The Algebra of Multi-Agent Dynamic Belief Revision

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Abstract

We refine our algebraic axiomatization in [8,9] of epistemic actions and epistemic update (notions defined in [5,6] using Kripke-style semantics), to incorporate a mechanism for dynamic belief revision in a multi-agent setting. We encode revision as a particular form of epistemic update, as a result of which we can revise with epistemic propositions as well as facts, we can also revise theories about actions as well as about states of the worlds, and we can do multi-agent belief revision. We show how our setting can be applied to a cheating version of the muddy children puzzle where by using this logic, after the cheating happens, honest children will not get contradictory beliefs.

Keywords: Belief Revision, Algebra, Quantale, Epistemic Update, Dynamic Logic, Epistemic Logic.

1 Introduction

We refine our algebraic axiomatization in [8,9] of epistemic actions and epistemic update (notions defined in [5,6] using a relational, Kripke-style semantics), to incorporate a mechanism for dynamic belief revision in a multi-agent setting. Our approach has a number of novel features, when compared with traditional belief revision systems such as AGM [2]. Firstly, while traditional belief revision was dealing only the revision of theories comprised of "facts", we can also revise theories comprised of epistemic/doxastic propositions. Thus, some of traditional AGM postulates have to be modified in order to deal with non-stable epistemic propositions, such as the ones generated by the so-called *Moore sentences*[3]. Secondly, ours is

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[3] E.g. the agent ("you") is informed that the "The cat is on the mat, but you don't know it".

a *dynamic* belief revision, i.e. a particular form of update: an action q happens, changing the original theory m to a revised theory $m * q$. Thirdly, this dynamic character is enhanced by the fact that we allow *revision of actions, and not just of "static" theories*. That is, we allow the agent to have theories about the current action, and these may also be revised by further actions. Fourthly, we do *multi-agent* belief revision, as opposed to traditional approaches that only revise the beliefs of one agent. Our approach, though related in aim, is different in flavor from the work of [14] on Kripke-style belief revision and from the more recent work of [3,15] on dynamic belief revision. These approaches are *semantic* and *"quantitative"*, that is they are based on having "degrees of belief" as the quantitative basis of belief revision. Our approach, on the contrary, is purely *qualitative* and *axiomatic*, and is thus closer in spirit to the traditional AGM approach. Indeed, our aim is simply to find the "correct" axiomatization of the dynamic and multi-agent version of AGM.

This paper is organized as follows: in the first section we introduce mathematical definitions of the concepts used in our algebra: our main mathematical object being the notion of a *dynamic revision system*, which is based on the epistemic systems as algebraic models of Dynamic Epistemic Logic; we also discuss the connection between our *dynamic revision* operator, which generalizes the update product, and the traditional AGM revision. Next section deals with introduction of agents and their views, to be able to do *multi-agent belief revision*, and enrichment of the structure with *positive and negative test* and with *leaning actions*, to deal with applications. Finally in the last section, we apply our *multi-agent learning systems*, that is *multi-agent dynamic revision systems with learning*, to encode and solve a cheating version of the muddy children puzzle and show how by using our logic, after the cheating, the honest children *learn that they are deceived*; so they revise their beliefs accordingly and will not have contradictory beliefs.

2 Dynamic Revision Systems

We introduce here a *dynamic analogue* of the classical AGM axioms of belief revision. For the moment, we neglect the agents doing the revision, concentrating (as in the AGM approach) on *a purely impersonal notion of rational belief revision*. In the next section, we will re-introduce the agents (and their views of the world) into the picture. Our setting is based on the notion of *system*, which is a structure composed of a pair quantale-module (M, Q) linked by an action

$$- \otimes - : M \times Q \to M\,,$$

subject to some conditions. Systems are very general algebraic settings for modelling dynamical phenomena. Quantales have been used to study different phenomena such as concurrent processes in [1] and physical properties of Quantum Mechanics in [10]. They have also been studied as models of Dynamic Logic for example in [12]. For a detailed mathematical study of quantales and their properties, we refer the reader to [13].

Definition 2.1 A system $(\mathbf{M}, \mathbf{Q}, \otimes)$ consists of:
(1) a *complete sup-lattice* $\mathbf{M} = (M, \bigvee)$, with $\bigvee$ as the supremum operator;
(2) a *quantale* $\mathbf{Q} = (Q, \bigvee, \bullet, 1)$, i.e. a complete sup-lattice $(Q, \bigvee)$ endowed with an

additional monoidal structure $(Q, \bullet, 1)$;
(3) a *right-module* structure on $\mathbf{M}$, i.e. an action $- \otimes - : M \times Q \to M$ on the quantale, which is *sup-preserving in both arguments* and satisfies:

$$m \otimes (q \bullet q') = (m \otimes q) \otimes q' \quad \text{for } m \in M \quad \text{and} \quad q, q' \in Q\,.$$

We call the elements of the module M, *theories* or *propositions* or also *situations*. These situations are not neccessarily deterministic, as our theories might not be complete. There exists exactly *one inconsistent theory*, namely $\bot := \bigvee \emptyset$. Similarly, there exists exactly one trivial (tautological) theory, namely $\top := \bigvee M$. In a possible-world model, elements of M can be modelled as *sets of possible worlds*, representing some theory (or belief) [4] about the real world: it is believed [5] that the real world belongs to this set. However, one can also represent theories, as it is standard in belief revision, in an *intensional way*, that is, as *sets of sentences of some language closed under logical consequence* [6] .

