HAUSDORFF AND GROMOV DISTANCES IN QUANTALE–ENRICHED CATEGORIES

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Abstract

This work studies Hausdorff and Gromov distances in quantale–enriched categories or $\mathcal{V}$–categories. We study those distances from classical and categorical perspectives. The classical study is conducted in the category of metric spaces and its results are mostly well known, but even in this setting we relax standard assumptions in several theorems. The categorical approach builds on the results of the classical one. The latter approach led to the discovery of several interesting properties of those distances in a categorical setting. A concise introduction to $\mathcal{V}$–categories is provided in the second chapter. The last chapter introduces the Vietoris topology and acts as a starting point for the treatment of this topology in a categorical setting.

To my parents

ჩემ მშობლებს

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1 Introduction

1.1 Overview

In 1973 F.W. Lawvere published his famous article *Metric spaces, generalized logic and closed categories*. The article proposed to view metric spaces as categories enriched over $[0, \infty]$. In 2002, the article along with a commentary by its author appeared in *Reprints in Theory and Applications of Categories*. In this commentary the author hints at some connections between Gromov and Hausdorff distances and enriched category theory. Discussions with my supervisor, W. Tholen, in the Summer and Fall of 2007 along with Lawvere's hints led to the establishment of the main goal of this work: to express and understand those distances in the more restrictive setting of quantale–enriched categories or the $\mathcal{V}$–categorical setting.

The work began with the study of Gromov and Hausdorff distances in their natural habitat: the category of metric spaces. It soon became clear that there are numerous restrictions that are unnecessarily imposed on the Hausdorff distance in its classical treatment, and consequently also on the Gromov distance. As the reader shall soon find out, our approach to those distances minimizes the number of assumptions, while producing the same results. Perhaps the most visible manifestation of this is our work with the more general notion of L-metrics instead of metrics. When compared to metric spaces, $\mathcal{V}$–categories have very little structure; surprisingly, many $\mathcal{V}$–categorical analogs of classical results hold even in this lax environment.

The $\mathcal{V}$–category setting is dynamic. By setting $\mathcal{V}$ to concrete models of quantales, we vary the category in which we work. In Chapter 2 we shall explain that for some choices of $\mathcal{V}$, $\mathcal{V}$–categories turn into L–metric spaces and for other choices they turn into ordered sets. Consequently, by defining Gromov and Hausdorff distances for $\mathcal{V}$–categories, we define them for many categories – one for each choice of $\mathcal{V}$. We shall take advantage of this and explicitly describe the Hausdorff order on the powerset of an ordered set and the Gromov order on the objects of Ord.

We use categorical tools to study concepts in the $\mathcal{V}$–categorical setting. The distances we deal with are no exception. For example, we define the Hausdorff functor, notice that it is a part of a monad and study the Eilenberg–Moore algebras of this monad. We describe the Gromov distance using $\mathcal{V}$–modules; we also express it as the colimit of a certain functor. We shall see that the categorical approach reveals some interesting interconnections between those two distances and exposes previously hidden properties.

The Vietoris topology is a generalization of the Hausdorff distance to Top – the topology induced by the Hausdorff metric is the Vietoris topology. Heaving dealt with hyperspaces in the metric setting, we begin working with them in the topological setting. The last chapter of this work deals with the Vietoris topology. We examine many results from classical theory; as before, we relax conventional, but unnecessary conditions. We also give a categorical treatment of the Vietoris topology by defining the Vietoris monad and describing its algebras.

Each subsequent chapter has a section with introductory and historical comments.

1.2 Notation

We list some of the standard notation we shall use throughout this work. We use capital letters X, Y, Z, ... to denote sets. We call a $(X, \leq)$ an *ordered set* provided that

$$\forall x, y, z \in X \;\; x \leq x \text{ and } \big(x \leq y, y \leq z \implies x \leq z\big).$$

A map f from an ordered set X to an ordered set Y is *monotone* if

$$x \leq x' \implies f(x) \leq f(x').$$

A function $d : X \times X \to [0, \infty]$ is a *metric* if d satisfies the following axioms:

1. $\forall x \in X \;\; d(x, x) = 0$ *(reflexivity)*;

2. $\forall x, y \in X \;\; d(x,y) = d(y,x)$ *(symmetry)*;

3. $\forall x, y \in X \;\; \big(d(x,y) = 0 \implies x = y\big)$ *(separation)*;

4. $\forall x, y \in X \;\; d(x,y) < \infty$ *(finiteness)*;

5. $\forall x, y, z \in X \;\; d(x,y) \leq d(x,z) + d(z,y)$ *(triangle inequality)*.

A *metric space* is a pair (X, d) where d is a metric. We often will relax the requirements on a metric: a function $d : X \times X \longrightarrow [0, \infty]$ is an L-*metric* if it satisfies only reflexivity and the triangle inequality. A set X equipped with an L-metric d is called an L-metric space. We shall often write X for an L-metric space when the L-metric d is either clear from the context or is not directly used.

Given any L-metric space (X, d) and some $x \in X$, we define the ε*–neighborhood* of x to be

$$\eta^d_\varepsilon(x) = \{y \in X \mid d(y,x) < \varepsilon\}.$$

If the L–metric d is clear from the context, we shall omit it from our notation and just write $\eta_\varepsilon(x)$ for $\eta^d_\varepsilon(x)$. Note that we define neighborhoods for non–symmetric d and hence the order of x and y in the above definition is crucial.

We call a function $f : (X, d_X) \longrightarrow (Y, d_Y)$ from an L-metric space X to an L-metric space Y *non–expansive* provided that

$$\forall x, x' \in X, \;\; d_Y\big(f(x), f(x')\big) \leq d(x, x').$$

We shall often work in the category LMet of L-metric spaces and non–expansive maps. A non–expansive map $f : (X, d_X) \longrightarrow (Y, d_Y)$ is an *isometry* provided that

$$\forall x, x' \in X, \ \ d_Y\big(f(x), f(x')\big) = d(x, x').$$

Isomorphisms in LMet are exactly the bijective isometries. We say that X and Y are isometric if they are isomorphic in LMet.

Given any set X, we denote its powerset by PX.

Let X be an ordered set. Let $A \subseteq X$ be a subset. The *down closure* of A is

$$\downarrow A := \{x \in X \mid \exists y \in A \ (x \leq y)\}.$$

We say that A is *down–closed* provided that $A = \downarrow A$. The set of all the down–closed subsets of X is denoted by DX.

A mapping between two ordered sets

$$\varphi : X \longrightarrow Y,$$

is a *right adjoint* or *has a left adjoint* provided that there exists a map

$$\psi : Y \longrightarrow X$$

such that

$$y \leq \varphi(x) \iff \psi(y) \leq x.$$

Set	sets and mappings
Met	metric spaces and non–expansive maps
Met_c	compact metric spaces and non–expansive maps
LMet	L–metric spaces and non–expansive maps
Ord	ordered sets and monotone maps
Top	topological spaces and continuous maps
CompHaus	compact Hausdorff spaces and continuous maps
$\mathcal{V}$-Rel	sets and $\mathcal{V}$–relations
$\mathcal{V}$-Cat	$\mathcal{V}$–categories and $\mathcal{V}$–functors
$\mathcal{V}$-Mod	$\mathcal{V}$–categories and $\mathcal{V}$–modules
C$\mathcal{V}$Lat	$\mathcal{V}$–categorical lattices and sup–preserving $\mathcal{V}$–functors

Table 1.1: List of categories

We call ψ the *the left adjoint of* φ. Both the left and the right adjoint are monotone. Furthermore,

$$1_Y \leq \varphi \cdot \psi \quad \text{and} \quad \psi \cdot \varphi \leq 1_X.$$

When φ is right adjoint and ψ is the corresponding left adjoint we write

$$\psi \dashv \varphi.$$

Finally we provide the following table of categories:

2 $\mathcal{V}$-Categories

2.1 Introduction

It is well known that groups are one–object categories and ordered sets are categories with rather simple hom–sets; category theory is thus used to study those objects. In 1973 W. Lawvere observed that metric spaces also can be described as certain categories. A category $\mathcal{C}$ consists of the following data:

1. a class ob$\mathcal{C}$ of objects;
2. a mapping $\hom : \mathrm{ob}\mathcal{C} \times \mathrm{ob}\mathcal{C} \longrightarrow \mathsf{Set}$ such that
   - there exists a composition mapping $\hom(A, B) \times \hom(B, C) \longrightarrow \hom(A, C)$; composition is associative.
   - For each $A \in \mathrm{ob}\mathcal{C}$ there exists a mapping $\{*\} \longrightarrow \hom(A, A)$ which chooses the identity with respect to composition.

An L–metric space is described by the following data:

1. a set X of points;

2. a distance mapping $d : X \times X \longrightarrow [0, \infty]$ such that

   - for all $x, y, z \in X$, $d(x, z) + d(z, y) \geq d(x, y)$;

   - for all $x \in X$, $0 \geq d(x, x)$

The connection between those two constructions becomes evident when one realizes that $[0, \infty] = (([0, \infty], \geq), +, 0)$ is a closed symmetric monoidal category (see Section 2.4), where $\longrightarrow$ in set is replaced by $\geq$, cartesian product by $+$ and the one–element set by 0. Thus an L–metric space is a category enriched over the closed symmetric monoidal category $[0, \infty]$. This realization allows us to study metric spaces using methods and results from enriched category theory.

Enriched category theory is often too general for many metric properties; quantale–enriched categories provide a more suitable environment for studying metric spaces. We shall soon see that $[0, \infty]$ is an example of a commutative quantale. This chapter serves as a very brief introduction to quantale–enriched categories. It closely follows (T1), but is less comprehensive, for we only present material that is used in subsequent chapters of this work.

We begin with a definition of a quantale and provide some important examples. We also define constructive complete distributivity for commutative quantales. Next, we give a definition of $\mathcal{V}$–categories and $\mathcal{V}$–functors and consider several examples. It turns out that $\mathcal{V}$ itself is a $\mathcal{V}$–category; we discuss the structure on $\mathcal{V}$. Important constructions in the category $\mathcal{V}$-Cat of $\mathcal{V}$–categories and $\mathcal{V}$–functors are also treated. W introduce $\mathcal{V}$–modules, which play an important role in the categorical definition of Gromov distance. The categorical approach often requires one to rethink standard definitions in order to work with them in a more general setting; this general setting is often independent from the context in which those definitions were originally conceived. One important example of this is the concept of Lawvere–completeness or L–completeness which generalizes Cauchy completeness to $\mathcal{V}$–categories. We discuss L–completeness in the last section.

2.2 Quantales

We say that the quadruple $\mathcal{V} = (\mathcal{V}, \bigvee, \otimes, k)$ is a *unital commutative quantale* provided that $(\mathcal{V}, \bigvee)$ is a complete lattice, $\otimes$ is an associative and commutative operation on $\mathcal{V}$ that preserves suprema in each variable:

$$u \otimes \bigvee_i v_i = \bigvee_i u \otimes v_i,$$

and k is the unit with respect to $\mathcal{V}$: for all $v \in \mathcal{V}$, $v \otimes k = k \otimes v = v$. $\mathcal{V}$ is non–trivial provided that $k \neq \bot$.

$\mathcal{V}$ is a *frame* when $\otimes = \wedge$ and $k = \top$.

Examples 2.2.1. 1. $2 = \{\bot < \top\}, \otimes = \wedge, k = \top$.

2. $\mathbb{P}_+ = ([0,\infty], \geq), \otimes = +, k = 0.$

$\mathcal{V}$ is *constructively completely distributive*, or ccd, provided that the mapping

$$\bigvee : D\mathcal{V} \to \mathcal{V}$$

has a left adjoint. This means, in particular, that there exists $A : \mathcal{V} \longrightarrow D\mathcal{V}$ such that for all $D \in D\mathcal{V}$ and $x \in \mathcal{V}$

$$A(\bigvee(D)) \subseteq D \quad \text{and} \quad x \leq \bigvee A(x)$$

or equivalently

$$A(x) \subseteq\downarrow S \iff x \leq \bigvee S.$$

Since $x = \bigvee \downarrow \{x\}$,

$$A(x) = A(\bigvee \downarrow \{x\}) \subseteq\downarrow \{x\}.$$

Therefore

$$\bigvee A(x) \leq \bigvee \downarrow \{x\} = x,$$

and consequently

$$x = \bigvee A(x).$$

Let's describe $A(x)$ explicitly: we define the *way–below* relation $\ll$ on $\mathcal{V}$ by

$$\begin{aligned} x \ll y \quad &\iff \quad \forall S \subseteq \mathcal{V} \left(y \leq \bigvee S \implies x \in\downarrow S\right) \\ &\iff \quad \forall S \subseteq \mathcal{V} \left(y \leq \bigvee S \implies \exists s \in S(x \leq s)\right). \end{aligned}$$

We immediately obtain the following useful property:

$$x \ll \bigvee S \implies \exists s \in S(x \leq s).$$

We can now describe A:

Proposition 2.2.2.

$$A(x) = \{u \in \mathcal{V} \mid u \ll x\}.$$

Proof. "$\subseteq$": Let $u \in A(x)$. Let $S \subseteq \mathcal{V}$ be such that $x \leq \bigvee S$. Then $u \in A(x) \subseteq \downarrow S$ and hence $u \ll x$.

"$\supseteq$": Say $u \ll x = \bigvee A(x)$. Then $u \in \downarrow A(x) = A(x)$. □

We proved one direction of the following Theorem; the other direction can be easily verified.

Theorem 2.2.3 ((Ho), Theorem 1.3)**.** *$\mathcal{V}$ is constructively completely distributive if, and only if, every $x \in \mathcal{V}$ can be written as*

$$x = \bigvee \{u \in \mathcal{V} \mid u \ll x\}.$$

The following proposition lists some basic properties of the relation $\ll$.

Proposition 2.2.4 ((Fl), Lemma 1.2)**.** *1. $x \ll y \implies x \leq y$*

2. $z \leq y, y \ll x \implies z \leq x$

3. $z \ll y, y \leq x \implies z \ll x$.

The most important example of a ccd quantale is $\mathbb{P}_+$ where the relation $\ll$ is given by $>$.

2.3 $\mathcal{V}$-relations and $\mathcal{V}$-categories

We define the category of $\mathcal{V}$–relations as follows. The object of this category are sets. A morphism between X and Y is a mapping

$$r : X \times Y \longrightarrow \mathcal{V}.$$

We write $X \nrightarrow Y$ instead of $X \times Y \longrightarrow \mathcal{V}$. Given two morphisms, $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ we define $s \cdot r : X \nrightarrow Z$ by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

The identity morphism on X is defined by

$$1_X(x, x') = \begin{cases} k, & \text{if } x = x'; \\ \perp, & \text{otherwise.} \end{cases}$$

There is a faithful functor

$$\mathsf{Set} \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}$$

defined by

$$(f : X \longrightarrow Y) \mapsto (f : X \nrightarrow Y)$$

with

$$f(x,y) = \begin{cases} k, & \text{if } f(x) = y; \\ \perp, & \text{else.} \end{cases}$$

and a mapping $(-)^\circ : \mathcal{V}\text{-}\mathsf{Rel} \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}$ that maps objects identically and

$$r^\circ(x,y) = r(y,x).$$

Examples 2.3.1.

2–Rel is just the category of sets and relations.

$\mathbb{P}_+$–Rel is the category of sets and fuzzy relations.

Next we define the category $\mathcal{V}\text{-}\mathsf{Cat}$ of $\mathcal{V}$–categories and $\mathcal{V}$–functors. Objects of $\mathcal{V}\text{-}\mathsf{Cat}$ are pairs (X,a) where X is a set and $a : X \nrightarrow X$ is a $\mathcal{V}$–relation that satisfies:

1. $1_X \leq a$ $(\forall x \in X : k \leq a(x,x))$ (reflexivity).

2. $a \cdot a \leq a$ $(\forall x,y,z \in X : a(x,y) \otimes a(y,z) \leq a(x,z))$ (triangle inequality).

$f : (X,a) \longrightarrow (Y,b)$ is a morphism in $\mathcal{V}\text{-}\mathsf{Cat}$ provided that

$$f \cdot a \leq b \cdot f \ \ \Big(\forall x,y \in X : a(x,y) \leq b(f(x),f(y))\Big).$$

We call morphisms in $\mathcal{V}\text{-}\mathsf{Cat}$ $\mathcal{V}$*–functors.*

We also have the endofunctor

$$(-)^{\mathrm{op}} : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$$

that maps

$$(f : (X,a) \longrightarrow (Y,b)) \mapsto (f : (X,a^\circ) \longrightarrow (Y,b^\circ)).$$

We write X^{op} for (X,a°).

Examples 2.3.2. 1. $2\text{-}\mathsf{Cat} = \mathsf{Ord}$. Indeed, given a 2–category (X,a) we define

$$x \le y \iff a(x,y) = k = \top.$$

Then $a(x,x) = \top \implies x \le x, \forall x \in X$ and $\big(a(x,y) \wedge a(y,z) \le a(x,z)\big) \implies \big(x \le y \text{ and } y \le z \implies x \le z\big)$ for all $x,y,z \in X$. 2–functors are easily seen to be monotone mappings.

2. $\mathbb{P}_+\text{-}\mathsf{Cat} = \mathsf{LMet}$. Given a $\mathbb{P}_+$–category (X,a) we have $a : X \times X \longrightarrow [0,\infty]$ such that $a(x,x) \ge 0$ for all $x \in X$ and $a(x,y) + a(y,z) \ge a(x,z)$ for all $x,y,z \in X$. $\mathbb{P}_+$–functors are non–expansive maps.

2.4 $\mathcal{V}$ as a $\mathcal{V}$-category, constructions in $\mathcal{V}$-Cat

$\mathcal{V}$ is itself a $\mathcal{V}$–category. For all $u \in \mathcal{V}$, the $\mathcal{V}$–functor

$$u \otimes (-) : \mathcal{V} \to \mathcal{V}$$

preserves $\bigvee$ and hence has a right adjoint. We denote this right adjoint by $u \multimap (-)$. Thus we have

$$z \le u \multimap v \iff u \otimes z \le v.$$

Also,

$$u \multimap v = \bigvee \{z \mid u \otimes z \leq v\}.$$

From $k \otimes v = v$ one gets $k \leq v \multimap v$ and from

$$u \otimes (u \multimap v) \otimes (v \multimap w) \leq v \otimes (v \multimap w) \leq w,$$

one obtains

$$(u \multimap v) \otimes (v \multimap w) \leq u \multimap w.$$

Hence, $(\mathcal{V}, \multimap)$ is a $\mathcal{V}$–category.

When $\mathcal{V} = \mathbb{P}_+$, we have

$$v \multimap u = \begin{cases} u - v, & \text{if } u \geq v; \\ 0, & \text{else.} \end{cases}$$

Given two $\mathcal{V}$–categories (X, a) and (Y, b), we set

$$X \otimes Y = (X \times Y, a \otimes b)$$

with

$$(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y').$$

We also define

$$Y^X = (\mathcal{V}\text{-}\mathsf{Cat}(X, Y), d)$$

with

$$d(f, g) = \bigwedge_{x \in X} b(f(x), g(x)).$$

We denote $\mathcal{V}^{X^{\mathrm{op}}}$ by $\hat{X}$ and the structure on $\hat{X}$ by $\hat{a}$.

There is a natural correspondence of $\mathcal{V}$–functors

$$\frac{Z \to Y^X}{Z \otimes X \to Y}.$$

Finally, we define the identity object $E = (\{*\}, 1_E)$ in $\mathcal{V}$-Cat. For more details see 1.6 in (T1).

2.5 $\mathcal{V}$–modules

$\mathcal{V}$–modules are $\mathcal{V}$–relations that are compatible with $\mathcal{V}$–categorical structures. Let (X, a) and (Y, b) be $\mathcal{V}$–categories and

$$\varphi : X \nrightarrow Y$$

a $\mathcal{V}$–relation. We say that φ is a $\mathcal{V}$–*module* provided that

$$\varphi \cdot a \leq \varphi \text{ and } \varphi \cdot b \leq \varphi.$$

In fact, since $1_X \leq a$ and $1_Y \leq b$, we actually have equality:

$$\varphi \cdot a = \varphi \text{ and } \varphi \cdot b = \varphi. \tag{2.1}$$

We write $\varphi : X \nrightarrow\!\!\circ\; Y$ to indicate that the relation φ is a $\mathcal{V}$–module.

In turns out that $\mathcal{V}$–modules are closed under relational composition. Equation (2.1) shows that the structure a is the identity $\mathcal{V}$–module on (X, a). $\mathcal{V}$–categories and $\mathcal{V}$–modules define a category

$$\mathcal{V}\text{-}\mathsf{Mod}.$$

We have the following useful proposition:

Proposition 2.5.1 ((T1), Proposition 1.8)**.** *For $\mathcal{V}$–categories X, Y and a $\mathcal{V}$–relation $\varphi : X \mathrel{\mapstochar\rightarrow} Y$, one has*

$$\varphi : X \mathrel{\circ\!\!\!\rightarrow} Y \text{ is a } \mathcal{V}\text{–module} \iff \varphi : X^{op} \otimes Y \longrightarrow \mathcal{V} \text{ is a } \mathcal{V}\text{–functor.}$$

2.6 L-completeness

L–completeness, or Lawvere–completeness, is a generalization of Cauchy completeness to $\mathcal{V}$–categories.

Applying the definition of adjointness to $\mathcal{V}$–modules, we see that $\varphi : X \mathrel{\circ\!\!\!\rightarrow} Y$ is left adjoint if there exists $\psi : Y \mathrel{\circ\!\!\!\rightarrow} X$ such that

$$a \leq \psi \cdot \varphi \text{ and } \varphi \cdot \psi \leq b.$$

Definition 2.6.1. A $\mathcal{V}$–category (X, a) is *L–complete* provided that either one of the following two equivalent conditions holds:

1. every left–adjoint $\mathcal{V}$–module $\varphi : E \mathrel{\circ\!\!\!\rightarrow} X$ is of the form $\varphi = a(x, -)$, for some $x \in X$;

2. every right–adjoint $\mathcal{V}$–module $\psi : X \nrightarrow E$ is of the form $\psi = a(-, x)$, for some $x \in X$.

Having a pair of adjoint modules $\varphi \dashv \psi : X \nrightarrow E$, we define

$$j = \varphi(*, -) : X \cong E^{\mathrm{op}} \otimes X \longrightarrow \mathcal{V}$$

and

$$h = \psi(-, *) : X^{\mathrm{op}} \cong X^{\mathrm{op}} \otimes E \longrightarrow \mathcal{V}$$

(see Proposition 2.5.1). The adjointness condition may be expressed in terms of j and h:

$$h(x) \otimes j(y) \leq a(x, y) \text{ and } k \leq \bigvee_{y \in X} h(y) \otimes j(y), \tag{2.2}$$

$(x, y \in X)$. The first inequality implies:

$$j(y) \leq \bigwedge_{x \in X} \big(h(x) \multimap a(x, y)\big)$$

and the second

$$k \leq \bigvee_{y \in X} \Big(h(y) \otimes \bigwedge_{y \in X} \big(h(x) \multimap a(x, y)\big)\Big) \tag{2.3}$$

We say that a $\mathcal{V}$–functor $h : X^{\mathrm{op}} \longrightarrow \mathcal{V}$ is a *tight $\mathcal{V}$–form* if h satisfies equation (2.3).

Theorem 2.6.2 ((T1), Theorem 1.16)**.** *A $\mathcal{V}$–category (X, a) is L-complete if, and only if, every tight $\mathcal{V}$-form on X is of the form $a(-, x)$ for some $x \in X$.*

We include the following results for completeness.

Theorem 2.6.3 ((T1), Corollary 1.18)**.** *$\mathcal{V}$ is L–complete.*

Theorem 2.6.4 ((T1), Theorem 1.19)**.** *$\hat{X}$ is L–complete, for every $\mathcal{V}$–category X.*

Example 2.6.5. Our goal is to show that when $\mathcal{V} = \mathbb{P}_+$, L–completeness and Cauchy completeness are equivalent. First, we establish a one–to–one relationship between equivalence classes of Cauchy sequences and pairs of adjoint modules $\big($given two Cauchy sequences (x_n) and (y_n) in an L–metric space (X, d), we say that they are equivalent provided that for each $\varepsilon > 0$, there exists a natural number N such that for all $n \geq N$, $d(x_n, y_n) < \varepsilon\ \big)$.

Given an equivalence class of Cauchy sequences $[(x_n)]$ in (X, d) we define

$$h : X^{\mathrm{op}} \longrightarrow [0, \infty] \text{ and } j : X \longrightarrow [0, \infty]$$

by

$$h(y) = \lim_n d(y, x_n) \text{ and } j(y) = \lim_n (x_n, y).$$

Then h and j are non–expansive: we need, for all $x, y \in X$,

$$h(x) \multimap h(y) \leq d(x, y).$$

That is

$$\lim_n \big(d(x, x_n) \multimap d(y, x_n)\big) \leq d(x, y),$$

but this immediately follows from

$$d(x, x_n) \multimap d(y, x_n) \leq d(x, y).$$

Similarly, j is non–expansive. From

$$d(x, y) \leq d(x, x_n) + d(x_n, y)$$

we conclude

$$d(x, y) \leq h(x) + j(y).$$

Finally, since for all $\varepsilon > 0$ there exists N such that for all $n \geq N$

$$d(x_n, x_N) < \varepsilon \text{ and } d(x_N, x_n) < \varepsilon,$$

we have

$$\bigwedge_y \big(h(y) + j(y)\big) = 0.$$

Hence h and j satisfy (2.2) and hence correspond to a pair of adjoint modules $\varphi \dashv \psi : X \nrightarrow E$.

Conversely, given such h and j, we define a sequence by choosing for each $n \in \mathbb{N}$ an $x_n \in X$ such that

$$h(x_n) + j(x_n) \leq \frac{1}{n}.$$

Then (x_n) is Cauchy:

$$d(x_n, x_m) \leq h(x_n) + j(x_m) \leq \frac{1}{m} + \frac{1}{n}.$$

Furthermore,

$$d(y, x_m) \leq h(y) + j(x_m) \leq h(y) + \frac{1}{m},$$

so that

$$\lim_n d(y, x_n) \leq h(y).$$

Since h is non–expansive

$$h(x) \multimap h(y) \leq d(x, y)$$

and hence

$$h(y) = \lim_m h(y) \multimap \lim_m h(x_m) \leq \lim_m d(y, x_m).$$

We thus get

$$h(y) = \lim_m d(y, x_m)$$

and similarly

$$j(y) = \lim_m d(x_m, y).$$

Finally, if (x_n) and (x'_n) are such that

$$\lim_m d(y, x_m) = h(y) = \lim_m d(y, x'_m),$$

then (x_n) and (x'_n) are equivalent: given any $\varepsilon > 0$, there exists a natural number N such that for all $n, m \geq N$, $d(x_m, x_n) < \varepsilon$. So, for all $m \geq N$, $\lim_n d(x_m, x'_n) < \varepsilon$. Hence, there exists N' such that for all $n \geq N'$ and $m \geq N$

$$d(x_n, x'_m) < \varepsilon.$$

So, for all $m \geq \max\{N, N'\}$,

$$d(x_m, x'_m) < \varepsilon.$$

Hence (x_n) and (x'_n) are equivalent and we established the desired one–to–one correspondence.

Next we establish a relationship between convergence and representability. If (x_n) is a Cauchy sequence that converges to x, then

$$h(y) = \lim d(y, x_n) = d(y, x),$$

so that $h = d(-, x)$. Conversely, if $h = d(-, x)$, then

$$0 = d(x, x) = \lim_n d(x, x_n),$$

so $x_n \longrightarrow x$.

We finally show that X is Cauchy–complete if, and only if, it is L–complete. Suppose that X is Cauchy complete. Let h be a tight $\mathcal{V}$–form on X and let $[(x_n)]$ the associated equivalence class of Cauchy sequences. Then there exists x such that $x_n \longrightarrow x$ and hence $h = d(-, x)$. Conversely, if X is L–complete, and (x_n) is a Cauchy sequence in X, we can write the associated tight $\mathcal{V}$–form h of $[(x_n)]$ as $h = a(-, x)$ for some x and hence $x_n \longrightarrow x$.

3 Hausdorff distance

3.1 Introduction

Contrary to what its name suggests, Hausdorff distance was introduced by Dimitrie Pompeiu his in 1905 PhD thesis. Pompeiu worked on a problem concerning singularities of uniform analytic functions. In his arguments in the complex plane he needed a notion of distance between closed curves. This was the reason for the introduction of distance between sets and what many call the birth of the theory of hyperspaces. Hausdorff studied this distance in his famous 1914 book. This, along with Hausdorff's popularity, is probably the reason the distance is named after him today. For more details regarding the birth of the Hausdorff distance see (BT) and (Mc).

It did not take long for mathematicians to discover some important properties of the Hausdorff distance. We discuss those properties in this chapter. The Hausdorff distance is also widely used in applications, especially to algorithms for image comparison. The Hausdorff distance is used to measure how different two images are from each other; using more familiar language: how far the images are from one another.

In this chapter we study the Hausdorff distance using classical and categorical tools. We begin with a classical study. We define the Hausdorff distance and its non-symmetric analog. As we shall soon find out, the non–symmetric version of the Hausdorff distance turns out to be more important and more natural than the standard concept (as pointed out by Lawvere). In addition to the classical definition, we also give other formulations. We prove several important results about the Hausdorff distance; namely, we show that completeness and total boundedness get transferred from the base space to the powerset. The chapter contains many examples. Our hope is that those examples will not only illustrate the concepts we introduce, but also highlight the intuitive properties of the Hausdorff distance and consequently convince the reader of the importance of this concept.

