public or private announcements to a subgroup of agents. The former appears as identity to all agents, whereas the latter is identity only to the insiders in the group, and either as nothing or all possible choices to the outsider agents. On an unsafe channel, the announcement actions are treated as separate send and receive actions.
- (vii) Honest agents may suspect the interception actions of the adversary. If they do so, these actions appear to them as either happened or not. If they do not, they appear to them as the neutral action in which nothing happens.

For example, the appearances of the projection action $P_1^{A,X}$ in our above example traces are as follows

$$f_A(P_1^{A,X}) = P_1^{A,X}, \quad f_B(P_1^{A,X}) = P_1^{A,X} \vee P_1^{A,-X} \vee P_1^{A,Z} \vee P_1^{A,-Z} .$$

The appearances of the communication actions depend on the safety assumptions of the channel in which they take place. For example, if the channel is safe, they are treated as broadcasts otherwise as separate send and receive actions. We present a

detailed example on those in the last section.

Due to space limits we cannot present the rewrite rules; they are similar to the system presented in [15]. By applying them, one first eliminates the logical connectives $\wedge, \vee, \Box_A, []$ and then the classical and quantum communication actions. The output is a set of *atomic expressions*, defined as follows

Definition 2.1 An expression $l \vdash r$ is atomic iff l is a quantum state followed by a sequence of atomic quantum actions and r is an atomic classical or quantum state.

For instance, for a safe quantum channel, the atomic form of the our sharing property is

$$(q_1 \otimes q_2)(N_1^C; N_2^C; E_{1,2}^C; P_1^{A,X}; P_2^{B,X}) \vdash s_1^0 \wedge s_2^0$$

These atomic expressions may contain new epistemic uncertainties and thus will need to be verified against our operational semantics. For this purpose, we introduce below the notion of a *well-defined* expression.

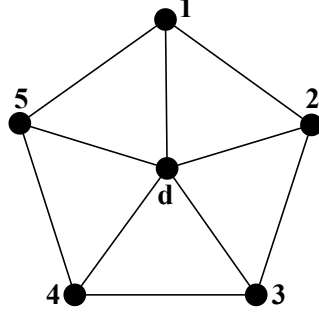
Definition 2.2 An atomic expression $l \vdash r$ is *well-defined* iff l is derivable within the operational semantics of DMC. It is true iff r holds in all configurations resulting from l . An epistemic property holds for a protocol whenever all its well-defined atomic expressions are true.

Proposition 2.3 For a protocol specification \mathcal{N} and an expression $l \vdash r$ which is built from the dynamic traces of \mathcal{N} , the process of deciding if the epistemic property in r holds for \mathcal{N} is terminating and sound with regard to the pair of an Epistemic System and a DMC model.

Proof. These follow from image finiteness of appearances of actions and propositions, together with soundness and termination of the rewrite system of Epistemic Systems and the DMC model [5,15].

3 Case study: quantum secret sharing

We apply our procedure to the quantum secret sharing (QSS) protocol recently established in [12]. In secret sharing a dealer holds a secret bit which he wants to send to n players, such that at least k players are needed to reconstruct the secret. The problem is well-known in the classical settings and solvable for all (n, k) . In the quantum case, only the (n, n) case has been solved for the GHZ-type entanglement [18]. The work in [12] uses instead graph states and thus is more suitable for modelling in our measurement-based setting. Moreover, it generalizes the quantum key distribution protocols and simplifies their proofs. We analyze and prove some of the epistemic properties of the QKS component of the $(3, 5)$ case, where a particular graph state is used to establish a secret key between three players and the dealer in one go (as opposed to via several 2-party QKD protocols). This key will then be used to distribute a secret using the other components of the protocol.



The resource required for the protocol is the graph state shown above, henceforward called $G(3, 5)$. It is prepared following the usual procedure for graph states, that is

$$G(3, 5) = (N_1; \dots; N_5; N_6; \prod_{e_{ij}} E_{ij}) \otimes_{i=1}^6 q_i,$$

where e_{ij} is the set of edges. The protocol proceeds as follows:

Step 1. The dealer prepares $G(3, 5)$, sends each agent a qubit q_i together with an agent identity i .

