Learning in a changing world via algebraic modal logic

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Abstract

We develop an algebraic modal logic that combines epistemic modalities with dynamic modalities with a view to modelling information acquisition (learning) by automated agents in a changing world. Unlike most treatments of dynamic epistemic logic, we have transitions that "change the state" of the underlying system and not just the state of knowledge of the agents. The key novel feature that emerges is the need to have a way of "inverting transitions" and distinguishing between transitions that "really happen" and transitions that are possible.

Our approach is algebraic, rather than being based on a Kripke-style semantics. The semantics are given in terms of quantales. We study a class of quantales with the appropriate inverse operations and prove soundness and completeness theorems. We illustrate the ideas with a simple game as well as a toy robot-navigation problem. The examples illustrate how an agent discovers information by taking actions.

1 Introduction

Epistemic logic has proved very important in the analysis of protocols in distributed systems (see, for example, [FHMV95]) and, more generally in any situation where there is some notion of cooperation or "agreement" between agents. The original work in distributed systems, by Halpern and Moses [HM84, HM90] and several

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others modelled the knowledge of agents using Kripke-style [Kri63] models. In these models there are a set of states (often called "possible worlds") in which the agent could be and, for each agent, an equivalence relation on the states. If two states are equivalent to an agent then that agent cannot "tell them apart". An agent "knows" a fact ϕ in the state s if, in all states t that the agent "thinks" is equivalent to s, the fact ϕ holds. The quoted words in the preceding sentences are, of course, unnecessary anthropomorphisms that are intended to give an intuition for the definitions.

A vital part of any analysis is how processes "learn" as they participate in the protocol. The bulk of papers in the distributed systems community treat this as a change in the Kripke equivalence relations and argue about these changes *only in the semantics*. The logic itself does not have the "dynamic" modalities that refer to updating of the state of knowledge. On the other hand, dynamic epistemic logic has indeed been studied; see, for example the recent book [vDvdHK08]. In the second author's doctoral dissertation an algebraic approach to dynamic epistemic logic was studied in depth [Sad06]. However, the bulk of the work concerns situations where the *state of knowledge* is changed by messages or broadcasts but not situations where the *state of the system* is changed. An example discovered by Caitlin Phillips [Phi09] showed that the usual formalisms can break down when this happens.

A deeper investigation of this example reveals that it is not a question of "patching up" the theory. There are some interesting fundamental changes that need to be made. First of all, one has to distinguish between transitions that exist in the agent's "mental model" of the system and actions that *actually occur*. Second, one has to introduce a converse dynamic modality in order correctly to formulate the axioms for updating knowledge. This paper presents the theory with these features.

The general feature of robot navigation protocols in AI is that an agent is given a description of a place, but cannot figure out where it is, but it can move to change its location and as a result acquires information and realizes where it is. Consider given to a robot, the map of a small computing laboratory with 5 rooms as follows

$$
\begin{tikzcd}
s_2 \arrow[r, "b"] \arrow[d, "a"'] & s_3 \arrow[r, "c"] & s_5 \\
s_1 \arrow[ur, "b"] \arrow[r, "a"'] & s_4 \arrow[ur, "c"] &
\end{tikzcd}
$$

Since the robot can do the same actions in the pairs s_1, s_2 and s_3, s_4, it cannot tell them apart, once in one of them thinks it is possible that it is in the other.

Location	Actions Available	Uncertainty
s_1, s_2	a, b	s_1, s_2
s_3, s_4	c	s_3, s_4

But if while in s_1 it performs an a action, then he reaches s_4, now it learns where it is. This property cannot be proved in [BCS07], in fact the main inequality there fails for this simple structure. The inequality under question requires that the uncertainty of a location after one has done an action on it to be included in the result of applying the action to the uncertainty of the location before the action, a property similar to *perfect recall* in protocol models of [HM84, HM90]. This fails here, since uncertainty of s_1 after performing an a is the same as uncertainty of s_4, consisting of s_3 and s_4. But performing a on uncertainty of s_1, consisting of s_1 and s_2, results in both s_4 and s_1, now observe that $\{s_4, s_1\}$ is not included in $\{s_3, s_4\}$. Moreover, after the robot moved to s_4, he can conclude that he *was* in s_1 before moving; the language of [BCS07] can simply not express these *past tense* properties.

Let us reason as the robot should, when he reaches s_4, he checks with its map and reasons that the only way it could have reached s_4 would be that it was originally in s_1. It rules out s_3 from its uncertainty set about s_4, because, according to the map, it could not have reached s_3 via an a action. We have two types of data here, the locations and actions described on the map and the ones in reality. The data on the map is hard coded in the robot and there is no uncertainty about it, the map fully describes the system. But the real locations and actions are partial, the robot is uncertain about locations and the actions he takes change his uncertainties. To be able to encode what actions could have led the robot to where it is, it needs to look back, so we need a converse operation. Now by moving from s_1 to s_4, the robot has removed its uncertainty, has acquired information, and has learned where it is located. This is exactly the manner in which our replacement for the main inequality of [BCS07] works. Our new *uncertainty reduction* inequality formalizes the elimination of past uncertainties: after performing a certain move in the real world, the robot consults its description, considers its possibilities and eliminates the ones that could not have been reached as a result of the action it just performed. Furthermore and with this converse operation, we can also derive information about past, that robot was in s_1 before doing action a.

We aim to formalize information acquisition from such navigation protocols, with applications to AI, mobile communication, and perhaps control theory. Of course, another natural situation where such ideas could be applied is the analysis of security protocols. We have not pursued these issues in this paper only because they are too interesting to be dismissed as an example, but deserve a thorough investigation in their own right.

We provide a modal algebraic setting to model to reason about such scenarios. The central algebraic notion of use here is that of a *second order converse di-System*. These are our extensions of the systems of [AV93], in that the quantale acts on the module via two actions that have Galois right adjoints and that are converse to one another. The corresponding logic is the positive fragment of Propositional Dynamic Logic (PDL) without test with converse; the Galois right adjoints correspond to dynamic modalities. The propositional part of the logic is the positive fragment of Prior's Tense logic. To this we add lax endomorphisms that encode uncertainties about locations and whose Galois right adjoints encode information (or belief). The first order actions of the quantale on the module encode the static description of a location, e.g. a map. The 2nd order actions of a di-System encodes real location changing actions, which change the uncertainties and as a result more information is acquired. This change of uncertainty is encoded in an *uncertainty reduction* axiom that relates the static and real actions to uncertainties. The logic of the setting with uncertainties would be a positive Dynamic Epistemic Logic with converse. These algebraic di-Systems, the second-order versions, and their lax morphisms are novel mathematical objects. The logic corresponding to the algebra is new as far as we know.

The main technical results are (1) completeness of the propositional part of the logic. i.e. Positive Tense Logic, with regard to Positive Tense Algebras and ordered Kripke frames, (2) a representation theorem for the version of the algebra used for grid applications, (3) completeness of positive PDL with converse without test with regard to converse di-Systems, and (4) that the algebraic setting of previous work [BCS07] on information acquisition from communication, embeds in our second-order converse di-Systems with lax endomorphisms. It would be interesting to relate our logic to axioms and rules of PDL with converse from [HKT00] and develop a logic for the positive Epistemic PDL with converse and its completeness theorem. For the latter we need a logical form for the *uncertainty reduction* axiom.