Our approach changes when we introduce the Hausdorff distance into the $\mathcal{V}$–categorical setting. In this setting, we concentrate on categorical rather than classical properties. Right from the beginning, we consider the functor

$$H : \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$$

that assigns to every $\mathcal{V}$–category the powerset with the Hausdorff structure. We discuss several formulations of the Hausdorff distance in the $\mathcal{V}$-category setting. The transfer of total boundedness also holds for $\mathcal{V}$–categories, but the transfer of

L–completeness is a more delicate matter. And although we did not prove that this transfer occurs, we discuss our attempts at this problem. The chapter concludes with a strictly categorical approach to the functor H: we first compare H to the generalized powerset functor and note that it is part of a monad which we investigate in the last section.

As the first development in the theory of hyperspaces, the Hausdorff distance is related to many other constructions. In particular, the Hausdorff distance has very intimate ties with both the Gromov distance and the Vietoris topology. For this reason this chapter serves not only as an introduction to the Hausdorff distance and its study in the $\mathcal{V}$–categorical setting, but also as the foundation for the rest of this work.

3.2 Classical definition

For an L–metric space (X, d), a subset $A \subseteq X$ and $\varepsilon > 0$ we define the *ε–neighborhood of A in X* by

$$N_\varepsilon^d(A) = \bigcup_{x \in A} \eta_\varepsilon(x).$$

We shall omit the superscript d when the metric d is clear from the context.

Definition 3.2.1. Let (X, d) be an L–metric space and $A, B \subseteq X$ be subsets. We define the non–symmetric Hausdorff distance $Hd : PX \times PX \longrightarrow [0, \infty]$ by

$$Hd(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq N_\varepsilon(B)\}.$$

Let us consider some examples.

1. Let $A = S^1$ be the unit circle in $\mathbb{R}^2$ equipped with the Euclidian metric and $B = \{(0, 0)\}$ be the origin. Then it is clear that $A \subseteq N_\varepsilon(B)$ if, and only if, $\varepsilon > 1$; we conclude that $Hd(A, B) = 1$. A similar argument shows that $Hd(B, A) = 1$.

2. Let A still denote the unit circle, and now let $B = D^2$ – the unit disk – still considered as subsets of $\mathbb{R}^2$. This time, the containment $A \subseteq B \subseteq N_\varepsilon(B)$, for any $\varepsilon > 0$ shows that $Hd(A, B) = 0$. Now, $B \subseteq N_\varepsilon(A)$ if, and only if, $\varepsilon > 1$. Thus $Hd(B, A) = 1$.

From the non–symmetric definition, we define the Hausdorff distance by symmetrizing:

Definition 3.2.2. Let (X, d) be an L–metric space and $A, B \subseteq X$ be subsets. We define the Hausdorff distance $(Hd)^{\mathrm{sym}} : PX \times PX \longrightarrow [0, \infty]$ by

$$(Hd)^{\mathrm{sym}}(A, B) = \max\{Hd(A, B), Hd(B, A)\}.$$

We have the following obvious formula:

$$(Hd)^{\mathrm{sym}}(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\}.$$

In order to simplify our notation we shall often write $H^s d$ for $(Hd)^{\text{sym}}$. We call (PX, Hd) the *non–symmetric Hausdorff space* of (X, d) and denote it by HX. Analogously, we call $(PX, H^s d)$ the *Hausdorff space* of X and denote it by $H^s X$.

We now compute some distances between familiar sets in $\mathbb{R}^2$.

1. Let (X, d) be a metric space and $A = \emptyset$ and $B \neq \emptyset$ be arbitrary. Then the condition $A \subseteq N_\varepsilon(B)$ holds trivially for any $\varepsilon > 0$, while the condition $B \subseteq N_\varepsilon(A)$ is never satisfied. So, we get that $H^s d(A, B) = \inf \emptyset = \infty$. If also $B = \emptyset$, then both inclusions in the definition hold trivially, and hence $H^s d(A, B) = 0$.

2. Let $A = \{(0, 1)\}$ and $B = \{(0, 0)\}$ be subsets of $\mathbb{R}^2$ with the Euclidian metric. $A \subseteq N_\varepsilon(B)$ and $B \subseteq N_\varepsilon(A)$ if, and only if, $\varepsilon > 1$. Taking the infimum, we see that $H^s d(A, B) = 1$.

3. Let $A = S^1$ be the unit circle in $\mathbb{R}^2$ and $B = \{0, 0\}$ be the origin considered as subsets of $\mathbb{R}^2$. Then $A \subseteq N_\varepsilon(B)$ and $B \subseteq N_\varepsilon(A)$ if, and only if, $\varepsilon > 1$, we conclude that $H^s d(A, B) = 1$, as our geometric intuition suggests.

4. Let A again denote the unit circle, and now let $B = D^2$ – the unit disk both considered as subsets of $\mathbb{R}^2$. Since, we always have the containment $A \subseteq B \subseteq N_\varepsilon(B)$, we just need to verify the other inclusion. Again, $B \subseteq N_\varepsilon(A)$, if, and only if, $\varepsilon > 1$. Taking the infimum, we get: $H^s d(A, B) = 1$.

5. For now we worked with closed and bounded sets. Let us now consider the case $A = D^2 \setminus S^1$ – the unit disk without its boundary, and $B = D^2$ subsets of $\mathbb{R}^2$. Again, we only need to worry about one containment: $B \subseteq N_\varepsilon(A)$, which holds if, and only if, $\varepsilon > 0$. Taking the infimum, we obtain: $H^s d(A, B) = 0$. Thus we see that $H^s d$ is not a metric on the powerset of $\mathbb{R}^2$.

6. If $A = \mathbb{R} \times \{0\}$ and $B = \{(0, 1)\}$ are subsets of $\mathbb{R}^2$ then $H^s d(A, B) = \infty$, since the infimum of the empty set is the top element.

7. We can have finite distance between two infinite sets. Let $A = \mathbb{R} \times 0$ and $B = \mathbb{R} \times 1$ be subsets of $\mathbb{R}^2$. Then both containments holds if, and only if, $\varepsilon > 1$, giving: $H^s d(A, B) = 1$.

Example 6 shows that $H^s d$ is never a metric. However, we shall see later that $H^s d$ does possess some of the essential properties of a metric. To be precise: Hd is reflexive and satisfies the triangle inequality; i.e. it is an L–metric.

Proposition 3.2.3. *Let (X, d) be an L–metric space. Then Hd is reflexive.*

Proof. Given any $A \subseteq X$, we have $Hd(A, A) = 0$, since, $A \subseteq N_\varepsilon(A)$, for all $\varepsilon > 0$. □

We have the following immediate consequence:

Corollary 3.2.4. *The Hausdorff distance $H^s d$ is reflexive.*

Proposition 3.2.5 (Theorem 2.2 in (IN)). *Let (X, d) be an L–metric space. Then Hd satisfies the triangle inequality.*

Proof. Let $A, B, C \subseteq X$ be arbitrary subsets of X. If $A = \emptyset$, then

$$0 = Hd(A, B) \leq Hd(A, C) + Hd(C, A)$$

since 0 is the bottom element in the lattice $[0, \infty]$.

If $A \neq \emptyset$ and $B = \emptyset$, we obtain:

$$\infty = Hd(A, B) \leq Hd(A, C) + Hd(C, B).$$

Indeed, if $C \neq \emptyset$, then $Hd(C, B) = \infty$, and if $C = \emptyset$, then $Hd(A, C) = \infty$.

From here on we assume that A, B and C are all non–empty. Let $\delta > 0$ and $\varepsilon_1 = Hd(A, C) + \delta/2$ and $\varepsilon_2 = Hd(C, B) + \delta/2$. Then

$$A \subseteq N_{\varepsilon_1}(C) \text{ and } C \subseteq N_{\varepsilon_2}(B).$$

From the later containment, we obtain $N_{\varepsilon_1}(C) \subseteq N_{\varepsilon_1}(N_{\varepsilon_2}(B))$. Finally, we apply, the triangle inequality for (X, d) to get $N_{\varepsilon_1}(N_{\varepsilon_2}(B)) \subseteq N_{\varepsilon_1+\varepsilon_2}(B)$. Indeed, given $x \in N_{\varepsilon_1}(N_{\varepsilon_2}(B))$, we can find $y \in N_{\varepsilon_2}(B)$ and $z \in B$ such that $d(x, y) < \varepsilon_1$ and $d(y, z) < \varepsilon_2$. The triangle inequality for d gives $d(x, z) < d(x, y) + d(y, z) < \varepsilon_1 + \varepsilon_2$, showing that $x \in N_{\varepsilon_1+\varepsilon_2}(B)$. We conclude that

$$A \subseteq N_{\varepsilon_1}(C) \subseteq N_{\varepsilon_1}(N_{\varepsilon_2}(B)) \subseteq N_{\varepsilon_1+\varepsilon_2}(B)$$

And so $Hd(A,B) \leq \varepsilon_1 + \varepsilon_2 = Hd(A,C) + Hd(C,B) + \delta$. Since this holds for all $\delta > 0$, we reached the desired conclusion. $\square$

Corollary 3.2.6. *The Hausdorff metric $H^s d$ satisfies the triangle inequality.*

Proof. This follows from Proposition 3.2.5:

$$\begin{aligned} H^s d(A,C) + H^s d(C,B) &\geq Hd(A,C) + Hd(C,B) \\ &\geq Hd(A,B). \end{aligned}$$

and

$$\begin{aligned} H^s d(A,C) + H^s d(C,B) &\geq Hd(C,A) + Hd(B,C) \\ &= Hd(B,C) + Hd(C,A) \\ &\geq Hd(B,A). \end{aligned}$$

So, $H^s d(A,C) + H^s d(C,B) \geq \max\{Hd(A,B), Hd(B,A)\} = H^s d(A,B)$. $\square$

We include the following result about separation of Hd:

Proposition 3.2.7. *Suppose that (X,d) is a metric space. Then $Hd(A,B) = 0$ if, and only if, $A \subseteq \overline{B}$.*

Proof. "$\Rightarrow$": Let $x \in A$ and $\varepsilon > 0$. Then $x \in N_\varepsilon(B)$. So, there exists $y \in B$ with $d(y,x) = d(x,y) < \varepsilon$, or, in other words, $\eta_\varepsilon(x) \cap B \neq \emptyset$. Since ε was chosen arbitrarily, $x \in \overline{B}$.

"$\Leftarrow$": We need to show that for all $\varepsilon > 0$, $A \subseteq N_\varepsilon(B)$. Given $x \in A$ and $\varepsilon > 0$, $\eta_\varepsilon(x) \cap B \neq \emptyset$. So there exists $y \in B$ with $d(y, x) < \varepsilon$. That is, for all $x \in A$, there exists $y_x \in B$ with $x \in \eta_\varepsilon(y_x)$. Hence,

$$A \subseteq \bigcup_{x \in A} \eta_\varepsilon(y_x) \subseteq N_\varepsilon(B).$$

$\square$

We conclude this section with the following classical result:

Theorem 3.2.8 (Theorem 2.2 in (IN))**.** *$H^s d$ is an extended metric on the set CLX of closed subsets of a metric space (X, d). That is, $H^s d$ possesses all the properties of a metric except finiteness.*

Proof. We already showed that $H^s d$ is reflexive (Proposition 3.2.4) and that it satisfies the triangle inequality (Proposition 3.2.6). Clearly it is symmetric. We just need to verify separation: $H^s d(A, B) = 0$ implies $Hd(A, B) = 0 = Hd(B, A)$. Thus, by Proposition 3.2.7, $A \subseteq \overline{B} = B$ and $B \subseteq \overline{A} = A$, obtaining $A = B$, as required. $\square$

Remark. The classical treatment of the Hausdorff distance defines CLX to be all the non–empty closed subsets of X, but, as we observed, the empty set can be included into CLX without difficulty.

3.3 Other formulations

There are several equivalent formulations of the Hausdorff distance; we shall discuss them in this section. As we shall see later, different formulations will be useful for different purposes; this is especially true in the $\mathcal{V}$-category setting.

Perhaps the most recognizable formulation of the Hausdorff distance is given by the following formula:

$$H_1^s d(A, B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{y\in B}\inf_{x\in A} d(x, y)\}.$$

The non-symmetric version of this definition is:

$$H_1 d(A, B) = \sup_{x\in A}\inf_{y\in B} d(x, y) \tag{3.1}$$

Proposition 3.3.1. *Let (X, d) be an L–metric space. For any $A, B \subseteq X$,*

$$Hd(A, B) = H_1 d(A, B).$$

Proof. "$\geq$": We need to show that for all $x \in A$, and for any ε such that $A \subseteq N_\varepsilon(B)$, $d(x, B) := \inf_{y\in B} d(x, y) \leq \varepsilon$. Indeed, given such x and ε, since $A \subseteq N_\varepsilon(B)$, there exists $y \in B$ with $d(x, y) < \varepsilon$. Thus, $d(x, B) \leq d(x, y) < \varepsilon$.

"$\leq$": Let $Hd(A, B) < \varepsilon$. Then $A \subseteq N_\varepsilon(B)$, so that for any $x \in A$ there exists $y \in B$ such that $d(x, y) < \varepsilon$. Thus, for all $x \in A$, $d(x, B) \leq \varepsilon$ and hence, $H_1 d(A, B) \leq \varepsilon$. Taking $\varepsilon = Hd(A, B) + \delta$ for some $\delta > 0$, we get

$$H_1 d(A, B) \leq Hd(A, B) + \delta;$$

since δ was chosen arbitrarily, we are done. $\square$

Corollary 3.3.2. *For any L–metric space (X, d), $H^s d = H_1 d^s$.*

From the proof of Proposition 3.3.1, we notice that we can rewrite (3.1) as:

$$Hd(A, B) = \sup_{x \in A} d(x, B)$$

$(d(x, B) = \inf_{y \in B} d(x, y))$.

We further notice that we can regard $d(-, B) : X \longrightarrow [0, \infty]$ as a non–expansive function from the L-metric space X to the L-metric space $[0, \infty]$. Here the L-metric on $[0, \infty]$ is given by

$$a \multimap b = \begin{cases} a - b, & \text{if } a \geq b; \\ 0, & \text{otherwise.} \end{cases}$$

$d(-, B)$ is indeed non–expansive:

$$\begin{aligned} d(x, x') + d(x', B) &= d(x, x') + \inf_{y \in B} d(x', y) \\ &= \inf_{y \in B} \big(d(x, x') + d(x', y)\big) \\ &\geq \inf_{y \in B} d(x, y) = d(x, B). \end{aligned}$$

Whence,

$$d(x, x') \geq d(x, B) \multimap d(x', B).$$

We call $d(-, B)$ the *distance functional induced by B*. When A and B are closed sets, we can identify them with distance functionals they induce. Indeed,

$d(-, A) = d(-, B)$ implies that for all $x \in A$, $d(x, B) = 0$ and hence, $x \in \overline{B} = B$. Similarly $B \subseteq A$. The following proposition shows that the non–symmetric Hausdorff distance is just the distance between distance functionals in the space $[0, \infty]^X$ with the metric of uniform convergence:

Proposition 3.3.3 (See section 3.2 in (Be)). *For an L–metric space (X, d), and $A, B \subseteq X$ subsets of X, we have*

$$Hd(A, B) = \sup_{x \in X} \big(d(x, B) \multimap d(x, A) \big).$$

Proof. Say $A = \emptyset$. Then $0 = Hd(A, B)$, and since the infimum of the empty set is the top element, $d(x, A) = \infty$. Whence

$$\sup_{x \in X} \big(d(x, B) \multimap d(x, A) \big) = \sup_{x \in X} 0 = 0.$$

If $B = \emptyset$ and A is non–empty, then $H(A, B) = \infty$ and $d(x, B) = \infty$. Thus

$$\sup_{x \in X} \big(d(x, B) \multimap d(x, A) \big) \geq d(x_0, B) \multimap d(x_0, A) = \infty,$$

with $x_0 \in A$.

From here on we assume that $A \neq \emptyset$ and $B \neq \emptyset$.

Let $\lambda = \sup_{x \in X} \big(d(x, B) \multimap d(x, A) \big)$. First we show that $Hd(A, B) \leq \lambda$. Let $x \in A$. Then $d(x, A) = 0$. Hence,

$$d(x, B) = d(x, B) \multimap d(x, A) \leq \lambda.$$

Since this holds for all $x \in A$, we conclude: $Hd(A, B) = \sup_{x \in A} d(x, B) \leq \lambda$.

We move on to the reverse direction. Let $\varepsilon > 0$ and note that for all $x \in X$ there exists $a \in A$ with $d(x, a) \leq d(x, A) + \varepsilon/2$. Also, there exists $b \in B$ with $d(a, b) \leq d(a, B) + \varepsilon/2 \leq Hd(A, B) + \varepsilon/2$. Thus,

$$d(x, B) \leq d(x, b) \leq d(x, a) + d(a, b) \leq d(x, A) + Hd(A, B) + \varepsilon.$$

Hence,

$$d(x, B) \multimap d(x, A) \leq Hd(A, B) + \varepsilon.$$

Taking the suprema over all $x \in X$, we get:

$$\lambda \leq Hd(A, B) + \varepsilon;$$

since $\varepsilon > 0$ was arbitrary, we get the desired conclusion. □

Remark. Notice that we could have proved the above Theorem right after the definition of the Hausdorff distance and from it derive that the non–symmetric Hausdorff distance is reflexive and satisfies the triangle inequality. We pursue this approach when we work with the Hausdorff distance in the $\mathcal{V}$–category setting.

The reader noticed that we did not restrict ourselves to closed sets in Proposition 3.3.3. The reason for this is simple: $Hd(A, B) = Hd(\overline{A}, \overline{B})$, for all $A, B \subseteq X$. Indeed, this follows immediately from

$$d(x, A) = d(x, \overline{A}),$$

for all $x \in X$ and $A \subseteq X$ and Proposition 3.3.1.

Corollary 3.3.4. *For any L–metric space (X, d) and any $A, B \subseteq X$,*

$$H^s d(A, B) = \sup_{x \in X} |d(x, B) - d(x, A)|$$

Proof.

$$\begin{aligned} H^s d(A, B) &= \max\{Hd(A, B), Hd(B, A)\} \\ &= \max\{\sup_{x \in X} d(x, B) \multimap d(x, A), \sup_{x \in X} d(x, A) \multimap d(x, B)\} \\ &= \sup_{x \in X} \max\{d(x, B) \multimap d(x, A), d(x, A) \multimap d(x, B)\} \\ &= \sup_{x \in X} |d(x, B) - d(x, A)|. \end{aligned}$$

□

3.4 Important theorems

In this section we list several important properties of the Hausdorff distance.

Proposition 3.4.1. *Any L–metric space (X, d) can be embedded into HX via the map $x \mapsto \{x\}$.*

Proof. We have

$$Hd(\{x\}, \{y\}) = \sup_{x \in \{x\}} \inf_{y \in \{y\}} d(x, y) = d(x, y).$$

□

There is a slight generalization of the previous result:

Proposition 3.4.2. *Let* (X, d) *and* (Y, ρ) *be* L*–metric spaces and* $f_i : X \longrightarrow Y$ *be an* I*–indexed family of non–expansive mappings. Then the map* $F : X \longrightarrow HY$ *defined by*

$$F(x) = \bigcup_{i \in I} \{f_i(x)\}$$

is also non–expansive.

Proof. It suffices to prove: $\forall a \in F(x), \rho(a, F(y)) \leq d(x, y)$. Given an $a = f_i(x) \in F(x)$, we have, by hypothesis, $\rho(f_i(x), f_i(y)) \leq d(x, y)$. Hence,

$$\rho(a, F(y)) \leq \rho(f_i(x), f_i(y)) \leq d(x, y),$$

as desired. □

The previous result hints at the importance of $\bigcup$ as a mapping $\bigcup : HHX \longrightarrow HX$. Indeed, we have

Theorem 3.4.3. *Let* (X, d) *be an* L*–metric space. Then*

$$\bigcup : HHX \longrightarrow HX$$

is non–expansive.

We shall prove the above result, and discuss its implications, in the general setting of $\mathcal{V}$-categories.

Next we shift our attention to completeness and compactness of H^sX. When dealing with completeness, we restrict ourselves to closed subsets of X. We also work with a symmetric metric d and the symmetric Hausdorff distance.

Theorem 3.4.4 (Proposition 7.3.7 in (BBI))**.** *If (X, d) is a complete metric space, then H^sX is also complete.*

Proof. Let (S_n) be a Cauchy sequence in H^sX. Set

$$S = \{x \in X \mid \forall U \text{ neigh. of } x, U \cap S_n \neq \emptyset \text{ for infinitely many } n\}.$$

We claim that $S_n \to S$. Fix $\varepsilon > 0$ and let n_0 be such that for all $m, n > n_0$ $H^sd(S_n, S_m) < \varepsilon$. We shall show that $H^sd(S, S_n) < 2\varepsilon$, for all $n \geq n_0$.

First we show that for all $x \in S$ and any $n \geq n_0$, $d(x, S_n) < 2\varepsilon$. Since $x \in S$, there exists $m \geq n_0$ such that $S_m \cap \eta_\varepsilon(x) \neq \emptyset$. That is, there exists a $y \in S_m$ with $d(y, x) \leq \varepsilon$. And because $d(y, S_n) \leq H^sd(S_m, S_n) < \varepsilon$, we conclude:

$$d(x, S_n) \quad \leq \quad d(x, y) + d(y, S_n) < 2\varepsilon,$$

where $\leq$ holds because of the triangle inequality Hd.

Fix $n \geq n_0$; we verify that for all $x \in S_n$, $d(x, S) < 2\varepsilon$. Fix $x \in S_n$ and let $n_1 = n$ and for every $k > 1$, choose n_k such that $n_k > n_{k-1}$ and $H^sd(S_p, S_q) < \frac{\varepsilon}{2^k}$, for all $p, q \geq n_k$. Then define the sequence (x_k) by $x_1 = x$; having defined $x_k \in S_{n_k}$, we define $x_{k+1} \in S_{n_{k+1}}$ as follows: we know

$$d(x_k, S_{n_{k+1}}) \leq H^s d(S_{n_k}, S_{n_{k+1}}) < \frac{\varepsilon}{2^k}.$$

Thus, there exists x_{k+1} in $S_{n_{k+1}}$ such that $d(x_k, x_{k+1}) < \frac{\varepsilon}{2^k}$. Notice that we then get

$$\sum_{k=1}^{\infty} d(x_k, x_{k+1}) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon < \infty. \tag{3.2}$$

Hence, for any $m \leq m'$ in $\mathbb{N}$ we can write $m' = m + r$ and get

$$\begin{aligned} d(x_m, x_{m'}) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{m+r-1}, x_{m+r}) \\ &\leq \sum_{k=m}^{\infty} d(x_k, x_{k+1}), \end{aligned}$$

which can be made as small as desired for a suitable choice of m by (3.2).

Thus the sequence (x_k) is a Cauchy sequence and hence converges to a point $y \in X$. Clearly, every neighborhood of y contains infinitely many of the x_k's and hence also intersects infinitely many of the S_n's. Hence, $y \in S$. Furthermore,

$$d(x, y) = \lim_{n \longrightarrow \infty} d(x_1, x_n) \leq \sum_{k=1}^{\infty} d(x_k, x_{k+1}) < \varepsilon.$$

Hence, $d(x, S) < d(x, y) < \varepsilon < 2\varepsilon$.

Thus, we showed that for all $x \in S$ and $n \geq n_0$, $d(x, S_n) < 2\varepsilon$. Hence,

$$Hd(S, S_n) = \sup_{x \in S} d(x, S_n) < 2\varepsilon.$$

Similarly, we saw that for all $x \in S_n$ with $n \geq n_0$, $d(x, S) < 2\varepsilon$. Thus,

$$Hd(S_n, S) = \sup_{x \in S_n} d(x, S) < 2\varepsilon.$$

We conclude that

$$H^s d(S, S_n) < 2\varepsilon,$$

whenever $n \geq n_0$, as required. $\square$

We consider some basic examples of convergence in HX.

1. Consider the sequence $S_n = \{(x, y) \in \mathbb{R}^2 \mid x^2 + \frac{y^2}{(1/n)^2} = 1\} \subseteq \mathbb{R}^2$. That is, S_n is a sequence of ellipses where the horizontal radius remains constant ($= 1$) and the vertical radius decreases. This is a Cauchy sequence in $H\mathbb{R}^2$. Thus, by Theorem 3.4.4 it has a limit. Furthermore, following the proof of Theorem 3.4.4, we can see that this limit is

$$S = \{(x, y) \in \mathbb{R}^2 \mid y = 0 \text{ and } x \in [-1, 1]\}.$$

2. Next, we consider the sequence (T_n), where

$$T_n = \{(\cos t, \sin t) \mid t \in [0, \pi - \frac{\pi}{n}]\} \cup \{(\cos t, \sin t) \mid t \in [\pi, 2\pi - \frac{\pi}{n}]\}$$

   So that $T_1 = \{(1, 0), (-1, 0)\}$, and T_2 is the set that contains two "quarters of a circle", etc. This is a Cauchy sequence; it's limit is the circle S^1, since every neighborhood of every point of the circle intersects infinitely many T_n's.

3. A more interesting example comes from the theory of fractals. For the sake of simplicity, we consider the case $X = \mathbb{R}^2$ with the standard Euclidean metric. Recall that a *proper contraction* $f : (X, d) \longrightarrow (Y, \rho)$ is a mapping with the property that for all $x, y \in X$,

$$\rho(f(x), f(y)) \leq rd(x, y)$$

with $0 < r < 1$. An *iterated function system* or IFS on (X, d) is a finite set of contractions $\{w_i : X \longrightarrow X \mid i = 1, \ldots, n\}$. Given an IFS on a complete metric space (X, d), we define a single mapping

$$W : H^s X \longrightarrow H^s X$$

by

$$W(A) = \bigcup_{i=1}^{n} w_i(A).$$

It turns out that W is a contraction (we know its non–expansive from Prop. 3.4.2). Since $H^s X)$ is also complete, W has a unique fixed point, by the Banach Fixed Point Theorem. This fixed point is called the *attractor* of the given IFS. Fractals are often described as attractors of given iterated function systems. One famous example of a fractal is the Sierpiński triangle. We do not concern ourselves with the definition of the IFS, or the details of the algorithm that is used to produce the output in Figure 3.1, but rather notice that the sequence of sets depicted in Figure 3.1 is a Cauchy sequence with

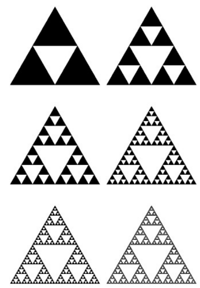


Figure 3.1: A sequence that converges to the Sierpiński triangle

respect to the Hausdorff metric: if we suppose that the length of the sides of the triangles is 1, then the distance between the first and the second triangle is less than $\frac{1}{4}$, the second and the third $\frac{1}{8}$ and so on. For a more detailed account of the relationship between fractals and the Hausdorff distance, see (Ba).

Total boundedness is another property that gets transferred from (X, d) to $H^s X$:

Theorem 3.4.5. *Let (X, d) be a metric space. If (X, d) is totally bounded, then so is $H^s X$*

Proof. This theorem follows from Proposition 3.8.3 when $\mathcal{V} = \mathbb{P}_+$. See the discussion of Hausdorff distance in $\mathcal{V}$-Cat for more details. $\square$

Corollary 3.4.6 (Theorem 7.3.8 in (BBI))**.** *If X is compact, so is $H^s X$.*

Proof. Since X is compact if, and only if, it is complete and totally bounded, the result follows from Theorem 3.4.4 and Theorem 3.4.5. $\square$

3.5 Hausdorff distance in $\mathcal{V}$-Cat

In this section, we modify our approach to the study of the Hausdorff distance. Whereas until now the emphasis was on classical results and examples, the sections that follow emphasize a categorical approach. We shall also enrich our collection of examples by studying Hausdorff distance in Ord. We begin with a definition. Consider the functor

$$H : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$$

Defined by

$$\big(f : (X, a) \longrightarrow (Y, b)\big) \mapsto \big(Hf : (PX, Ha) \longrightarrow (PY, Hb)\big)$$

where $Hf(A) = f(A)$, for all $A \subseteq X$ and

$$Ha(A, B) = \bigwedge_{x \in A} \bigvee_{y \in B} a(x, y).$$

We prove that Ha is a structure on PX later. To see that H is in fact a functor, we need to show that Hf is a $\mathcal{V}$-functor for every $\mathcal{V}$–functor f. This is indeed the case, since

$$\begin{aligned} Ha(A, B) &= \bigwedge_{x \in A} \bigvee_{y \in B} a(x, y) \\ &\leq \bigwedge_{x \in A} \bigvee_{y \in B} b(f(x), f(y)) \\ &= \bigwedge_{x' \in f(A)} \bigvee_{y' \in f(B)} b(x', y') = Hb(Hf(A), Hf(B)). \end{aligned}$$

When $\mathcal{V} = \mathbb{P}_+$, we obtain a functor

$$\mathsf{Met} \longrightarrow \mathsf{Met}$$

that sends (X, d) to $(PX, Hd) = HX$. We shall write HX for (PX, Ha), when the structure a is clear from the context.

If $\mathcal{V} = 2$, we get an order on PX; namely, an application of H to the 2–category (X, a) gives rise to the following order on PX:

$$\begin{aligned} A \leq B \quad &\iff \quad Ha(A, B) = \top \\ &\iff \quad \bigwedge_{x \in A} \bigvee_{y \in B} a(x, y) = \top \\ &\iff \quad \forall x \in A \exists y \in B \ \ (x \leq y). \end{aligned}$$

We also define the symmetric Hausdorff functor, H^{sym} or H^s by

$$H^{\text{sym}}(X, a) = H^{\text{sym}} X = (PX, H^s a),$$

with

$$H^s a(A, B) = Ha(A, B) \wedge Ha(B, A).$$

H and H^s coincide on morphisms.