The lattice order $m \leq m'$ on the module M represents the *consequence relation*, that is the logical entailment $T \vdash T'$ between theories. In terms of theories as sets of possible worlds, this is simply *set inclusion* $\subseteq$. But in terms of deductively closed theories as sets of sentences, it is *reversed inclusion* $T \supseteq T'$. So in particular our join $m \vee m'$ corresponds to union in terms of sets of possible worlds, i.e. it is logical disjunction of propositions, but notice that it also corresponds to *intersection* $T \cap T'$ of sets of sentences. Dually, our meet $m \wedge m'$ is the logical *conjunction* of propositions, i.e. the intersection of the two sets of worlds, but it also corresponds to the *deductive closure of the union* $\mathcal{C}l(T \cup T')$ of the sets of sentences of the two theories.

We call the elements of the quantale Q *epistemic actions*, or *experiments*, or also *announcements*. They are information-changing actions, which *do not change the objective facts of the world, but only uncover, discover, or communicate them.* One can think of elements of quantale as scientific experiments, or even better as actions of *communicating the results of an experiment*. More generally, they may cover *any communication* of, for example *facts* or *hypothesis about the world, or even about what is (not) known about the world*, etc. Since these actions are not necessarily deterministic, they might also be thought of as "theories about the current action". That is there might be uncertainty about what exactly has been discovered or announced. The lattice order $q \leq q'$ on the quantale Q is the order of non-determinism: q' contains less information about the current action (and thus is more non-determined) than q. The quantale multiplication $q \bullet q'$ represents *sequential composition* of actions: *first do* q*, then do* q'. The unit 1 of multiplication stands for the *skip* or the "do-nothing" action.

Observe the lack of a Boolean structure: there is no negation in M, or in Q. One can of course define some pseudo-complement, as is usually done in complete lattices, but this will not necessarily behave as a Boolean negation. This lack is the result of a conscious choice: we think this algebraic setting is simpler, more transparent and

[4] This theory may be of course held by some (arbitrary) agent. This agent is left implicit here, but in the next section we will introduce it explicitly.

[5] See the previous footnote.

[6] Algebraically, theories in this sense correspond to *filters* in a complete lattice, or Boolean algebra.

more general than the one that one would obtain by adding negation [7]. Moreover, it is applicable to cases of partial information. In particular, when thinking of M as consisting of "theories", there is no meaningful notion of "negation of a theory"; similarly, there is no action that can be called "the negation" of action q. However, as we show bellow, one can define in some sense a "propositional negation" of an action.

The operation $\otimes$ is called *update product* and it encodes the way in which an action *changes* a situation or theory. That is, if a theory m correctly describes the current situation and q is a correct description of the current action, then the theory $m \otimes q$ will correctly describe the situation after the action.

Kernel. An inconsistent situation: $m \otimes q = \bot$ simply means that the *action q is impossible in (the situation described by) m*. The condition of impossibility of a given action can be encoded as a proposition in M, by defining the *kernel* of action q as:

$$ker(q) = \bigvee \{m \in M \mid m \otimes q = \bot\} .$$

This is the weakest theory/situation that makes this action impossible. The kernel can be thought of as a "strong negation of an action": $ker(q)$ is the proposition asserting the *impossibility of action q*. [8]

Image Maps. Any action $q \in Q$ "acts" on M via the update product $\otimes$, and "acts" on Q itself via multiplication $\bullet$. As a consequence, we can define two notions of *image of an action*:

$$Im^M(q) = \{m \otimes q : m \in M\}$$
$$Im^Q(q) = \{q' \bullet q : q' \in Q\}$$

Atoms. In a lattice M, the set of *atoms* is defined as

$$Atm(M) = \{a \in M : \forall m \in M,\ (m \neq \bot, m \leq a) \ \Rightarrow\ m = a\} .$$

If existing, the *atoms* of the module M are called "states", and can be thought of as *complete (fully determined) situations*, i.e. complete descriptions of the world. In terms of sets of states (or possible worlds), they simply are the states (or rather the singleton-sets consisting of only one state). In terms of theories, they are the *complete consistent* (i.e. maximally consistent) theories. [9] Similarly, the *atoms of the quantale Q* represent *deterministic* actions (or, alternatively, complete theories about actions).

Atomicity. A lattice M is *atomistic* if every element is the supremum of all the atoms bellow it, i.e. for all $m \in M$ we have

$$m = \bigvee \{s \in Atm(M) : s \leq m\}$$

A system is called *atomistic* if both the module M and the quantale Q are atomistic lattices and if any update product or multiplication of atoms is either an atom or

[7] However, it would be easy to extend our setting to intuitionistic or Boolean settings, by requiring the module to be a Heyting or a Boolean Algebra.

[8] In the "concrete" Kripke-model-based setting of [5,6], the role of the kernel was played by its complement, the *precondition* $pre(q)$ of an action, defining its conditions of possibility.