2 The Algebra of di-Systems

We need to model "actions" and "formulas". The actions are modelled by a quantale while the propositions are a module over the quantale; in other words the actions modify the propositions.

Definition 2.1 *A **quantale** $(Q, \bigvee, \bullet, 1)$ is a sup-lattice equipped with a unital monoid structure satisfying:*

$$q \bullet \bigvee_i q_i = \bigvee_i (q \bullet q_i) \qquad \bigvee_i q_i \bullet q = \bigvee_i (q_i \bullet q)$$

Instead of an arbitrary sup-lattice we will take it to be a completely distributive prime-algebraic lattice.

Recall that a prime element, or simply "prime", p in lattice has the property that for any x, y in the lattice, $p \leq x \vee y$ implies that $p \leq x$ or $p \leq y$; "prime algebraic" means that every element is the supremum of the primes below it. The restriction to prime algebraic lattices is not a serious restriction for the logical applications that we are considering; it would be a restriction for extensions to probabilistic systems; we will address such issues in future work. The use of algebraicity is to be able to use simple set-theoretic arguments via the representation theorem for such lattices [Win09]. For finite distributive lattices it is not a restriction at all because of Birkhoff's classical representation theorem. Henceforth, we will not explicitly state that we are working with (completely) distributive prime-algebraic lattices.

Our actions need to modify propositions.

Definition 2.2 *A **right-module** over Q is a sup-lattice M with an action of Q on M, $- \cdot -\colon M \times Q \longrightarrow M$ satisfying*

$$(m{\cdot}q){\cdot}q' = m{\cdot}(q{\bullet}q') \qquad m{\cdot}\bigvee_i q_i = \bigvee_i (m{\cdot}q_i) \qquad \bigvee_i m_i{\cdot}q = \bigvee_i (m_i{\cdot}q) \qquad m{\cdot}1 = m$$

We call the collection of actions and propositions a *system*. We do not have an explicit notion of "state".

Definition 2.3 *A **system** is a pair consisting of a quantale Q and a right-module M over Q. We write $(M, Q, \cdot)$ for a system.*

This is closely related to a definition of Abramsky and Vickers who also studied quantales of actions, see [AV93].

Since the action preserves all the joins of its module, the map $- \cdot q\colon M \longrightarrow M$, obtained by fixing the module argument, has a Galois right adjoint that preserves the meets. This is denoted by $- \cdot q \dashv [q]-$ and is canonically defined as

$$[q]m := \bigvee \left\{ m' \in M \mid m' \cdot q \leq m \right\}$$

Proposition 2.4 *The following inequalities hold in any system* $(M, Q, \cdot)$*:*

(1)	$([q]m) \cdot q \leq m$	$m \leq [q](m \cdot q)$
(2)	$(m \wedge m') \cdot q \leq m \cdot q \wedge m' \cdot q$	$m \cdot (q \wedge q') \leq m \cdot q \wedge m \cdot q'$
(3)	$[q](m \vee m') \geq [q]m \vee [q]m'$	
(4)	$q \leq q' \implies [q']m \leq [q]m$	$[\bot]m = \top$
(5)	$[q \vee q']m = [q]m \wedge [q']m$	$[q \vee q']m \leq [q]m \vee [q']m$
(6)	$[q \wedge q']m \geq [q]m \vee [q']m$	$[q \wedge q']m \geq [q]m \wedge [q']m$

Proof: The first two are immediate consequnces of the definition of $[q]$ as a right adjoint. The second and the third easily follow from monotonicity. For the first item of (4), we note that $m' \cdot q' \leq m$ implies that $m' \cdot q \leq m' \cdot q' \leq m$ so $\{m' \mid m' \cdot q'\} \subseteq \{m'' \mid m'' \cdot q \leq m\}$ so clearly

$$\bigvee \{m' \mid m' \cdot q'\} \leq \bigvee \{m'' \mid m'' \cdot q \leq m\}.$$

For the second item of (4), the direction $[\bot]m \leq \top$ is trivial, the other direction $\top \leq [\bot]m$ is equivalent to $\top \cdot \bot \leq m$, which holds since $\top \cdot \bot = \bot$. For the first item of (5), the $\leq$ direction follows from (4) and definition of meet, for the $\geq$ direction we have to show $[q]m \wedge [q']m \leq [q \vee q']m$, which is by adjunction equivalent to $([q]m \wedge [q']m) \cdot (q \vee q') \leq m$. By join preservation of action, this is equivalent to $([q]m \wedge [q']m) \cdot q \vee ([q]m \wedge [q']m) \cdot q' \leq m$. To show this, we have to show that both disjuncts are less than or equal to m. Consider the first one, by the first item of (2) and transitivity, it suffices to show $[q]m \cdot q \wedge [q']m \cdot q \leq m$, by the definition of meet and transitivity it suffices to show either of the conjuncts satisfy the inequality, now $[q]m \cdot q \leq m$, is true by the first item of (1). The proofs of the remaining items follow easily from (4) and definitions of meet and join. ∎

Corollary 2.5 *In any system* $(M, Q, \cdot)$ *we have that* $[\bigvee_i q_i]m = \bigwedge_i [q_i]m$.

Proof: This follows immediately from the second item of (4) and the first item of (5) in Prop. 2.4. ∎

Definition 2.6 *A sup lattice* M *is a right di-module of the quantale* Q *whenever there are two right actions* $- \cdot -\colon M \times Q \longrightarrow Q$ *and* $- \times -\colon M \times Q \longrightarrow Q$*. We call the pair of a quantale and its di-module* $(M, Q, \cdot, \times)$ *a* ***di-System****.*

Among di-Systems, we focus on the ones in which the two actions are converse to each other; these have nice properties and are the ones that we shall use for the applications. Accordingly, we spend some time discussing their properties.

Definition 2.7 *If the two actions – written* $\cdot and \cdot^{-1}$ *for the purposes of this defini-*

tion – of a di-system are related as follows: for $m, m' \in M, q \in Q$

$$
\begin{array}{rccc}
(i) & m \cdot q \leq m' & \implies & m \leq m' \cdot^{-1} q, \\
(ii) & m \cdot^{-1} q \leq m' & \implies & m \leq m' \cdot q, \\
(iii) & m \cdot^{-1} (q \bullet q') & = & (m \cdot^{-1} q') \cdot^{-1} q,
\end{array}
$$

*then we refer to the di-system as a **converse di-System** and denote it by* $(M, Q, \cdot, \cdot^{-1})$.

Lemma 2.8 *In a converse di-System* $(M, Q, \cdot, \cdot^{-1})$ *we have*

$$m \leq (m \cdot q) \cdot^{-1} q \qquad\qquad m \leq (m \cdot^{-1} q) \cdot q$$

Definition 2.9 *A converse di-System is **past-deterministic** whenever* $m \leq m' \cdot q \implies m \cdot^{-1} q \leq m'$. *A converse di-System is **future-deterministic** whenever* $m \leq m' \cdot^{-1} q \implies m \cdot q \leq m'$.

Lemma 2.10 *In a past-deterministic converse di-System we have* $m \leq m' \cdot q \iff m \cdot^{-1} q \leq m'$ *and in a future-deterministic converse di-System we have* $m \leq m' \cdot^{-1} q \iff m \cdot q \leq m'$.