3.6 Other formulations in $\mathcal{V}$-Cat

Recall that in the classical setting we realized the Hausdorff distance between subsets as distance between certain functions (see 3.3.3). This result carries into the

$\mathcal{V}$–category setting.

For a $\mathcal{V}$-category (X, a), given any $A \subseteq X$, we define the $\mathcal{V}$-functor

$$a(-, A) : X^{\mathrm{op}} \to \mathcal{V}$$

by

$$a(-, A)(x) := a(x, A) = \bigvee_{y \in A} a(x, y).$$

$a(-, A)$ is in fact a $\mathcal{V}$-functor: for any $x, y \in X$,

$$\begin{aligned} a(y, x) \otimes a(x, A) &= a(y, x) \otimes \bigvee_{z \in A} a(x, z) \\ &= \bigvee_{z \in A} \big(a(y, x) \otimes a(x, z)\big) \\ &\leq \bigvee_{z \in A} a(y, z) = a(y, A). \end{aligned}$$

thus,

$$a(y, x) = a^{\circ}(x, y) \leq a(x, A) \multimap a(y, A),$$

as required. Thus, for all $A \subseteq X$, $a(-, A) \in \hat{X} = \mathcal{V}^{X^{\mathrm{op}}}$. This allows us to state the following theorem:

Theorem 3.6.1. *Let (X, a) be a $\mathcal{V}$-category. Then for any $A, B \subseteq X$,*

$$Ha(A, B) = \bigwedge_{x \in X} (a(x, A) \multimap a(x, B)) = \hat{a}(a(-, A), a(-, B)).$$

Proof. "$\leq$": We need to show that $Ha(A, B) \leq a(x, A) \multimap a(x, B)$ for all $x \in X$, or, equivalently,

$$a(x, A) \otimes Ha(A, B) \leq a(x, B).$$

But,

$$a(x,A) \otimes Ha(A,B) = \bigvee_{y \in A} \big(a(x,y) \otimes Ha(A,B)\big) \leq a(x,B)$$

if, and only if, for all $y \in A$,

$$a(x,y) \otimes Ha(A,B) \leq a(x,B).$$

Fixing $y \in A$,

$$\begin{aligned}
a(x,y) \otimes Ha(A,B) &\leq a(x,y) \otimes a(y,B) \\
&= a(x,y) \otimes \bigvee_{x' \in B} a(y,x') \\
&= \bigvee_{x' \in B} \big(a(x,y) \otimes a(y,x')\big) \\
&\leq \bigvee_{x' \in B} a(x,x') = a(x,B),
\end{aligned}$$

as required. Thus,

$$Ha(A,B) \leq \hat{a}(a(-,A), a(-,B)).$$

"$\geq$": We need to show for all $x \in A$,

$$\hat{a}(a(-,A), a(-,B)) \leq a(x,B).$$

But for any fixed $x \in A$,

$$\begin{aligned}
\hat{a}(a(-,A), a(-,B)) &= \bigwedge_{x' \in X} (a(x',A) \multimap a(x',B)) \\
&\leq a(x,A) \multimap a(x,B) \\
&\leq k \multimap a(x,B) = a(x,B);
\end{aligned}$$

where $a(x, A) = \bigvee_{y \in A} a(x, y) \geq a(x, x) \geq k$ implies the last inequality.

Since this holds for all $x \in A$, we conclude that

$$\hat{a}(a(-, A), a(-, B)) \leq Ha(A, B).$$

$\square$

Corollary 3.6.2. *For any $\mathcal{V}$-category (X, a), HX is a $\mathcal{V}$-category.*

Proof. Since $\hat{a}$ is a structure on $\hat{X}$, we have immediately:

$$Ha(A, A) = \hat{a}(a(-, A), a(-, A)) \geq k,$$

and

$$\begin{aligned} Ha(A, B) \otimes Ha(B, C) &= \hat{a}(a(-, A), a(-, B)) \otimes \hat{a}(a(-, B), a(-, C)) \\ &\leq \hat{a}(a(-, A), a(-, C)) = Ha(A, C). \end{aligned}$$

$\square$

Let us briefly come back to the category Met. Then upon close examination of Definition 3.2.2, we notice that for an L-metric space (X, d),

$$Hd(A, B) = \inf\{\varepsilon \geq 0 \mid A \subseteq N_\varepsilon(B)\} = \inf\{\varepsilon \geq 0 \mid \forall x \in A \exists y \in B : d(x, y) \leq \varepsilon\}.$$

This observation leads us to ask whether we have a similar formula for the Hausdorff distance in $\mathcal{V}$-Cat. We answer this question with the following Proposition:

Proposition 3.6.3 (See Section 4 in (CT2)). *Let $\mathcal{V}$ be ccd and (X,a) be a $\mathcal{V}$-category. Then*

$$Ha(A,B) = \bigvee\{v \in \mathcal{V} \mid \forall x \in A \exists y \in B \ \ (v \leq a(x,y))\} =: \bigvee \mathcal{H}(A,B).$$

Proof. Given v such that $\forall x \in A \exists y \in B \ \ (v \leq a(x,y))$ and $x \in A$,

$$v \leq a(x,y) \leq a(x,B).$$

Thus $v \leq Ha(A,B)$. And consequently,

$$\bigvee \mathcal{H}(A,B) \leq Ha(A,B).$$

On the other hand, for $v \ll Ha(A,B)$, and every $x \in A$,

$$v \ll Ha(A,B) \leq a(x,B);$$

hence, $v \ll a(x,B)$. Thus, there exists $y \in B$ such that $v \leq a(x,y)$, so that $v \in \mathcal{H}(A,B)$. Thus, $v \leq \bigvee \mathcal{H}(A,B)$ and

$$Ha(A,B) = \bigvee\{v \mid v \ll Ha(A,B)\} \leq \bigvee \mathcal{H}(A,B).$$

□

Having the above proposition in mind, we finally can justify the notation Ha. Given any relation $r : X \nrightarrow Y$, we define $Hr : PX \nrightarrow PY$ by

$$Hr(A,B) := \bigvee\{v \in \mathcal{V} \mid \forall x \in A \exists y \in B \ \ (v \leq r(x,y))\}.$$

Of course the notation is only partly justified since, $H : \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$ is defined only when $\mathcal{V}$ is ccd.

Proposition 3.6.4 (see Section 4 in (CT2))**.** *When $\mathcal{V}$ is ccd, $H : \mathcal{V}\text{-}\mathsf{Rel} \to \mathcal{V}\text{-}\mathsf{Rel}$ is a lax functor; that is:*

$$Hs \cdot Hr \leq H(s \cdot r)$$

and

$$1_{PX} \leq H1_X,$$

for all $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$.

Proof. We have for any $B \subseteq Y$

$$Hr(A, B) \otimes Hs(B, C) =$$

$$\bigvee \mathcal{H}(A, B) \otimes \bigvee \mathcal{H}(B, C) =$$

$$\bigvee \{v \otimes \bigvee \mathcal{H}(A, B) \mid \forall y \in B \exists z \in C(v \leq s(y, z))\} =$$

$$\bigvee \{v \otimes \bigvee \{w \mid \forall x \in A \exists y \in B(w \leq r(x, y))\} \forall y \in B \exists z \in C(v \leq s(y, z))\} =$$

$$\bigvee \{v \otimes w \mid \forall x \in A \exists y \in B(w \leq r(x, y)), \forall y \in B \exists z \in C(v \leq s(y, z))\}. \tag{3.3}$$

Also we have:

$$H(s \cdot r)(A, C) = \bigvee \{u \mid \forall x \in A \exists z \in C(u \leq \bigvee_{y \in Y} \big(r(x, y) \otimes s(y, z)\big)\},$$

Given $v \otimes w$ as in (3.3), there exists $x \in A$, $y \in B$ and $z \in C$ such that

$$(w \otimes v) \leq r(x, y) \otimes s(y, z) \leq \bigvee_{y \in Y} r(x, y) \otimes s(y, z)),$$

we conclude that

$$Hr(A,B) \otimes Hs(B,C) \leq H(s \cdot r)(A,C),$$

for any $B \subseteq Y$. Thus $(Hs \cdot Hr)(A,C) \leq H(s \cdot r)(A,C)$, for all $A, C \subseteq X$.

Since for all $x \in A$, $k = 1_X(x,x)$, we have, trivially:

$$k \leq \bigvee \{v \mid \forall x \in A \exists y \in A (v \leq 1_X(x,y))\} = H(1_X)(A,A).$$

Thus $1_{PX} \leq H(1_X)$. $\square$

Remark. We note that we never use the fact that a is a structure in the proof of Proposition 3.6.3 . Indeed, one easily verifies that the statement of the Proposition holds for an arbitrary relation $r : X \otimes Y \longrightarrow \mathcal{V}$. That is, given any relation $r : X \otimes Y \longrightarrow \mathcal{V}$ and $A \subseteq X$, $B \subseteq Y$,

$$Hr(A,B) = \bigwedge_{x \in A} \bigvee_{y \in B} r(x,y).$$

We also have the symmetric lax Hausdorff functor

$$H^s : \mathcal{V}\text{-}\mathsf{Rel} \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}$$

defined by symmetrizing $H : \mathcal{V}\text{-}\mathsf{Rel} \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}$.

Lemma 3.6.5. *For $\mathcal{V}$–relations $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$,*

$$(Hr \cdot Hs)^{\mathrm{sym}} \geq (H^s r) \cdot (H^s s).$$

Proof. Let $C \subseteq Y$ be arbitrary. Then

$$\begin{aligned}
& \big(Hr(A,C) \wedge Hr(C,A)\big) \otimes \big(Hs(C,B) \wedge Hs(B,C)\big) \\
\leq\ & \big(Hr(A,C) \otimes Hs(C,B)\big) \wedge \big(Hr(C,A) \otimes Hs(B,C)\big) \\
\leq\ & \bigvee_{C' \subseteq Y} \big(Hr(A,C') \otimes Hs(C',B)\big) \wedge \bigvee_{C'' \subseteq Y} \big(Hs(B,C'') \otimes Hr(C'',A)\big) \\
=\ & \big((Hr \cdot Hs)(A,B)\big) \wedge \big((Hr \cdot Hs)(B,A)\big).
\end{aligned}$$

Taking the supremum over all $C \subseteq Y$,

$$\begin{aligned}
(H^s r) \cdot (H^s s)(A,B) &= \bigvee_{C \subseteq Y} \big(H^s r(A,C) \otimes H^s s(C,B)\big) \\
&\leq (Hr \cdot Hs)^{\mathrm{sym}}(A,B).
\end{aligned}$$

$\square$

Proposition 3.6.6. *When $\mathcal{V}$ is a ccd quantale,*

$$H^s : \mathcal{V}\text{-}\mathsf{Rel} \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}$$

is a lax functor.

Proof. Let $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ be $\mathcal{V}$–relations. Then for $A \subseteq X$, $B \subseteq Z$,

$$\begin{aligned}
H^s(r \cdot s)(A,B) &= H(r \cdot s)(A,B) \wedge H(r \cdot s)(B,A) \\
&\geq (Hr \cdot Hs)(A,B) \wedge (Hr \cdot Hs)(B,A) \\
&= (Hr \cdot Hs)^{\mathrm{sym}}(A,B) \\
&\geq \big((H^s r) \cdot (H^s s)\big)(A,B).
\end{aligned}$$

Also,

$$H^s(1_X)(A,A) = H(1_X)(A,A) \geq k.$$

$\square$

3.7 L-completeness of HX

In this section we discuss the problem of transfer of L–completeness from X to HX.

We recall Theorem 1.16 of (T1):

Theorem 3.7.1. *A $\mathcal{V}$–category (X,a) is L-complete if, and only if, every tight $\mathcal{V}$-form on X is of the form $a(-,x)$ for some $x \in X$.*

Thus, in order to show that HX is L–complete, we need to show that any tight $\mathcal{V}$–form $h : (HX)^{\mathrm{op}} \longrightarrow \mathcal{V}$ is representable, i.e., there exist some $A \subseteq X$ such that for any $B \subseteq X$,

$$h(B) = Ha(B,A).$$

We managed to reduce this problem: it suffices to show the above equality only for singleton B's. We prove this in the next two lemmas.

Lemma 3.7.2. *Let (X,a) be a $\mathcal{V}$–category. Suppose that $h : (HX)^{\mathrm{op}} \longrightarrow \mathcal{V}$ is a $\mathcal{V}$-functor and $h(\{x\}) = a(x,A)$, for some $A \subseteq X$ and all $x \in X$. Then*

$$h = Ha(-,A) \iff h(A) \geq k.$$

Proof. "$\Rightarrow$": $h(A) = Ha(A, A) \geq k$.

"$\Leftarrow$": Observe,

$$a(x, B) = Ha(\{x\}, B) \leq h(B) \multimap h(\{x\}) = h(B) \multimap a(x, A).$$

So,

$$h(B) \leq a(x, B) \multimap a(x, A), \forall x \in X.$$

Hence,

$$h(B) \leq \bigwedge_{x \in X} \big(a(x, B) \multimap a(x, A)\big) = Ha(B, A).$$

The reverse inequality follows from:

$$Ha(B, A) \leq h(A) \multimap h(B) \leq k \multimap h(B) = h(B).$$

$\square$

Lemma 3.7.3. *Let (X, a) be a $\mathcal{V}$–category and $h : HX^{\mathrm{op}} \to \mathcal{V}$ be a tight $\mathcal{V}$-form. Suppose that $h(\{x\}) = a(x, A)$, for some $A \subseteq X$. Then $k \leq h(A)$.*

Proof. We know that

$$k \leq \bigvee_{B \subseteq X} h(B) \otimes \Big(\bigwedge_{C \subseteq X} \big(h(C) \multimap Ha(C, B)\big)\Big).$$

Observe that

$$\bigwedge_{C \subseteq X} \big(h(C) \multimap Ha(C, B)\big) \leq h(\{x\}) \multimap a(x, B),$$

since $C = \{x\}$ is a subset of X. Since this holds for all $x \in X$,

$$\begin{aligned}
\bigwedge_{C \subseteq X} h(X) \multimap Ha(C, B) &\leq \bigwedge_{x \in X} h(\{x\}) \multimap a(x, B) \\
&= \bigwedge_{x \in X} a(x, A) \multimap a(x, B) \\
&= Ha(A, B).
\end{aligned}$$

Hence,

$$k \leq \bigvee_{B \subseteq X} h(B) \otimes Ha(A, B).$$

Since h is a $\mathcal{V}$-functor,

$$H(A, B) \leq h(B) \multimap h(A)$$

giving $Ha(A, B) \otimes h(B) \leq h(A)$, for all $B \subseteq X$. Thus $k \leq f(A)$, as claimed. $\square$

Corollary 3.7.4. *To show that HX is L-complete, it suffices to show that for every tight $\mathcal{V}$-form $h : HX^{\mathrm{op}} \to \mathcal{V}$ there exists a set $A \subseteq X$, such that for all $x \in X$, $h(\{x\}) = a(x, A)$.*

It seems that a natural candidate for A in the above Corollary is

$$A = \{x \in X \mid h(x) \geq k\}.$$

especially when one analyzes the meaning of this set in Met. Furthermore, Proposition 2.1.7 in (Be), gives us good indication that the above choice of A is correct. Unfortunately, we did not manage to prove this conjecture yet.

3.8 Total boundedness of HX

In defining total boundedness in the $\mathcal{V}$-categorical setting, we must take into account some of the essential properties of the classical definition.

We say that a metric space (X, d) is *totally bounded* if, and only if, for every $\varepsilon > 0$, there is a finite set $F \subseteq X$ such that for all $x \in X$ there exists $y \in F$ with $d(x, y) \leq \varepsilon$.

Our first observation is that ε is strictly greater than 0 and hence, we need to use the "way–below" relation which imposes complete distributivity on $\mathcal{V}$. Hence, from here on and until the end of this section we assume $\mathcal{V}$ to be constructively completely distributive. Further more, we have to make sure that $\varepsilon < \infty$. Hence, we assume that in $\mathcal{V}$, $k \gg \perp$.

Next, we ask: how essential is the symmetry of d in the definition of total boundedness? In other words, we want to know whether the above definition remains meaningful in a non–symmetric environment. So, suppose that (X, d) is a non–symmetric space and is totally bounded. Consider the space HX – the induced (non–symmetric) Hausdorff space. Let $\varepsilon > 0$ and $F \subseteq X$ be the associated finite ε–net. Now, for any $A \subseteq X$ we have:

$$\forall x \in A \exists y \in F : d(x, y) \leq \varepsilon.$$

Hence, $Hd(A, F) \leq \varepsilon$. And thus, (PX, Hd) is bounded by a (non–symmetric) "ball" of radius ε. Note that the choice of ε was arbitrary, and hence we conclude that HX is contained in an ε–"ball" for all $\varepsilon > 0$ – a property that is counter–intuitive.

The above observation leads us to consider total boundedness only in the case the distance function is symmetric.

Definition 3.8.1. Let (X, a) be a symmetric $\mathcal{V}$-category. Then we say that (X, a) is *totally bounded* provided that for all $v \in \mathcal{V}$ with $\perp \ll v \ll k$, there exists a finite set $F \subseteq X$ such that for all $x \in X$ there exists $y \in F$ such that $a(x, y) \geq v$.

Based on the discussion preceding the definition, it is clear that total boundedness in $\mathbb{P}_+$-Cat is just the standard notion we know from the category of metric spaces. In 2-Cat, the way–below relation is given by

$$x \ll y \iff y = \top;$$

thus, the only choice for v in our definition is $v = \top$. Next, we want a finite subset $F \subseteq X$ such that for all $x \in X$ exists $y \in F$ such that $a(x, y) = \top$. Translating this to the language of ordered sets, we have that $(X, \leq)$ is totally bounded if, and only if, first the order is symmetric: $(x \leq y \iff y \leq x)$ and second: there exists a finite set $F \subseteq X$ such that

$$\forall x \in X \exists y \in F : x \leq y.$$

Of course the definition of total boundedness in the category of ordered sets is given just as an illustration of this concept in $\mathcal{V}$-Cat, since we are not too interested in symmetric ordered sets.

It is well known that a metric space is compact if, and only if, it is totally bounded and complete. This leads us to the definition of compactness in $\mathcal{V}$-Cat:

Definition 3.8.2. $(X, a) \in \mathcal{V}$-Cat is *compact* provided that it is L-complete and totally bounded.

Compactness in $\mathbb{P}_+$-Cat is just the standard notion of compactness familiar from the theory of metric spaces. Since every ordered set is L-complete, compactness and total boundedness in 2-Cat coincide.

Proposition 3.8.3. *Let $\mathcal{V}$ be a ccd quantale. If a symmetric $\mathcal{V}$-category (X, a) is totally bounded, then so is $H^s X$.*

Proof. Let $\bot \ll v \ll k$. There exists a finite set $F \subseteq X$ such that for all $x \in X$, there exists $y \in F$ with $a(x, y) \geq v$. We claim that for all $A \subseteq X$ exist $F_A \subseteq F$ such that $H^s a(A, F_A) \geq v$. Indeed suppose we are given $A \subseteq X$. Set

$$F_A = \{y \in F \mid a(y, A) \geq v\}.$$

For any $x \in A$, there exists $y \in F$ such that $a(x, y) \geq v$. Since

$$a(y, A) \geq a(y, x) = a(x, y) \geq v,$$

$y \in F_A$. Thus, for all $x \in A$ there exists $y \in F_A$ such that

$$a(x, F_A) \geq a(x, y) \geq v.$$

Taking the infimum over all the x's in A, we obtain:

$$Ha(A, F_A) = \bigwedge_{x \in A} a(x, F_A) \geq v.$$

Next, let $y \in F_A$. Then $a(y, A) \geq v$. Since this holds for all $y \in F_A$, we obtain,

$$Ha(F_A, A) = \bigwedge_{y \in F_A} a(y, A) \geq v.$$

Consequently,

$$(Ha)^{\text{sym}}(A, F_A) = Ha(A, F_A) \wedge Ha(F_A, A) \geq v,$$

as desired. $\square$

3.9 Functorial connections of H

In this section we ask: what are the functorial properties of H and how does H interact with other functors?

The powerset functor appears right in the definition of $H : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$. While the similarities between it and the Hausdorff functor exists and will be explored in the next section, there is also an important difference.

The contravariant powerset functor $P : \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is self adjoint. That is, there is a natural correspondence

$$\frac{\varphi : X \longrightarrow PY}{\psi : Y \longrightarrow PX},$$

defined by

$$\psi(y) = \{x \in X \mid y \in \varphi(x)\} \tag{3.4}$$

Unfortunately, this property does not get transferred to the Hausdorff functor. We provide the following counter–example:

Example 3.9.1. Let $\mathcal{V} = 2$. We are then working in Ord. Let $X = \{1, 2, 3\}$ have the smallest order such that $1 \leq 2$ and $1 \leq 3$. Let $Y = \{1, 2\}$ have the smallest order such that $1 \leq 2$. Consider $\varphi : (X, \leq) \longrightarrow (PY, \leq)$ defined by

$$\varphi(1) = \emptyset, \ \ \varphi(2) = \{1\}, \ \ \varphi(3) = \{2\}.$$

Then φ is monotone, since the Hausdorff order on PY is

$$\emptyset \leq \{1\} \leq \{2\} \leq Y.$$

We define ψ as in (3.4):

$$\psi(1) = \{2\} \text{ and } \psi(2) = \{3\}.$$

If ψ was monotone, then we would have $\{2\} \leq \{3\}$ in $(PX, \leq)$, which holds if, and only if, $2 \leq 3$ in X and this certainly is not true.

Next, we consider the relationship of the Hausdorff functor and the $\mathcal{V}$-*powerset* functor $\mathcal{P} : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$. Following the observations made by D. Hofmann, we define a functor $\mathcal{P} : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ by

$$\mathcal{P}(X, a) = \mathcal{V}^{X^{\mathrm{op}}} = \{\text{all } \mathcal{V}\text{–modules } X \nrightarrow 1\}$$

and for $f : (X, a) \longrightarrow (Y, b)$, define

$$\mathcal{P}f(\alpha) = \alpha \cdot f^* := \alpha \cdot f^\circ \cdot b.$$

with $\alpha : X \nrightarrow 1$ a $\mathcal{V}$–module.

Proposition 3.9.2. *Let $\mathfrak{y}_X : HX \longrightarrow \mathcal{P}X$ be defined by $\mathfrak{y}_X(A) = a(-, A)$. Then $\mathfrak{y} : H \longrightarrow \mathcal{P}$ is a lax natural transformation in the following sense:*

$$\begin{tikzcd} HX \arrow[r, "\mathfrak{y}_X"] \arrow[d, "Hf"'] \arrow[dr, phantom, "\leq"] & \mathcal{P}X \arrow[d, "\mathcal{P}f"] \\ HY \arrow[r, "\mathfrak{y}_Y"'] & \mathcal{P}Y \end{tikzcd}$$

Proof. For any $A \subseteq X$, we have

$$\mathcal{P}f(\mathfrak{y}_X(A)) = a(-, A) \cdot f^*.$$

And

$$
\begin{aligned}
a(-,A)\cdot f^*(y,*) &= \bigvee_{x\in X} f^*(y,x)\otimes a(x,A)\\
&= \bigvee_{x\in X} f^\circ\cdot b(y,x)\otimes a(x,A)\\
&= \bigvee_{x\in X} b(y,f(x))\otimes a(x,A)\\
&\geq \bigvee_{x\in A} b(y,f(x))\otimes a(x,A)\\
&\geq \bigvee_{x\in A} b(y,f(x))\otimes k\\
&= \bigvee_{x\in A} b(y,f(x))\\
&= b(y,f(A)) = (\mathfrak{y}_Y\cdot Hf(A))(y,*).
\end{aligned}
$$

□

3.10 The Hausdorff monad

As promised, we prove that the union map is a $\mathcal{V}$-functor.

Proposition 3.10.1. *Let $\mathcal{V}$ be constructively completely distributive. Then given any $\mathcal{V}$-category (X,a), the map*

$$\bigcup : HHX \to HX$$

is a $\mathcal{V}$-functor.

Proof. First we note that for any $x \in X$, any $\mathcal{B} \subseteq PX$ and $B \in \mathcal{B}$,

$$a(x, B) = \bigvee_{y \in B} a(x, y) \leq \bigvee_{y \in \bigcup \mathcal{B}} a(x, y) = a(x, \bigcup \mathcal{B}).$$

We have

$$HHa(\mathcal{A}, \mathcal{B}) = \bigvee \{v \in \mathcal{V} \mid \forall A \in \mathcal{A} \exists B \in \mathcal{B} (v \leq Ha(A, B))\}. \tag{3.5}$$

Suppose that $v \in \mathcal{V}$ satisfies the condition in (3.5). We need to show that

$$v \leq Ha(\bigcup \mathcal{A}, \bigcup \mathcal{B}). \tag{3.6}$$

For this, it suffices to show for all $x \in \bigcup \mathcal{A}$

$$v \leq Ha(x, \bigcup \mathcal{B}).$$

Given any $x \in \bigcup \mathcal{A}$, there exists $A \in \mathcal{A}$ with $x \in A$ and by (3.5) there exists $B \in \mathcal{B}$ such that

$$v \leq Ha(A, B) \leq a(x, B) \leq a(x, \bigcup \mathcal{B}).$$

Consequently, (3.6) holds for every v that satisfies the condition in (3.5). We conclude that $\bigcup$ is indeed a $\mathcal{V}$–functor. $\square$

We also have:

Proposition 3.10.2. *For any $\mathcal{V}$-category (X, a), the map $(X, a) \hookrightarrow HX$ $x \mapsto \{x\}$ is an an embedding.*

Proof. $Ha(\{x\}, \{y\}) = a(x, y)$. $\square$

Recall that the *power set monad* is the triple $\mathbb{P} = (P, m, e)$ with

$$P : \mathsf{Set} \longrightarrow \mathsf{Set}$$

the power set functor, and for any set X,

$$m_X = \bigcup : PPX \longrightarrow PX, \quad e_X : X \longrightarrow PX, x \mapsto \{x\}$$

natural transformations.

It is well known (and easy to check) that $\mathbb{P}$ is in fact a monad. Keeping this in mind, we define a triple

$$\mathbb{H} = (H : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}, m, e)$$

with m and e as in the powerset monad. Since, at the set level m and e satisfy all the requirements for a monad and Propositions 3.10.1 and 3.10.2 tell us that for all X, m_X and e_X are $\mathcal{V}$-functors, we conclude that $\mathbb{H}$ is a monad too.

We turn our attention to the study of the Eilenberg–Moore algebras for this monad; as we shall see momentarily, the algebras for this monad are complete lattices that are also $\mathcal{V}$-categories.

Let $\big((X, a), \varphi : HX \longrightarrow X\big) \in \mathcal{V}\text{-}\mathsf{Cat}^{\mathbb{H}}$; it corresponds to a sup–lattice $(X, \bigvee)$ with

$$\bigvee A = \varphi(A).$$

Furthermore, the map $\bigvee : HX \to X$ is a $\mathcal{V}$–functor. Conversely, a given sup–lattice that is also a $\mathcal{V}$–category $(X, \bigvee, a)$ such that $\bigvee$ is a $\mathcal{V}$-functor, gives rise to an $\mathbb{H}$–algebra $\bigvee : HX \to X$.

If $\mathcal{V} = \mathbb{P}_+$, $\mathbb{H}$-algebras have a nice geometric description: they become sup–lattices $(X, \bigvee, d)$ with an L–metric structure d such that for all $A, B \subseteq X$

$$d(\bigvee A, \bigvee B) \leq Hd(A, B);$$

that is: the distance between suprema of two sets is bounded by the distance between those sets.

If $\mathcal{V} = 2$, those algebras are sup–lattices with an additional order structure that satisfy:

$$\forall A, B \subseteq X: \;\; (A \leq B \implies \bigvee A \leq \bigvee B).$$

It is easy to see that any sup–lattice $(X, \bigvee)$ with the order defined by $(x \leq y \iff x \vee y = y)$ satisfies the above requirement. Indeed, $A \leq B \iff \forall x \in A \; \exists y_x \in B \; (x \leq y_x)$. So for all $x \in A$, $x \leq y_x \leq \bigvee B \implies \bigvee A \leq \bigvee B$.

There is a better description for $\mathbb{H}$-algebras:

Proposition 3.10.3. *$\mathbb{H}$-algebras are triples $(X, \bigvee, a)$ where $(X, \bigvee)$ is a sup–lattice, (X, a) is a $\mathcal{V}$-category and for all $A \subseteq X$*

$$\bigvee_{y \in A} a(x, y) \leq a(x, \bigvee A);$$
$$\bigwedge_{y \in A} a(y, x) \leq a(\bigvee A, x).$$

Proof. We need to prove that the two conditions above hold, if, and only if, $\bigvee : HX \longrightarrow X$ is a $\mathcal{V}$-functor.

"if": We have for any $A, A' \subseteq X$, $Ha(A, A') \leq a(\bigvee A, \bigvee A')$. In particular, taking $A = \{x\}$, we get

$$\bigvee_{y \in A'} a(x, y) \leq a(x, \bigvee A').$$

Similarly, taking $A' = \{x\}$ gives the second inequality.