[9] Algebraically, these can be the *ultrafilters* in a Boolean algebra.

inconsistent, i.e.:

$$s \in Atm(M) \quad \text{and} \quad \sigma, \sigma' \in Atm(Q) \Rightarrow$$
$$(s \otimes \sigma \in Atm(M) \cup \{\bot\} \quad \text{and} \quad \sigma \bullet \sigma' \in Atm(Q) \cup \{\bot\})$$

This condition expresses a natural property of determinism: a deterministic action acts as a partial function on states, i.e. it transforms any (fully determined) state into a (at most one, fully determined) state (or else, it fails); and the sequential composition of two deterministic actions (if not failing) is itself a deterministic action.

Extensionality. A system is called *extensional* if actions are uniquely determined by their behavior on situations, i.e.:

$$\text{if} \quad \forall m \in M,\ m \otimes q = m \otimes q' \quad \text{then} \quad q = q'\,.$$

As a consequence of this, we have that:

$$\text{if} \quad ker(q) = \top \quad \text{then} \quad q = \bot\,.$$

This is because if the kernel of an action q is $\top$ then the action cannot be applied to any proposition. That is if $ker(q) = \top$ then for all $m \in M$ we have $m \otimes q = \bot$. But epistemic update is bottom-preserving in the sense that $\bot = m \otimes \bot$. So we have $m \otimes q = m \otimes \bot$ and if the system is extensional we get $q = \bot$.

Facts. Since we think of our actions as "purely epistemic" (i.e. actions of discovery, belief-change or communication), they do not affect the "objective" features of the world. In other words, they do not change the "facts" of the world. We can turn things around, by definining "facts" to be the propositions that are invariant under any actions, i.e. the ones that are "stable" under any update. In other words, the set of "facts" is defined as the "stabilizer" of all actions:

$$Stab(Q) = \{m \in M : \forall q \in Q,\ m \otimes q \leq m\}\,.$$

Note that in our system any proposition that is invariant under actions is called a fact. For example since update preserves all joins including the empty join $\vee\emptyset = \bot$, we have $\bot \otimes q = \bot$, which says bottom is invariant under any action and thus a fact. But it is a fact in which we are not interested, since it is wrong!

Dynamic Modalities. Since the update product preserves joins on both of its arguments, it has a (Galois) right adjoint, defined as

$$[q]m' := \bigvee\{m \in M : m \otimes q \leq m'\}\,.$$

This adjoint $[q]m$ is the standard *dynamic (action) modality* of Propositional Dynamic Logic or PDL [12]. It reads as 'after action q proposition m holds' and denotes the *weakest precondition* of an action. That is, the weakest proposition that should be true before q, so that proposition m becomes true after q. The adjunction $- \otimes q \dashv [q]-$ implies the equivalence:

$$m \otimes q \leq m' \quad \text{iff} \quad m \leq [q]m'$$

Residuals. Since the multiplication $- \bullet - : Q \times Q \to Q$ preserves joins in both arguments, it has two right adjoints, called *left and right residuals*. The *right residual* is the right adjoints of $- \bullet q' : Q \to Q$ and is denoted by

$$q/q' := \bigvee \{q'' \in Q : q'' \bullet q' \leq q\} .$$

The *left residual* is the right adjoint of $q' \bullet - : Q \times Q \to Q$, is denoted by $q' \backslash q$ and is defined symmetrically.

We now enrich the notion of system with a way to *revise old theories both about the world and about actions in the view of new experiments*:

Definition 2.2 A *dynamic revision operator* on a system $(\mathbf{M}, \mathbf{Q}, \otimes)$ is a pair $* = (*^M, *^Q)$ of maps

$$- *^M - : M \times Q \to M, \qquad - *^Q - : Q \times Q \to Q$$

satisfying a list of conditions (to be given). In practice, we skip the superscripts whenever possible. The required conditions are the following:

(i) $m * q \in Im^M(q)$ and $q * q' \in Im^Q(q')$

(ii) $m \otimes q \leq m * q$ and $q \bullet q' \leq q * q'$

(iii) if $m \otimes q \neq \bot$ then $m * q \leq m \otimes q$, and if $q \bullet q' \neq \bot$ then $q * q' \leq q \bullet q'$

(iv) if $m * q = \bot$ then $q = \bot$; similarly: if $q * q' = \bot$ then $q' = \bot$

(v) $m * (q * q') = (m * q) * q'$ and $q * (q' * q'') = (q * q') * q''$

A system enriched with this revision operator is:

Definition 2.3 A *dynamic revision system* is an extensional, atomistic system endowed with a dynamic revision operator.

Explanation of Conditions. In this section we explain how the above conditions are dynamic equivalents of the classical AGM clauses. We start with the intuition about $m * q$:

> *If m is the current (possibly incorrect) theory about the world, then $m*q$ represents the ("rationally") revised theory after the experiment q is performed.*

We call this "dynamic" revision, since $m * q$ is the actual theory *about the state of the world after the action* q: it is *not* a revised theory about the original state of the world before the action [10] . Similarly, the intuition about $q * q'$ is

> *If q is a (possibly incorrect) theory about what action is going on, then $q * q'$ represents the revised theory (about what is going on) after the experiment q' is performed.*

First part of condition (i) says that the revised theory $m * q$ *has to be consistent with the experiment* q, that is

$$m * q \in Im^M(q) \quad \text{means} \quad \exists m' \in M', m * q = m' \otimes q .$$

This says that the new theory can be thought of as the result of updating some previously existing situation m' with the actual experiment q. The situation m' expresses some tentative theory about the original state of the world. This tentative

[10] In contrast, classical "static" belief revision deals with revised theories about the original state of the world.