The converse action preserves all the joins of the module, thus similar to the action, it has a Galois right adjoint denoted by $- \cdot^{-1} q \dashv [q]^{-1}-$, canonically defined using the converse action,

$$[q]^{-1}m := \bigvee \{m' \in M \mid m' \cdot^{-1} q \leq m\}$$

Proposition 2.11 *In any converse di-System we have*

$$[q](l \vee l') \leq [q]l \ \vee \ l' \cdot^{-1} q \qquad\qquad [q]^{-1}(l \vee l') \leq [q]^{-1}l \ \vee \ l' \cdot q$$

Proof: Since we have a prime-algebraic lattice, every element is the sup of the primes below it. Consider the first inequality, we take an arbitrary prime $l'' \in M$ below $\leq [q](l \vee l')$ and show $l'' \leq [q]l \vee l' \cdot^{-1} q$. From this assumption by adjunction and definition of join we obtain $l'' \cdot q \leq l$ or $l'' \cdot q \leq l'$ because l'' is a prime. The former is equivalent to $l'' \leq [q]l$ by the definition of adjunction, the latter implies $l'' \leq l \cdot^{-1} q$ by part (i) of definition 2.7. Thus we have that $[q]l \vee l' \cdot^{-1} q$ is an upper bound for the primes below $[q](l \vee l')$, since $[q](l \vee l')$ is the least upper bound of such primes we have the desiered inequality.

Consider the second inequality and take a prime $l'' \leq [q]^{-1}(l \vee l')$, from this, by adjunction and the definition of join, we obtain $l'' \cdot^{-1} q \leq l$ or $l'' \cdot^{-1} q \leq l'$, again by the fact that l'' is a prime. By the properties of an adjunction, the former is equivalent to $l'' \leq [q]^{-1}l$, the latter implies $l'' \leq l' \cdot q$ by part (ii) of definition 2.7 and we can complete the argument as at the end of the previous paragraph. ∎

Proposition 2.12 *In a future-deterministic converse di-System we have*

$$l \cdot^{-1} q \ \wedge \ [q]l' \leq (l \wedge l') \cdot^{-1} q$$

Proof: We take an arbitrary prime $l'' \in M$ to be $\leq l \cdot^{-1} q \ \wedge \ [q]l'$ and show $l'' \leq (l \wedge l') \cdot^{-1} q$. That is, $l'' \leq l \cdot^{-1} q$ and $l'' \leq [q]l'$. The first one implies $l'' \cdot q \leq l$ by the future-deterministic condition and the latter is equivalent to $l'' \cdot q \leq l'$ by adjunction. Thus we have $l'' \cdot q \leq l \wedge l'$, which implies $l'' \leq (l \wedge l') \cdot^{-1} q$ by part (i) of Definition 2.7. ■

Proposition 2.13 *In a past-deterministic converse di-System we have*

$$l \cdot q \ \wedge \ [q]^{-1}l' \leq (l \wedge l') \cdot q$$

Proof: We take an arbitrary prime $l'' \in M$ to be $\leq l \cdot q \ \wedge \ [q]^{-1}l'$ and show $l'' \leq (l \wedge l') \cdot q$. That is $l'' \leq l \cdot q$ and $l'' \leq [q]^{-1}l'$. The former implies $l'' \cdot^{-1} q \leq l$ by the past-deterministic condition, the latter is equivalent to $l'' \cdot^{-1} q \leq l'$ by adjunction. Thus we have $l'' \cdot^{-1} q \leq (l \wedge l')$, which implies $l'' \leq (l \wedge l') \cdot q$ by part (ii) of definition 2.7. ■

Example 2.14. Consider the following transition system

$$\begin{tikzcd} & x \arrow[dl, "a"'] \arrow[dr, "a"] & & y \arrow[dl, "a"] \\ z_1 & & z_2 & \end{tikzcd}$$

We model this in a di-System $(M, Q, \cdot, \cdot^{-1})$ by assuming $x, y, z_1, z_2 \in M, a \in Q$, and $x \cdot a = z_1 \vee z_2, y \cdot a = z_2, z_1 \cdot^{-1} a = x$, and $z_2 \cdot^{-1} a = x \vee y$. It is easy to check that these satisfy the inequalities of definition 2.7, but not their converses: the transition system is neither past-deterministic nor future-deterministic. A counterexample for the converse of part (i) is $x \leq z_2 \cdot^{-1} a$ but $x \cdot a \not\leq z_2$. If we eliminate the leftmost edge, then the system becomes future-deterministic and the converse of (i) holds. A counterexample for the converse of part (ii) is $z_2 \leq y \cdot a$ but $z_2 \cdot^{-1} a \not\leq y$. If we eliminate the rightmost edge, then the system becomes past-deterministic and the converse of (i) holds.

Example 2.15. The set of subsets $\mathcal{P}(S)$ of a set S is the right di-module of the quantale of all the relations thereon $\mathcal{P}(S \times S)$, where relational composition is the monoid multiplication and the diagonal relation is its unit, and the join is set union. The action and its converse are the pointwise images of the relation and its converse, defined as follows for $W \subseteq S$ and $R \subseteq S \times S$

$W \cdot R := \{s \in S \mid w \in W, (w, s) \in R\}$ and $W \cdot^{-1} R := \{s \in S \mid w \in W, (s, w) \in R\}$

In other words

$$W \cdot R = \bigcup_{w \in W} R[w] \qquad\qquad W \cdot^{-1} R = \bigcup_{w \in W} R^{-1}[w]$$

It is easy to see that $W \cdot^{-1} R = W \cdot R^{-1}$. If R^{-1} is a singleton then this di-system becomes a past-deterministic one, and if R is a singleton, then it becomes a future-deterministic one.

Example 2.16. Consider a division ring $(R, 0, +, -, 1, \cdot)$, i.e. a ring where $\forall r \neq 0, \exists r', r \cdot r' = r' \cdot r = 1$, denote the r' corresponding to r by r^{-1}. An example of a right module of a ring is the ring itself with the action $R \times R \longrightarrow R$ being the ring multiplication $r \cdot r$, see [Lam86]. One can define a converse operation for this example as $r \cdot^{-1} r' := r \cdot (r'^{-1})$ [1]. A partial order is defined using the ring addition as $r \leq r'$ iff $r + r' = r'$. It is easy to verify that these satisfy conditions of definitions 2.7 and 2.9. Consider the conditions for a past-deterministic system, here is the calculation for the first condition of 2.7

$$r \cdot r' \leq r'' \iff r \cdot r' + r'' = r'' \implies (r \cdot r' + r'') \cdot r'^{-1} = r'' \cdot r'^{-1} \implies$$
$$r \cdot r' \cdot r'^{-1} + r'' \cdot r'^{-1} = r'' \cdot r'^{-1} \implies r + r'' \cdot r'^{-1} = r'' \cdot r'^{-1} \iff r \leq r'' \cdot r'^{-1}$$

Here is the calculation for first condition of 2.9

$$r \leq r' \cdot r''^{-1} \iff r + r' \cdot r''^{-1} = r' \cdot r''^{-1} \implies (r + r' \cdot r''^{-1}) \cdot r'' = r' \cdot r''^{-1} \cdot r'' \implies$$
$$r \cdot r'' + r' \cdot r''^{-1} \cdot r'' = r' \implies r \cdot r'' + r' = r' \iff r \cdot r'' \leq r'$$

Example 2.17. In the same system as above, a second action, which is not a converse action but is defined based on the converse action can be defined as follows

$$W \times R = \begin{cases} \emptyset & W \cdot^{-1} R = \emptyset \\ W & o.w. \end{cases}$$

To see that this is indeed an action we check the axioms.