"only if":

$$\begin{aligned} Ha(A, A') &= \bigwedge_{x \in A} \bigvee_{y \in A'} a(x, y) \\ &\leq \bigwedge_{x \in A} a(x, \bigvee A') \\ &\leq a(\bigvee A, \bigvee A'). \end{aligned}$$

$\square$

Proposition 3.10.3 describes the category of $\mathbb{H}$–algebras without reference to the Hausdorff distance; the objects of this category are given by proposition 3.10.3. We call those objects $\mathcal{V}$*–categorical lattices*. A morphism between $\mathcal{V}$–categorical lattices is a map $f : (X, \bigvee, a) \longrightarrow (Y, \widetilde{\bigvee}, b)$ that is a $\mathcal{V}$–functor which preserves the join operation:

$$f(\bigvee A) = \widetilde{\bigvee} f(A),$$

for all $A \subseteq X$. Let us denote the category of $\mathcal{V}$–catategorical lattices and the corresponding morphisms by $\mathcal{V}\text{-}\mathsf{CatLat}$.

We know from general theory that the forgetful functor $U^{\mathbb{H}} : \mathcal{V}\text{-}\mathsf{CatLat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ has a left adjoint $F^{\mathbb{H}} : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{CatLat}$ given by $F^{\mathbb{H}}(X, a) = (HX, m) = (HX, \bigcup)$. But, the $\mathcal{V}$-functor $\bigcup$ is hidden in the definition of H: it comes with the powerset. Further, since for any V–functor $f : X \longrightarrow Y$, Hf preserves unions, we conclude that $\bar{H} = F^{\mathbb{H}}$, where $\bar{H} : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{CatLat}$ is just H with a different codomain. This simple observation allows us to give a categorical characterization of $\bar{H}$:

Proposition 3.10.4. $\bar{H} \dashv U^{\mathbb{H}} : \mathcal{V}\text{-}\mathsf{CatLat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$.

From the above proposition, we have the following correspondence:

$$\frac{\varphi : (X, a) \longrightarrow (Y, b, \bigvee)}{\Phi : HX \longrightarrow (Y, b, \bigvee)}.$$

We can describe Φ explicitly: for all $A \subseteq X$,

$$\Phi(A) = \bigvee H\varphi(A).$$

Indeed, both $H\varphi$ and $\bigvee$ are $\mathcal{V}\text{-}\mathsf{CatLat}$–morphisms (that fact that the latter is just follows from the definition of morphisms between $\mathbb{H}$–algebras). Thus Φ is a $\mathcal{V}\text{-}\mathsf{CatLat}$–morphism too. And, for any $x \in X$,

$$\Phi(\{x\}) = \bigvee H\varphi e_X(x) = \bigvee H\varphi(\{x\}) = \bigvee \{\varphi(x)\} = \varphi(x).$$

4 Gromov–Hausdorff distance

4.1 Introduction

Given a finitely generated group G, with a fixed set of generators $\Gamma = \{\gamma_1, \dots, \gamma_n\}$, we define a norm on G as follows: each $x \in G$ can be represented by a word

$$x = \gamma_{i_1} \cdot \dots \cdot \gamma_{i_n}.$$

The number n is called the length of the word. The norm of x, $\|x\|$, is the length of a shortest word representing x.

A ball of radius r around the identity is then defined by

$$B(r) = \{x \in G \mid \|x\| \leq r\}.$$

We say the G has *polynomial growth* provided that there exist two positive numbers d and C such that

$$|B(r)| \leq Cr^d,$$

for all $r \in \mathbb{R}_+$ (it turns out that this definition is independent of the choice of generators).

One of the landmarks of geometric group theory was M. Gromov's proof of the following Theorem:

Theorem 4.1.1 ((Gr1), Main Theorem)**.** *If a finitely generated group G has polynomial growth, then G contains a nilpotent subgroup of finite index.*

This theorem, along with known results, completely characterized finitely generated groups of polynomial growth as exactly those groups that contain a nilpotent subgroup of finite index.

One of the key ideas in the proof of Theorem 4.1.1 was the convergence of a sequence of groups (with metric structure induced by the above norm) to another group. To establish this convergence, Gromov introduced a distance on the class of all metric spaces – the Gromov–Hausdorff distance.

This chapter is devoted to the study of Gromov–Hausdorff (G–H) distance in both the metric and $\mathcal{V}$–categorical settings. We begin with a definition of Gromov–Hausdorff distance. Next, we demonstrate the difficulty that arises when we attempt to compute the G–H distance between simple sets . We describe the G–H distance using correspondences; this simplifies our work with this distance and allows us to prove that G–H distance is a metric on isometry classes of compact metric spaces. We then discuss G–H convergence. Numerous examples of G–H convergence are provided. We also introduce length spaces and discuss their G–H limits. The classical treatment of G–H distance concludes with Gromov's compactness criterion and a result about completeness of the class of compact metric spaces.

In the second part of the chapter we introduce the Gromov–Hausdorff structure on the objects of $\mathcal{V}$-Cat. Following Lawvere's hint, we establish ties between $\mathcal{V}$–modules and the G–H structure. We also describe the G–H structure using set functions instead of correspondences. Next, we show that the Hausdorff functor and the tensor product are both $\mathcal{V}$–functors with respect to the G–H structure. The chapter concludes with a description of the G–H structure as a colimit.

4.2 Definition and examples

We define the *Gromov–Hausdorff distance* on the class of all L–metric spaces by

$$G((X, d_X), (Y, d_Y)) = \inf_{(Z,d_Z)} H^s d_Z(f(X), g(Y))$$

where the infimum is taken over all L–metric spaces (Z, d_Z) such that there exist embeddings $f : X \hookrightarrow Z$ and $g : Y \hookrightarrow Z$.

It turns out that G is reflexive and satisfies the triangle inequality i.e. it is an L–metric. We shall prove this later. We define the *Gromov–Hausdorff distance* on the class of all metric spaces analogously: just replace L–metric spaces by metric spaces in the above definition. We shall often refer to the Gromov–Hausdorff distance as the Gromov distance.

Examples 4.2.1. 1. Let (X,d) be an L–metric space and $A, B \subseteq X$ be subsets. Then since as L–metric spaces on their own, A and B can be embedded into (X,d) and $G(A,B) \leq H^s d(A,B)$. Thus, the Gromov distance between subspaces of an L–metric space is at most the Hausdorff distance between those subspaces.

2. If X and Y are isometric, then the Gromov distance between them is 0. In fact, we shall see that when we deal with compact metric spaces, the converse also holds. Of course, we can have zero Gromov distance between non–isometric spaces: let $X = [0,1]$ and $Y = \mathbb{Q} \cap [0,1]$. Then since $\overline{Y} = X$, $H^s d(X,Y) = 0$, where d denotes the standard metric on $\mathbb{R}$. From example (1), we then get

$$G(X,Y) \leq H^s d(X,Y) = 0.$$

3. Let (X,d) be a metric space and $N \subseteq X$ be an ε–net for X. Then

$$G(X,Y) \leq H^s d(X,Y) = \varepsilon.$$

As the simple examples above suggest, it is not easy to compute the exact Gromov distance between metric spaces X and Y directly from the definition. The problem is that there are too many metric spaces into which X and Y can be embedded. Luckily, it is possible to reduce the metric spaces under consideration to only those whose underlying set is the disjoint union $X \cup Y$ of X and Y. More precisely:

Proposition 4.2.2 (Remark 7.3.12 in (BBI))**.** *Let (X, d_X) and (Y, d_Y) be L–metric spaces. Define*

$$G'(X,Y) = \inf_d H^s d(X,Y),$$

where d ranges over all L–metrics on $X \cup Y$ such that $d|_{X\times X} = d_X$ and $d|_{Y\times Y} = d_Y$. Then $G(X,Y) = G'(X,Y)$.

Proof. The inequality $G(X,Y) \leq G'(X,Y)$ is clear, since we can take $Z = (X \cup Y, d)$ and f, g the inclusion maps.

Conversely, we prove that for any L–metric space (Z, d_Z) and embeddings $f : X \hookrightarrow Z$, $g : Y \hookrightarrow Z$, $G'(X,Y) \leq H^s d_Z(f(X), g(Y))$. Indeed, given such a Z, define $d : (X \cup Y) \times (X \cup Y) \longrightarrow [0, \infty]$ by $d|_{X\times X} = d_X$, $d|_{Y\times Y} = d_Y$ and for any $x \in X$ and $y \in Y$

$$d(x,y) = d_Z(f(x), g(y)).$$

Then d is reflexive, since d_X and d_Y are. It satisfies the triangle inequality: for any $x' \in X$,

$$\begin{aligned} d(x,x') + d(x',y) &= d_Z(f(x), f(x')) + d_Z(f(x'), g(y)) \\ &\geq d_Z(f(x), g(y)) = d(x,y); \end{aligned}$$

a similar argument works if $x' \in Y$. Further,

$$\begin{aligned} Hd(X,Y) &= \sup_{x\in X}\inf_{y\in Y} d(x,y) \\ &= \sup_{x\in X}\inf_{y\in Y} d_Z(f(x),g(y)) = Hd_Z(f(X),g(Y)). \end{aligned}$$

So,

$$\begin{aligned} H^s d(X,Y) &= \max\{Hd_Z(X,Y), Hd(Y,X)\} \\ &= \max\{Hd_Z(f(X),g(Y)), Hd_Z(g(Y),f(X))\} = H^s d_Z(f(X),g(Y)). \end{aligned}$$

Thus,

$$G'(X,Y) \le H^s d(X,Y) = H^s d_Z(f(X),g(Y)).$$

Since this holds for all Z, f and g, we conclude:

$$G'(X,Y) \le G(X,Y).$$

$\square$

Definition 4.2.3. Let (X,d_X) and (Y,d_Y) be L–metric spaces. A metric d on $X \cup Y$ is *admissible* provided that $d|_{X\times X} = d_X$ and $d|_{Y\times Y} = d_Y$.

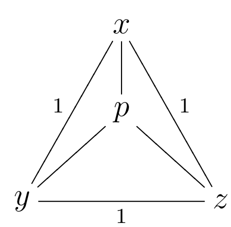
As the following example illustrates, the above proposition allows us to compute the Gromov distance between some simple sets.

Example 4.2.4. Let $X = \{x, y, z\}$, with the distance function

$$d(r, s) = \begin{cases} 0, & \text{if } r = s; \\ \\ 1, & \text{otherwise.} \end{cases}$$

be a metric space and $Y = \{p\}$ denote the one point space.

Let d be some admissible metric on the disjoint union of X and Y. The following diagram depicts $X \cup Y$:



We obtain the following inequalities:

$$1 = d(x, y) \leq d(x, p) + d(p, y)$$

$$1 = d(x, z) \leq d(x, p) + d(p, z).$$

Hence,

$$1 - d(x, p) \leq d(p, y) \text{ and } 1 - d(x, p) \leq d(p, z). \tag{4.1}$$

A similar argument shows:

$$1 - d(y, p) \leq d(p, z) \text{ and } 1 - d(y, p) \leq d(p, x). \tag{4.2}$$

Also,

$$Hd(X, Y) = \max\{d(x, p), d(y, p), d(z, p)\}$$

and

$$Hd(Y,X) = \min\{d(p,x), d(p,y), d(p,x)\}.$$

Now, if $Hd(X,Y) \leq \frac{1}{2}$, then $d(x,p), d(y,p) \leq \frac{1}{2}$, thus we see from equations (4.1) and (4.2) that

$$\frac{1}{2} \leq d(p,x), d(p,y), d(p,z);$$

consequently, $\frac{1}{2} \leq Hd(Y,X) \leq H^s d(X,Y)$. And if $\frac{1}{2} \leq Hd(X,Y)$, then, of course, $\frac{1}{2} \leq H^s d(X,Y)$. Since, d was arbitrarily chosen,

$$\frac{1}{2} \leq G(X,Y).$$

Next, define an admissible metric $\delta : (X \cup Y) \times (X \cup Y) \longrightarrow [0,\infty]$ by

$$\delta(x,p) = \frac{1}{2} = \delta(p,x),$$

for all $x \in X$. Then δ is a metric. Hence,

$$G(X,Y) \leq H\delta^s(X,Y) = \frac{1}{2}.$$

We conclude that $G(X,Y) = \frac{1}{2}$.

Observe that the embedding of X and Y into $\mathbb{R}^n$ with $H^s d(X,Y)$ minimal is one where p is equidistant from the vertices of X, i.e., as is depicted in the above figure. In this case, $H^s d(X,Y) = \frac{1}{\sqrt{3}}$. So, the value for $G(X,Y)$ we obtained was achieved by embedding X and Y into a non–Euclidean space.

Remark. In chapter 3 we observed that the non–symmetric Hausdorff distance is more natural that its symmetric counterpart. Many statements about the Hausdorff distance hold without the presence of symmetry. This is no longer the case with the Gromov distance. In particular, if we replace H^s by its non–symmetric analog in the above example, by picking $Z = X$ and sending $p \mapsto x$ we will be forced to conclude $Hd_Z(Y, X) = 0$. This will guarantee $G(X, Y) = 0$ with X and Y not isometric; but this rids the Gromov distance of one of its key properties.

4.3 Other formulations

In the preceding section we saw that it is not easy to work with the Gromov distance in general. Even for simple examples we had to construct a new metric and verify the triangle inequality. We can avoid much of this difficult work by using the theory of correspondences.

Definition 4.3.1. Let X and Y be sets. A subset $R \subseteq X \times Y$ is called a *correspondence between X and Y* provided that for all $x \in X$ exists $y \in Y$ such that $(x, y) \in R$, and for all $y \in Y$ there exists $x \in X$ such that $(x, y) \in R$.

For example, any surjective mapping $f : X \longrightarrow Y$ gives rise to a correspondence: $R = \{(x, f(x)) \mid x \in X\}$. We say in this case that R is *induced* by f.

Definition 4.3.2. Let (X, d_X) and (Y, d_Y) be L–metric spaces and R be a correspondence between X and Y. We define the *distortion* of R by

$$\mathrm{dis}R = \sup\{|d_X(x,x') - d_Y(y,y')| \mid (x,y),(x',y') \in R\}.$$

The distortion of a correspondence R between X and Y measures how far R is from being incuded by an isometry. More precisely:

Proposition 4.3.3 (Exercise 7.3.24 in (BBI))**.** *Let R be a correspondence between metric spaces (X, d_X) and (Y, d_Y). Then* $\mathrm{dis}R = 0$ *if, and only if, R is induced by an isometry.*

Proof. "if": Let $f : X \longrightarrow Y$ be an isometry. Then $d_Y(f(x), f(x')) = d_X(x,x')$, and hence

$$\mathrm{dis}R = \sup\{|d_X(x,x') - d_Y(f(x), f(x')) \mid x, x' \in X\} = 0.$$

"only if": Suppose $\mathrm{dis}R = 0$. Define $f : X \longrightarrow Y$ by $f(x) = y$ whenever $(x,y) \in R$. Then f is well defined: if $(x,y),(x,y') \in R$, then

$$\begin{aligned} 0 &= \sup\{|d_X(x,x') - d_Y(y,y')| \mid (x,y),(x',y') \in R\} \\ &\geq |d_X(x,x) - d_Y(y,y')| \\ &= d_Y(y,y'). \end{aligned}$$

Thus $y = y'$. We clearly have $0 = |d_X(x,x') - d_Y(f(x), f(x'))|$ for all $x, x' \in X$. Surjectivity follows from the fact that for all $y \in Y$ there exist $x \in X$ with $(x,y) \in R$. $\square$

Using correspondences, we can view Gromov distance from a different angle:

Theorem 4.3.4 (Theorem 7.3.25 in (BBI))**.** *For any L–metric spaces (X, d_X) and (Y, d_Y),*

$$G(X, Y) = \frac{1}{2} \inf_R (\mathrm{dis} R)$$

where the infimum is taken over all the correspondences between X and Y.

Before we give the proof of this theorem, we demonstrate its strength. Let us again have X and Y as in Example 4.2.4. The only correspondence between X and Y is

$$R = \{(x, p), (y, p), (z, p)\}.$$

Then

$$\mathrm{dis} R = \sup\{|d_X(x, x') - d_Y(y, y') \mid (x, y), (x', y') \in R\} = 1.$$

Thus from the theorem we conclude: $G(X, Y) = \frac{1}{2}$. We can also calculate the Gromov distance between more complex metric spaces. For any metric space (X, d_X) and $Y = \{p\}$, we have again have only one correspondence between X and Y, namely:

$$R = \{(x, p) \mid x \in X\},$$

and

$$\mathrm{dis} R = \sup\{d_X(x, x') \mid x, x' \in X\} =: \mathrm{diam} X.$$

So,

$$G(X, \{p\}) = \frac{1}{2}\mathrm{diam}X.$$

Proof. First we show: $2G(X,Y) \geq \inf_R(\mathrm{dis}R)$.

Let $r > G(X,Y)$. Then there exists an L–metric space (Z, d_Z) such that X and Y can be embedded into Z and $r > H^s d_Z(X,Y)$. Define

$$R = \{(x,y) \mid d_Z(x,y) \leq r\}.$$

Then R is a correspondence. Indeed, since

$$\begin{aligned} r > H^s d_Z(X,Y) &\geq H d_Z(X,Y) \\ &= \inf\{r' > 0 \mid \forall x \in X \exists y \in Y\ (d_Z(x,y) \leq r')\}, \end{aligned}$$

for each $x \in X$ there exist $y \in Y$ such that $d_Z(x,y) \leq r$. A symmetric argument shows that for all $y \in Y$, there exists $x \in X$ with $(x,y) \in R$. Next we calculate dis$R$. For pairs $(x,y),(x',y') \in R$,

$$\begin{aligned} |d_X(x,x') - d_Y(y,y')| &= |d_Z(x,x') - d_Z(y,y')| \\ &= |d_Z(x,x') - d_Z(y,x') + d_Z(y,x') - d_Z(y,y')| \\ &\leq |d_Z(x,x') - d_Z(y,x')| + |d_Z(y,x') - d_Z(y,y')| \\ &\leq |d_Z(x,y)| + |d_Z(y',x')| \\ &= d_Z(x,y) + d_Z(y',x') \leq 2r. \end{aligned}$$

Hence dis$R \le 2r$.

Since, for all $\varepsilon > 0$,

$$G(X,Y) < G(X,Y) + \varepsilon,$$

we have

$$\inf_R(\mathrm{dis}R) \le 2G(X,Y) + 2\varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have reached the desired conclusion.

Next, we show $2G(X,Y) \le \inf_R(\mathrm{dis}R)$.

Let dis$R = 2r$. We construct an admissible L–metric d on the disjoint union of X and Y as follows: for any $x \in X$ and $y \in Y$, set

$$d(x,y) = \inf\{d_X(x,x') + r + d_Y(y',y) \mid (x',y') \in R\}$$

and

$$d(y,x) = \inf\{d_Y(y,y') + r + d_X(x',x) \mid (x',y') \in R\}.$$

Reflexivity of d follows from the reflexivity of d_X and d_Y. We verify the triangle inequality: For $\bar{x} \in X$ we have

$$\begin{aligned} d(x,\bar{x}) + d(\bar{x},y) &= d_X(x,\bar{x}) + \inf\{d_X(\bar{x},x') + r + d_Y(y',y) \mid (x',y') \in R\} \\ &= \inf\{d_X(x,\bar{x}) + d_X(\bar{x},x') + r + d_Y(y',y) \mid (x',y') \in R\} \\ &\le \inf\{d_X(x,x') + r + d_Y(y',y) \mid (x',y') \in R\} = d(x,y). \end{aligned}$$

A similar argument shows that the triangle inequality holds if $\bar{x}$ would be in Y.

Next, observe that $Hd(X, Y) \leq r$. Indeed, given any $x \in X$ pick $y \in Y$ such that $(x, y) \in R$. Then $d(x, y) = r$. Similarly, $Hd(Y, X) \leq r$. Thus we get

$$H^s d(X, Y) \leq r = \frac{1}{2} \mathrm{dis} R.$$

Since this holds for any correspondence R between X and Y, we conclude,

$$H^s d(X, Y) \leq \frac{1}{2} \inf_R (\mathrm{dis} R),$$

as required. □

Observe that this proof also works for the Gromov distance defined on obMet: just replace "L–metric" by "metric" everywhere in the proof and notice that the admissible L–metric d on $X \cup Y$ we construct in the second half of the proof is actually a metric when d_X and d_Y are metrics.

The above characterization of the Gromov metric allows to easily show that the triangle inequality holds for the Gromov distance:

Proposition 4.3.5 (Exercise 7.3.26 in (BBI)). *For any L–metric spaces X, Y and Z,*

$$G(X, Z) \leq G(X, Y) + G(Y, Z).$$

Proof. The proof consists of several simple components.

1. Let R_1 be a correspondence between X and Y and R_2 a correspondence between Y and Z. Then $R_1 \cdot R_2$ is a correspondence between X and Z. Indeed, given and $x \in X$ there exists $y \in Y$ with $(x, y) \in R_1$ and exists $z \in Z$ with $(y, z) \in R_2$, so that $(x, z) \in R_1 \cdot R_2$. Similarly, for each $z \in Z$ there is an $x \in X$ with $(x, z) \in R_1 \cdot R_2$.

2. $\mathrm{dis}(R_1 \cdot R_2) \leq \mathrm{dis}R_1 + \mathrm{dis}R_2$

We shall the use the symbol $\mathcal{C}$ to denote the condition $\exists y \in Y((x, y) \in R_1, (y, z) \in R_2), \exists y' \in Y((x', y') \in R_1, (y', z') \in R_2)$.

$$\begin{aligned}
\mathrm{dis}(R_1 \cdot R_2) &= \sup\{|d_X(x, x') - d_Z(z, z')| \mid \mathcal{C} \text{ holds}\} \\
&= \sup\{|d_X(x, x') - d_Y(y, y') + d_Y(y, y') - d_Z(z, z')| \mid \mathcal{C} \text{ holds}\} \\
&\leq \sup\{|d_X(x, x') - d_Y(y, y')| + |d_Y(y, y') - d_Z(z, z')| \mid \mathcal{C} \text{ holds}\} \\
&\leq \sup\{|d_X(x, x') - d_Y(y, y')| \mid (x, y) \in R_1, (x', y') \in R_1\} \\
&\qquad + \sup\{|d_Y(y, y') - d_Z(z, z')| \mid (y, z) \in R_2, (y', z') \in R_2\} \\
&= \mathrm{dis}R_1 + \mathrm{dis}R_2.
\end{aligned}$$

3. $G(X,Z) \leq G(X,Y) + G(Y,Z)$:

$$\begin{aligned}
2G(X,Y) &= \inf_{R} \mathrm{dis} R \quad (R \text{ corr. b/w } X \text{ and } Z) \\
&\leq \inf_{R_1,R_2} \mathrm{dis}(R_1 \cdot R_2) \quad (R_1 \text{ corr. b/w } X \text{ and } Y,\ R_2 \text{ b/w } Y \text{ and } Z) \\
&\leq \inf_{R_1,R_2} (\mathrm{dis} R_1 + \mathrm{dis} R_2) \\
&= \inf_{R_1} \mathrm{dis} R_1 + \inf_{R_2} \mathrm{dis} R_2 \\
&= 2(G(X,Y) + G(Y,Z)).
\end{aligned}$$

□

4.4 G as a metric on isometry classes of compact metric spaces

We saw that the Gromov metric satisfies the triangle inequality, and it is clearly reflexive and symmetric. It turns out that if X and Y are compact metric spaces and $G(X,Y) = 0$, then X and Y are isometric. Hence G is a metric on the class of isometry classes of compact metric spaces. The goal of this section is to establish this result.

Given two metric spaces, (X, d_X) and (Y, d_Y) and any set mapping $f : X \longrightarrow Y$, we define the *distortion* of f analogously to a distortion of a correspondence between X and Y:

$$\mathrm{dis} f = \sup\{|d_X(x,x') - d_Y(f(x), f(x'))| \mid x, x' \in X\}.$$

It is clear that when R is a correspondence induced by a a surjection f, then $\mathrm{dis}f = \mathrm{dis}R$.

Definition 4.4.1. Let X and Y be metric spaces and $\varepsilon > 0$. A (set!) mapping $f : X \to Y$ is an *ε-isometry* provided that $\mathrm{dis}f \leq \varepsilon$ and $f(X) \subseteq Y$ is an ε-net.

As with correspondences, ε–isometries are approximations of isometric maps. Indeed, regular isometries are just "0-isometries". So in what follows a good way to interpret ε–isometries is to think of them as maps that are ε distance from being an isometry.

The next proposition establishes a connection between ε–isometries and the Gromov distance.

Proposition 4.4.2 (Corollary 7.3.28 in (BBI))**.** *Let (X, d_X) and (Y, d_Y) be metric spaces and let $\varepsilon > 0$. Then*

1. *$G(X, Y) < \varepsilon \implies \exists\, 2\varepsilon$–isometry from X to Y;*
2. *If there exists an ε–isometry from X to Y, then $G(X, Y) < 2\varepsilon$.*

Proof. 1. By Theorem 4.3.4, we can write

$$2G(X, Y) = \inf_R \mathrm{dis}R < 2\varepsilon,$$

giving us a correspondence R with $\mathrm{dis}R < 2\varepsilon$. For $x \in X$ pick $f(x) \in Y$ such that $(x, f(x)) \in R$. This defines a mapping from X to Y with $\mathrm{dis}f \leq \mathrm{dis}R < 2\varepsilon$. It remains to verify that $f(X)$ is an 2ε–net in Y. Given any $y \in Y$ there exists $x \in X$ such that $(x, y) \in R$. Thus,

$$d(y, f(x)) = |d_X(x, x) - d_Y(y, f(x))| \leq \text{dis}R \leq 2\varepsilon,$$

since $(x, y), (x, f(x)) \in R$.

2. Let f be an ε–isometry and define R by

$$R = \{(x, y) \mid d_Y(y, f(x)) \leq \varepsilon\}.$$

Then R is a correspondence: for any $x \in X$ there exists $y = f(x)$ in Y such that $d_Y(y, f(x)) = 0 \leq \varepsilon$. For any $y \in Y$ there exists $f(x)$ such that $d_Y(y, f(x)) \leq \varepsilon$, since $f(X)$ is an ε–net in Y.

Next, observe that for all $(x, y), (x', y') \in R$

$$\begin{aligned}
|d_Y(y, y') - d_X(x, x')| &\leq |d_Y(y, f(x)) + d_Y(f(x), y') - d_X(x, x')| \\
&\leq |d_Y(y, f(x)) + d_Y(f(x), f(x')) + d_Y(f(x'), y') - d_X(x, x')| \\
&\leq |d_Y(f(x), f(x')) - d_X(x, x')| + |d_Y(y, f(x))| + |d_Y(f(x'), y')| \\
&\leq \text{dis}f + 2\varepsilon \leq 3\varepsilon
\end{aligned}$$

Thus,

$$G(X, Y) \leq \frac{1}{2}\text{dis}R \leq \frac{3}{2}\varepsilon < 2\varepsilon.$$

□

We are now in the position to prove the main theorem of this section.

Theorem 4.4.3 (Theorem 7.3.30 in (BBI))**.** *G defines a metric on the class of isometry classes of compact metric spaces.*

Proof. First we show that this metric is finite. Let X and Y be compact metric spaces, then both are totally bounded. Let F_X and F_Y be 1–nets for X and Y, respectively. Then by Proposition 4.3.5, we have

$$G(X,Y) \leq G(X,F_X) + G(F_X,F_Y) + G(F_Y,Y) \leq 2 + G(F_X,F_Y).$$

Each correspondence between F_X and F_Y has finite cardinality. And thus, for any such correspondence R, we have the implication:

$$\forall\ (x,y),(x',y') \in R\ \ (|d_X(x,x') - d_Y(y,y')| < \infty) \implies \mathrm{dis}R < \infty.$$

Whence, $G(F_X,F_Y) < \infty$ and consequently G is finite.

Next, we show that if $G(X,Y) = 0$, then X and Y are isometric. From $G(X,Y) < \frac{1}{2n}$ for all $n \in \mathbb{N}$ and Proposition 4.4.2, we obtain a sequence $\{f_n\}$ of maps such that for every $n \in \mathbb{N}$, f_n is a $\frac{1}{n}$–isometry. Consequently, $\mathrm{dis}f_n \longrightarrow 0$, as $n \longrightarrow \infty$. Let $D \subseteq X$ be a countable dense subset of X (such a subset exists because X is totally bounded). Since Y is compact, we can apply Cantor's diagonal process (see (Ca), pg. 90), to obtain a subsequence $\{f_{n_k}\}$, such that for all $x \in D$, $\{f_{n_k}(x)\}$) converges in Y. Without loss of generality, we assume that this actually holds for the original sequence $\{f_n\}$.

We define $f : D \longrightarrow Y$ by

$$f(x) = \lim_n f_n(x),$$

for all $x \in D$. Now,

$$|d_Y(f_n(x), f_n(x')) - d_X(x, x')| \leq \mathrm{dis} f_n \longrightarrow 0.$$

So,

$$d_Y(f(x), f(x')) = \lim_n d_Y(f_n(x), f_n(x')) = d_X(x, x'),$$

for all $x, x' \in D$. Since X is complete, we can extend $f : D \longrightarrow Y$ to an isometry $\tilde{f} : X \longrightarrow Y$. Similarly, there exists an isometry $\tilde{g} : Y \longrightarrow X$. Thus, $\tilde{f} \cdot \tilde{g} : Y \longrightarrow Y$ is distance preserving. Since Y is compact, $\tilde{f} \cdot \tilde{g}$ is surjective (see (BBI), Theorem 1.6.14). Thus $\widetilde{f}$ is surjective too and so X and Y are isometric.

The triangle inequality and reflexivity were already demonstrated. Symmetry of G follows directly from the definition. Hence, we showed that G is a metric on the class of all equivalence classes of compact metric spaces. □

We conclude this section with a proposition that shows the relationship between Gromov distance and products of metric spaces.