theory is consistent with the result of the new experiment q, that is $m' \otimes q \neq \bot$. In other words, this expresses a possible "static belief revision" of m (with the information provided by q). This is expressed in the following proposition, equivalent to first part of condition (i):

Proposition 2.4 *For the tentative theory m' that is consistent with experiment q and for which we have $m * q = m' \otimes q$, the following is true:*

$$m' \leq [q](m * q)$$

The proof follows directly by applying the adjunction $- \otimes q \dashv [q]-$ to the equality $m' \otimes q = m * q$. The proposition says that m' is the *weakest* tentative belief consistent with q. As a result, the tentative belief that an agent might have about the prior situation (the situation before q) *after action* q, will be $([q](m * q)) \otimes q$. This can be seen as a tentative retroactive justification of the particular dynamic belief revision $m * q$, in terms of a static belief revision m'.

The second part of condition (i) is interpreted similarly to the first part. It says that the revised theory has to be consistent with the experiment, that is

$$q * q' \in Im^Q(q') \quad \text{means} \quad \exists q'' \in Q, q * q' = q'' \bullet q' \, .$$

A similar version of the above proposition can be stated for the second part.

Conditions (ii) and (iii) together, say that in the case the original theory is consistent with the experiment q, that is when $m \otimes q \neq \bot$, the theory is simply "updated" with the action q using the update product:

$$\text{if} \quad m \otimes q \neq \bot \quad \text{then} \quad m * q = m \otimes q$$

Otherwise, the theory m *has been refuted* by the experiment q, and it has to be revised. Similarly, in the case the original action theory is consistent with the experiment, the two action theories are simply composed sequentially in $q \bullet q'$: the new theory says that action q followed by action q' has been going on:

$$\text{if} \quad q \bullet q' \neq \bot \quad \text{then} \quad q * q' = q \bullet q'$$

Condition (iv) corresponds to the AGM clause of "success of revision"; it says that revision with some consistent new information q such that $q \neq \bot$, is always successful, in the sense that the revised theory is consistent $m * q \neq \bot$.

Finally, condition (v) imposes two types of transitivity of revision [11]. It is a dynamic version of the classic AGM condition about revision with a conjunction $\phi \wedge \psi$. Since we have two types of revisions (one for propositions and one for actions), we need to relate the two. In particular, the first clause of the fifth axiom is about the consistency of revision of actions with revision of propositions. It says that if we revise a proposition m with an action q that has itself been revised by another action q', we get the same revision as when we first revise m with q and then with q'. It can also be seen as a way of defining revision of actions $q * q'$ in terms of revision of propositions.

[11] Technically speaking, we will not need this last condition, since all its instances that are relevant to us will follow from the axioms on belief revision endomorphisms in the next section.

3 Multi-Agent Dynamic Belief Revision

We want to introduce in the picture *agents* and their views about the world and about actions.

Definition 3.1 A *dynamic-revision endomorphism*[12] on a dynamic revision system $(M, Q, \otimes, *)$ is a pair $f = (f^M, f^Q)$ of maps

$$f^M : M \to M, \qquad f^Q : Q \to Q\,,$$

satisfying some extra-conditions (to be given below). As before, we skip the superscripts whenever possible. The conditions are:

(i) Both f^M and f^Q are sup-preserving maps, and $f^Q(1) = 1$

(ii) For atoms $s \in Atm(M), \sigma \in Atm(Q)$ such that $s \otimes \sigma \neq \bot$, we have

$$f(s \otimes \sigma) = f(s) * f(\sigma)$$

(iii) For atoms $\sigma, \sigma' \in Atm(Q)$ such that $\sigma \bullet \sigma' \neq \bot$, we have

$$f(\sigma \bullet \sigma') = f(\sigma) * f(\sigma')$$

Intuitively, we think of $f(m)$ as the theory that an arbitrary agent has about the situation described by m: when the real situation is given m, the agent believes that this situation is given by (his theory) $f(m)$. Similarly, we think of $f(q)$ as the theory that the (unspecified) agent has about the action q.

Definition 3.2 A dynamic-revision endomorphism f is said to be *doxastic* (or $D45$) if it satisfies the following additional conditions:

D. ("Consistency of beliefs"): $ker(f) = \bot$

45. ("Introspection"): $f \circ f = f$

A doxastic dynamic-revision endomorphism is also called an *appearance map*.

The intuition is that the theory $f(m)$ that the arbitrary agent has about the situation m, gives the "appearance" of this situation to the agent (or the "view" that the agent on this situation). Axiom D says that, in any consistent situation $m \neq \bot$, the agent has *consistent* theory $f(m) \neq \bot$. Axiom 45 says that the apearance mp is idempotent, i.e. the agent is introspective: the theory $f(f(m))$ that he has about (the situation represented by) his own theory $f(m)$ coincides with his theory, that is $f(f(m)) = f(m)$.

Note: The notion of epistemic endomorphism in [8,9] was not required to satisfy the $D45$ conditions. Moreover, in that context, condition D was *not satisfied* by the examples, since (in specific examples) it was contradicted by the nature of update product: agents may indeed come to inconsistent believes when using the update product of [5,6] to update their beliefs! The reason, again, was the absence of a mechanism for belief revision.