Step 2. The dealer measures his qubit in the Y or Z basis randomly and broadcasts his measurement basis.

Step 3. Each participating player measures his qubit in the X , Y or Z basis randomly, then broadcasts his identity and measurement basis.

Step 4. Depending on these messages, each agent determines if the run was successful as follows:

- If the participating agents are neighbours, then we have $ijk = i(i+1)(i+2)$; this is the case for the following measurement combinations

$$M_6^Z M_i^Z M_j^X M_k^Z \quad \text{and} \quad M_6^Y M_i^X M_j^Y M_k^X.$$

- If they are in a so-called T-shape, we have $ijk = i(i+1)(i+3)$; this is the case for the following measurement combinations

$$M_6^Z M_i^X M_j^Y M_k^Y \quad \text{and} \quad M_6^Y M_i^Y M_j^Z M_k^Z.$$

Step 5. For a successful run, measurement outcomes are correlated as $s_6 = s_i \oplus s_j \oplus s_k$. Players use their secure classical channels to exchange measurement outcomes and determine if $s = s_6$, hence establishing a shared key with the dealer.

We refrain from giving the full specification of the QSS network and move straight on to its traces, where we treat all the communication actions as broadcasts and later on break them to separate send and receive actions, as necessary and according to the safety assumptions of their channels. Whenever the subscript of a broadcast action is missing, e.g. in $c!^D a$ it means that the broadcast is a public action that can be listened to by everyone. A typical trace for a successful run of QSS is as follows

$$\begin{aligned}
 \pi &= N_1^D; \dots; N_6^D; \prod_{e_{ij}} E_{ij}^D && \text{(preparation)} \\
 &(\mathbf{qc}!^D_1 q_1) \dots (\mathbf{qc}!^D_5 q_5) && \text{(private broadcast of qubits)} \\
 &P_6^{D,\pm a} P_i^{A_i,\pm b} P_j^{A_j,\pm c} P_k^{A_k,\pm f} && \text{(measurement projections)} \\
 &\mathbf{c}!^D a; \mathbf{c}!^{A_i} b; \mathbf{c}!^{A_j} c; \mathbf{c}!^{A_k} c && \text{(public broadcast of measurement bases)} \\
 &(\mathbf{c}!^{A_i}_{A_j, A_k} s_i)(\mathbf{c}!^{A_j}_{A_i, A_k} s_j)(\mathbf{c}!^{A_k}_{A_i, A_j} s_k) && \text{(private broadcast of player's mes. outcomes).}
 \end{aligned}$$

Here $a \in \{X, Y\}, b, c, f \in \{X, Y, Z\}$ are measurement basis, $\mathbf{qc}!^D_i$ is the quantum message passing from D to $A_i \in \{A_1, \dots, A_5\}$ denoting the 5 players, and $\mathbf{c}!^{A_i}_\beta$ is the private announcement from player A_i to the group $\beta \subseteq \{A_1, \dots, A_5\}$. We omit the calculation of the secret key, which is determined by the following exclusive-or formula

$$s = s_i \oplus s_j \oplus s_k$$

Successful traces only depend on the chosen values for a, b, c and f ; one example of such a trace for adjoining agents A_1, A_2 and A_3 , owning qubits 1, 2 and 3 respectively, is as follows

$$\begin{aligned}
 \pi &= \dots \\
 &P_6^{D,+Z} P_1^{A_1,-Z} P_2^{A_2,-X} P_3^{A_3,+Z}; \\
 &Z!^D; Z!^{A_1}; X!^{A_2}; Z!^{A_3}; \\
 &(\mathbf{c}!^{A_1}_{A_2, A_3} 1)(\mathbf{c}!^{A_2}_{A_1, A_3} 1)(\mathbf{c}!^{A_3}_{A_1, A_2} 0)
 \end{aligned}$$

3.1 Epistemic Properties

We consider three cases: agents' heaven, adversary's heaven, and adversary's hell. In the first case the quantum channel is safe, in the second case it is not and the honest agents do not suspect it, in the third case it is not and the honest agents do suspect it. The other channels are assumed to be safe in all three cases. For each case, we show how the appearances of agents of actions in the dynamic traces are set. This is done according to the safety assumptions on the channel and our rules. Then we present some of the related epistemic security properties of each case.