Proposition 4.4.4. *Let X, Y, Z and W be L–metric spaces. Then the following formula holds:*

$$G(X \times Y, Z \times W) \leq G(X, Z) + G(Y, W).$$

(Here the metric on $X \times Y$ is given by $d((x,y),(x',y')) = d_X(x,x') + d_Y(y,y')$).

Proof. We denote the metric on $X \times Y$ by d and the metric on $Z \times W$ by d'. Let R be a correspondence from X to Z, S a correspondence from Y to W. Define a relation T from $X \times Y$ to $Z \times W$ by

$$((x,y),(z,w)) \in T \iff (x,z) \in R \text{ and } (y,w) \in S.$$

Then T is a correspondence since R and S are. We compute the distortion of T:

Let $\mathcal{C}'$ denote the condition $\big(((x,y),(z,w)) \in T, ((x',y'),(z',w')) \in T\big)$ and $\mathcal{C}''$ denote $\big((x,z) \in R, (x',z') \in R, (y,w) \in S, (y',w') \in S\big)$.

$$\begin{aligned}
\mathrm{dis}T &= \sup\{|d((x,y),(x',y')) - d'((z,w),(z',w'))| \mid \ \mathcal{C}' \text{ holds}\} \\
&= \sup\{|d_X(x,x') + d_Y(y,y') - d_Z(z,z') - d_W(w,w'))| \mid \ \mathcal{C}' \text{ holds}\} \\
&\leq \sup\{|d_X(x,x') - d_Z(z,z')| + |d_Y(y,y') - d_W(w,w')| \mid \ \mathcal{C}' \text{ holds}\} \\
&= \sup\{|d_X(x,x') - d_Z(z,z')| + |d_Y(y,y') - d_W(w,w')| \mid \ \mathcal{C}'' \text{ holds}\} \\
&\leq \sup_{(x,z)\in R,(x',z')\in R} \big(|d_X(x,x') - d_Z(z,z')|\big) \\
&\qquad + \sup_{(y,w)\in S,(y',w')\in S} \big(|d_Y(y,y') - d_W(w,w')|\big) \\
&= \mathrm{dis}R + \mathrm{dis}S.
\end{aligned}$$

Now, fix correspondences R between X and Z and S between Y and W. Then

$$2G(X \times Y, Z \times W) - \mathrm{dis}R \leq \mathrm{dis}T - \mathrm{dis}R \leq \mathrm{dis}S.$$

Keeping R fixed and varying S we conclude

$$2G(X \times Y, Z \times W) - \mathrm{dis}R \leq 2G(Y, W).$$

A similar trick shows:

$$G(X \times Y, Z \times W) \leq G(X, Z) + G(Y, W).$$

$\square$

4.5 Gromov convergence

In this section we concentrate on convergence in the space of compact metric spaces. We are in a position to make the following simple observations:

1. Convergence of a sequence of subsets of a metric space X with respect to the Hausdorff distance always implies Gromov convergence of those sets considered as stand–alone metric spaces. This is immediate from the inequality

   $$G(A, B) \leq H^s d_X(A, B)$$

   for any subsets $A, B \subseteq X$.

2. Consider a sequence of metrics on a fixed space X: $\big(d_n : X \times X \longrightarrow [0, \infty]\big)_{n \in \mathbb{N}}$ that uniformly converges to a metric $d : X \times X \longrightarrow [0, \infty]$. Then the sequence

$X_n = (X, d_n)$ converges to (X, d) with respect to the Gromov distance. Indeed let $\varepsilon > 0$ and consider the correspondence

$$R_n = \{(x, x) \mid x \in X\}$$

between (X, d) and (X, d_n). Then

$$G(X, X_n) \leq \mathrm{dis} R_n = \sup_{x,y \in X} |d(x, y) - d_n(x, y)| \leq \varepsilon,$$

for sufficiently large n.

3. We saw earlier that $G(X, \{p\}) = \frac{1}{2}\mathrm{diam} X$. Thus a sequence of metric spaces converges to a point, if, and only if, their diameter tends to 0. In particular, let (X, d) be a metric space with $\mathrm{diam} X < \infty$ and define $\lambda_i X = (X, \lambda_i d)$. If $\lambda_i \longrightarrow 0$, then

$$\mathrm{diam}(\lambda_i X) = \sup_{x,x' \in X} \lambda_i d(x, x') = \lambda_i \sup_{x,x' \in X} d(x, x') = \lambda_i(\mathrm{diam} X) \longrightarrow 0;$$

consequently, $\lambda_i X \longrightarrow \{p\}$.

4. Let $X_n = (\mathbb{Z}, \frac{1}{n}d)$ where d denotes the standard Euclidian distance restricted to the integers: $d(x, y) = |x - y|$. We claim that $X_n \longrightarrow (\mathbb{R}, d) =: X$. Given any $n \in \mathbb{N}$, define the relation

$$R_n = \{(x, \lfloor nx \rfloor) \mid x \in \mathbb{R}\}.$$

Then R_n is a correspondence because for any real x, the pair $(x, \lfloor nx \rfloor)$ is in R_n. For any integer $m \in \mathbb{Z}$, $m/n \in \mathbb{R}$ and hence $(\frac{m}{n}, m) \in R_n$. We calculate the distortion of R_n:

$$\begin{aligned}
\mathrm{dis}R_n &= \sup_{x,y\in\mathbb{R}} |d(x,y) - \frac{1}{n}d(\lfloor nx \rfloor, \lfloor ny \rfloor)| \\
&= \sup_{x,y\in\mathbb{R}} \big||x-y| - \frac{1}{n}|\lfloor nx \rfloor - \lfloor ny \rfloor|\big| \\
&\leq \sup_{x,y\in\mathbb{R}} |x - y - \frac{1}{n}\lfloor nx \rfloor + \frac{1}{n}\lfloor ny \rfloor| \\
&\leq \sup_{x,y\in\mathbb{R}} |x - \frac{1}{n}\lfloor nx \rfloor| + |y - \frac{1}{n}\lfloor ny \rfloor|.
\end{aligned}$$

Now, we always have the inequalities:

$$x - 1 \leq \lfloor x \rfloor \leq x$$

and hence

$$x - \frac{1}{n} \leq \frac{1}{n}\lfloor nx \rfloor \leq x;$$

consequently

$$x \leq \lim \frac{1}{n}\lfloor nx \rfloor \leq x$$

and the limit is independent of x. We conclude that

$$G(X_n, X) \leq \frac{1}{2}\mathrm{dis}R_n \leq \frac{1}{2}\sup_{x,y\in\mathbb{R}} |x - \frac{1}{n}\lfloor nx \rfloor| + |y - \frac{1}{n}\lfloor ny \rfloor| \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Visually, the spaces X_n are integers with size of the gaps between the numbers shrinking to the point where those gaps virtually vanish. It is interesting to note that if we would take any finite set of integers in place of $\mathbb{Z}$ in the definition of X_n, the limit of the sequence would then be a single point space. So, any finite subset of integers with decreasing metrics converges to a point, but the set of *all* integers with decreasing metrics converges to the continuum!

5. We keep the notation of the previous example: $X_n = (\mathbb{Z}, \frac{1}{n}d)$, where d is the usual Eucledian metric on $\mathbb{R}$ restricted to $\mathbb{Z}$ and $X = (\mathbb{R}, d)$. Let $Y_n = (X_n \times X_n) = (\mathbb{Z} \times \mathbb{Z}, \frac{1}{n}d')$ where d' denotes the product metric (i.e. the taxicab metric on $\mathbb{R}^2$ restricted to $\mathbb{Z}^2$). We claim that (Y_n) converges to $X \times X := (\mathbb{R}^2, d')$. Indeed Proposition 4.4.4 gives:

$$G(X_n \times X_n, X \times X) \leq G(X_n, X) + G(X_n, X) \longrightarrow 0,$$

as $n \longrightarrow \infty$.

Visually, the spaces Y_n are grids in $\mathbb{R}^2$ that become finer as $n \longrightarrow \infty$.

Interestingly, this example of Gromov convergence played a role in Gromov's proof of the Milnor conjecture for groups of polynomial growth. For more details see (Gr1) and (Gr2).

6. Given a compact metric space X and an $\varepsilon > 0$, there exists a *finite* ε–net $F \subseteq X$. Then, as we already observed, $G(X, F) \leq \varepsilon$. Consequently, the set of finite metric spaces is dense in the class of compact metric spaces equipped

with the Gromov distance. This observations will greatly simplify our work with Gromov convergence.

Inspired by the last example, we shall create a framework that will allow us to reduce convergence with respect to the Gromov distance to convergence of finite metric spaces. We begin with a definition:

Definition 4.5.1. Let (X, d_X) and (Y, d_Y) be compact metric spaces, $\varepsilon > 0$ and $\delta > 0$. We say that X and Y are (ε, δ)*–approximations* of each other provided that there exist finite sets $A := \{x_1, \ldots, x_N\}$ and $B := \{y_1, \ldots, y_N\}$ such that

1. A is an ε–net for X; B is a ε–net for Y.

2. $|d_X(x_i, x_j) - d_Y(y_i, y_j)| < \delta$, for all i, j.

We call X and Y ε*–approximations* of each other if they are $(\varepsilon, \varepsilon)$–approximations of each other.

We establish the following useful, but somewhat technical result:

Proposition 4.5.2 (Proposition 7.4.11 in (BBI))**.** *Let X and Y be compact metric spaces*

1. *If Y is an (ε, δ)–approximation of X, the $G(X, Y) < 2\varepsilon + \delta$.*

2. *If $G(X, Y) < \varepsilon$, then Y is a 5ε–approximation of X.*

Proof. (1) Let $X_0 = \{x_1, \dots, x_N\}$ and $Y_0 = \{y_1, \dots, y_N\}$ be chosen as in the definition. Define the correspondence R between X_0 and Y_0 by

$$R = \{(x_i, y_i) \mid 1 \le i \le N\}.$$

Then $\mathrm{dis} R < \delta$ and hence $G(X_0, Y_0) < \frac{\delta}{2}$. Since X_0 and Y_0 are ε–nets for X and Y, respectively, we conclude:

$$G(X, Y) \le G(X, X_0) + G(X_0, Y_0) + G(Y, Y_0) < 2\varepsilon + \delta.$$

(2) By Proposition 4.4.2, there exists a 2ε–isometry $f : X \longrightarrow Y$. Set $X_0 = \{x_1, \dots, x_N\}$ – an arbitrary ε–net in X – and $Y_0 = \{y_1 = f(x_1), \dots, y_N = f(x_N)\}$. Then

$$|d_X(x_i, x_j) - d_Y(y_i, y_j)| \le \mathrm{dis} f \le 2\varepsilon < 5\varepsilon.$$

It remains to show that Y_0 is a 5ε–net in Y. Let $y \in Y$. Then there exists $x \in X$ such that $d_Y(y, f(x)) \le 2\varepsilon$ (since $f(X)$ is a 2ε–net in Y). There exists $x_i \in X_0$ such that $d_X(x, x_i) < \varepsilon$. Hence,

$$\begin{aligned} d(y, y_i) &\le d(y, f(x)) + d(f(x), f(x_i)) \\ &\le 2\varepsilon + d(x, x_i) + \big(d(f(x), f(x_i)) - d(x, x_i)\big) \\ &\le 3\varepsilon + 2\varepsilon = 5\varepsilon. \end{aligned}$$

□

We can now reduce Gromov–convergence to convergence of finite metric spaces:

Proposition 4.5.3 (Proposition 7.4.12 in (BBI))**.** *For compact metric spaces X and X_n, $X_n \longrightarrow X$ with respect to the Gromov distance, if, and only if, for all $\varepsilon > 0$, there exists a finite ε–net $S \subseteq X$ and ε–nets $S_n \subseteq X_n$ such that $S_n \longrightarrow S$ with respect to the Gromov distance. Moreover, those nets can be chosen so that for all sufficiently large n, S_n has the same cardinality as S.*

Proof. If such ε–nets exist, then for large enough n, the cardinality of S_n and S are the same and we can increase n further to assure that $G(S_n, S) < \frac{\varepsilon}{2}$, so that we get a correspondence R_n between S_n and S with $\mathrm{dis}R_n < \varepsilon$. Thus, writing

$$S = \{x_1, \dots, x_N\}$$

and

$$S_n = \{y_1, \dots, y_N\}$$

with y_i such that $(x_i, y_i) \in R$, we get

$$|d_X(x_i, x_j) - d_{X_n}(y_i, y_j)| \leq \mathrm{dis}R < \varepsilon.$$

Thus, for large enough n, X and X_n are ε–approximations of each other. From Proposition 4.5.2 we conclude that $G(X, X_n) < 3\varepsilon$, for large enough n. Thus, $X_n \longrightarrow X$.

Conversely, suppose that $X_n \longrightarrow X$. So there exists N such that for all $n \geq N$, $G(X_n, X) < \frac{\varepsilon}{5}$ and hence, by Proposition 4.5.2, for all such n, X_n is an ε–

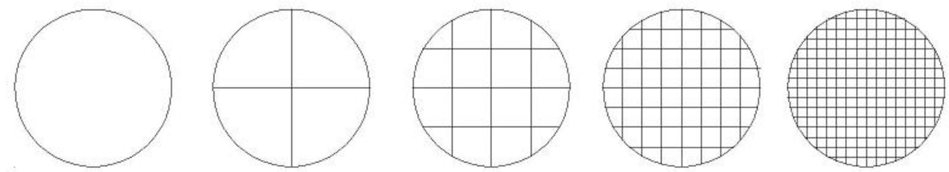


Figure 4.1: A sequence that converges to the unit disk

approximation of X. So there exist finite ε–nets S_n and S of X_n and X, respectively, such that their cardinalities equal and $G(S_n, S) < \varepsilon$. $\square$

Consider the sequence (X_n) of compact subsets of $\mathbb{R}^2$, where X_n is a grid contained in the unit circle such that the distance between the lines in the grid in $\frac{1}{2^n}$; this sequence is depicted in Figure 4.1. Note that the circle is not part of the sequence, but is drawn in the figure for reference.

We shall apply Proposition 4.5.3 to show that (X_n) converges to the unit disk. Given any $\varepsilon > 0$, we can create a grid in the unit disk such that the points where the lines of the grid intersect (we call those points the points of intersection of the grid) form a finite ε–net in the disk. In particular, we can pick $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \varepsilon$ and then the points of intersection of X_N form an ε–net in D^2. Let S denote this ε–net. Now, define S_n to be an arbitrary finite ε–net in X_n if $n < N$ and for $n \geq N$ set $S_n = S$. This definition makes sense because $X_n \subseteq X_{n+1}$ for all n. Then it is clear that the sequence (S_n) converges to S. Since ε was chosen arbitrarily, we conclude that (X_n) converges to D^2 as claimed.

In fact, the above argument can be easily modified to show that an increasing sequence of sets $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$ converges in the Gromov distance to their union $\cup_i X_i$.

4.6 Example of convergence: length spaces

In this section we give a brief introduction to length spaces, omitting some proofs, and then prove a result about limits of length spaces.

Let (X, d) be a metric space. Think of X as the surface of the earth and d the distance on a map between points in X. Thus, the distance between two mountain peeks is the distance a bird has to travel to get from one peek to another. For practical purposes, we would like to know what is the distance on foot from one peek to another. That is, we would like to know the distance of the shortest path on land between those peeks. This intuitive idea gives rise to the notion of length spaces which we define in what follows.

Let (X, d) be a metric space. A *path* in X is a continuous map $\gamma : [0, 1] \longrightarrow X$. Given a path γ in X, we define its *length* by

$$L(\gamma) = \sup_P \{\sum_{i=1}^{N} d(\gamma(x_{i-1}), \gamma(x_i))\}$$

where $P = \{x_0 = 0 < x_1 < \ldots < x_N = 1\}$ is a partition of $[0, 1]$ and the supremum is taken over all such partitions.

Next, we define a new metric on X by

$$\hat{d}(x,x') = \inf\{L(\gamma) \mid \gamma \text{ is a path from } x \text{ to } x'\}$$

If there are no paths between x and x' we set $\hat{d}(x,x') = \infty$. This new metric is called *the intrinsic metric induced by d.*

We say that (X,d) is a *length space* or *is intrinsic* if d coincides with the intrinsic metric induced by d, that is: $d = \hat{d}$.

We say that $z \in X$ is a *midpoint* between x and y if $d(x,z) = d(y,z) = \frac{1}{2}d(x,y)$. z is a ε*–midpoint* provided that $|2d(x,z) - d(x,y)| \leq \varepsilon$, and $|2d(y,z) - d(x,y)| \leq \varepsilon$.

We state the following theorem without proof:

Theorem 4.6.1. *Let (X,d) be a complete metric space. (X,d) is a length space if, and only if, for every $x, y \in X$ and a positive ε, there exists an ε–midpoint between x and y.*

Proof. See (BBI), Theorem 2.4.16 and Lemma 2.4.10. $\square$

Finally, we have the following result regarding limits of length spaces:

Theorem 4.6.2 (Theorem 7.5.1 in (BBI))**.** *Let $(X_n) = (X_n, d_n)$ be a sequence of length spaces, $X = (X,d)$ a complete metric space such that $X_n \to X$ in the Gromov metric. Then X is a length space.*

Proof. By the above theorem, it suffices to prove that given any two points $x, y \in X$, we can find an ε–midpoint for any $\varepsilon > 0$. Let n be such that $G(X, X_n) \le \frac{\varepsilon}{10}$. Then by Theorem 4.3.4 there is a correspondence R between X and X_n with $\mathrm{dis}R \le \frac{\varepsilon}{5}$. Take $\bar{x}, \bar{y} \in X_n$ such that $(x, \bar{x}), (y, \bar{y}) \in R$. Since, X_n is a length space, there exists $\bar{z} \in X_n$ that is an $\frac{\varepsilon}{5}$–midpoint for $\bar{x}$ and $\bar{y}$. Pick $z \in Z$ such that $(z, \bar{z}) \in R$. We claim that z is a ε–midpoint for x and y. Indeed,

$$
\begin{aligned}
|d(x,z) - \frac{1}{2}d(x,y)| &= |d(x,z) - d_n(\bar{x},\bar{z}) + d_n(\bar{x},\bar{z}) - \frac{1}{2}d(x,y)| \\
&\le |d(x,z) - d_n(\bar{x},\bar{z})| + \\
&\quad |d_n(\bar{x},\bar{z}) - \frac{1}{2}d_n(\bar{x},\bar{y}) + \frac{1}{2}d_n(\bar{x},\bar{y}) - \frac{1}{2}d(x,y)| \\
&\le \mathrm{dis}R + |d_n(\bar{x},\bar{z}) - \frac{1}{2}d_n(\bar{x},\bar{y})| + \frac{1}{2}\mathrm{dis}R \\
&< 2\mathrm{dis}R + |d_n(\bar{x},\bar{z}) - \frac{1}{2}d_n(\bar{x},\bar{y})| \\
&< \frac{2\varepsilon}{5} + \frac{\varepsilon}{5} < \varepsilon.
\end{aligned}
$$

□

4.7 Gromov compactness theorem

The next result sheds some light on compact classes of metric spaces with respect to Gromov distance.

Definition 4.7.1. We say that a class $\mathfrak{X}$ of compact metric spaces is *uniformly totally bounded* provided that

1. there exists a constant D such that $\operatorname{diam} X \leq D$ for all $X \in \mathfrak{X}$;

2. for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that every $X \in \mathfrak{X}$ contains an ε–net consisting of at most $N(\varepsilon)$ points.

Example 4.7.2. Let (X, d) be any compact metric space and let $\mathfrak{X}$ be the set of all compact subsets of X. We claim that $\mathfrak{X}$ is uniformly totally bounded. Indeed, the diameters of the subsets are bounded by $\operatorname{diam} X$. Let $A \in \mathfrak{X}$ be arbitrary. Now, given any $\varepsilon > 0$, we can construct a $\frac{\varepsilon}{2}$–net F in X. For any $y \in F$, let $a_y \in \eta_{\frac{\varepsilon}{2}}(y) \cap A$, if this intersection in non–empty. Then $F' = \{a_y \mid y \in F \text{ and } \eta_{\frac{\varepsilon}{2}}(y) \cap A \neq \emptyset\}$ is an ε–net for A. Indeed, given any $a \in A$ there exists y_0 in F such that $d(a, y_0) < \frac{\varepsilon}{2} (\implies \eta_{\frac{\varepsilon}{2}}(y) \cap A \neq \emptyset\ )$ and thus $a_{y_0} \in F'$ and,

$$d(a, a_{y_0}) \leq d(a, y_0) + d(y_0, a_{y_0}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We conclude that given any $\varepsilon > 0$, setting $N(\varepsilon) = |F|$ where F is a $\frac{\varepsilon}{2}$–net for X, will guarantee that any $A \in \mathfrak{X}$ contains a ε–net consisting of at most $N(\varepsilon)$ elements.

Theorem 4.7.3 (Theorem 7.4.15 in (BBI))**.** *Any uniformly totally bounded class $\mathfrak{X}$ of compact metric spaces is pre–compact in the Gromov topology. That is, the closure of $\mathfrak{X}$ is compact.*

Proof. Let $\mathfrak{X}$ be a uniformly totally bounded class of compact metric spaces. Using the notation of Definition 4.7.1, we set $N_1 = N(1)$ and for any $k \geq 2$, $N_k = N_{k-1} + N(\frac{1}{k})$. Let (X_n, d_n) be a sequence in $\mathfrak{X}$. Let $S_n = \{x_{i,n}\}_{i=1}^{\infty}$ be the union of $\frac{1}{k}$–nets in X_n such that the first N_k elements of S_n form a $\frac{1}{k}$–net in X_n. Such S_n can be constructed because of hypothesis on $\mathfrak{X}$. Further, by condition (1) of Definition 4.7.1, there exists a $D > 0$ such that for all n and all i, j,

$$d_n(x_{i,n}, x_{j,n}) \in [0, D].$$

Thus, using Cantor's diagonal argument (see (Ca), pg. 90), we can extract a subsequence (X_{n_m}) of (X_n) such that

$$d_{n_m}(x_{i,n_m}, x_{j,n_m})$$

converges for all i and j (since $[0, D]$ is compact). Without loss of generality, we suppose that those sequences converge without moving to the subsequence – this will only simplify our notation.

Set $X = \{x_i \mid i \in \mathbb{N}\}$ be an arbitrary countable set and define a metric d on X by

$$d(x_i, x_j) = \lim_{n \longrightarrow \infty} d_n(x_{n,i}, x_{n,j}).$$

d is clearly reflexive and satisfies the triangle inequality. Define a relation $R \subseteq X \times X$ by

$$(x_i, x_j) \in R \iff d(x_i, x_j) = 0.$$

Then R is an equivalence relation.

Let X/d denote the *metric* space $(X/R, d)$ with $d(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ ($\bar{x}$ denotes the equivalence class of x). Observe that d is well defined. Write $\overline{X}$ for the completion of X/d. We abuse notation and continue to denote the metric on $\overline{X}$ by d.

Our goal is to prove that (X_n) converges to $\overline{X}$ with respect to G. First we show that $\overline{X}$ is compact. Let

$$S^{(k)} = \{\bar{x_i} \mid 1 \leq i \leq N_k\} \subseteq \overline{X}$$

We claim that $S^{(k)}$ is a $\frac{1}{k}$–net for $\overline{X}$. Indeed, the set

$$S_n^{(k)} = \{x_{i,n} \mid 1 \leq i \leq N_k\}$$

is a $\frac{1}{k}$–net in X_n, for all n. Thus, for all $x_{i,n} \in S_n$, there exists a $j \leq N_k$ such that

$$d_n(x_{i,n}, x_{j,n}) \leq \frac{1}{k}.$$

Further, there is a fixed $j_0 \leq N_k$ such that

$$d_n(x_{i,n}, x_{j_0,n}) \leq \frac{1}{k}$$

for infinitely many n (by the pigeonhole principle). Thus, for every $\bar{x_i} \in \overline{X}$, we can pick $\overline{x_{j_0}} \in \overline{X}$ such that

$$d(\overline{x_i}, \overline{x_{j_0}}) = \lim_n d_n(x_{i,n}, x_{j_0,n}) \leq \frac{1}{k}.$$

Thus $\overline{X}$ is totally bounded. Since $\overline{X}$ is complete by construction, it is compact.

Next, we observe that $S^{(k)}$ is the Gromov limit of $S_n^{(k)}$. Indeed, define a correspondence between $S^{(k)}$ and $S_n^{(k)}$ by

$$R = \{(\bar{x}_i, x_{i,n}) \mid 1 \leq n \leq N_k\}.$$

Certainly, R is a correspondence. Let $\varepsilon > 0$. Then for all $1 \leq i, j \leq N_k$ there exists $N_{i,j} \in \mathbb{N}$ such that for all $n \geq N_{i,j}$,

$$|d(\bar{x_i}, \bar{x_j}) - d_n(x_{i,n}, x_{j,n})| \leq \varepsilon.$$

Let $N = \max\{N_{i,j} \mid 1 \leq i, j \leq N_k\}$. Then for all $n \geq N$

$$\begin{aligned} G(S^{(k)}, S_n^{(k)}) &\leq \operatorname{dis} R \\ &= \sup_{1 \leq i,j \leq N_k} |d(\bar{x_i}, \bar{x_j}) - d_n(x_{i,n}, x_{j,n})| < \varepsilon. \end{aligned}$$

By Proposition 4.5.3, we conclude that (X_n) converges to $\overline{X}$, as required. $\square$

Corollary 4.7.4. *The class of all compact metric spaces endowed with the Gromov metric is complete.*

Proof. Let (X_n) be a Cauchy sequence of compact metric spaces. We show that it has a convergent subsequence. By the previous Theorem, it suffices to show that the set

$$\mathfrak{X} = \{X_n \mid n \in \mathbb{N}\}$$

is uniformly totally bounded. Indeed, there exists an N_0 such that for all $n, m \geq N_0$, $G(X_n, X_m) \leq 1$. Let

$$D = \max\{\mathrm{diam}X_1, \ldots, \mathrm{diam}X_{N_0} + 1\}.$$

Then given any $n \in \mathbb{N}$, if $n \leq N_0$, then clearly $\mathrm{diam}X_n \leq D$. If $n \geq N_0$, then

$$\begin{aligned} \mathrm{diam}X_n = G(X_n, \{p\}) &\leq G(X_n, X_{N_0}) + G(X_{N_0}, \{p\}) \\ &\leq 1 + \mathrm{diam}X_{N_0} \leq D. \end{aligned}$$

Also, for all $\varepsilon > 0$, there exists N_1 such that $\forall\, n \geq N_1$, $G(X_n, X_{N_1}) \leq \frac{\varepsilon}{5}$. Then by Proposition 4.5.2 X_{N_1} is an ε–approximation of X_n. That is, there exist ε–nets $F_n \subseteq X_n$ and $F_{N_1} \subseteq X_{N_1}$ such that $|F_n| = |F_{N_1}|$, for all $n \geq N_1$. Since, all X_n are compact, for $1 \leq n < N_1$, there exist ε–nets $F_n \subseteq X_n$. Set

$$N(\varepsilon) = \max\{|F_1|, |F_2|, \ldots, |F_{N_1}|\}.$$

Then $N(\varepsilon)$ satisfies condition (2) in the definition of a uniformly totally bounded class. Whence, (X_n) has a convergent subsequence as claimed and since it is a Cauchy sequence, it is itself convergent. $\square$

4.8 Gromov distance in the $\mathcal{V}$-Cat

We shift our attention from classical results concerning Gromov distance to a categorical investigation of this concept. In particular, we define Gromov distance in the setting of $\mathcal{V}$–categories and investigate it and related concepts in $\mathcal{V}$-Cat.

We begin with a very general definition. Let $K : \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ be a functor defined by

$$K(X,a) = (PX, Ka)$$

with Ka some structure on PX. We define the *Gromov structure on* $\mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat})$ *with respect to* K by

$$G(K)(X,Y) = \bigvee_{(Z,c)} Kc(f(X), g(Y))$$

where the supremum is taken over all embeddings $f : X \hookrightarrow Z$ and $g : Y \hookrightarrow Z$.

We shall be primarily concerned with two functors: $K = H$ and $K = H^{\mathrm{sym}}$. We call $G(H^{\mathrm{sym}})$ the *Gromov–Hausdorff* or simply *Gromov* structure on $\mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat})$ and denote this structure by G. That is: $G = G(H^{\mathrm{sym}})$.

It is clear that in case $\mathcal{V} = \mathbb{P}_+$, G coincides with the Gromov distance between metric spaces. When $\mathcal{V} = 2$, we obtain an order on the class of ordered sets. In particular:

$$\begin{aligned}
X \leq_G Y \quad &\iff \quad G(X,Y) = \top \\
&\iff \quad \bigvee_{(Z,c)} H^s c(f(X), g(Y)) = \top \\
&\iff \quad \exists\ (Z,c) \text{ such that } X \hookrightarrow Z, Y \hookrightarrow Z \text{ and } H^{\mathrm{sym}}c(X,Y) = \top \\
&\iff \quad \exists (Z,c) \text{ with } X, Y \hookrightarrow Z \text{ and } \forall x \in X \exists y \in Y (x \leq_c y) \\
&\qquad\qquad \text{and } \forall y \in Y \exists x \in X (y \leq_c x).
\end{aligned}$$

It is easily seen that $\mathbb{R} \leq_G \mathbb{Z}$. Since the G is symmetric, we trivially have $\mathbb{Z} \leq_G \mathbb{R}$. A symmetric order does not seem too interesting. We ask: is it worth to strip G of its symmetry for order theoretic purposes? It turns out the the answer to this is negative. Indeed, setting

$$\begin{aligned} X \leq_{G'} Y \iff & \exists Z \text{ into which } X \text{ and } Y \text{ can be embedded} \\ & \text{and } \forall x \in X \exists y \in Y (x \leq y) \end{aligned}$$

gives us a trivial order in the sense that $X \leq_{G'} Y$ for any ordered sets X and Y. To see this, just take $Z = X \cup Y$ – the disjoint union of X and Y and order it by keeping the existing orders on X and Y and setting $x \leq y$ for all $x \in X$ and $y \in Y$. This indeed is an order on $X \cup Y$; reflexivity in trivially satisfied and the triangle inequality holds since $x \leq y$ and $y \leq y'$ implies $x \leq y'$ trivially. Thus it is obvious that $X \leq Y$ as subsets of Z.