[12] Note that this notion differs essentially from the notion of *epistemic endomorphism* in [8,9]: in the conditions corresponding to the last two conditions above, the epistemic endomorphisms had update product $\otimes$, and respectively composition $\bullet$, instead of dynamic revision $*$.

Definition 3.3 A *multi-agent dynamic belief revision system*

$$(M, Q, \otimes, *, \{f_A\}_{A \in \mathcal{A}})$$

is a dynamic revision system $(M, Q, \otimes, *)$ endowed with a family of appearance maps (=doxastic dynamic-revision system endomorphisms), indexed by a set of "agents" $A \in \mathcal{A}$.

The element $f_A(m) \in M$ is called *the appearance of (situation) m to agent A*, or the *theory of A about m*. Different agents may have different *views* of the same situation, or different *interpretations* of the same theory. Similarly, $f_A(q)$ is *the appearance of action q to agent A*. In other words, $f_A(q)$ represents what agent A thinks is going on when in reality action q is going on. For instance different appearance maps $f_A(q)$ might represent different interpretations or different views of the experiment q. An experiment might be public or private, its results might be communicated only to some of the agents, or some outsiders might be deluded by their dogmatic beliefs into rejecting, misunderstanding or misinterpreting the experiment.

The essential difference between this notion and the notion of *epistemic system* in [8,9] comes from the clauses for the appearance of an updated situation, and of a composition of actions, which is

$$f_A(s \otimes \sigma) = f_A(s) * f_A(\sigma)$$

This means that the theory agent A has about the world after the experiment σ, is obtained by *dynamically revising* A's old theory about the world $f_A(s)$ with A's theory $f_A(\sigma)$ about the experiment. That is, in case the experiment appears to contradict the old theory, the contradiction is solved by the agent in favor of the experiment (using the dynamic revision operator). Similarly, the identity

$$f_A(\sigma \bullet \sigma') = f_A(\sigma) * f_A(\sigma')$$

means that the theory A has about the composed action $\sigma \bullet \sigma'$ is obtained by dynamically revising A's theory about the first action with A's theory about the second action.

Belief. The *belief* modality $\Box_A$ can be defined as the *right-adjoint of the appearance map*: $f_A(-) \dashv \Box_A -$. Indeed, since the appearance maps are join-preserving maps, they have meet-preserving Galois right adjoints which are defined as:

$$\Box_A m' = \bigvee \{m \in M : f_A(m) \leq m'\} .$$

Formally that is to say

$$f_A(m) \leq m' \quad \text{iff} \quad m \leq \Box_A m' .$$

We read $\Box_A m$ as "agent A believes theory m ". A similar notion $\Box_A q$ can be defined for actions. One can easily see that the belief modality has the properties of a *normal modality*, and moreover (due to the $D45$ conditions above), it satisfies "Introspection" ($\Box_A\, m = \Box_A \Box_A\, m$) and "Consistency of beliefs" ($\Box_A \top = \top$).

Multi-Agent Learning Systems. To deal with applications, it is useful to enrich our structure a bit, allowing it to deal with *positive and negative tests*, and with *learning actions* (and as a consequence, with public/private announcements and refutations).

Definition 3.4 A *multi-agent learning system* is a multi-agent dynamic belief revision system

$$(M, Q, \otimes, *, \{f_A\}_{A \in \mathcal{A}}),$$

endowed with maps

$$-? : M \times M \to Q \qquad \text{and} \qquad L_{\mathcal{B}} : Q \to Q,$$

for each set $\mathcal{B} \subseteq \mathcal{A}$ of agents. These maps are required to satisfy certain conditions, to be given below. As a notation, we put $m?\overline{n} := ?(m, n)$. We read it as

$m?\overline{n}$ is the experiment of "testing" m and (simultaneously) refuting n.

This is an objective *non-epistemic*, PDL-like test: theory m is "tested", while theory n is "refuted", but *without any of the relevant agents being announced* (of the result of the test/refutation).

The action $L_{\mathcal{B}}\, q$ is read as *learning of action q by group $\mathcal{B}$*. More precisely, it can be described as: "while an action q is happening, the agents in group $\mathcal{B}$ privately get together and learn (in common, by mutual update) that q is happening". This learning action is so private that none of outsiders $C \in \mathcal{A} \setminus \mathcal{B}$ suspect that it is happening. The required conditions are:

- If $s \in Atm(M)$ then $s \otimes (m?\overline{n}) = \begin{cases} s & \text{if } s \leq m \text{ and } s \not\leq n \\ \bot & \text{otherwise} \end{cases}$
- $f_A(m?\overline{n}) = 1$
- $ker(L_{\mathcal{B}}\, q) = ker(q)$
- $f_A(L_{\mathcal{B}}\, q) = \begin{cases} L_{\mathcal{B}}\, q & \text{for } A \in \mathcal{B} \\ 1 & \text{for } A \notin \mathcal{B} \end{cases}$

The first clause says that a state survives a test/refutation $m?\overline{n}$ iff it satisfies the tested property but it doesn't satisfy the refuted property; in which case the state is left unchanged by the state. The second clause says that the appearance of a pure test/refutation action to all agents is 1, i.e. the action *skip* in which "nothing happens". This expresses the fact that such a PDL-like test is non-epistemic: agents do not learn anything from it. The third clause says that learning an action q is (im)possible iff the learned action q is (im)possible. Finally, the fourth clause says that the learning action appears as "learning" to all the agents involved in it, and it appears as *skip* to all the outsiders.