(i) Agents' heaven

The appearance of the projections are set according to the rule *(iii)* of appearances. Since the channels are safe, the communication actions on the quantum channel are treated as public broadcasts and by rule *(vi)* and for σ an agent they are set as follows

$$f_\sigma(\mathbf{qc}!^D q_i) = \mathbf{qc}!^D q_i$$

That is, all the agents are fully aware of the broadcast action and thus have only one possibility in their appearance, the broadcast action itself. The communication actions on the classical channels are private announcements and by rule *(vi)* their appearances are set as follows, for β a subset of players

$$f_{\sigma}(c!_{\beta}^{A_i} s_i^j) = \begin{cases} c!_{\beta}^{A_i} s_i^j & \sigma \in \beta \\ c!_{\beta}^{A_i} s_i^j \vee c!_{\beta}^{A_i} \bar{s}_i^j & \sigma \notin \beta \end{cases}$$

This says that the insider agents $\sigma \in \beta$ who receive the sent bit s_i^j , which is either equal to 1 or 0, are fully aware what has happened and thus have only one possibility about the private broadcast action, that is the broadcast action itself and thus their appearance is identity. But by rule (i) of appearances, the outsider agents $\sigma \notin \beta$ are only aware that a bit has been privately broadcasted to the subgroup β and are uncertain about the value of that bit. So they consider it possible that either a bit with value 1 or a bit with value 0 has been privately broadcasted to insider agents in β . Thus their appearance is the choice of these two possibilities.

Some of the epistemic properties of interest for our trace π , allied players $A_i \in \{A_1, A_2, A_3\}$, joined with dealer $\sigma \in \{D, A_1, A_2, A_3\}$ are as follows

- The dealer knows his bit and binary sum of allied players' bits, i.e.

$$\Box_D (s_6^0 \wedge (s_1^{b_1} \oplus s_2^{b_2} \oplus s_3^{b_3})).$$

- Allied players moreover know the value of each single measurement, i.e.

$$\Box_{A_i} (s_6^0 \wedge s_1^1 \wedge s_2^1 \wedge s_3^0).$$

- The dealer knows that the players know his bit and the players know that the dealer knows the sum of their bits, i.e.

$$\Box_D \Box_{A_i} s_6^0 \quad \text{and} \quad \Box_{A_i} \Box_D (s_1^{b_1} \oplus s_2^{b_2} \oplus s_3^{b_3}).$$

- The adversary does not know any of the above, i.e.

$$\neg \Box_E (s_6^0 \wedge (s_1^{b_1} \oplus s_2^{b_2} \oplus s_3^{b_3})).$$

- The dealer and the agents know the above, i.e.

$$\Box_{\sigma} \neg \Box_E (s_6^0 \wedge (s_1^{b_1} \oplus s_2^{b_2} \oplus s_3^{b_3})).$$

(ii) **Adversary's heaven**

In this case, the quantum channel is not safe and by rule (iv) the adversary can intercept the channel. By rule (vi) since the channel is not safe, we must break its broadcasts to separate send and receive actions. The appearances of these actions to the agents involved in them (e.g. the appearance of the sent action to the agents who received it) are not identities any more. The appearances for the send of a qubit are set as follows, where q_j is a new qubit with $j \geq 7$

$$f_{\sigma'}(\mathbf{qc}_i^D q_i) = \begin{cases} \mathbf{qc}_i^D q_i; \mathbf{qc}_D^E q_i; P_i^{E,e}; N_j^{E,e}; \mathbf{qc}_i^E q_j & \sigma' = E \\ \mathbf{qc}_i^D q_i & o.w. \end{cases}$$