When $\mathcal{V} = \mathbb{P}_+$, and $G(X,Y) = 0$, X and Y are isomorphic in Met_c – the full subcategory of Met consisting of all compact metric spaces. For general $\mathcal{V}$ this result is no longer true. Possibly the simplest counter–example exists in Ord. $G(X,Y) = k$ means precisely $X \leq_G Y$. Recall that an ordered set $(X, \leq)$ is compact if and only if the order is symmetric and there is a finite set $F \subseteq X$ with

$$\forall x \in X \exists y \in F (x \leq y)$$

(see the discussion following Definition 3.8.1). Take $X = \{1, 2\}$ with $1 \leq 2$ and

$2 \leq 1$ and $Y = \{3\}$. Then X and Y are compact. Let $Z = \{1, 2, 3\}$ with $\leq$ defined by $z \leq z'$ for all $z, z' \in Z$. Then X and Y can be embedded into Z and from there it easily follows that $X \leq_G Y$. But clearly X is not order isomorphic to Y.

As in the classical setting, we need more convenient tools for working with the Gromov distance. The first slight simplification of the definition allows to restrict the class of $\mathcal{V}$–categories Z to the disjoint union of X and Y:

Definition 4.8.1. Let (X, a) and (Y, b) be $\mathcal{V}$–categories. We say that a structure d on the disjoint union of X and Y is *admissible* provided that $d|_{X\times X} = a$ and $d|_{Y\times Y} = b$.

Proposition 4.8.2. *Let (X, a) and (Y, b) be $\mathcal{V}$-categories and $X \cup Y$ be their disjoint union. Then*

$$G(X, Y) = \bigvee_d H^{\mathrm{sym}} d(X, Y)$$

where the supremum is taken over all admissible structures on $X \cup Y$.

Proof. "$\geq$": For any admissible structure d on $X \cup Y$, take $Z = (X \cup Y, d)$. Then

$$G(X, Y) \geq H^{\mathrm{sym}} d(X, Y).$$

Since this holds for all d, we get the desired conclusion.

"$\leq$": Let (Z, c) be a $\mathcal{V}$–category into which X and Y can be embedded via $\mathcal{V}$–functors f and g, respectively . We define an admissible structure d on $X \cup Y$

by

$$d(x,y) = c(f(x), g(y)).$$

Then d is indeed a structure on $X \cup Y$. It is reflexive since its admissible.

For $x, x' \in X$, $y \in Y$:

$$\begin{aligned} d(x,x') \otimes d(x',y) &= a(x,x') \otimes c(f(x'), g(y)) \\ &= c(f(x), f(x')) \otimes c(f(x'), g(y)) \\ &\leq c(f(x), g(y)) = d(x,y), \end{aligned}$$

and similarly for $x \in X$ and $y, y' \in Y$.

Next, observe that $Hc(f(X), g(Y)) = Hd(X,Y)$:

$$\begin{aligned} Hd(X,Y) &= \bigwedge_{x \in X} \bigvee_{y \in Y} d(x,y) \\ &= \bigwedge_{x \in X} \bigvee_{y \in Y} c(f(x), g(y)) = Hc(f(X), g(Y)) \end{aligned}$$

Thus, for all (Z, c)

$$H^{\text{sym}} c(f(X), g(Y)) = H^{\text{sym}} d(X,Y) \leq \bigvee_d H^{\text{sym}} d(X,Y)$$

and the desired result follows. $\square$

Our next formulation of the Gromov distance is based on Lawvere's observation that admissible structures on $X \cup Y$ are just pairs of $\mathcal{V}$–modules $\varphi : X \nrightarrow Y$ and

$\psi : Y \nrightarrow X$. Indeed, given any admissible structure d on $X \cup Y$, we define

$$\varphi : (X, a) \nrightarrow (Y, b)$$

by setting $\varphi(x, y) = d(x, y)$. Then φ is a $\mathcal{V}$–module:

$$\begin{aligned} (\varphi \cdot a)(x, y) &= \bigvee_{x' \in X} a(x, x') \otimes \varphi(x', y) \\ &= \bigvee_{x' \in X} d(x, x') \otimes d(x', y) \\ &\leq d(x, y) = \varphi(x, y). \end{aligned}$$

A similar argument shows that $b \cdot \varphi \leq \varphi$. We define $\psi : Y \nrightarrow X$ by $\psi(y, x) = d(y, x)$; as before, it is easy to see that ψ is a $\mathcal{V}$–module.

Conversely, given any two $\mathcal{V}$–modules $\varphi : X \nrightarrow Y$ and $\psi : Y \nrightarrow X$, we define an admissible structure d on $X \cup Y$ by $d(x, y) = \varphi(x, y)$ and $d(y, x) = \psi(y, x)$ for all $x \in X, y \in Y$. Then d is a indeed a structure: it is reflexive since its admissible. The triangle inequality also holds: given any $x, x' \in X$ and $y \in Y$, we have

$$\begin{aligned} d(x, x') \otimes d(x', y) &= a(x, x') \otimes \varphi(x', y) \\ &\leq \bigvee_{x'' \in X} a(x, x'') \otimes \varphi(x'', y) \\ &\leq \varphi(x, y) = d(x, y). \end{aligned}$$

The other cases follow similarly.

In order to define the Gromov structure in terms of modules, we need to be able to apply the (lax) Hausdorff functor to modules. We already defined $H : \mathcal{V}\text{-}\mathsf{Rel} \longrightarrow$

$\mathcal{V}$-Rel when $\mathcal{V}$ is ccd (see discussion preceding Proposition 3.6.4). We extend H to a lax functor

$$H : \mathcal{V}\text{-}\mathsf{Mod} \longrightarrow \mathcal{V}\text{-}\mathsf{Mod}$$

as follows: for $\varphi : X \nrightarrow Y$ in $\mathcal{V}$-Mod an application of H yields:

$$H\varphi : HX \nrightarrow HY$$

where $H\varphi$ is a V–module:

$$H\varphi \cdot Ha \leq H(\varphi \cdot a) \leq H\varphi,$$

and the second inequality follows similarly.

We are now in the position to show that Lawvere's observations and Proposition 4.8.2 lead to a categorical description of the Gromov structure on $\mathcal{V}$-Cat:

Theorem 4.8.3. *Let $\mathcal{V}$ be ccd. For any $\mathcal{V}$–categories (X, a) and (Y, b)*

$$G(X,Y) = \bigvee_{\varphi,\psi} (H\varphi(X,Y) \wedge H\psi(Y,X))$$

where the supremum is taken over all $\mathcal{V}$–modules $\varphi : X \nrightarrow Y$ and $\psi : Y \nrightarrow X$.

Proof.

$$\begin{aligned}
G(X,Y) &= \bigvee_d H^{sym} d(X,Y) \qquad (d \text{ admissible structure on } X \cup Y) \\
&= \bigvee_d \big(Hd(X,Y) \wedge Hd(Y,X)\big) \\
&= \bigvee_d \Big(\big(\bigwedge_x \bigvee_y d(x,y)\big) \wedge \big(\bigwedge_y \bigvee_x d(y,x)\big)\Big) \\
&= \bigvee_{\varphi,\psi} \Big(\big(\bigwedge_x \bigvee_y \varphi(x,y)\big) \wedge \big(\bigwedge_y \bigvee_x \psi(y,x)\big)\Big) \\
&= \bigvee_{\varphi,\psi} \Big(H\varphi(X,Y) \wedge H\psi(Y,X)\Big)
\end{aligned}$$

$\square$

The proof of the following result is almost identical to the proof of the above Theorem:

Proposition 4.8.4. *Let $\mathcal{V}$ be ccd. For any $\mathcal{V}$–categories (X,a) and (Y,b)*

$$G(H)(X,Y) = \bigvee_\varphi H\varphi(X,Y)$$

where the supremum is taken over all $\mathcal{V}$–modules $\varphi : X \nrightarrow Y$.

Theorem 4.8.3 and Proposition 4.8.4 allow us to prove that G and $G(H)$ are structures on ob$\mathcal{V}$-Cat.

Theorem 4.8.5. *Let $\mathcal{V}$ be a ccd quantale. Then $G(H)$ is a $\mathcal{V}$–category structure on* ob$\mathcal{V}$-Cat*.*

Proof. Let $\varphi : X \rightarrow\!\!\!\!\!\circ\; Y$ and $\psi : Y \rightarrow\!\!\!\!\!\circ\; Z$ be arbitrary $\mathcal{V}$–modules. Since

$$H(\psi \cdot \varphi)(X, Z) \geq (H\psi \cdot H\varphi)(X, Z),$$

we have in particular

$$\begin{aligned} G(H)(X, Z) &\geq H(\psi \cdot \varphi)(X, Z) \\ &\geq H\varphi(X, Y) \otimes H\psi(Y, Z) \end{aligned}$$

and since $\otimes$ respects suprema in each variable

$$\begin{aligned} G(H)(X, Z) &\geq \bigvee_{\varphi} H\varphi(X, Y) \otimes \bigvee_{\psi} H\psi(Y, Z) \\ &= G(H)(X, Y) \otimes G(H)(Y, Z). \end{aligned}$$

Reflexivity is clear. $\square$

Theorem 4.8.6. *Let $\mathcal{V}$ be a ccd quantale. Then G is a $\mathcal{V}$–category structure on* ob$\mathcal{V}$-Cat.

Proof. We pick four arbitrary $\mathcal{V}$–modules:

$$\varphi : X \rightarrow\!\!\!\!\!\circ\; Y, \quad \psi : Y \rightarrow\!\!\!\!\!\circ\; Z,$$

$$\psi' : Z \rightarrow\!\!\!\!\!\circ\; Y, \quad \varphi' : Y \rightarrow\!\!\!\!\!\circ\; X.$$

Then

$$
\begin{aligned}
G(X,Z) &\geq H(\psi\cdot\varphi)(X,Z)\wedge H(\varphi'\cdot\psi')(Z,X)\\
&\geq \big((H\psi\cdot H\varphi)(X,Z)\big)\wedge\big((H\varphi'\cdot H\psi')(Z,X)\big)\\
&\geq \big(H\varphi(X,Y)\otimes H\psi(Y,Z)\big)\wedge\big(H\psi'(Z,Y)\otimes H\varphi'(Y,X)\big)\\
&\geq \big(H\varphi(X,Y)\wedge H\varphi'(Y,X)\big)\otimes\big(H\psi(Y,Z)\wedge H\psi'(Z,Y)\big).
\end{aligned}
$$

Taking suprema over all φ, φ' and ψ, ψ' leads us to the desired conclusion.

Reflexivity is clear. □

The last formulation of the Gromov structure we provide is based on correspondences. We slightly modify the results of the classical theory in the hope that this will shed a bit more light on the object of our study.

Given any set mapping $f : (X, a) \longrightarrow (Y, b)$ define the *distortion* of f by

$$\mathrm{dis} f = \bigwedge_{x,x'\in X} \big(b(f(x), f(x')) \multimap a(x, x')\big).$$

‘

For any $n \in \mathbb{N}$ and $v \in \mathcal{V}$ we shall write nv for $v \otimes \ldots \otimes v$, n times.

In order mimic the statement of Theorem 4.3.4 in the $\mathcal{V}$–categorical setting, we need the notion of "$v/2$" for an arbitrary element $v \in \mathcal{V}$. One may achieve this by making sure that $\mathcal{V}$ is a value quantale in the sense of R. Flagg (see (Fl)). $\mathcal{V}$ is a value quantale if it is ccd and satisfies the following

$$v \ll k, w \ll k \implies v \vee w \ll k. \tag{4.3}$$

This condition allows us, for each $v \ll k$ to find $w \ll k$ such that $v \ll w \otimes w = 2w \ll k$. Thus, abusing notation we have

$$\frac{v}{2} \ll w.$$

In case $\mathcal{V} = \mathbb{P}_+$ this translates to

$$\forall x > 0 \; \exists y > 0 \; (\frac{x}{2} > y > 0).$$

One problem with the above approach is that we do not get halves but only a value strictly less that a half. The following approach fixes this problem. Consider the monotone map

$$\varphi : \mathcal{V} \to \mathcal{V}$$

given by

$$\varphi(v) = v \otimes v.$$

If we impose the following requirement on φ,

$$\varphi(\bigvee_i v_i) = \bigvee_i \varphi(v_i), \tag{4.4}$$

then φ has a right adjoint: ψ. We set

$$\frac{v}{2} := \psi(v) = \bigvee \{w \in \mathcal{V} \mid w \otimes w \leq v\}.$$

From general theory we get

$$v \leq \psi(\varphi(v)) = \frac{v \otimes v}{2}$$

and

$$\frac{v}{2} \otimes \frac{v}{2} = \varphi(\psi(v)) \leq v.$$

If we further ask

$$\forall\, v, w \in V, \quad (w \otimes w \leq v \otimes v \implies w \leq v), \tag{4.5}$$

that is, we ask φ to be a full functor, then the unit is an isomorphism, and hence

$$v = \frac{v \otimes v}{2}.$$

And if we also require ψ to be a full functor, that is,

$$\forall\, v, w \in V, \quad (\frac{w}{2} \leq \frac{v}{2} \implies w \leq v), \tag{4.6}$$

then

$$\frac{v}{2} \otimes \frac{v}{2} = v,$$

since the co–unit is then an isomorphism.

When $\mathcal{V} = \mathbb{P}_+$, φ is given by $\varphi(x) = x + x$ and it satisfies (4.4):

Proposition 4.8.7. *When $\mathcal{V} = \mathbb{P}_+$, φ preserves $\bigwedge$:*

$$\bigwedge_{i \in I}(x_i + x_i) = (\bigwedge_{j \in I} x_j) + (\bigwedge_{k \in I} x_k).$$

Proof. For all $i \in I$,

$$x_i + x_i \geq \big(\bigwedge_{j \in I} x_j\big) + \big(\bigwedge_{k \in I} x_k\big)$$

and hence "$\geq$" holds. Let $\alpha = \bigwedge_{i \in I}(x_i + x_i)$ and $\beta = \big(\bigwedge_{j \in I} x_j\big) + \big(\bigwedge_{k \in I} x_k\big)$. Suppose

$$\alpha > \beta = \bigwedge_{k,j \in I} (x_j + x_k).$$

Then there exists $j, k \in I$ such that for all $i \in I$

$$2x_i \geq \alpha > x_j + x_k.$$

Hence, in particular $x_j > x_k$ and $x_k > x_j$ which is a contradiction. Whence $\alpha = \beta$. □

Furthermore, both (4.5) and (4.6) hold in $\mathbb{P}_+$.

When $\mathcal{V} = 2$, φ is just the identity and hence (4.4), (4.5) and (4.6) all hold trivially.

We will also have to consider the scenario where ψ preserves $\bigvee$, that is, ψ has a right adjoint. When $\mathcal{V} = \mathbb{P}_+$ this certainly holds:

$$\frac{\bigwedge_i x_i}{2} = \frac{1}{2} \bigwedge_i x_i = \bigwedge_i \frac{x_i}{2};$$

it holds trivially when $\mathcal{V} = 2$.

Theorem 4.8.8. *Let $\mathcal{V}$ be a ccd quantale that satisfies (4.4). Let (X,a) and (Y,b) be any $\mathcal{V}$–categories. Then*

$$G(H)(X,Y) \geq \bigvee_{f:X \longrightarrow Y} \frac{\mathrm{dis} f}{2}$$

and

$$G(H)(X,Y) \leq \frac{\bigvee_f \mathrm{dis} f}{2}$$

with

$$G(H)(X,Y) = \bigvee_{f:X \longrightarrow Y} \frac{\mathrm{dis} f}{2}$$

when ψ preserves $\bigvee$.

Proof. "$\geq$": Let f be any set mapping $f : X \longrightarrow Y$ and define an admissible structure d on $X \cup Y$ by

$$d(x,y) = \bigvee_{x' \in X} a(x,x') \otimes \delta \otimes b(f(x'),y),$$

where $\delta = \frac{\mathrm{dis} f}{2}$.

Then d is indeed a structure. It is reflexive since its admissible. It also satisfies the triangle inequality: for any $\bar{x} \in X$

$$\begin{aligned} d(x,\bar{x}) \otimes d(\bar{x},y) &= a(x,\bar{x}) \otimes \bigvee_{x'} \big(a(\bar{x},x') \otimes \delta \otimes b(f(x'),y)\big) \\ &= \bigvee_{x'} \big(a(x,\bar{x}) \otimes a(\bar{x},x') \otimes \delta \otimes b(f(x'),y)\big) \\ &\leq \bigvee_{x'} \big(a(x,x') \otimes \delta \otimes b(f(x'),y)\big) \\ &= d(x,y). \end{aligned}$$

A similar argument shows that the triangle inequality also holds when $\bar{x} \in Y$.

We claim that $Hd(X,Y) \geq \delta$. Indeed for any $x \in X$, $f(x) \in Y$ and

$$\begin{aligned} d(x, f(x)) &= \bigvee_{x'} (a(x,x') \otimes \delta \otimes b(f(x'), f(x))) \\ &\geq (a(x,x) \otimes \delta \otimes b(f(x), f(x))) \\ &\geq \delta. \end{aligned}$$

Thus, by Proposition 3.6.3

$$Hd(X,Y) \geq \delta$$

and hence

$$G(H)(X,Y) \geq Hd(X,Y) \geq \delta = \frac{\mathrm{dis} f}{2}.$$

Since f was arbitrary, we have reached the desired conclusion.

"$\leq$": Take $v \in \mathcal{V}$ such that $v \ll k$ and $G(H)(X,Y) \gg v$. Then there exists an admissible structure c on $X \cup Y$ such that $Hc(X,Y) \gg v$.

We define a function

$$f : X \longrightarrow Y$$

by $f(x) = y$ where $y \in Y$ is such that $c(x,y) \gg v$ (here we assume the axiom of choice). Then f is a function. Indeed,

$$Hc(X,Y) = \bigvee \{ w \mid \forall x \in X \exists y \in Y (c(x,y) \geq w) \} \gg v$$

Thus, there exists $w \in \mathcal{V}$ such that

$$\forall x \in X \exists y = f(x) \in Y \ (c(x, y) \geq w \gg v).$$

Next, we show that $\mathrm{dis} f \geq 2v$.

$$\begin{aligned}
\mathrm{dis} f &= \bigwedge_{x, x' \in X} b(f(x), f(x')) \multimap a(x, x') \\
&= \bigwedge_{x, x' \in X} c(f(x), f(x')) \multimap c(x, x') \\
&\geq \bigwedge_{x, x' \in X} \big(c(f(x), f(x')) \multimap c(x, f(x'))\big) \otimes \big(c(x, f(x')) \multimap c(x, x')\big) \\
&\geq \bigwedge_{x, x' \in X} c(x, f(x)) \otimes c(x', f(x')) \\
&\geq 2v.
\end{aligned}$$

where the middle inequality and the one that follows it hold because of the triangle inequality for $\multimap$ and c, respectively. Hence,

$$\bigvee_{f : X \longrightarrow Y} \mathrm{dis} f \geq \mathrm{dis} f \geq 2v$$

Thus, we have the following inclusion of sets

$$\{v \in \mathcal{V} \mid v \ll G(H)(X, Y)\} \subseteq \{v \in \mathcal{V} \mid 2v \leq \bigvee_f \mathrm{dis} f\}.$$

Consequently,

$$\begin{aligned}
G(H)(X, Y) &= \bigvee \{v \in \ V \mid v \ll G(H)(X, Y)\} \\
&\leq \bigvee \{v \mid 2v \leq \bigvee_f \mathrm{dis} f\} = \frac{\bigvee_f \mathrm{dis} f}{2}.
\end{aligned}$$

When ψ preserves $\bigvee$, $\frac{\bigvee_f \operatorname{dis} f}{2} = \bigvee_f \frac{\operatorname{dis} f}{2}$ and we get equality. $\square$

The following result demonstrates the relationship between the Gromov distance G and distortions of set functions. Since the proof of this result is similar to the proof of the above Theorem, we do not provide as much detail.

Theorem 4.8.9. *Let $\mathcal{V}$ be a ccd quantale that satisfies (4.4). Let (X, a) and (Y, b) be $\mathcal{V}$–categories. Then*

$$G(X,Y) \geq \bigvee_{f,g} \Big(\frac{\operatorname{dis} f}{2} \wedge \frac{\operatorname{dis} g}{2}\Big)$$

and

$$G(X,Y) \leq \Big(\frac{\bigvee_f \operatorname{dis} f}{2}\Big) \wedge \Big(\frac{\bigvee_g \operatorname{dis} g}{2}\Big),$$

where the supremum is taken over all mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$. If ψ preserves $\bigvee$ and $(\mathcal{V}, \bigvee, \wedge)$ is a frame, then

$$G(X,Y) = \bigvee_{f,g} \Big(\frac{\operatorname{dis} f}{2} \wedge \frac{\operatorname{dis} g}{2}\Big).$$

Proof. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be arbitrary. Define admissible structures d' and d'' on $X \cup Y$ by

$$d'(x,y) = \bigvee_{x' \in X} a(x,x') \otimes \delta' \otimes b(f(x'),y)$$

and

$$d''(x,y) = \bigvee_{y' \in Y} a(x,g(y')) \otimes \delta'' \otimes b(y',y)$$

where $\delta' = \frac{\mathrm{dis} f}{2}$ and $\delta'' = \frac{\mathrm{dis} g}{2}$. Set

$$d = d' \vee d''.$$

Then, by arguments from the proof of Theorem 4.8.8, d is an admissible structure on $X \cup Y$ and for all $x \in X$, $f(x) \in Y$ is such that

$$d(x, f(x)) \geq d'(x, f(x)) \geq \delta'.$$

For all $y \in Y$,

$$d(g(y), y) \geq d''(g(y), y) \geq \delta''.$$

Whence $Hd(X, Y) \geq \delta'$ and $Hd(Y, X) \geq \delta''$. We conclude:

$$H^s d(X, Y) = Hd(X, Y) \wedge Hd(Y, X) \geq \delta' \wedge \delta'' = \frac{\mathrm{dis} f}{2} \wedge \frac{\mathrm{dis} g}{2}.$$

Taking the supremum over all such f and g, we obtain

$$G(X, Y) \geq \bigvee_{f,g} \Big(\frac{\mathrm{dis} f}{2} \wedge \frac{\mathrm{dis} g}{2} \Big).$$

This proves the first inequality.

For the second inequality, from $G(X, Y) \gg v$, we obtain an admissible structure c on $X \cup Y$ such that $H^s c(X, Y) \gg v$. In particular this implies $Hc(X, Y) \gg v$ and $Hc(Y, X) \gg v$. As before, we construct $f : X \longrightarrow Y$ by choosing $f(x) \in Y$ such that $c(x, f(x)) \gg v$. Similarly, we define $g : Y \longrightarrow X$ by picking $g(y) \in X$ such that $c(g(y), y) \gg v$. A calculation identical to the one we carried in the proof of 4.8.8

shows that $\mathrm{dis}f \geq 2v$ and $\mathrm{dis}g \geq 2v$. Thus,

$$G(X,Y) \leq \frac{\bigvee_f \mathrm{dis}f}{2}$$

and

$$G(X,Y) \leq \frac{\bigvee_g \mathrm{dis}g}{2}$$

giving us

$$G(X,Y) \leq \Big(\frac{\bigvee_f \mathrm{dis}f}{2}\Big) \wedge \Big(\frac{\bigvee_g \mathrm{dis}g}{2}\Big).$$

If ψ preserves $\bigvee$, then the second inequality turns into

$$G(X,Y) \leq \Big(\bigvee_f \frac{\mathrm{dis}f}{2}\Big) \wedge \Big(\bigvee_g \frac{\mathrm{dis}g}{2}\Big).$$

When $(\mathcal{V}, \bigvee, \wedge)$ is a frame, $\wedge$ distributes over $\bigvee$ and the above inequality becomes

$$G(X,Y) \leq \bigvee_{f,g} \Big(\frac{\mathrm{dis}f}{2} \wedge \frac{\mathrm{dis}g}{2}\Big),$$

giving the last statement of the Theorem. $\square$

We define the *diameter* of a $\mathcal{V}$–category (X,a) by

$$\mathrm{diam}X = \bigwedge_{x,x' \in X} a(x,x').$$

Corollary 4.8.10. *Let $\mathcal{V}$ be a ccd quantale that satisfies (4.4) such that $k = \top$. Then for $P = \{*\}$ and any $\mathcal{V}$–category (X,a),*

$$G(X,P) \leq \frac{\mathrm{diam}X}{2}.$$

Proof. The only map from X to P is the constant map, we call it f.

$$\mathrm{dis} f = \bigwedge_{x,x' \in X} k \multimap a(x,x') = \bigwedge_{x,x' \in X} a(x,x') = \mathrm{diam} X.$$

For each $x \in X$ we have the map $g_x(*) = x$ from P to X with $\mathrm{dis} g_x = a(x,x) \multimap k \leq k \multimap k \leq k$. Thus,

$$\begin{aligned} G(X,P) &\leq \left(\frac{\bigvee_f \mathrm{dis} f}{2}\right) \wedge \left(\frac{\bigvee_g \mathrm{dis} g}{2}\right) \\ &= \frac{\mathrm{diam} X}{2} \wedge \frac{k}{2} \\ &= \frac{\mathrm{diam} X}{2} \wedge \top = \frac{\mathrm{diam} X}{2}, \end{aligned}$$

since $\frac{\top}{2} = \top$. $\square$

4.9 Some $\mathcal{V}$–functors on $(\mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat}), G)$

Proposition 4.9.1. *Let $\mathcal{V}$ be ccd. Let (X,a) be any $\mathcal{V}$–category and consider the functor*

$$X \otimes (-) : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}.$$

Then $X \otimes (-)$ is a $\mathcal{V}$–functor from $(\mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat}), G)$ to itself.

Proof. Let (Y,b) and (Y',b') be any $\mathcal{V}$–categories. We need to show

$$G(Y,Y') \leq G(X \otimes Y, X \otimes Y').$$

Let (Z,c) be such that Y can be embedded into Z via a $\mathcal{V}$–functor f and Y' can be embedded into Z via a $\mathcal{V}$–functor f'. Define $F : X \otimes Y \longrightarrow X \otimes Z$ by

$F(x,y) = (x, f(y))$ and $F' : X \otimes Y' \longrightarrow X \otimes Z$ by $F'(x,y) = (x, f'(y))$. One easily verifies that F and F' are also embeddings. We claim

$$Hc(f(Y), f'(Y')) \leq H(a \otimes c)(F(X \otimes Y), F'(X \otimes Y')).$$

Indeed, let $v \in \mathcal{V}$ such that for all $y \in Y$ there exists $y' \in Y'$ such that

$$v \leq c(f(y), f'(y')).$$

Pick any $x_0 \in X$. Then $(x_0, y) \in X \otimes Y$ and $(x_0, y') \in X \otimes Y'$. Thus,

$$\begin{aligned} v &\leq c(f(y), f'(y')) \\ &= k \otimes c(f(y), f'(y')) \\ &\leq a(x_0, x_0) \otimes c(f(y), f'(y')) \\ &= (a \otimes c)((x_0, f(y)), (x_0, f'(y'))). \end{aligned}$$

Hence the claim, and consequently the Proposition, hold. □

Proposition 4.9.2. *Let $\mathcal{V}$ be ccd. The the Hausdorff functor*

$$H : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$$

is a $\mathcal{V}$–functor from $(\mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat}), G)$ *to itself.*

Proof. We need to show

$$G(X,Y) \leq G(HX, HY).$$

Suppose that X and Y can be embedded into (Z, c). Then HX and HY can be embedded into HZ. Indeed, if $g : X \to Z$ is an embedding, then $Hg : HX \to HZ$ is also one. Without loss of generality suppose that the embeddings are inclusions. Next we show

$$Hc(X, Y) \leq HHc(HX, HY).$$

Suppose that $v \in \mathcal{V}$ is such that for all $x \in X$ there exists $y \in Y$ with $v \leq c(x, y)$. We thus want

$$v \leq HHc(HX, HY) = \bigwedge_{A \subseteq X} \bigvee_{B \subseteq Y} Hc(A, B).$$

So it is enough to show, for all $A \subseteq X$

$$v \leq \bigvee_{B \subseteq Y} Hc(A, B).$$

So, let $A \subseteq Y$ be arbitrary. If A is empty, the above inequality holds trivially. If $A \neq \emptyset$, the for each $x \in A$ there exists $y_x \in Y$ with $v \leq c(x, y_x)$. Let

$$B_A = \bigcup_{x \in A} y_x.$$

Then

$$v \leq c(x, y_x) \leq c(x, B_A)$$

and since this holds for all $x \in A$,

$$v \leq \bigwedge_{x \in A} c(x, B_A) = Hc(A, B_A) \leq \bigvee_{B \subseteq Y} Hc(A, B),$$

as required.