- To show that the action preserves joins of the module, we have to show that $(W \cup T) \times R = (W \times R) \cup (T \cup R)$ and $\emptyset \times R = \emptyset$. The latter is obvious since $\emptyset \cdot^{-1} R = \emptyset$. For the former start from $(W \cup T) \times R$, this is $\emptyset$ iff $(W \cup T) \times R = (W \cup T) \cdot^{-1} R = (W \cdot^{-1} R) \cup (T \cdot^{-1} R) = \emptyset$, which is the case iff $W \cdot^{-1} R = T \cdot^{-1} R = \emptyset$, which are the conditions that make $(W \times R) \cup (T \times R)$ equal to emptyset. The non-empty case is $W \cup T$ for both sides.

[1] This converse operation cannot be seen as an action, since it is partial, being only defined for $r' \neq 0$. Our goal here is mainly to demonstrate a case where the converse axioms hold.

- That the action preserves joins of the quantale is shown similarly to above.
- To show that the action preserves the identity relation $\triangle_S$, we have to show $W \times \triangle_S = W$. This is shown by observing that $W \cdot^{-1} \triangle_S$ is always equal to W and will never get equal be to $\emptyset$.
- To show associativity of the action wrt to the quantale multiplication we have to show $(W \times R) \times R' = W \times (R \bullet R')$. Left hand side is the $\emptyset$ whenever either $R' = \emptyset$, or $W \cdot^{-1} R = \emptyset$, or $W \cdot^{-1} R \cdot^{-1} R' = \emptyset$. The right hand side is equal to $\emptyset$ whenever we have $W \times (R \bullet R') = W \cdot^{-1} R \cdot^{-1} R' = \emptyset$, which happens exactly when either of the same conditions as above are satisfied. The non-empty case will be equal to W for both sides, when the conditions for emptyness are not satisfied.

2.1 Iteration

On the quantale of a converse di-System, we define $q^* = \bigvee_{i=0}^{n} q^i$, it is known that this satisfies the axioms of the Kleene-star operation [HKT00], i.e.

$$\textit{computation} : 1 \vee (q^* \bullet q) = 1 \vee (q \bullet q^*)$$

and

$$\textit{induction} : q \bullet q' \leq q' \implies q^* \bullet q' \leq q'$$

and

$$q' \bullet q \leq q' \implies q' \bullet q^* \leq q'.$$

It is also known that the Kleene star is the least fixed point of the function $id \vee (q \bullet -)$, similarly that its meet dual $q^\flat = q \wedge q \bullet q \wedge q \bullet q \bullet q \wedge \cdots$, is the greatest fixed point of $id \wedge (q \bullet -)$, e.g. see [vK98]. In the setting of *sets of observations*, fully studied in *Temporal Algebras* of [vK98], these are used to recover the fixed-point operators of LTL and CTL, and it is shown how Pnueli's 17 axioms of the proof system of LTL are recovered from them. Here we can use the Kleene star operation to define a star operation for the action and its adjoint as follows.

Definition 2.18 *Given a System $(M, Q, \cdot)$ we define $- \cdot^* q, [q]^\flat -\colon M \longrightarrow M$ as follows*

$$m \cdot^* q := m \vee m \cdot q \vee m \cdot (q \bullet q) \vee \cdots \qquad [q]^\flat m := m \wedge [q]m \wedge [q \bullet q]m \wedge \cdots$$

Corollary 2.19 *In a System $(M, Q, \cdot)$ the star operator of definition 2.18 satisfies the following*

$$m \cdot^* q = m \cdot q^* \qquad\qquad m \cdot^* q = m \vee m \cdot q \vee m \cdot q \cdot q \vee \cdots$$
$$[q]^\flat m = [q^*]m \qquad\qquad [q]^\flat m = m \wedge [q]m \wedge [q][q]m \wedge \cdots$$

Proof: First one follows from associativity of action over the quantale multiplication. Second one follows from corollary 2.5 and associativity of action over the quantale multiplication. ∎

Proposition 2.20 *In a converse di-System $(M, Q, \cdot, \cdot^{-1})$, we have the following adjunctions*

$$- \cdot^* q \dashv [q]^\flat - \qquad\qquad - \cdot^{-1*} q \dashv [q]^{-1\flat} -$$

Proof: Consider the first one, we have to show $m \cdot^* q \leq m' \iff m \leq [q]^\flat m'$. For the right direction, assuming $m \cdot^* q \leq m'$, we show $m \leq [q]^\flat m'$, i.e. that $m \leq m' \wedge [q]m' \wedge [q \bullet q]m' \wedge \cdots$, by definition of meet it suffices to show that $m \leq m'$ and $m \leq [q]m'$ and $m \leq [q \bullet q]m'$ and $\cdots$, by the adjunction $- \cdot q \dashv [q]-$, this is equivalent to showing that $m \leq m'$ and $m \cdot q \leq m'$ and $m \cdot (q \bullet q) \leq m'$ and $\cdots$, by the module associativity axiom this is equivalent to $m \leq m'$ and $m \cdot q \leq m'$ and $m \cdot q \cdot q \leq m'$ and $\cdots$, which is, by definition of join being the least upper bound, implied by our assumption that $m \cdot^* q = m \vee m \cdot q \vee m \cdot q \cdot q \vee \cdots \leq m'$. The proofs of the left direction and the second adjunction are similar. ∎

2.2 Boolean Modules

From propositions 2.12 and 2.13 it follows that in a past-deterministic and future converse di-system the following four inequalities hold

$$\begin{array}{llcl}
(1) & [q](l \vee l') & \leq & [q]l \;\vee\; l' \cdot^{-1} q \\
(2) & [q]^{-1}(l \vee l') & \leq & [q]^{-1}l \;\vee\; l' \cdot q \\
(3) & l \cdot^{-1} q \;\wedge\; [q]l' & \leq & (l \wedge l') \cdot^{-1} q \\
(4) & l \cdot q \;\wedge\; [q]^{-1}l' & \leq & (l \wedge l') \cdot q
\end{array}$$

These four axioms are meant to axiomatize de Morgan duality in the absence of negation; they are special cases (product,co-product) of their more general peers (tensor, par) in [BCS02], referred to as *linear strength*. A version of these can also be found in [Dun05], referred to as *interaction axioms*, to show the de Morgan duality of box and diamond of positive classical modal logic. The same result holds in our setting.

Proposition 2.21 *If the module of a past-deterministic and future converse di-System is an atomic Boolean algebra with negation operator $\neg -: M \longrightarrow M$, we have $l \cdot q = \neg[q]^{-1}\neg l$ and $l \cdot^{-1} q = \neg[q]\neg l$.*

Proof: Consider the first equation, we unfold the definitions of $l \cdot q$ as the left adjoint to $[q]-$ and the definition of $[q]^{-1}\neg l$ as the right adjoint to $- \cdot^{-1} q$, hence equivalently show the following

$$\bigwedge\{l' \in M \mid l \leq [q]l'\} \quad = \quad \neg \bigvee\{l' \in M \mid l' \cdot^{-1} q \leq \neg l\}$$

Boolean negation turns joins to meets and negates the argument, hence the rhs is equivalent to $\bigwedge\{l' \in M \mid l' \cdot^{-1} q \not\leq \neg l\} = \bigwedge\{l' \in M \mid l' \cdot^{-1} q \leq \neg\neg l\}$. This equal to lhs, since Boolean negation is involutive and by adjunction $l \leq [q]l'$ is equivalent to $l \cdot q \leq l'$, which by definitions 2.7 and 2.9 is equivalent to $l' \cdot^{-1} q \leq l$. Here we are using the part of the definition based on the fact that the di-System is future-deterministic. The proof of the second equation above is similar and uses the fact that the di-System is past-deterministic. ∎

Corollary 2.22 *If the module of a past-deterministic and future converse di-System is a Boolean algebra with negation operator* $\neg - \colon M \longrightarrow M$ *we have* $m \cdot^{*} q = \neg[q]^{-1\flat}\neg m \qquad m \cdot^{-1*} q = \neg[q]^{\flat}\neg m$.