This says that neither the agents nor the dealer suspect that E intercepted dealer's sent qubit and thus their appearance of dealer's sent action is (wrongly) identity. Where as in reality, E did the following sequence of interception events, but they only appear to him as identity.

$$\mathbf{qc}_i^D q_i; \mathbf{qc}_D^E q_i; P_i^{E,e}; N_j^{E,e}; \mathbf{qc}_i^E q_j$$

According to this sequence of events, E received the dealer's sent qubit q_i that was meant to be received by agent i , measured it, then prepared a corresponding new qubit q_j and sent it to agent i . For the corresponding receive action, it appears to the dealer that players received the qubit that he sent to them, $f_D(\mathbf{qc}_D^{A_i} q_i) = \mathbf{qc}_D^{A_i} q_i$, whereas in reality they receive the qubit sent to them by adversary, $f_{A_i}(\mathbf{qc}_D^{A_i} q_i) = \mathbf{qc}_E^{A_i} q_j$. In case the eavesdropper is lucky and chooses the right projection for all three qubits he intercepts, he is able to derive the value of the key. In this case some of the epistemic properties of interest are

- The adversary knows the shared key, i.e. $\Box_E s_6^0$.
- The players and the dealer wrongly think that he does not know this, i.e.

$$\Box_\sigma \neg \Box_E s_6^0.$$

Note that here the adversary has to be more lucky than in Ekert'91. This is because he has to intercept the qubits of three allied players instead of one, and has to choose from three measurement bases.

(iii) Adversary's hell

This is the same as above, but the players suspect adversary's actions, that is according to rule (vii), it appears to them either there was no interception or there was one and the above sequence of actions took place by the adversary. Thus we obtain

$$f_{A_i}(\mathbf{qc}_i^D q_i) = \mathbf{qc}_i^D q_i \vee (\mathbf{qc}_i^D q_i; \mathbf{qc}_D^E q_i; P_i^{E,e}; N_j^{E,e}; \mathbf{qc}_i^E q_j)$$

Similarly, the dealer suspect adversary's actions on the receipt of his sent qubit

$$f_D(\mathbf{qc}_D^{A_i} q_i) = \mathbf{qc}_D^{A_i} q_i \vee (\mathbf{qc}_D^E q_i; P_i^{E,e}; N_j^{E,e}; \mathbf{qc}_i^E q_j; \mathbf{qc}_E^{A_i} q_j)$$

In this case, an interesting epistemic property would be the following

The dealer and the players are not sure anymore if the adversary knows their secret bit, and thus if the bit can be treated as a secret i.e.

$$\neg \Box_\sigma \neg \Box_E s_6^0.$$

3.2 Verifying Epistemic Properties

As examples, we verify two properties: one from the agents' heaven and one from the adversary's hell.

- **Agents' heaven**

From this scenario, we verify the following property

$$\otimes_{i=1}^6 q_i \vdash [\pi] \Box_D \Box_i s_6^0$$

The atomic expressions are generated via the following rewritings, where α_i 's denote the juxtaposed actions of π

$$\begin{aligned} \otimes_{i=1}^6 q_i \vdash [\pi] \Box_D \Box_{A_i} s_6^0 &\rightsquigarrow \otimes_{i=1}^6 q_i; \pi \vdash \Box_D \Box_{A_i} s_6^0 \rightsquigarrow \\ f_{A_i} f_D (\otimes_{i=1}^6 q_i; \pi) \vdash s_6^0 &\rightsquigarrow f_{A_i} f_D (\otimes_{i=1}^6 q_i); f_{A_i} f_D (\pi) \vdash s_6^0 \\ &\rightsquigarrow f_{A_i} f_D (\otimes_{i=1}^6 q_i); f_{A_i} f_D (\alpha_1); \dots; f_{A_i} f_D (\alpha_n) \vdash s_6^0. \end{aligned}$$