We conclude

$$
\begin{aligned}
G(X,Y) &= \bigvee_{(Z,c)} H^s c(X,Y) \\
&= \bigvee_{(Z,c)} \big(Hc(X,Y) \wedge Hc(Y,X)\big) \\
&\leq \bigvee_{(Z,c)} \big(HHc(HX,HY) \wedge HHc(HY,HX)\big) \\
&= \bigvee_{(Z,c)} H^s Hc(HX,HY) \\
&\leq G(HX,HY).
\end{aligned}
$$

□

4.10 $(\mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat}), G)$ as a colimit

For a $\mathcal{V}$–category (X,a) and $A \subseteq X$, $(A, a|_{A\times A})$ is a $\mathcal{V}$–category in itself. We have the $\mathcal{V}$–functor

$$(A, a|_{A\times A}) \hookrightarrow (X,a).$$

We call such a $\mathcal{V}$–functor an *inclusion*. Let $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{inc}}$ denote the category of all $\mathcal{V}$–categories and inclusions. With $\mathcal{V}\text{-}\mathsf{CAT}$ denoting the category of large $\mathcal{V}$–categories, we have the following result:

Theorem 4.10.1 (Section 6 in (CT2))**.** $\mathcal{G} = (\mathrm{ob}(\mathcal{V}\text{-}\mathsf{Cat}), G)$ *is a colimit of*

$$\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{inc}} \xrightarrow{H^s} \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{inc}} \hookrightarrow \mathcal{V}\text{-}\mathsf{CAT}$$

in $\mathcal{V}$-CAT.

Proof. We define the mappings $\lambda_X : (HX, Ha) \longrightarrow \mathcal{G}$ by

$$\lambda_X(A) = (A, a|_{A\times A}).$$

Then,

$$H^s a(A,B) \leq G(A,B) = G(\lambda_X(A), \lambda_X(B)),$$

showing that for all X, λ_X is a $\mathcal{V}$-functor.

Now, given any inclusion $i : A \hookrightarrow X$ in $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{inc}}$, and any $B \subseteq A$,

$$\lambda_X(H^s i(B)) = \lambda_X(B) = B = \lambda_A(B),$$

showing that $\lambda : H^s \longrightarrow \mathcal{G}$ is a co-cone.

Next, we show that this cocone is universal. Let $\alpha : H^s \longrightarrow (\mathcal{J}, J)$ be another cocone. We define $F : \mathcal{G} \longrightarrow \mathcal{J}$ by

$$F(X) = \alpha_X(X).$$

Then F is a $\mathcal{V}$–functor: for any admissible structure c on $X \cup Y$,

$$\begin{aligned} H^s c(X,Y) &\leq J(\alpha_{X\cup Y}(X), \alpha_{X\cup Y}(Y)) \\ &= J(\alpha_X(X), \alpha_Y(Y)) \\ &= J(F(X), F(Y)). \end{aligned}$$

Consequently, $G(X, Y) \leq J(F(X), F(Y))$. Also, for any X and $A \subseteq X$,

$$F \cdot \lambda_X(A) = F(A) = \alpha_A(A) = \alpha_X(A).$$

If $K : \mathcal{G} \longrightarrow \mathcal{J}$ is such that $K \cdot \lambda = \alpha$, then

$$F(X) = \alpha_X(X) = K\lambda_X(X) = K(X).$$

$\square$

5 The Vietoris topology

5.1 Introduction

In an attempt to create a notion of a manifold, in 1922 Leopold Vietoris introduced a topology on the set of non–empty closed subsets of a topological space; this topology now bears his name. The topological space (CLX, τ_V), where τ_V is the Vietoris topology is usually called the hyperspace of X. Vietoris proved that if X is a T_3 compact topological space, then so is its hyperspace. He further studied the relationship between the base space X and its hyperspace. For example, he showed that the set of all connected and closed subsets of X is compact in the hyperspace of X.

Some topologies on the powerset are called "hit–and–miss" topologies. This essentially means that the open sets consist of sets that intersect certain type of sets (hit them) and do not intersect other types of sets (miss them). The Vietoris topology is the archetype of "hit–and–miss" topologies – many familiar "hit–and–miss" topologies can be obtained by modifying the Vietoris topology. Perhaps then the title of the 2002 paper by S. Naimpally, "All hypertopologies are hit–and–miss" demonstrates the importance of the Vietoris topology.

This chapter serves as a primer for the study of the Vietoris topology in the $(\mathbb{T}, \mathcal{V})$-$\mathsf{Cat}$ setting. We study the Vietoris topology mainly in the classical setting, but do not forget to mention some of its categorical properties. As in the previous two chapters, we motivate our study with many examples; the notable ones are geometric models of hyperspaces that are given in examples 5 and 6 of 5.2.3. The Vietoris topology comes with its own notation. We explore this notation and establish a toolkit of formulas for working with it. The relationship between the Vietoris topology and the Hausdorff distance we establish in Section 5.4 ties this chapter with the rest of the work. In the same section we show that the Vietoris topology can be viewed as a weak topology with respect to a family of distance functionals – this sheds new light on hyperspace topologies and suggests new approaches to further research in the subject. We conclude the chapter with a categorical study: we introduce the Vietoris monad and study its Eilenberg–Moore algebras. We show that with slight modifications the Vietoris monad extends the Hausdorff monad.

5.2 Definition and examples

Definition 5.2.1. Let (X, τ) be a topological space. The *Vietoris topology*, τ_V, on PX is the smallest topology on PX such that for any open set $U \in \tau$ the set

$$\{A \mid A \subseteq U\}$$

is open in PX, and for any closed set C in X, the set

$$\{A \mid A \subseteq C\}$$

is closed in PX. We shall write VX for (PX, τ_V).

The above definition gives us a natural description of the Vietoris topology – it allows us to easily identify some open and closed sets in it, but it is not easy to work with. The following notation will simplify our work with the Vietoris topology:

$$\langle U_1, \ldots, U_n \rangle := \{A \subseteq X \mid A \subseteq \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset\}.$$

Figure 5.1 aids in understanding the above notation. The hatched area is a set that is contained in the union of $U_1, \ldots, U_4$ and intersect all those sets.

Theorem 5.2.2 ((IN), Theorem 1.2)**.** *Let (X, τ) be a topological space and*

$$\mathcal{B} = \{\langle U_1, \ldots, U_n \rangle \mid U_i \in \tau, n < \infty\}.$$

Then $\mathcal{B}$ is a base for τ_V.

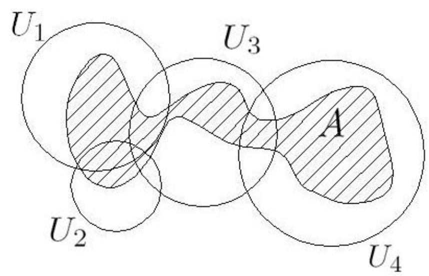


Figure 5.1: A visualization of $\langle U_1, \ldots, U_4 \rangle$

Proof. We have

$$\langle U \rangle = \{A \mid A \subseteq U\}$$

and

$$\begin{aligned} \neg\langle X, X \setminus B \rangle &= \neg\{A \subseteq X \mid A \cap (X \setminus B) \neq \emptyset\} \\ &= \{A \subseteq X \mid A \cap (X \setminus B) = \emptyset\} \\ &= \{A \subseteq X \mid A \subseteq B\}. \end{aligned}$$

Thus, τ_V is the smallest topology containing the sets $\langle U \rangle$ and $\langle X, U \rangle$ with $U \in \tau$ arbitrary. In other words

$$\mathcal{S} = \{\langle U \rangle \mid U \in \tau\} \cup \{\langle X, U \rangle \mid U \in \tau\}$$

is a subbase for τ_V. Let $\mathcal{S}^*$ denote the set of all finite intersections of elements of $\mathcal{S}$. Then $\mathcal{S}^*$ forms a base for the Vietoris topology on PX. Our goal is to show that $\mathcal{S}^* = \mathcal{B}$.

"$\mathcal{B} \subseteq \mathcal{S}^*$": Let $U_1, \ldots, U_n \in \tau$ and observe:

$$\langle U_1, \ldots, U_n \rangle = \langle \bigcup_{i=1}^{n} U_i \rangle \cap \big(\bigcap_{i=1}^{n} \langle X, U_i \rangle \big)$$

Indeed, if $A \in \langle U_1, \ldots, U_n \rangle$, then $A \in \langle \bigcup_{i=1}^{n} U_i \rangle$ and $A \cap U_i \neq \emptyset$, for all i. So, for all i, $A \subseteq \langle X, U_i \rangle$. The reverse inclusion follow similarly.

This proves "$\subseteq$".

"$\mathcal{B} \supseteq \mathcal{S}^*$: First, we show that U, W in $\mathcal{B}$ imply that their intersection is also in $\mathcal{B}$. To prove this, let $U = \langle U_1, \ldots, U_k \rangle$ and $W = \langle W_1, \ldots, W_m \rangle$ and let

$$U' = \bigcup_{i=1}^{k} U_i, \quad W' = \bigcup_{i=1}^{m} W_i.$$

Then

$$U \cap W = \langle U_1 \cap W', \ldots U_k \cap W', W_1 \cap U', \ldots, W_m \cap U' \rangle.$$

Indeed, $A \subseteq \bigcup_{i=1}^{k} U_i$ and $A \subseteq \bigcup_{i=1}^{m} W_i$ implies that

$$\begin{aligned} A &\subseteq \big(\bigcup_{i=1}^{k} U_i\big) \cap \big(\bigcup_{i=1}^{m} W_i\big) \\ &= \bigcup_{i=1}^{k} (U_i \cap W') \cup \bigcup_{i=1}^{m} (W_i \cap U'). \end{aligned}$$

$A \cap U_i \neq \emptyset$ gives us

$$A \cap (U_i \cap W') = (A \cap \bigcup_{j=1}^{m} W_j) \cap U_i = A \cap U_i \neq \emptyset.$$

A similar argument shows that $A \cap W_j \cap U' \neq \emptyset$, for all j.

For the reserve inclusion, it is clear from previous arguments that $A \in \langle U_1 \cap W', \ldots, U_k \cap W', W_1 \cap U', \ldots, W_m \cap U' \rangle$ implies $A \subseteq \bigcup_{i=1}^{k} U_i$ and $A \subseteq \bigcup_{i=1}^{m} W_i$. And for all i

$$A \cap (U_i \cap W) \neq \emptyset \implies A \cap U_i \neq \emptyset.$$

Similarly, for all j

$$A \cap (W_j \cap U') \neq \emptyset \implies A \cap W_j \neq \emptyset.$$

Thus, $\mathcal{B}$ is closed under intersections. Since $\mathcal{S} \subseteq \mathcal{B}$, it follows that $\mathcal{S}^* \subseteq \mathcal{B}$, as required. $\square$

We defined the Vietoris topology in the most general setting: on PX. One is often interested in the the following subspaces of VX:

1. CLX – the subspace of all closed subsets of VX;
2. KX – the subspace of all compact subsets of VX;
3. CX – the subspace of all connected compact subsets of VX.

Even though the topology on the above subspaces is the subspace topology inherited from VX, we shall abuse notation and refer to this topology again as the Vietoris topology. Let $\mathcal{R}$ denote any of the above subspaces. We define

$$\langle U_1, \ldots, U_n \rangle_{\mathcal{R}} = \langle U_1, \ldots, U_n \rangle \cap \mathcal{R}$$

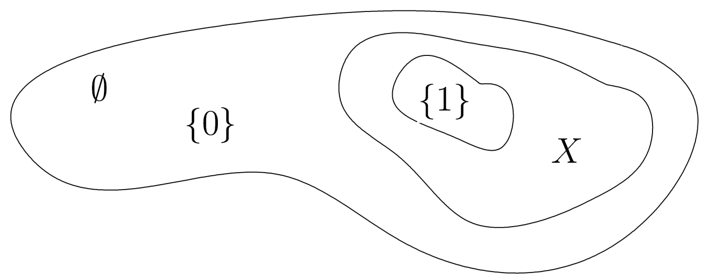


Figure 5.2: A visualization of the Vietoris topology on $P2$

We shall sometimes restrict ourselves to one of those subspaces. In this case, we will write $\langle U_1, \ldots, U_n \rangle$ for $\langle U_1, \ldots, U_n \rangle_{\mathcal{R}}$.

We are now in the position to consider several examples.

Examples 5.2.3. 1. Take X to be the Sierpiński space: $X = \{0, 1\}$. Then a basis for PX with the Vietoris topology consists of the following sets:

$$\langle \{1\} \rangle = \{\{1\}\}, \langle X, \{1\} \rangle = \{\{1\}, X\}, \langle \emptyset \rangle = \emptyset, \langle X \rangle = PX.$$

Those open sets are depicted in Figure 5.2.

2. Let $X = \mathbb{R}^2$ with the Euclidian metric d and let τ_d denote the induced topology. Let $x = \{(0, 0)\}$ and consider the ball $\eta_1^{H^s d}(\{x\})$. We claim that this ball is open with respect to the Vietoris topology on KX. Indeed, let $A \in \eta_1^{H^s d}(\{x\})$. Then

$$A \subseteq \bigcup_{y \in A} \eta^d_{\frac{1}{2}}(y)$$

is an open cover of A, and since A is compact we can pick $y_1, \dots, y_n$ in A such that

$$\eta^d_{\frac{1}{2}}(y_i), \quad i = 1, \dots, n$$

is a minimal finite subcover of A. Then

$$A \subseteq \langle \eta^d_{\frac{1}{2}}(y_1), \dots, \eta^d_{\frac{1}{2}}(y_n) \rangle$$

and it turns out that

$$\langle \eta^d_{\frac{1}{2}}(y_1), \dots, \eta^d_{\frac{1}{2}}(y_n) \rangle \subseteq \eta^{H^s d}_1(\{x\}).$$

The exact relationship between the Vietoris topology on KX and the Hausdorff distance on the same set is captured in Theorem 5.4.1. The proof of this theorem provides the details we omitted in this example.

3. Recall that in a topological space X, a sequence (x_n) converges to a point x provided that for any neighborhood U of x, there exists a natural number N such that for all n greater than N, $x_n \in U$.

   Take X to be the real line with the topology induced by the Euclidian metric. Let $A = [0, 1]$ and $A_n = [0-\frac{1}{n}, 1+\frac{1}{n}]$. We claim that the sequence A_n converges to A in the Vietoris topology. Given any open set $\mathcal{V}$ in VX containing A, there exist $U_1, \dots, U_n$ open sets in $\mathbb{R}$ with $A \in \langle U_1, \dots U_n \rangle \subseteq \mathcal{V}$. We show that for n large enough $A_n \in \langle U_1, \dots U_n \rangle$. Indeed, one of the $U_i's$ contains 0, call this set V_0. One of the $U_i's$ contains 1. We call this set V_1. Since both V_0 and V_1 are open, there exist $N \in \mathbb{N}$ such that $\eta_{\frac{1}{N}}(0) \subseteq V_0$ and $\eta_{\frac{1}{N}}(1) \subseteq V_1$. Let $n \geq N$. Then

$$A_n \subseteq V_0 \cup U_1 \cup \ldots \cup U_n \cup V_1 = U_1 \cup \ldots \cup U_n.$$

Since $A \subseteq A_n$, $A_n \cap U_i \neq \emptyset$, for all i. Thus for all $n \geq N$, $A_n \in \mathcal{V}$ and hence

$$A_n \longrightarrow A,$$

as claimed.

4. Recall that an increasing sequence of subspaces A_n converges to their union in the Gromov distance. If we now start with a topological space (X, τ) and let A_n be any increasing sequence (i.e. $A_n \subseteq A_{n+1}, \forall n \in \mathbb{N}$) in VX, then

$$A_n \longrightarrow A := \bigcup_{n=1}^{\infty} A_n.$$

To see this, let $\mathcal{V}$ be an arbitrary neighborhood of A in VX. Then there exist $U_1, \ldots U_k$ open in X such that

$$A \in \langle U_1, \ldots, U_k \rangle \subseteq \mathcal{V}.$$

Clearly for all n, $A_n \subseteq A \subseteq \bigcup_{i=1}^{k} U_i$. For every $i \in \{1, \ldots k\}$, there exists $x_i \in A \cap U_i$. Then there exists A_{n_i} with $x_i \in A_{n_i}$. Set $N = \max\{n_i \mid i = 1, \ldots k\}$.

Then for all i, $x_i \in A_N$ and consequently, for all i, $A_N \cap U_i \neq \emptyset$. Thus, for all $n \geq N$ and all i, $A_n \cap U_i \neq \emptyset$. We conclude that for all $n \geq N$,

$$A_n \in \langle U_1, \ldots, U_n \rangle.$$

Hence $A_n \longrightarrow A$.

5. [(IN), Example 5.1] In this example, we aim to construct a simple geometric model for CX with $X = [0,1]$ equipped with the Euclidian metric. The compact connected subsets of X are just intervals $[a,b]$ with $0 \leq a \leq b \leq 1$. We can identify each such interval with a tuple $(a,b) \in \mathbb{R}$. Let

$$T = \{(a,b) \mid 0 \leq a \leq b \leq 1\}.$$

We show that the bijection $h : CX \longrightarrow T$ defined by $h([a,b]) = (a,b)$ is in fact a homeomorphism. To this end, it suffices to prove:

$$[a_n, b_n] \longrightarrow [a,b] \text{ in } VX \iff (a_n, b_n) \longrightarrow (a,b) \text{ in } T.$$

where $[a_n, b_n]$ is a sequence in VX and $(a_n, b_n) = h([a_n, b_n])$ is a sequence in T.

We shall write $\langle U_1, \ldots, U_n \rangle$ for $\langle U_1, \ldots, U_n \rangle_{CX}$.

"$\Rightarrow$": Take $\varepsilon > 0$ and let $V = \eta_\varepsilon((a,b)) \cap T$. Let

$$\begin{aligned} U_1 &= (a - \frac{\varepsilon}{\sqrt{2}}, a + \frac{\varepsilon}{\sqrt{2}}) \cap X, \\ U_2 &= (a,b), \\ U_3 &= (b - \frac{\varepsilon}{\sqrt{2}}, b + \frac{\varepsilon}{\sqrt{2}}) \cap X. \end{aligned}$$

Then all the U_i's are open in X and $[a,b] \in \langle U_1, U_2, U_3 \rangle$. So there exists an N such that for all $n \geq N$, $[a_n, b_n] \in \langle U_1, U_2, U_3 \rangle$. This in turn implies that $a_n \in U_1$ and $b_n \in U_3$ and hence $d(a, a_n) < \frac{\varepsilon}{\sqrt{2}}$, $d(b, b_n) < \frac{\varepsilon}{\sqrt{2}}$. Thus, for all $n \geq N$,

$$d((a_n, b_n), (a,b)) = \sqrt{d(a,a_n)^2 + d(b,b_n)^2} < \sqrt{\frac{2\varepsilon^2}{2}} = \varepsilon.$$

Hence, for all $n \geq N$, $(a_n, b_n) \in \eta_\varepsilon((a,b)) \cap T$ and consequently $(a_n, b_n) \longrightarrow (a,b)$.

"$\Leftarrow$": Suppose that $(a_n, b_n) \longrightarrow (a,b)$ in T. Take $U_1, \ldots, U_m$ open in X such that $[a,b] \in \langle U_1, \ldots, U_m \rangle$. Set $V_a = \bigcap_{a \in U_i} U_i$ and $V_b = \bigcap_{b \in U_i} U_i$. That is, $\mathcal{V}_a$ is the intersection of all the sets from $\{U_1, \ldots, U_n\}$ that contain a; similarly for V_b. Pick $\varepsilon > 0$ small enough such that $\eta_\varepsilon(a) \cap X \subseteq V_a$, and $\eta_\varepsilon(b) \subseteq V_b$. Say there exists j such that $U_j \subseteq \eta_\varepsilon(a)$. We can reduce ε as needed to assure that $U_j \nsubseteq \eta_\varepsilon(a)$. We repeat this process until for all j, $U_j \nsubseteq \eta_\varepsilon(a)$ and $U_j \nsubseteq \eta_\varepsilon(b)$. There exists an N such that for all $n \geq N$, $(a_n, b_n) \in \eta_\varepsilon(a) \times \eta_\varepsilon(b)$. We claim that for all $n \geq N$, $[a_n, b_n] \in \langle U_1, \ldots, U_m \rangle$. It is easy to see that

$[a_n, b_n] \subseteq \bigcup_{i=1}^{n} U_i$. Indeed, there are three cases one needs to consider to prove this, the most extreme one of them being when $a_n < a \leq b < b_n$. But even in this case,

$$[a_n, a] \subseteq \eta_\varepsilon(a) \subseteq U_i$$

for some U_i containing a, and similarly

$$[b, b_n] \subseteq \eta_\varepsilon(b) \subseteq U_{i'}.$$

So,

$$[a_n, b_n] \subseteq [a_n, a] \cup [a, b] \cup [b, b_n] \subseteq U_i \cup \bigcup_{i=1}^{n} U_i \cup U_{i'} = \bigcup_{i=1}^{n} U_i.$$

Next, suppose that there exist j such that $U_j \cap [a_n, b_n] = \emptyset$. Without loss of generality, suppose that for all $x \in U_j$, $x < a_n$. If $a \in U_j$, then $a_n \in \eta_\varepsilon(a) \subseteq U_j$ – a contradiction. Otherwise a not in U_j implies that for all $x \in U_j$, $a < x$ and hence

$$U_j \subseteq [a, a_n] \subseteq \eta_\varepsilon(a);$$

but this too is a contradiction.

Thus, for all i, $[a_n, b_n] \cap U_i \neq \emptyset$. We conclude that for all $n \geq N$,

$$[a_n, b_n] \in \langle U_1, \ldots, U_m \rangle$$

and hence, $[a_n, b_n] \longrightarrow [a, b]$, as claimed.

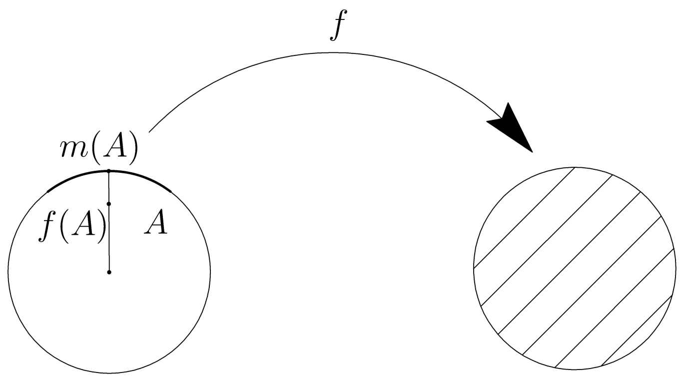


Figure 5.3: A depiction of f

6. [(IN), Example 5.2] In our final example, we construct a geometric model for CX with X the unit circle in $\mathbb{R}^2$. As we shall momentarily see, CX is homeomorphic to the unit disk. We describe the homeomorphism $f : CX \to D^2$ as follows: the elements of CX are arcs in the unit circle. Given an arc A, let $l(A)$ denote it's length and $m(A)$ it's midpoint, as depicted in Figure 5.3.

   We denote $m(\vec{A})$ the vector from the origin to the point $m(A)$ on the unit circle. The length of single points is 0, and the length of the whole circle is 2π. We take $m(X)$ to be any predetermined point in X. Then for any $A \in CX$,

$$f(A) = \text{the endpoint of } \left(1 - \frac{l(A)}{2\pi}\right)m(\vec{A}).$$

   We give a geometric sketch to show that f is a homeomorphism. First, two different arcs in X either have different lengths, or different midpoints, and hence will be mapped to different points of D^2. The distance of $f(A)$ from the origin is determined by $l(A)$; the midpoint of A determines the angle between the ray from the origin to $f(A)$ and the x–axis. Since any angle can be achieved by picking an appropriate point on the circle and then any distance between 1 and 0 from the origin can be achieved by picking an arc of appropriate length at our midpoint, f is surjective. The continuity of f and of its inverse follow from similar geometric arguments.

5.3 Some formulas and their consequences

The notation we introduced in the previous section becomes powerful when it is coupled with a toolkit of formulas. In this section we develop this toolkit.

Theorem 5.3.1. *Let X and Y be any sets, $f : X \to Y$ any function and $U_1, \ldots, U_n \subseteq X$, $V_1, \ldots V_n \subseteq Y$.*

1. $f^{-1}(\langle V_1, \ldots, V_n \rangle) = \langle f^{-1}(V_1), \ldots, f^{-1}(V_n) \rangle$

2. $\langle V_1 \cap Y, \ldots, V_n \cap Y \rangle = \langle V_1, \ldots, V_n \rangle \cap PY$

3. $\langle U_1 \rangle \cap \langle U_2 \rangle = \langle U_1 \cap U_2 \rangle$

4. $U_i \subseteq U_i'$ *for all i implies*

$$\langle U_1, \ldots, U_n \rangle \subseteq \langle U_1', \ldots, U_n' \rangle$$

5. *If X is a topological space, then*

$$\langle \overline{U_1}, \dots \overline{U_n} \rangle = \overline{\langle U_1, \dots, U_n \rangle},$$

with the closure on the right-hand-side taken with respect to the Vietoris topology on PX.

Proof. (1) "$\subseteq$": If $f(A) \in \langle V_1, \dots, V_n \rangle$, then

$$f(A) \subseteq \bigcup_{i=1}^{n} V_i, \text{ and for all } i, \ f(A) \cap V_i \neq \emptyset.$$

Hence,

$$A \subseteq \bigcup_{i=1}^{n} f^{-1}(V_i) \text{ and for all } i, \ A \cap f^{-1}(V_i) \neq \emptyset.$$

"$\supseteq$": From $A \cap f^{-1}(V_i) \neq \emptyset$, we immediately have $f(A) \cap V_i \neq \emptyset$ for all i. From $A \subseteq \bigcup_{i=1}^{n} f^{-1}(V_i)$, we get for any $y = f(x) \in f(A)$

$$y = f(x) \in f\big(\bigcup_{i=1}^{n} f^{-1}(V_i)\big) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(V_i)) \subseteq \bigcup_{i=1}^{n} V_i.$$

So, $f(A) \subseteq \bigcup_{i=1}^{n} V_i$.

(2) Note that when $A \subseteq Y$ then $A \cap V_i \cap Y \neq \emptyset \iff A \cap V_i \neq \emptyset$. Then the claim follows from the definition of $\langle V_1, \dots, V_n \rangle$.

(3) It is clear that ($A \subseteq U_1$ and $A \subseteq U_2$) if, and only if, $A \subseteq U_1 \cap U_2$.

(4) $A \subseteq \bigcup_{i=1}^{n} U_i \subseteq \bigcup_{i=1}^{n} U_i'$ and $A \cap U_i \neq \emptyset$ implies $A \cap U_i' \neq \emptyset$, since $U_i \subseteq U_i'$.

(5) Let $R := \langle \overline{U_1}, \dots \overline{U_n} \rangle$ and $S := \overline{\langle U_1, \dots, U_n \rangle}$.

"$S \subseteq R$": We show that R contains $\langle U_1, \ldots, U_n \rangle$ and is closed.

Let $A \in \neg R$.

Case 1: $A \nsubseteq \bigcup_{i=1}^n \overline{U_i}$. Then there exists an $x \in A$ such that for all i, x is not in $\overline{U_i}$. That is, there are neighborhoods W_i of x with $W_i \cap U_i = \emptyset$. Let $W = \bigcap_{i=1}^n W_i$. Then W is a neighborhood of x and for all i, $W \cap U_i = \emptyset$. We claim that

$$A \in \langle X, W \rangle \subseteq \neg R.$$

Indeed, for any $B \in \langle X, W \rangle$, $B \cap W \neq \emptyset$, by definition. If $B \subseteq \bigcup_{i=1}^n \overline{U_i}$, then there exists $x \in W$ with $x \in \overline{U_{i_0}}$ for some i_0 giving us

$$W \cap U_{i_0} \neq \emptyset$$

a contradiction! Thus $B \in \neg R$, as required and consequently $\neg R$ is open.

Case 2: $\exists i$ such that $A \cap \overline{U_i} = \emptyset$. Then for all $x \in A$ there exists neighborhood W_x of x such that $W_x \cap U_i = \emptyset$. Let

$$W = \bigcup_{x \in A} W_x.$$

We claim

$$A \in \langle W \rangle \subseteq \neg R.$$

Indeed, suppose that $B \subseteq W$. Then if $x \in B \cap \overline{U_i}$, we have that $W \cap U_i \neq \emptyset$, since $x \in W$ and W is open. Hence, we again get a contradiction. Thus $\neg R$ is again open.

We conclude that R is closed and since $U_i \subseteq \overline{U_i}$ for all i,

$$\langle U_1, \ldots, U_n \rangle \subseteq \langle \overline{U_1}, \ldots, \overline{U_n} \rangle.$$

Consequently we get $S \subseteq R$.

"$S \supseteq R$": Let $A \in \langle \overline{U_1}, \ldots \overline{U_n} \rangle$. Let $\mathcal{W}$ be any open set in VX containing A. We wish to show that $\mathcal{W} \cap \langle U_1, \ldots, U_n \rangle \neq \emptyset$. There exist open sets $V_1, \ldots, V_k$ in X such that

$$A \in \langle V_1, \ldots, V_k \rangle \subseteq \mathcal{W}.$$

Set $V = \bigcup_{j=1}^k V_j$ and $U = \bigcup_{i=1}^n U_i$. We showed in the proof of Theorem 5.2.2 that

$$\langle U_1, \ldots, U_n \rangle \cap \langle V_1, \ldots, V_k \rangle = \langle U_1 \cap V, \ldots, U_n \cap V, V_1 \cap U, \ldots, V_k \cap U \rangle.$$

All that remains is to show that for all i, $U_i \cap V \neq \emptyset$ and for all j, $V_j \cap U \neq \emptyset$. Let i be arbitrary; then there exists $x_i \in A \cap \overline{U_i}$ and j_i such that $x_i \in V_{j_i}$. Since V_{j_i} is open,

$$\emptyset \neq U_i \cap V_{i_j} \subseteq U_i \cap V.$$

A similar argument shows that for all j, $V_j \cap U \neq \emptyset$. Thus $\mathcal{W} \cap \langle U_1, \ldots, U_n \rangle \neq \emptyset$, as desired. $\square$

Corollary 5.3.2. *Let X and Y be topological spaces and $f : X \longrightarrow Y$ be a continuous function between them. Define $Vf : VX \longrightarrow VY$ by*

$$Vf(A) = f(A).$$

Then Vf is continuous.