Public/Private Announcements/Refutations. In a multi-agent learning system, we can define a *mutual announcement-and-refutation to a group $\mathcal{B}$* of agents, by putting:

$$m!_{\mathcal{B}}\overline{n} := L_{\mathcal{B}}(m?\overline{n}).$$

Notice that this is a "combined" action, in which something (m) is mutually announced, while something else (n) is mutually refuted. More "pure" special cases of this are *mutual announcements* and *mutual refutations*:

$$m!_{\mathcal{B}} := m!_{\mathcal{B}}\overline{\bot} \qquad \text{and} \qquad !_{\mathcal{B}}\overline{m} := \top !_{\mathcal{B}}\,\overline{m}\,.$$

Even more special cases are *public announcements* and *public refutations*:

$$m! := m!_{\mathcal{A}} \qquad \text{and} \qquad !\overline{m} := !_{\mathcal{A}}\,\overline{m}\,.$$

Also *private announcements* and *private refutations* to one agent

$$m!_A := m!_{\{A\}} \qquad \text{and} \qquad !_A\overline{m} := !_{\{A\}}\,\overline{m}\,.$$

Examples: Secret, Secure Communication and Secret Interception. The *passing of a secret (truthful) message m from A to B* (over a secure channel) is represented by the action $m!_{A,B}$. If the channel is not really secure and in fact *the message is secretely intercepted (and read, but allowed to go further) by an agent C*, then the action is represented by

$$L_C(m!_{A,B})\,.$$

Using the definition of the learning action, we can calculate the kernels of the actions defined above. For example we can prove that the refuted proposition of a public refutation constitutes its kernel. That is, a public refutation cannot be applied on the states where the refuted proposition holds. In other words:

Lemma 3.5 *For a public refutation $!\overline{m}$ and an atom $s \in Atm(M)$ we have*

$$s \otimes !\overline{m} = \bot \quad \textit{iff} \quad s \leq m\,.$$

On the other hand, for a public announcement, every proposition but the announced one is in the kernel. That is a public announcement action can only be applied on the states where the announced proposition holds. In other words:

Lemma 3.6 *For a public announcement $m!$ and an atom $s \in Atm(M)$ we have*

$$s \otimes m! \neq \bot \quad \textit{iff} \quad s \leq m\,.$$

Proofs are easy and follow directly from the definition of the kernel of the learning action and how public announcements and refutation actions are defined in terms of the learning action.

4 "Cheating Muddy Children" with Dynamic Belief Revision.

In this section we apply our setting to the muddy children puzzle, for a full discussion of its original form we refer the reader to [11]. The puzzle has been solved using the update product in kripke structures in [5,6] and using the update product in the algebraic setting of epistemic systems in [8,9]. Here, we first go through the algebraic analysis of the original puzzle, however, since our interest is in a cheating version of the puzzle, which has been originally presented in [4] and also discussed in [8], we encode and solve the cheating version in the algebra. We show where the epistemic update approach of [4,8] fails and how our current version with the revision operator solves the problem; this being the contribution of the dynamic belief revision of this paper to the cheating version of the muddy children puzzle: honest children *learn that they are deceived* and thus will not have contradictory beliefs after the cheating.

The Original Puzzle. For the purpose of this paper, we deal with with four children three of them dirty. Suppose children 1, 2, and 3 are the dirty ones and child 4 is clean. We assume given a learning system and encode the puzzle in it as follows. The children are in the set of agents:

$$\{1, 2, 3, 4\} \subseteq \mathcal{A}\,,$$

the module M includes some atomic situations (”states”) $s_{\mathcal{B}}$ for $\mathcal{B} \subseteq \mathcal{A}$. Each $s_{\mathcal{B}}$ represents the situation in which the dirty children are precisely the ones in the group $\mathcal{B}$. For instance the following state represents the real state, that is the state in which the first three children have dirty foreheads:

$$s_{1,2,3} \in Atm(M)$$

We also have some stable propositions in $Stab(Q) \subseteq M$, some saying ‘child i is dirty’, and some ‘child i is clean’:

$$D_i, \overline{D}_i \in Stab(Q) \quad \text{for } i = 1, 2, 3, 4\,.$$

These are “facts” that cannot be changed by epistemic actions, so that is why we assume them to belong to $Stab(Q)$. Each state satisfies its corresponding facts, i.e. we put

$$s_{\mathcal{B}} \leq D_i \quad \text{iff} \quad i \in \mathcal{B}\,,$$

and

$$s_{\mathcal{B}} \leq \overline{D}_i \quad \text{otherwise}\,.$$

In any state $s_{\mathcal{B}}$, each child sees the faces of other children but not his own, so he doesn’t know if he’s dirty or not. This is encoded in the appearance maps, by putting

$$f_i(s_{\mathcal{B}}) = s_{\mathcal{B}\cup\{i\}} \vee s_{\mathcal{B}\setminus\{i\}}\,,$$

for example in the real state $s_{1,2,3}$ where child one is dirty, he thinks either he is or he is not dirty: the real state appears to him as $f_1(s_{1,2,3}) = s_{1,2,3} \vee s_{2,3}$.