By rule (ii) of appearances we have $f_{A_i} f_D (\otimes_{i=1}^6 q_i) = \otimes_{i=1}^6 q_i$. By rule (iv) and our assumptions on channels, we have $f_D (\alpha_i) = \alpha_i$ for α_{A_i} a quantum or broadcast communication action. By rule (vi) for communication between players we have $f_D (c!_{\beta}^{A_i} s_i^j) = c!_{\beta}^{A_i} s_i^j \vee c!_{\beta}^{A_i} \bar{s}_i^j$. Similarly for the projection actions we have

$$f_D (P_6^{D,+Z}) = P_6^{D,+Z}$$

and

$$f_D (P_1^{A_1,-Z}) = P_1^{A_1,-Z} \vee P_1^{A_1,+Z} \vee P_1^{A_1,-X} \vee P_1^{A_1,+X} \vee P_1^{A_1,-Y} \vee P_1^{A_1,+Y} .$$

The values for the f_{A_i} 's are similarly set. Substituting these values in the above expression, we first eliminate the traces in which the bases of projections do not match the announced bases. Next we eliminate the communication actions from these traces whose content do not match the projections. As a result, we obtain a set of atomic expressions, of which only those satisfying $s_6^0 = s_1^{b_1} \oplus s_2^{b_2} \oplus s_3^{b_3}$ are well-defined in DMC. An example (out of four) is

$$\otimes_{i=1}^6 q_i; N_1^D; \dots; N_6^D; \prod_{e_{ij}} E_{ij}^D; P_6^{D,+Z}; P_1^{A_1,+Z}; P_2^{A_2,-X}; P_3^{A_3,-Z} \vdash s_6^0 .$$

This atomic expression is true, since in all its final configurations s_6 is 0, and thus our epistemic property holds for the secret sharing protocol.

- **Adversary's hell**

On the contrary, in the adversary's hell, one shows that the epistemic property $\Box_D \neg \Box_E s_6^0$ does not hold and thus s_6^0 is not treated a secret anymore. Moreover, we also discover paths of an intercept-change attack for each agent, for example the one for the player A_1 contains the following sequence of actions

$$\dots; \mathbf{qc}_1^D q_1; \mathbf{qc}_1^E q_1; P_1^{E,+Z}; N_7^{E,+Z}; \mathbf{qc}_1^E q_7; \mathbf{qc}_E^{A_1} q_7; P_7^{A_1,+Z}; \dots$$

In this path, the adversary receives dealer's original qubit q_1 that was meant to be received by agent 1, then measures it in basis Z by doing projection $P_1^{E,+Z}$, then prepares a new qubit q_7 according to his measurement result and sends it to agent 1. The adversary turns out to be lucky and agent 1 picks the same measurement basis as him, that is Z and does the same projections. The classical result of this projection will for sure be the same as adversary's but might not the same as the dealer's.

4 Conclusion

In this article we proposed a new framework for formal analysis of security issues in quantum cryptographic protocols. Our framework combines an algebraic rewrite system with a specification language for quantum distributed computations. The former provides machinery to work with uncertainties of agents in a protocol in a compositional way, while the latter inherently encodes the rules of quantum mechanics. Our framework was put to test in the analysis of a recent quantum secret sharing protocol based on graph states, where we proved some epistemic properties of the protocol in the presence and absence of an active adversary and discovered paths of an intercept-exchange attack. For a full analysis one needs to generate many more epistemic traces and the need for automation and software implementation is gravely felt. A software implementation of the algebra [15] is already in place to handle part of the verification. The construction of a tool that automatically derives the traces and semantics of a protocol is currently underway.

Acknowledgments

We had fruitful discussions with V. Danos, P. Panangaden, D. Markham. The second author has given talks on a version of the algebra with some quantum axioms encoded in it, which was partly based on joint work with E. Kashefi.

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