Proof. For any element $\langle V_1, \ldots, V_n \rangle$ of the basis of VY we have

$$f^{-1}(\langle V_1, \ldots, V_n \rangle) = \langle f^{-1}(V_1), \ldots, f^{-1}(V_n) \rangle$$

by (1) of Theorem 5.3.1. The result then follows from the continuity of f. $\square$

Corollary 5.3.3. *Let X be a topological space and Y a subspace. Then VY is a subspace of VX.*

Proof. We have the inclusion map

$$i : Y \to X.$$

For all $A \subseteq Y$, $Vi(A) = i(A) = A$. Hence,

$$Vi : VY \to VX$$

is continuous. Next, take $W_1, \ldots, W_n$ open in Y and consider

$$\langle W_1, \ldots, W_n \rangle \subseteq VY.$$

We wish to show that $\langle W_1, \ldots, W_n \rangle = \langle U_1, \ldots, U_n \rangle \cap VY$, for some U_i open in X. Indeed, since Y is a subspace, for each i exists U_i such that $W_i = U_i \cap Y$. So,

$$\langle W_1, \ldots, W_n \rangle = \langle U_1 \cap Y, \ldots, U_n \cap Y \rangle = \langle U_1, \ldots, U_n \rangle \cap PY,$$

by item (2) of the above theorem. $\square$

5.4 When VX is metrizable

The main goal of this section is to study the relation between Hausdorff distance and the Vietoris topology.

Theorem 5.4.1 ((IN), Theorem 3.1)**.** *Let (X,d) be a metric space and $\tau = \tau_d$ be the topology induced by d. Then (KX, τ_V), with τ_V the subspace topology inherited from VX, is metrizable and $H^s d$ induces τ_V.*

Proof. We denote the topology induced by $H^s d$ by $\tau_{H^s d}$. Since we are working in CLX, we abuse notation and write $\langle U_1, \ldots, U_n \rangle$ for $\langle U_1, \ldots, U_n \rangle_{KX}$

"$\tau_{H^s d} \supseteq \tau_V$": Let $U \in \tau_d$. If $U = X$, then $\langle U \rangle = KX \in \tau_{H^s d}$. Suppose $U \neq X$. Let $A \subseteq U$ and

$$\varepsilon = \inf\{d(a,x) \mid a \in A, x \in X \setminus U\}.$$

Then since A is compact and U open, $\varepsilon > 0$. Also,

$$\eta_\varepsilon^{H^s d}(A) \subseteq \langle U \rangle.$$

Indeed, $B \in \eta_\varepsilon^{H^s d}(A)$ means that $H^s d(A,B) < \varepsilon$ and thus for any $b \in B$ there exists $a \in A$ such that $d(a,b) < \varepsilon$. Now if for some $b \in B$, b is not in U, then for all $a \in A$ $d(a,b) \geq \varepsilon$ – a contradiction. Hence,

$$A \in \eta_\varepsilon^{H^s d}(A) \subseteq \langle U \rangle.$$

Since this holds for any $A \in \langle U \rangle$, we conclude

$$\langle U \rangle \in \tau_{H^s d}$$

Next, suppose that $A \cap U \neq \emptyset$. Let $p \in A \cap U$. Then since in particular $p \in U$ there exists $\delta > 0$ such that

$$p \in \eta_\delta^d(p) \subseteq U.$$

We prove

$$\eta_\delta^{H^s d}(A) \subseteq \langle X, U \rangle.$$

Indeed, let $B \in \eta_\delta^{H^s d}(A)$. Then $H^s d(A, B) < \delta$ and hence for every $a \in A$ exists $b \in B$ with $d(a, b) < \delta$. In particular, there exists $b \in B$ such that

$$d(b, p) < \delta.$$

Thus $b \in \eta_\delta^d(p) \subseteq U$, and $B \cap U \neq \emptyset$.

So far we proved that sets of the form $\langle U \rangle$ and $\langle X, U \rangle$ are in $\tau_{H^s d}$. Since those sets form a subbasis for τ_V, $\tau_V \subseteq \tau_{H^s d}$.

"$\tau_{H^s d} \subseteq \tau_V$": It suffices to show that for every $A \in KX$ and any $r > 0$, there exist $U_1, \ldots, U_n$ open sets in X such that

$$A \in \langle U_1, \ldots U_n \rangle \subseteq \eta_r^{H^s d}(A).$$

So fix $A \in KX$ and $r > 0$. If A is empty we have

$$\emptyset \in \langle \emptyset \rangle \subseteq \eta_r^{H^s d}(\emptyset) = \emptyset.$$

If $A \neq \emptyset$, consider the cover

$$A \subseteq \bigcup_{x \in A} \eta^d_{\frac{r}{2}}(x).$$

Since A is compact we can produce a finite subcover consisting of the open balls above. Let $U_1, \dots, U_n$ be a minimal subcover of A. This subcover has the following properties:

1. $A \subseteq \bigcup_{i=1}^n U_i$
2. $A \cap U_i \neq \emptyset \quad \forall i$
3. $\text{diam} U_i < r, \forall i.$

Clearly $A \in \langle U_1, \dots, U_i \rangle$. Let $K \in \langle U_1, \dots, U_n \rangle$. Then for any $k \in K$ there exists i such that $k \in U_i$. Since $A \cap U_i \neq \emptyset$, there exists $a \in U_i \cap A$ such that,

$$d(k, a) < r.$$

Thus,

$$K \subseteq N^d_r(A).$$

Similarly,

$$A \subseteq N^d_r(K).$$

Hence,

$$\langle U_1, \dots, U_n \rangle \subseteq \eta^{H^s d}_r(A).$$

Thus the sets $\eta_r^{H^s d}(A)$ are open in τ_V. Since those sets for a basis of $\tau_{H^s d}$, we have reached the desired conclusion. $\square$

Next we turn our attention to a different question: what can be said about X when VX is metrizable? The following theorem answers this question:

Theorem 5.4.2 ((IN), Theorem 2.4)**.** *If VX is metrizable, then X is compact and metrizable.*

We need the following lemmata:

Lemma 5.4.3 ((IN), Lemma 2.3)**.** *If Y is an infinite discrete space, then the Vietoris topology on PY does not have a countable basis.*

Proof. Let $\mathfrak{B}$ be a basis for VY. Since Y is discrete, for any $A \subseteq Y$, $\langle A \rangle$ is open. Thus, for each $A \subseteq Y$ there exists $\mathcal{B}_A \in \mathfrak{B}$ such that

$$A \in \mathcal{B}_A \subseteq \langle A \rangle.$$

Furthermore, it is clear that

$$\bigcup \mathcal{B}_A = A.$$

Hence $\mathcal{B}_A = \mathcal{B}'_A$ implies that $A = A'$ and consequently we can identify a subset of $\mathfrak{B}$ with the uncountable set PY. $\square$

Recall that a topological space in *separated* provided that it has a countable dense subset.

Lemma 5.4.4. *If X is a separated topological space, then so is VX.*

Proof. Let $D \subseteq X$ be a countable dense subset. Set

$$\hat{D} = \{\text{all finite subsets of } D\}.$$

Then $\hat{D}$ is a countable subset of VX. Furthermore, given any subset A of X and open sets $U_1, \ldots, U_n$ of X such that $A \in \langle U_1, \ldots U_n \rangle$, there exists, for each i, $x_i \in U_i \cap D$. Set

$$F = \{x_i \mid i = 1 \ldots n\}.$$

Then $F \in \hat{D} \cap \langle U_1, \ldots, U_n \rangle$. Hence,

$$\langle U_1, \ldots, U_n \rangle \cap \hat{D} \neq \emptyset$$

and we conclude that $\hat{D}$ is dense in VX. □

We are now ready to prove Theorem 5.4.2.

Proof. We can identify X with the subspace of one elements subsets of X in VX. Hence, when VX is metrizable, so is X.

Suppose X is not compact. Then since X is metrizable, there exists an infinite subset Y of X that has no limit points. That is, Y is discrete. Without loss of generality, we can assume Y to be countable (if Y is not countable, we can consider a countable subset of Y). Since Y is a subspace of X, VY is a subspace of VX, and hence metrizable. Y is clearly separable. Thus VY is also separable. Then, since VY is also metrizable, it has a countable basis. But this contradicts our first lemma. Hence X must be compact. $\square$

We finish this section with a result that allows us to think of the Vietoris topology as an initial topology. Recall that given a topological space Y, a set X and a family $f_i : X \longrightarrow Y$ of mappings, the *initial topology determined by* $(f_i)_{i \in I}$ is the weakest topology on X making all the f_i's continuous. Given two metrics d and d' on a set X we say that d and d' are *compatible* provided that $\tau_d = \tau_{d'}$.

Theorem 5.4.5 ((Be), Theorem 2.2.5)**.** *Let X be a metrizable topological space and $\mathcal{D}$ denote the set of all compatible metrics for X. Then the Vietoris topology, τ_V, on CLX is the weak topology determined by the following family*

$$d(x, -) : CLX \longrightarrow [0, \infty]$$

where $d \in \mathcal{D}$ and $x \in X$.

Proof. Let τ_W denote the initial topology determined by the family $\{d(x, -) \mid d \in \mathcal{D}, x \in X\}$.

"$\tau_W \subseteq \tau_V$": It suffices to show that the subbase of τ_W is contained in τ_V. The subbase of τ_W consists of sets

$$(1)\ \{B \in CLX \mid d(x, B) < \alpha\} = d(x, -)^{-1}([0, \alpha))$$

and

$$(2)\ \{B \in CLX \mid d(x, B) > \alpha\} = d(x, -)^{-1}((\alpha, \infty]).$$

Fix $d \in \mathcal{D}$, $x \in X$ and $\alpha > 0$.

(1): We claim

$$\{B \in CLX \mid d(x, B) < \alpha\} = \langle X, \eta_\alpha^d(x)\rangle$$

Indeed, if $d(x, B) < \alpha$, then there exists some $y \in B$ such that $d(x, y) < \alpha$ and hence $\eta_\alpha^d(x) \cap B \neq \emptyset$.

If $\eta_\alpha^d(x) \cap B \neq \emptyset$, then there exists a $y \in B$ with $d(x, y) < \alpha$ and hence $d(x, B) < \alpha$.

Hence, sets of the form (1) are all members of τ_V.

(2): Let $A \in \{B \in CLX \mid d(x, B) > \alpha\}$. Then $A \cap \overline{\eta_\alpha(x)} = \emptyset$ and hence

$$A \in \langle X \setminus \overline{\eta_\alpha(x)}\rangle \subseteq \{B \in CLX \mid d(x, B) > \alpha\}.$$

Indeed,

$$\begin{aligned} U \subseteq X \setminus \overline{\eta_\alpha(x)} \quad &\implies \quad U \cap \overline{\eta_\alpha(x)} = \emptyset \\ &\implies \quad d(x, U) > \alpha \\ &\implies \quad U \in d(x, -)^{-1}((\alpha, \infty]). \end{aligned}$$

Thus set of the form (2) are in τ_V.

Since d, x and α were arbitrary, we conclude that $\tau_W \subseteq \tau_V$.

"$\tau_W \supseteq \tau_V$": Let V be an open set in X. We first show that the set $\langle X, V\rangle$ is in τ_W. Let $A \in \langle X, V\rangle$. Then $A \cap V \neq \emptyset$. So there exists $x \in A \cap V$. Fix $d \in \mathcal{D}$.

There there exists $\varepsilon > 0$ such that

$$x \in \eta_\varepsilon^d(x) \subseteq V.$$

Thus, $x \in A \cap \eta_\varepsilon^d(x)$ and hence $A \in \langle X, \eta_\varepsilon^d(x)\rangle$. Also $U \in \langle X, \eta_\varepsilon^d(x)\rangle \implies U \cap \eta_\varepsilon^d(x) \neq \emptyset$ and hence $U \cap V \neq \emptyset$. This gives us

$$A \in \langle X, \eta_\varepsilon^d(x)\rangle \subseteq \langle X, V\rangle.$$

But we already showed that

$$\langle X, \eta_\varepsilon^d(x)\rangle = d(x, -)^{-1}([0, \varepsilon)).$$

Thus,

$$A \in d(x, -)^{-1}([0, \varepsilon)) \subseteq \langle X, V\rangle,$$

and consequently, sets of the form $\langle X, V\rangle$ are in τ_W.

It remains to show that that sets of the form $\langle W\rangle$ are open in τ_W when W is open in X.

First, take $W = X$. Then $\langle W\rangle = CLX \in \tau_W$. If $W = \emptyset$, then $\langle W\rangle = \emptyset \in \tau_W$. So suppose that W is an arbitrary proper non–empty open subset of X, and let $x_0 \in X \setminus W$. Fix $A \in \langle W\rangle$. We produce a compatible metric ρ on X with the following property:

$$A \in \{B \in CLX \mid \rho(x_0, A) - \frac{1}{4} < \rho(x_0, B)\} \subseteq \langle W\rangle.$$

Note that if we set $\alpha = \rho(x_0, A) - \frac{1}{4}$, then the above equation translates to

$$A \in \rho(x_0, -)^{-1}((\alpha, \infty]) \subseteq \langle W \rangle.$$

So once we produce this metric, we are going to be done.

Since A and $X \setminus W$ are closed, $(X \setminus W) \cap A = \emptyset$ and every metric space is normal, we find, using Uryshon's Lemma, a continuous function

$$\varphi : X[0, \infty]$$

such that

$$\varphi(A) = 0 \text{ and } \varphi(X \setminus W) = 1.$$

Define

$$\rho(x, y) = \min\{\frac{1}{2}, d(x, y)\} + |\varphi(x) - \varphi(y)|.$$

Then, as the Lemma that follows this theorem shows, ρ is a metric on X that is compatible with d. Suppose $\{B \in CLX \mid \rho(x_0, A) - \frac{1}{4} < \rho(x_0, B)\} \nsubseteq \langle W \rangle$. Then there exists B such that $B \nsubseteq W \iff B \cap (X \setminus W) \neq \emptyset$ and $\rho(x_0, A) - \frac{1}{4} < \rho(x_0, B)$. Let $y \in B \cap (X \setminus W)$. Then , since both x_0 and y are not in W,

$$\rho(x_0, y) = \min\{\frac{1}{2}, d(x_0, y)\} \leq \frac{1}{2}.$$

Now, since for every $x' \in A$,

$$\rho(x_0, x') \geq |\varphi(x_0) - \varphi(x')| = 1,$$

we get

$$
\begin{aligned}
1 \le \rho(x_0, A) &\le \rho(x_0, B) + \frac{1}{4} \\
&\le \rho(x_0, y) + \frac{1}{4} \\
&\le \frac{1}{2} + \frac{1}{4} \le \frac{3}{4},
\end{aligned}
$$

which is most certainly a contradiction. Hence,

$$A \in \rho(x_0, -)^{-1}((\alpha, \infty]) \subseteq \langle W \rangle,$$

and $\langle W \rangle \in \tau_W$.

We have shown that the subbasis of τ_V is contained in τ_W. Hence $\tau_V \subseteq \tau_W$, as claimed. $\square$

Lemma 5.4.6. *Let (X, d) be a metric space and $\varphi : X \longrightarrow \mathbb{R}$ a continuous function. Define*

$$d'(x, y) = d(x, y) + |\varphi(x) - \varphi(y)|.$$

Then d' is a metric compatible with to d.

Proof. d' is a metric since both d and $|(-) - (-)| : \mathbb{R}^2 \longrightarrow [0, \infty]$ are.

"$\tau_d \subseteq \tau_{d'}$": Take $x \in X$, $\varepsilon > 0$ and $y \in \eta^d_\varepsilon(x)$. We wish to find $\delta > 0$ such that $y \in \eta^{d'}_\delta(y) \subseteq \eta^d_\varepsilon(x)$. Take $\delta = \varepsilon - d(x, y)$. Then for any $z \in \eta^{d'}_\delta(y)$,

$$d(x, z) \le d(x, y) + d(y, z) \le d(x, y) + d'(y, z) < d(x, y) + \varepsilon - d(x, y) = \varepsilon.$$

"$\tau_{d'} \subseteq \tau_d$": Take any $x \in X$ and $\varepsilon > 0$. Pick $y \in \eta_\varepsilon^{d'}(x)$. We wish to find $\delta > 0$ such that

$$y \in \eta_\delta^d(y) \subseteq \eta_\varepsilon^{d'}(x).$$

Since φ is continuous, there exists a $\delta' > 0$ such that $d(x,y) < \delta'$ implies that $|\varphi(x) - \varphi(y)| < \frac{\varepsilon - d'(x,y)}{2}$. Set

$$\delta = \min\{\frac{\varepsilon - d'(x,y)}{2}, \delta'\}.$$

Thus for any $z \in \eta_\delta^d(y)$,

$$\begin{aligned} d'(x,z) \leq d'(x,y) + d'(y,z) &\leq d'(x,y) + d(y,z) + |\varphi(x) - \varphi(y)| \\ &< d'(x,y) + \frac{\varepsilon - d'(x,y)}{2} + \frac{\varepsilon - d'(x,y)}{2} = \varepsilon. \end{aligned}$$

□

5.5 Compactness and Hausdorffness of VX

Recall that if (X,d) is a compact metric space, then so is H^sX. In light of Theorem 5.4.1, we expect the same transfer of properties to occur with the Vietoris topology. The following theorem shows that our expectations are well justified:

Theorem 5.5.1 ((IN), Exercise 3.12). *(X,τ) is a compact topological space if, and only if, VX is compact.*

Proof. " $\implies$ ": Suppose that

$$VX = \bigcup_{i\in I}\langle U_i\rangle \cup \bigcup_{j\in J}\langle X, V_j\rangle,$$

where the U_i and V_j are all open in X. If $X \subseteq \bigcup_{j\in J} V_j$, there exists a finite collection $V_1,\dots,V_n$ such that $X \subseteq \bigcup_{i=1}^n V_i$. Hence, $VX \subseteq \bigcup_{i=1}^n \langle X, V_i\rangle$. If $X \neq \bigcup_{j\in J} V_j$, set $V = \bigcup_{j\in J} V_j$, and $K = X \setminus V$. Then K is a closed subset of X and $K \cap V_j = \emptyset$, $\forall j \in J$. Thus, there exists i_0 such that $K \subseteq U_{i_0}$,by hypothesis. Thus,

$$X \subseteq (\bigcup_{j\in J} V_j) \cup U_{i_0}.$$

Consequently, there exists a finite subcover

$$X \subseteq V_1 \cup \ldots \cup V_m \cup U_{i_0}.$$

We claim that $VX \subseteq \langle U_{i_0}\rangle \cup \bigcup_{i=1}^n \langle X, V_i\rangle$. If $A \subseteq X$ and $A \cap \bigcup_{i=1}^n V_i \neq \emptyset$, then for some j between 1 and n, $A \in \langle X, V_j\rangle$. Otherwise, $A \subseteq U_{i_0}$ and $A \in \langle U_{i_0}\rangle$. Thus any cover of VX by elements of its subbasis has a finite subcover and hence by Alexander's Subbase Lemma, VX is compact.

"$\Leftarrow$": Suppose that VX is compact. Let $X \subseteq \bigcup U_i$ with $U_i \subseteq X$ open sets. Then $VX \subseteq \bigcup \langle X, U_i\rangle$. So

$$VX \subseteq \bigcup_{i=1}^n \langle X, U_i\rangle.$$

Let $x \in X$. There is some $i \in \{1,\dots,n\}$ such that $\overline{\{x\}} \cap U_i \neq \emptyset$. So, there exists $y \in \overline{\{x\}} \cap U_i$. Since U_i is an open neighborhood of y, $U_i \cap \{x\} \neq \emptyset$. Hence, $x \in U_i$. Thus, $U_1,\dots,U_n$ is a finite subcover of X. Hence X is compact. $\square$

Remark. Note that we can replace VX by (CLX, τ_V) in the above theorem and the proof still holds. (This is the reason we took the closure of $\{x\}$ in the second part of the proof). We record this result below.

Theorem 5.5.2. *(X, τ) is a compact topological space if, and only if, (CLX, τ_V) is compact.*

Hausdorffness also gets transferred from the base space, but with certain additional conditions.

Theorem 5.5.3 ((Wy), Proposition 5.3)**.** *If X is compact Hausdorff, so is (CLX, τ_V).*

Proof. (CLX, τ_V) is compact when X is compact. If $A \neq B$ are closed subsets of X, then, without loss of generality, there exists $x \in A \setminus B$. Thus $\{x\}$ and B are disjoint closed subsets of X. Since X is normal, there exist U and V open subsets of X such that $x \in U$ and $B \subseteq V$. Then $A \in \langle X, U \rangle$, $B \in \langle V \rangle$ and $\langle X, U \rangle$ and $\langle V \rangle$ are disjoint. $\square$

Proposition 5.5.4. *If X is Hausdorff, then so is (KX, τ_V).*

Proof. Let $A \neq B$ be two compact subsets of X. Without loss of generality we suppose that there exists $x \in A \setminus B$. For all $y \in B$ we can choose U_y and V_y open subsets of X such that $x \in U_y$, $y \in V_y$ and $V_y \cap U_y = \emptyset$. Since

$$B \subseteq \bigcup_{y \in B} V_y,$$

there exist a finite set $\{y_1, \ldots, y_n\}$ such that

$$B \subseteq \bigcup_{i=1}^{n} V_{y_i}.$$

Set $U = \bigcap_{i=1}^{n} U_{y_i}$ and $V = \bigcup_{i=1}^{n} V_{y_i}$. Then $A \in \langle X, V\rangle$, $B \in \langle U\rangle$ and $\langle X, U\rangle \cap \langle V\rangle = \emptyset$. $\square$

5.6 The Vietoris monad

So far we viewed the Vietoris topology only from a classical point of view. Now we examine its categorical properties. We define the *Vietoris functor*

$$V : \mathsf{Top} \longrightarrow \mathsf{Top};$$

as follows: on objects

$$V(X, \tau) = (PX, \tau_V) = VX$$

and for a morphism $f : X \longrightarrow Y$ between spaces X and Y

$$Vf(A) = f(A).$$

It immediately follows from Corollary 5.3.2 that V is indeed a functor.

As the following two propositions show, more is true: $\bigcup = \bigcup_X : VVX \longrightarrow VX$ and $e_X : X \hookrightarrow VX$ are natural transformations, so that

$$\mathbb{V} = (V, e, \bigcup)$$

is an extension of the powerset monad to Top. Indeed,

Proposition 5.6.1. *Define* $e_X : X \to VX$ *by*

$$e_X(x) = \{x\}.$$

Then e_X *is continuous.*

Proof. For any finite set $\{U_1, \ldots, U_n\}$ of open sets in X we have

$$\begin{aligned} e_X^{-1}(\langle U_1, \ldots, U_n\rangle) &= \{x \mid \{x\} \subseteq \bigcup_{i=1}^{n} U_i,\ x \in U_i \quad, \forall i\} \\ &= \{x \mid x \in U_i,\ \forall i\} \\ &= \bigcap_{i=1}^{n} U_i. \end{aligned}$$

□

Proposition 5.6.2. *The union map*

$$\bigcup : VVX \to VX$$

is continuous.

Proof. Let $\{U_1, \ldots, U_n\}$ be a finite collection of open sets in X. The result follows from the following formula:

$$(\bigcup)^{-1}(\langle U_1, \ldots, U_n\rangle) = \big\langle \langle \bigcup_{i=1}^{n} U_i\rangle \big\rangle \cap \big(\bigcap_{i=1}^{n} \big\langle VX, \langle X, U_i\rangle \big\rangle \big).$$

"$\subseteq$": let $\mathcal{A} \in \bigcup^{-1}(\langle U_1, \ldots, U_n\rangle)$. Then

$$\begin{aligned} \forall A \in \mathcal{A}, A \subseteq \bigcup \mathcal{A} \subseteq \bigcup_{i=1}^{n} U_i \quad &\Longleftrightarrow \quad \mathcal{A} \subseteq \langle \bigcup_{i=1}^{n} U_i \rangle \\ &\Longleftrightarrow \quad \mathcal{A} \in \big\langle \langle \bigcup_{i=1}^{n} U_i\rangle \big\rangle, \end{aligned}$$

gives us

$$\mathcal{A} \in \langle\langle \bigcup_{i=1}^{n} U_i \rangle\rangle.$$

Given an arbitrary i, $\bigcup \mathcal{A} \cap U_i \neq \emptyset$. Thus, there exists $A \in \mathcal{A}$ such that $A \cap U_i \neq \emptyset$. So, $A \in \langle X, U_i \rangle$ and consequently $\mathcal{A} \cap \langle X, U_i \rangle \neq \emptyset$. Thus $\mathcal{A} \in \big\langle VX, \langle X, U_i \rangle \big\rangle$.

"$\supseteq$": From the above arguments it is clear that

$$\mathcal{A} \in \langle\langle \bigcup_{i=1}^{n} U_i \rangle\rangle \implies \bigcup \mathcal{A} \subseteq \bigcup_{i=1}^{n} U_i$$

and

$$\mathcal{A} \in \big\langle VX, \langle X, U_i \rangle \big\rangle \implies \Big(\bigcup \mathcal{A}\Big) \cap U_i \neq \emptyset.$$

$\square$

The Eilengerg–Moore algebras of the Vietoris monad are easily seen to be exactly those sup–lattices for which the map

$$\bigvee : PX \to X$$

is continuous with respect to the Vietoris topology.

We can define an endofunctor

$$\tilde{V} : \mathsf{CompHaus} \to \mathsf{CompHaus},$$

on the category of compact Hausdorff spaces by

$$\tilde{V}X = (CLX, \tau_V).$$

and for any morphism f,

$$\tilde{V}f = Vf.$$

$\tilde{V}$ is well defined by Theorem 5.5.3. Furthermore, $\tilde{\mathbb{V}} = (\tilde{V}, e, \bigcup)$ is a monad (the proofs of Propositions 5.6.1 and 5.6.2 continue to hold if PX is replaced by CLX).

$\tilde{V}$ is no longer an extension of the powerset monad, but rather extends a modified version of the Hausdorff monad. In particular, we have the following diagram

$$\begin{tikzcd} \mathsf{CompHaus} \arrow[r, "\tilde{V}"] & \mathsf{CompHaus} \\ \mathsf{Met}_c \arrow[u] \arrow[r, "\tilde{H}"'] & \mathsf{Met}_c \arrow[u] \end{tikzcd}$$

where $\tilde{H}(X, d) = (CLX, H^s d)$.

In (Wy) O. Wyler investigates the algebras of $\tilde{\mathbb{V}}$. He calls them compact meet semilattices with continuous inf structure and relates them to the algebras of the filter monad.

6 Further work

In this chapter we briefly describe several interesting projects that we did not have a chance to fully pursue in this work.

1. The transfer of Cauchy completeness from X to HX is an important property of the Hausdorff distance. Since Cauchy completeness is defined in $\mathcal{V}$-Cat as L–completeness, we can ask: does L–completeness get transferred from X to HX in $\mathcal{V}$-Cat? As we saw in Section 3.7, the problem is reduced to representability of certain $\mathcal{V}$–functors. This problem is not easy because in the classical approach the validity of this result relies heavily on convergence of certain well-known sequences of real numbers. It might be possible to prove the classical result without resorting to known facts about the reals and hence show that L–completeness indeed gets transferred to HX.

2. There is a connection between the Vietoris monad and the filter monad (see (Wy)). Can we establish a connection between the Hausdorff monad and the filter monad? Maybe we need to consider a metric version of the filter

monad: a variation of the sequence monad. Since there is a connection between algebras of the filter monad and the Vietoris monad, we would expect a connection between $\mathbb{H}$–algebras and algebras of the sequence monad. This connection might allow us to relate the category of $\mathbb{H}$–algebras to a familiar category.

3. We defined $\frac{v}{2}$ with $v \in \mathcal{V}$. Another way of defining $\frac{v}{2}$ might allow us to to relax the conditions on $\mathcal{V}$ in Theorem 4.8.8. Alternatively, a modification of the Theorem can reduce our dependance on the concept of halves and therefore make the Theorem more fitting for $\mathcal{V}$-Cat.

4. In Section 4.9 we describe some functors that are also $\mathcal{V}$–functors with respect to G. What other functors have this property? What makes those functors special? It would be interesting to characterize the endofunctors on $\mathcal{V}$-Cat that are also $\mathcal{V}$–functors with respect to G.

5. We saw that G is defined on objects of Met and $\mathcal{V}$-Cat. How does G interact with morphisms?

6. We studied the Vietoris topology mainly in the classical setting. We would like to study it in the categorical setting of $(\mathbb{T}, \mathcal{V})$–categories. In order to define the Vietoris topology in this setting, we need to describe it using ultrafilter convergence. So, we would like, given a convergence structure on X, to be able

to give a formula for the convergence structure on PX in terms of the one on X. The next step should be to define the "Vietoris approach structure". After those two questions are settled, one should introduce the Vietoris structure to $(\mathbb{T}, \mathcal{V})$-$\mathsf{Cat}$ and create a theory of hyperspaces in this setting.

7. The Gromov distance naturally gives rise to the Gromov topology on obMet. Can we create a topology on obTop such that its relationship with the Vietoris topology is similar to the relationship of the Gromov distance and the Hausdorff distance? The restriction of this topology to obMet probably should coincide with the topology induced by the Gromov distance. What properties will this topology share with the topology induced by G?

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