Father’s first announcement that “At least some one is dirty” can be represented as a *public announcement*:

$$q_0 := (\vee_{i=1}^{4} D_i)!$$

which is assumed to be an atomic element of our quantale Q. By the above definition of public announcements, we have

$$f_i(q_0) = q_0 \qquad \text{for all} \quad 1 \leq i \leq 4$$

This says that every child hears this announcement. Every round in which all children answer “I don’t know that I am dirty” is represented by a public refutation, which is also an atomic element of the quantale:

$$q := !\overline{(\vee_{i=1}^{4} \Box_i D_i)}$$

with the refuted proposition as its kernel. With these assumptions, we have formalized the original statement of the puzzle, that after two rounds of no answers each dirty child knows that he is dirty, and prove it in [8,9]. But here we are interested in a cheating version of the puzzle, which we are going to deal with below.

The Cheating Version. The muddy children puzzle has several cheating versions in which different cheating and lying actions happen between the children, for example for a lying version has been solved in [8]. In this paper we deal with the scenario discussed in [4]: this scenario is the same as the original version until before the second round of answers. That is, father makes his announcement and children answer no in the first round of answers, but after the first round, children 2 and 3 cheat by *secretly communicating* to each other that they are dirty. The state of the system after the announcement and the first round of no answers is:

$$s' = s_{1,2,3} \otimes q_0 \otimes q\,,$$

the cheating is encoded as a secret message passing:

$$\pi := (D_2 \wedge D_3)!_{\{2,3\}}\,.$$

Notice that, as discussed before, this action appears to children 1 and 4 as *skip*: $f_1(\pi) = f_4(\pi) = 1$. The new state of the system after this cheating is

$$s'' := s' \otimes \pi\,.$$

Now the cheating children know they are dirty! Thus, in the second round of answers, while children 1 and 4 proceed as usual (that is they refute that they know they are dirty

$$!\overline{(\Box_1 D_1 \vee \Box_4 D_4)}\,,$$

the cheating children announce that they know they are dirty

$$(\Box_2 D_2 \wedge \Box_3 D_3)!$$

So this second round of answers is a combination of yes and no anwers, i.e. a public refutation by children 1 and 4 and a public announcement by children 2 and 3 at the same time. This can be encoded as a *mutual announcement-and-refutation to a group* action

$$q' = (\Box_2 D_2 \wedge \Box_3 D_3)!\overline{\Box_1 D_1 \vee \Box_4 D_4}\,.$$

After q', child 1 will wrongly conclude that she is clean! In our system, this statement is expressed by the following inequality:

Proposition 4.1

$$s_{1,2,3} \leq [q_0 \bullet q \bullet \pi \bullet q']\Box_1 \overline{D}_1\,.$$

Proof (Sketch) We first use the adjunction between update and dynamic modality to take $[q_0 \bullet q \bullet \pi \bullet q']$ to the left hand side:

$$s_{1,2,3} \otimes (q_0 \bullet q \bullet \pi \bullet q') \leq \Box_1 \overline{D}_1\,.$$

Similarly, use the knoweldge-appearance adjunction to take the $\Box_1$ to the left hand side:

$$f_1\left(s_{1,2,3} \otimes (q_0 \bullet q \bullet \pi \bullet q')\right) \leq \overline{D}_1\,.$$

Since the sequential composition and update of atoms is an atom, we have

$$(q_0 \bullet q \bullet \pi \bullet q') \in Atm(Q)$$

and also

$$(s_{1,2,3} \otimes (q_0 \bullet q \bullet \pi \bullet q')) \in Atm(M)\,.$$

Conditions two and three of the dynamic revision endomorphism (definition 3.1) tell us that for atoms $s \in Atm(M)$ and $\sigma, \sigma' \in Atm(Q)$ we have:

$$f(s \otimes \sigma) = f(s) * f(\sigma) \quad \text{and} \quad f(\sigma \bullet \sigma') = f(\sigma) * f(\sigma')\,.$$

So we can distribute our f_1 on the tensor and replace the updates and sequential compositions with our revisions operator to get:

$$f_1\left(s_{1,2,3} \otimes (q_0 \bullet q \bullet \pi \bullet q')\right) = f_1(s_{1,2,3}) * f_1(q_0) * f_1(q) * f_1(\pi) * f_1(q')\,,$$

and it would be enough to show

$$f_1(s_{1,2,3}) * f_1(q_0) * f_1(q) * f_1(\pi) * f_1(q') \leq \overline{D_1}\,.$$

We replace the f_1 maps with their values (introduced above as assumptions) and we have to show

$$(s_{1,2,3} \vee s_{2,3}) * q_0 * q * 1 * q' \leq \overline{D_1}\,.$$

According to the lemma 3.6 for the public announcement of the father $q_0 = (\vee_{i=1}^4 D_i)!$ we have

$$s_{2,3} \otimes q_0 \neq \perp$$

this is because by assumption $s_{2,3} \leq D_2$ and $D_2 \notin ker(q_0)$, so we get

$$s_{2,3} \otimes q_0 \neq \perp \quad \text{and consequently} \quad (s_{1,2,3} \otimes q_0) \vee (s_{2,3} \otimes q_0) \neq \perp$$

which is equal to

$$(s_{1,2,3} \vee s_{2,3}) \otimes q_0 \neq \perp\,.$$

Now by axioms 2 and 3 of definition 2.2:

$$(s_{1,2,3} \vee s_{2,3}) * q_0 = (s_{1,2,3} \vee s_{2,3}) \otimes q_0\,,$$

and by the same line of argument for actions q and q' we get

$$(s_{1,2,3} \vee s_{2,3}) * q_0 * q * q' = (s_{1,2,3} \vee s_{2,3}) \otimes q_0 \otimes q \otimes q'$$

and it would be enough to show

$$(s_{1,2,3} \vee s_{2,3}) \otimes q_0 \otimes q \otimes q' \leq \overline{D_1}\,.$$