Proof: C onsider the first equation, after unfolding the definitions we have to show

$$m \vee m \cdot q \vee m \cdot q \cdot q \vee \cdots = \neg(\neg m \wedge [q]^{-1}\neg m \wedge [q]^{-1}[q]^{-1}\neg m \wedge \cdots)$$

By de Morgan duality between meet and join the rhs is equal to $m \vee \neg[q]^{-1}\neg m \vee \neg[q]^{-1}[q]^{-1}\neg m \vee \cdots$, by equation one of proposition 2.21 this is equal to $m \vee m \cdot q \vee m \cdot q \cdot q \vee \cdots$, equal to the lhs. ∎

As an amusing aside – and to demonstrate the nice accord between adjoint and de Morgan dual maps – we leave it to the reader to verify that Boolean negation can be axiomatized via the adjunction $m \wedge - \dashv \neg m \vee -$, leading to the *Dedekind law*, spelled out below; the proofs are exactly the same as the ones in [vK98].

Proposition 2.23 *If the module of a past-deterministic and future converse di-System is a Boolean algebra with negation operator* $\neg - \colon M \longrightarrow M$*, then we have* $(m \cdot q) \wedge m' \leq (m \wedge (m' \cdot^{-1} q)) \cdot q$.

Perhaps one may find instances of the Boolean case via the examples of a *converse Kleene Algebra*, e.g. developed in [DMS06]. The rough idea here is that a Kleene Algebra is an idempotent semi-ring with a Kleene star operation and complete Kleene Algebras, referred to as $*$-*continuos*, are examples of quantales. If a KA has tests, they can be extracted as propositions, hence forming a Boolean module. So complete Kleene Algebras with converse and test should be examples of Boolean converse di-systems.

3 Logic: Completeness and Representation Theorems

The logic corresponding to the algebra of di-Systems is positive Propositional Dynamic Logic with converse, without tests. Its propositional part is new and consists of positive Tense Logic. PDL with converse [HKT00] and Tense Logic [Pri68] have been studied before. The absence of negation is novel and makes a considerable difference: one needs new axioms, rules, models, and completeness and representation theorems. The proves of some of these becomes harder in the absence of negation. In this section we first develop such a theory for positive tense logic, then move on to prove completeness of positive PDL with converse with regard to the algebra of converse di-Systems.

3.1 Positive Tense Logic

Positive tense logic (PTL) is generated over a set of atoms $p \in P$ via the following grammar:

$$m ::= \bot \mid \top \mid p \mid m \wedge m \mid m \vee m \mid \Box m \mid \Diamond m \mid \Box^{-1} m \mid \Diamond^{-1} m$$

The axioms and rules of the logic are obtained from those of a distributive lattice from [GNV05], of adjunction from [SD09], and two new axioms for converse modalities. Since the language does not include implication, the sequents are of the form $m \vdash m'$ for $m, m' \in M$, as defined in [GNV05, Dun05] .

Axioms.

$$m \vdash m, \qquad \bot \vdash m, \qquad m \vdash \top$$

$$m \wedge (m' \vee m'') \vdash (m \wedge m') \vee (m \wedge m'')$$

$$m \vdash m \vee m', \qquad m' \vdash m \vee m', \qquad m \wedge m' \vdash m, \qquad m \wedge m' \vdash m'$$

$$\Diamond(m \vee m') \vdash \Diamond m \vee \Diamond m', \qquad \Diamond^{-1}(m \vee m') \vdash \Diamond^{-1} m \vee \Diamond^{-1} m'$$

$$\Box m \wedge \Box m' \vdash \Box(m \wedge m'), \qquad \Box^{-1} m \wedge \Box^{-1} m' \vdash \Box^{-1}(m \wedge m')$$

$$\top \vdash \Box^{-1}\top, \qquad \top \vdash \Box\top, \qquad \Diamond\bot \vdash \bot, \qquad \Diamond^{-1}\bot \vdash \bot$$

Adjunction	$\Diamond^{-1}\Box m \vdash m, \qquad m \vdash \Box\Diamond^{-1} m, \qquad \Diamond\Box^{-1} m \vdash m, \qquad m \vdash \Box^{-1}\Diamond m$
Converse	$m \vdash \Diamond\Diamond^{-1} m, \qquad m \vdash \Diamond^{-1}\Diamond m$

Rules.

$$\frac{m \vdash m' \quad m' \vdash m''}{m \vdash m''}\ cut$$

$$\frac{m \vdash m'' \quad m' \vdash m''}{m \vee m' \vdash m''}\ \vee \qquad \frac{m \vdash m' \quad m \vdash m''}{m \vdash m' \wedge m''}\ \wedge$$

$$\frac{m \vdash m'}{\Diamond m \vdash \Diamond m'}\ \Diamond \qquad \frac{m \vdash m'}{\Diamond^{-1} m \vdash \Diamond^{-1} m'}\ \Diamond^{-1} \qquad \frac{m \vdash m'}{\Box m \vdash \Box m'}\ \Box \qquad \frac{m \vdash m'}{\Box^{-1} m \vdash \Box^{-1} m'}\ \Box^{-1}$$

Definition 3.1 *A Positive Tense Algebra (PTA)*[2] *is a bounded distributive lattice* $(L, \top, \bot)$ *with four order-preserving maps* $\Diamond, \Diamond^{-1}, \Box, \Box^{-1} \colon L \longrightarrow L$*, where* $\Diamond^{-1}$ *is left adjoint to* $\Box$ *and* $\Diamond$ *is left adjoint to* $\Box^{-1}$*, moreover* $\Diamond$ *and* $\Diamond^{-1}$ *are converses, i.e. for all* $l, l' \in L$ *the following hold:*

Monotonicity	$l \le l'$	$\Longrightarrow$	$\Diamond l \le \Diamond l'$	$l \le l'$	$\Longrightarrow$	$\Diamond^{-1} l \le \Diamond^{-1} l'$
Monotonicity	$l \le l'$	$\Longrightarrow$	$\Box l \le \Box l'$	$l \le l'$	$\Longrightarrow$	$\Box^{-1} l \le \Box^{-1} l'$
Adjunction	$\Diamond l \le l'$	$\Longleftrightarrow$	$l \le \Box^{-1} l'$	$\Diamond^{-1} l \le l'$	$\Longleftrightarrow$	$l \le \Box l'$
Converse	$\Diamond l \le l'$	$\Longrightarrow$	$l \le \Diamond^{-1} l'$	$\Diamond^{-1} l \le l'$	$\Longrightarrow$	$l \le \Diamond l'$