Since update distributes over joins, we now have to prove two cases:

$$s_{1,2,3} \otimes q_0 \otimes q \otimes q' \leq \overline{D_1} \qquad \text{and} \qquad s_{2,3} \otimes q_0 \otimes q \otimes q' \leq \overline{D_1}\,.$$

The second disjunct is true since by assumptions we have $s_{2,3} \leq \overline{D_1}$ and $\overline{D_1}$ is a fact. The first disjunct is proven by induction on the number of dirty children k. For the proof of induction we refer the reader to the detailed proofs of [8,9]. The difference is that there we showed

$$s_{2,3} \otimes q_0 \otimes q \otimes q = \perp$$

because until the third round everybody answered no, but here we have to show

$$s_{1,2,3} \otimes q_0 \otimes q \otimes q' = \perp$$

because in the second round children 2 and 3 answer yes. This induction is done in the similar lines, that is by showing $(s_{1,2,3} \otimes q_0 \otimes q) \in ker(q')$. $\square$

We take this proof to be a "success" of the epistemic system formalism: we think it accurately predicts the (most likely) behavior of child 1; moreover, child 1's belief, though wrong, is justified by the appearance of the actions to her.

The Role of Belief Revision: discovering that you have been deceived. Observe that at this stage of the cheating muddy children, there is a problem with child 4: he sees that there are at least three dirty children, so children 2 and 3 *cannot* have come to know so early that they were dirty. We can see that there is a contradiction between the announcement q' and child 4's beliefs before this announcement. The state of the system *before* q' is $s'' = s' \otimes \pi$ where $s' = s_{1,2,3} \otimes q_0 \otimes q$. But the appearance of s'' to child 4 is the same as the appearance of s' to him:

$$f_4(s'') = f_4(s' \otimes \pi) = f_4(s') * f_4(\pi) = f_4(s') * 1 = f_4(s') \otimes 1 = f_4(s')\,,$$

So s'' is indistinguishable to child 4 from the previous state s', in which the dirty children did not know they were dirty. Hence, child 4 believes that e.g. child 2 does not know that he is dirty: $f_4(s'') = f_4(s') \not\leq \Box_2 D_2$. But this is contradicted by the announcement q', by which child 2 says he knows he is dirty. In other words, the appearance of the state s'' to child 4 is incompatible with action q' happening next:

$$f_4(s'') \otimes q' = \bot\,.$$

Thus, after hearing the announcement q', child 4 must engage in belief revision, otherwise he will be lead to have inconsistent beliefs. This is precisely where the epistemic updates of [4,8] fail: according to the these settings we would have

$$f_4(s'' \otimes q') = f_4(s'') \otimes f_4(q') = f_4(s'') \otimes q' = \bot\,,$$

and hence

$$s'' \otimes q' \leq \Box_4 \bot\,,$$

which says at the next state, child 4 will believe the impossible. However, as will be shown in the following proposition, in our dynamic revision setting this is not the case:

Proposition 4.2 *In the dynamic belief revision setting, child 4 successfully revises his beliefs and will not believe in the impossible:*

$$s'' \otimes q' \not\leq \Box_4 \bot$$

Proof. As before, we use the adjunction to take the $\Box_4$ to the left hand side to get

$$f_4(s'' \otimes q') \not\leq \bot\,.$$

Now observe that

$$f_4(s'' \otimes q') = f_4(s'') * f_4(q') = f_4(s'') * q' \neq \bot$$

since $q' \neq \bot$ and we are done. $\Box$

So child 4 has succesfuly revised his beliefs; in fact, our revision axioms imply that he can also form a new hypothesis $m'' \not\leq ker(q')$ about the previous state s'', hypothesis that can explain his new beliefs:

$$f_4(s'') * q' = m'' \otimes q'\,.$$

5 Conclusion and Future Work

We have presented an algebraic setting for dynamic belief revision in multi-agent systems. Our main mathematical object is a Belief Revision System based on Epistemic Systems as algebraic models of Dynamic Epistemic Logic. We axiomatize a notion of multi-agent dynamic revision, which generalizes the update product to inconsistent pairs of a theory and an experiment, and which revises the theory to accommodates the experiment. In our setting, agents can revise with complex epistemic propositions as well as with facts. They can also revise past actions in the view of new experiments. We have applied our setting to a cheating version of the muddy children puzzle, and show that by using this logic, after the cheating the honest children would not face any dangerous consequences, in terms of contradiction and confusion.

We are currently working on a complete sequent calculus for this logic, adapting the work in [8,9] to the present setting. Also, for simplicity we have chosen here to follow the AGM approach in postulating a uniform revision rule: "the rational" revision operator. But there exists of course the possibility of having "personalised revision" operators: by introducing labelled revision operators $*_A$ for each agent, we can allow different agents to use different revision rules, subject only to minimal rationality constraints. We are planning to investigate this possibility in future work.

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