Proposition 3.2 *In any PTA the following inequalities hold:*

[2] Perhaps it is worth noting that a PTA is the bounded distributive lattice version of *Classical Galois algebras* of [vK98]. These are Boolean Algebras with a pair of adjoint modalities and are used to develop algebraic semantics for temporal logic.

$$
\begin{array}{lll}
(1) & \Diamond(l \vee l') = \Diamond l \vee \Diamond l' & \Diamond^{-1}(l \vee l') = \Diamond^{-1} l \vee \Diamond^{-1} l' \\
(2) & \Box(l \wedge l') = \Box l \wedge \Box l' & \Box^{-1}(l \wedge l') = \Box^{-1} l \wedge \Box^{-1} l' \\
(3) & \Diamond(l \wedge l') \leq \Diamond l \wedge \Diamond l' & \Diamond^{-1}(l \wedge l') \leq \Diamond^{-1} l \wedge \Diamond^{-1} l' \\
(4) & \Box l \vee \Box l' \leq \Box(l \vee l') & \Box^{-1} l \vee \Box^{-1} l' \leq \Box^{-1}(l \vee l') \\
(5) & \Diamond \bot = \bot, \quad \Diamond^{-1} \bot = \bot & \Box \top = \top, \quad \Box^{-1} \top = \top \\
(6) & \Diamond^{-1} \Box l \leq l \leq \Box \Diamond^{-1} l & \Diamond \Box^{-1} l \leq l \leq \Box^{-1} \Diamond l \\
(7) & l \leq \Diamond \Diamond^{-1} l & l \leq \Diamond^{-1} \Diamond l
\end{array}
$$

Proof: Proofs are straightforward; we leave most of them to the reader, e.g. first part of property (6) $\Diamond^{-1}\Box l \leq l$ is proved by applying the adjunction rule to $\Box l \leq \Box l$, and its second part $l \leq \Box\Diamond^{-1} l$ is obtained by applying adjunction to $\Diamond^{-1} l \leq \Diamond^{-1} l$. The first part of property (7) is proved by applying the converse rule to $\Diamond^{-1} l \leq \Diamond^{-1}$ and its second part is proved by applying the converse rule to $\Diamond l \leq \Diamond l$. ∎

Theorem 3.3 *PTL is sound and complete with respect to PTA.*

Proof: One defines, in an evident fashion, an interpretation map from the atoms of a PTL to a PTA $[\![-]\!] \colon At \longrightarrow L$ and extends it to formulae by induction on the structure, i.e.

$$
\begin{array}{rl}
[\![m_1 \vee m_2]\!] = [\![m_1]\!] \vee [\![m_2]\!], & [\![m_1 \wedge m_2]\!] = [\![m_1]\!] \wedge [\![m_2]\!], \\
[\![\Diamond m]\!] = \Diamond [\![m]\!], [\![\Diamond^{-1} m]\!] = \Diamond^{-1} [\![m]\!] & [\![\Box m]\!] = \Box [\![m]\!], [\![\Box^{-1} m]\!] = \Box^{-1} [\![m]\!], \\
[\![\top]\!] = \top, & [\![\bot]\!] = \bot\,.
\end{array}
$$

Soundness states that any derivable sequent is valid, where a sequent $m \vdash m'$ is *true* in an interpretation $[\![-]\!]$ in L iff $[\![m]\!] \leq [\![m']\!]$; it is *true* in L iff true in all interpretations in L, and it is *valid* if true in every PTA. It is proved by showing that the axioms of PTL are valid and its rules are truth-preserving. An example is to show that the adjunction and converse axioms of a PTL are valid, i.e.

$$
[\![\Diamond^{-1}\Box m]\!] \leq [\![m]\!] \leq [\![\Box\Diamond^{-1} m]\!] \qquad [\![\Diamond\Box^{-1} m]\!] \leq [\![m]\!] \leq [\![\Box^{-1}\Diamond m]\!]
$$

and

$$
[\![l]\!] \leq [\![\Diamond^{-1}\Diamond l]\!] \qquad [\![l]\!] \leq [\![\Diamond\Diamond^{-1} l]\!]
$$

for any $[\![\,]\!]$ in any PTA L. These follow from properties (6) and (7) of proposition 3.2.

Completeness states any valid sequent is derivable. We prove it by the Lindenbaum-Tarski method. First we define an equivalence relation on the formulae of a PTL using the logical equivalence $\vdash\dashv$ relation, it is easy to show that the order defined as $\vdash$ on these classes is a partial order. Next we define the logical operations of a

PTL on these classes and show that they are well-defined, e.g. we define $\Diamond[m]$ to be $[\Diamond m]$ and show

$$[m] \vdash\dashv [m'] \qquad \Longrightarrow \qquad [\Diamond m] \vdash\dashv [\Diamond m']$$

This is true via one application of the $\Diamond$ rule as follows

$$\dfrac{[m] \vdash [m']}{[\Diamond m] \vdash [\Diamond m']}\,\Diamond \qquad\qquad \dfrac{[m'] \vdash [m]}{[\Diamond m'] \vdash [\Diamond m]}\,\Diamond$$

Finally we prove that these equivalence classes form a PTA, by showing that they satisfy its axioms. We provide some example proofs. Consider $\Diamond$ preserving joins and $\Box$ preserveing meets, one directions of these are stipulated as axioms of PTL, the proof trees of the other directions (we drop the equivalence class brackets) are as follows

$$\dfrac{\dfrac{\dfrac{}{m \vdash m \vee m'}\,Ax.}{\Diamond m \vdash \Diamond(m \vee m')}\,\Diamond \quad \dfrac{\dfrac{}{m' \vdash m \vee m'}\,Ax.}{\Diamond m' \vdash \Diamond(m \vee m')}\,\Diamond}{\Diamond m \vee \Diamond m' \vdash \Diamond(m \vee m')}\,\vee \qquad \dfrac{\dfrac{\dfrac{}{m \wedge m' \vdash m}\,Ax.}{\Box(m \wedge m') \vdash \Box m}\,\Box \quad \dfrac{\dfrac{}{m \wedge m' \vdash m'}\,Ax.}{\Box(m \wedge m') \vdash \Box m'}\,\Box}{\Box(m \wedge m') \vdash \Box m \wedge \Box m'}\,\wedge$$

The proof tree for the direction of distributivity that is not stipulated as a PTL axiom is

$$\dfrac{\dfrac{\dfrac{}{m \wedge m' \vdash m}\,Ax. \quad \dfrac{\dfrac{}{m \wedge m' \vdash m'}\,Ax. \quad \dfrac{}{m' \vdash m' \vee m''}\,Ax.}{m \wedge m' \vdash m' \vee m''}\,cut}{m \wedge m' \vdash m \wedge (m' \vee m'')}\,\wedge \quad \dfrac{\dfrac{}{m \wedge m'' \vdash m}\,Ax. \quad \dfrac{\dfrac{}{m \wedge m'' \vdash m''}\,Ax. \quad \dfrac{}{m'' \vdash m' \vee m''}\,Ax.}{m \wedge m'' \vdash m' \vee m''}\,cut}{m \wedge m'' \vdash m \wedge (m' \vee m'')}\,\wedge}{(m \wedge m') \vee (m \wedge m'') \vdash m \wedge (m' \vee m'')}\,\vee$$

The proof trees for the converse are

$$\dfrac{\dfrac{}{m \vdash \Diamond^{-1}\Diamond m}\,conv. \quad \dfrac{\Diamond m \vdash m'}{\Diamond^{-1}\Diamond m \vdash \Diamond^{-1} m'}\,\Diamond^{-1}}{m \vdash \Diamond^{-1} m'}\,cut \qquad \dfrac{\dfrac{}{m \vdash \Diamond\Diamond^{-1} m}\,conv. \quad \dfrac{\Diamond^{-1} m \vdash m'}{\Diamond\Diamond^{-1} m \vdash \Diamond m'}\,\Diamond}{m \vdash \Diamond m'}\,cut$$

The proof trees for the $\Diamond \dashv \Box^{-1}$ adjunction are as follows and those for $\Diamond^{-1} \dashv \Box$ are similar

$$\dfrac{\dfrac{}{m \vdash \Box^{-1}\Diamond m'}\,Adj. \quad \dfrac{\Diamond m \vdash m'}{\Box^{-1}\Diamond m \vdash \Box^{-1} m'}\,\Box^{-1}}{m \vdash \Box^{-1} m'}\,cut \qquad \dfrac{\dfrac{m \vdash \Box^{-1} m'}{\Diamond m \vdash \Diamond\Box^{-1} m'}\,\Diamond \quad \dfrac{}{\Diamond\Box^{-1} m' \vdash m'}\,Adj.}{\Diamond m \vdash m'}\,cut$$

$\blacksquare$

The results in the rest of this section use general theorems of [GNV05] for distributive modal logics.

Proposition 3.4 *PTL is Sahlqvist.*

Proof: It suffices to show that the adjunction and converse axioms are Sahlqvist. For the former see [SD09]. For the latter, we apply definitions 3.1 and 3.2 of [GNV05]. a modal sequent is Sahlqvist iff its left hand side is left Sahlqvist and its right hand side is right Sahlqvist. It is obvious that m is left Sahlqvist. To show that $\Diamond_q \Diamond_q^{-1} m$ and $\Diamond_q^{-1} \Diamond_q m$ are Sahlqvist, we have to look at their negative generation trees

$$- \longrightarrow (\Diamond_q^{-1}, -) \longrightarrow (\Diamond_q, -) \longrightarrow (m, -) \qquad - \longrightarrow (\Diamond_q, -) \longrightarrow (\Diamond_q^{-1}, -) \longrightarrow (m, -)$$

Neither of these has a choice node as all the diamond formulae are negative and thus universal, so obviously no choice occurs in the scope of a universal node and both trees become right Sahlqvist. ∎

Generalizing definition 2.4 of [GNV05] we define

Definition 3.5 *A* Kripke frame *for PTL is a tuple $(S, \leq, R, R^{-1})$, where S is a set of worlds, R is a binary relation on S, R^{-1} is its converse, and $\leq$ is a partial order on S satisfying*[3]

$$\leq \circ R \quad \subseteq \quad R \circ \leq \qquad \textit{and} \qquad \geq \circ R \quad \subseteq \quad R \circ \geq$$

The above are equivalent to $R^{-1} \circ \geq \ \subseteq \ \geq \circ R^{-1}$ and $R^{-1} \circ \leq \ \subseteq \ \leq \circ R^{-1}$, respectively.

Definition 3.6 *A* Kripke structure *for PTL is a pair $\mathbb{M} = (\mathbb{F}, V)$ where $\mathbb{F}$ is a Kripke frame for PTL and $V \subseteq S \times P$ is a valuation.*

Given such a Kripke structure, a satisfaction relation $\models$ is defined on S and formulae of PTL in the routine fashion. The clauses for the modalities are as follows:

- $\mathbb{M}, w \models \Diamond m$ iff $\exists v \in S,\ wRv$ and $\mathbb{M}, v \models m$
- $\mathbb{M}, w \models \Box m$ iff $\forall v \in S,\ wRv$ implies $\mathbb{M}, v \models m$
- $\mathbb{M}, w \models \Diamond^{-1} m$ iff $\exists v \in S,\ wR^{-1}v$ and $\mathbb{M}, v \models m$
- $\mathbb{M}, w \models \Box^{-1} m$ iff $\forall v \in S,\ wR^{-1}v$ implies $\mathbb{M}, v \models m$

[3] Because of the use of one name for R and R^{-1}, we do not need conditions corresponding to interaction axioms, as in [GNV05].

Theorem 3.7 *PTL is sound and complete with respect to Kripke structures for PTL.*

Proof: By theorem 3.8 of [GNV05], any distributive modal logic that is Salhqvist is sound and complete with regard to its Kripke structures. We have shown that PTL is distributive in the proof of theorem 3.3 and that it is Sahlqvist in proposition 3.4. ∎

Definition 3.8 *The dual algbera of an ordered frame $(S, \leq)$ is the collection of downward closed subsets of S with regards to $\leq$.*

Definition 3.9 *A distributive lattice is called perfect whenever it is complete, completely distributive, and join generated by (i.e. each element of it is equal to the join of) the set of all of its completely join irreducible elements.*

Lemma 3.10 *The dual algebra of a PTL frame $(S, \leq, R, R^{-1})$ is closed under intersection, union, and the following modal operators defined for $W \subseteq S$*

$$\begin{aligned}
\Box W &= \{w \in S \mid \forall v \in S, (w, v) \in R \textit{ implies } v \in W\} \\
\Diamond W &= \{w \in S \mid \exists v \in S, (w, v) \in R \textit{ and } v \in W\} \\
\Box^{-1} W &= \{w \in S \mid \forall v \in S, (w, v) \in R^{-1} \textit{ implies } v \in W\} \\
\Diamond^{-1} W &= \{w \in S \mid \exists v \in S, (w, v) \in R^{-1} \textit{ and } v \in W\}
\end{aligned}$$

Proof: Downward closure under intersection and union is routine. For closure under modalities, consider $\Box^{-1}W$, we show that for a downward closed W, $\Box^{-1}W$ is also downward closed, i.e. $x \in \Box^{-1}W$ and $y \leq x$ implies $y \in \Box^{-1}W$. That $y \in \Box^{-1}W$ means that for all $z \in S, (y, z) \in R^{-1}$ implies $z \in W$. If y has no R^{-1} successors, this is true vaciously, else we have that $x \geq y$ and $(y, z) \in R^{-1}$; these are equivalent to $(z, y) \in R$ and $y \leq x$, i.e. that $(z, x) \in \leq \circ R$. By the PTL frame condition we have $\leq \circ R \ \subseteq R \circ \leq$, hence $(z, x) \in R \circ \leq$. So there should be an $r \in S$ such that $z \leq r$ and $(r, x) \in R$, i.e. $(x, r) \in R^{-1}$, but from this and that $x \in \Box^{-1}W$, we obtain $r \in W$, since W is downward closed and $z \leq r$ we obtain that $z \in W$. The proofs for other modalities are similar. ∎

We get a representation theorem with regard to special kinds of PTL frames defined below. These correspond to the frames of our grid navigation applications.

Definition 3.11 *A PTL frame $(S, \leq, R, R^{-1})$ is serial and co-serial whenever $\forall s \in W$ we have $R(s) \neq \emptyset$ and $R^{-1}(s) \neq \emptyset$, where $R(s) = \{t \in W \mid (s, t) \in R\}$ and $R^{-1}(s) = \{t \in W \mid (s, t) \in R^{-1}\}$.*

Lemma 3.12 *The dual algebra of a serial and co-serial PTL frame is a perfect PTA.*

Proof: Once we have shown that the dual algebra is a PTA, it is routine to show that it is perfect. To show the former, we need to show that the adjunction and converse axioms hold for the modalities of the dual algebra. Consider the adjunction $\Diamond \dashv \Box^{-1}$, we have to show $\Diamond W \subseteq W' \iff W \subseteq \Box^{-1}W'$. For the right direction, assume $\Diamond W \subseteq W'$ and take $x \in W$ and $(x,y) \in R^{-1}$, we show $y \in W'$. From $(x,y) \in R^{-1} \cong (y,x) \in R$ and $x \in W$ we obtain that $y \in \Diamond W$, and by assumption $y \in W'$. For the left direction, assume $W \subseteq \Box^{-1}W'$ and take $x \in \Diamond W$, we show $x \in W'$. That $x \in \Diamond W$ means $\exists y \in S, (x,y) \in R$ and $y \in W$. From $y \in W$ and the assumption we obtain that $y \in \Box^{-1}W'$, from this and $(x,y) \in R \cong (y,x) \in R^{-1}$ we obtain that $x \in W'$. The proof for the $\Diamond^{-1} \dashv \Box$ is similar. Now consider the converse axiom $\Diamond W \subseteq W' \implies W \subseteq \Diamond^{-1}W'$. We assume $\Diamond W \subseteq W'$, take $x \in W$, and show $x \in \Diamond^{-1}W'$, i.e. that $\exists y \in S$ such that $(x,y) \in R^{-1}$ and $y \in W'$. By co-seriality there exists a y such that $(x,y) \in R^{-1} \cong (y,x) \in R$, from this and that $x \in W$ we obtain that $y \in \Diamond W$, from this and the assumption $y \in W'$. The proof of the other converse axiom is similar and uses seriality. ∎

Theorem 3.13 *Given a perfect PTA $\mathcal{M}$, there is a frame whose dual algebra is isomorphic to $\mathcal{M}$.*

Proof: By lemma 3.12, it suffices to construct a frame from $\mathcal{M}$ in a way that the dual algebra of the frame is isomorphic to $\mathcal{M}$. As shown in lemma 2.26 and proposition 2.25 of [GNV05], the atom structure of a perfect PTA is a such a frame. ∎

3.2 Positive PDL with Converse without Test

Positive Propositional Dynamic Logic with converse without test, for short PDL^{+-}, is generated over a set of atomic propositions $p \in P$ and atomic actions $\sigma \in Ac$ via the following grammar:

$$
\begin{aligned}
m &::= \bot \mid \top \mid p \mid m \wedge m \mid m \vee m \mid [q]m \mid < q > m \mid [q]^{-1}m \mid < q >^{-1} m \\
q &::= \top \mid \bot \mid \sigma \mid 1 \mid q \bullet q \mid q \vee q \mid q^*
\end{aligned}
$$

Axioms and rules are the ones of PTL, with q-indexed modalities, plus the following for box and diamond and their converses, and axioms and rules for programs and fixed points.

Axioms.

$$q \vdash q \qquad \bot \vdash q \qquad q \vdash \top$$

$$m \vdash <1> m \qquad m \vdash [1]m \qquad m \vdash <1>^{-1} m \qquad m \vdash [1]^{-1}m$$

$$[q \bullet q']m \vdash\dashv [q][q']m \qquad < q \bullet q' > m \vdash\dashv < q >< q' > m$$

$$[q \bullet q']^{-1}m \vdash\dashv [q']^{-1}[q]^{-1}m \qquad < q \bullet q' >^{-1} m \vdash\dashv < q' >^{-1}< q >^{-1} m$$

$$< q \vee q' > m \vdash < q > m \vee \; < q' > m \qquad < q \vee q' >^{-1} m \vdash < q >^{-1} m \vee \; < q' >^{-1} m$$

$$1 \vee q \bullet q^* \vdash\dashv q^* \qquad 1 \vee q^* \bullet q \vdash\dashv q^* \qquad 1 \vee q \bullet^{-1} q^* \vdash\dashv q^* \qquad 1 \vee q^* \bullet^{-1} q \vdash\dashv q^*$$

Rules.

$$\frac{q \vdash q'}{[q']m \vdash [q]m}\, anti \qquad \frac{q \vdash q'}{< q > m \vdash < q' > m}\, mono$$

$$\frac{q \vdash q'}{[q']^{-1}m \vdash [q]^{-1}m}\, anti^{-1} \qquad \frac{q \vdash q'}{< q >^{-1} m \vdash < q' >^{-1} m}\, mono^{-1}$$

$$\frac{q \bullet q' \vdash q'}{q^* \bullet q' \vdash q'}\, *l \qquad \frac{q' \bullet q \vdash q'}{q' \bullet q^* \vdash q'}\, *r \qquad \frac{q \bullet^{-1} q' \vdash q'}{q^* \bullet^{-1} q' \vdash q'}\, *^{-1}l \qquad \frac{q' \bullet^{-1} q \vdash q'}{q' \bullet^{-1} q^* \vdash q'}\, *^{-1}r$$

Definition 3.14 *A finite *-converse di-System is a pair (M, Q) of M a bounded lattice and Q a bounded lattice monoid, with two monotone residuated operations $\cdot, \cdot^{-1} \colon M \times Q \longrightarrow M$ such that $- \cdot q \dashv [q]-$ and $- \cdot^{-1} q \dashv [q]^{-1}-$ satisfying the finite versions of the axioms of a converse di-System, and a $* \colon Q \longrightarrow Q$ operator satisfying axioms of a Kleene star, as follows*

$$1 + q \bullet q^* = 1 + q^* \bullet q = q^* \qquad 1 + q \bullet^{-1} q^* = 1 + q^* \bullet^{-1} q = q^*$$
$$q \bullet q' \leq q' \implies q^* \bullet q' \leq q' \qquad q \bullet^{-1} q' \leq q' \implies q^* \bullet^{-1} q' \leq q'$$

Theorem 3.15 *PDL^{+-} is sound and complete with regard to finite *-converse di-Systems.*

Proof: Follows by using the same methods used for soundness and completeness of a PTL with regard to a PTA, and by taking $< q > m$ to be $m \cdot q$ and $< q >^{-1} m$ to be $m \cdot^{-1} q$. ∎

Theorem 3.16 *The completion of the Lindenbaum-Tarski Algebra of PDL^{+-} is a converse di-system.*

Proof: This follows by routine extension of ideal construction and definitions of [Sad06, BCS07]. ∎

Corollary 3.17 *PDL^{+-} is sound and complete with regard to converse di-systems.*

Proof: Follows from theorems 3.15 and 3.16 by using the same method as in [Sad06, BCS07] to show that the embedding of a finite converse di-system into a converse di-system is a homomorphism. ∎

It would be nice to relate our PDL^{+-} to PDL with converse of [HKT00], where each action q has a converse q^- and axioms have implications in them. One way is to take $[q]^{-1}m$ to be $[q^-]m$ and $< q >^{-1} m$ to be $< q^- > m$ and show that positive axioms and rules of [HKT00] are derivable from ours. For instance, it is the case that [HKT00]'s axioms for converse $m \longrightarrow [q] < q^- > m$ and $m \longrightarrow [q^-] < q > m$ are halves of the corollaries of adjunction, expressed in two of our four PTA axioms for adjunction. The proof tree for one direction of axiom (iv) of PDL from p. 173 of [HKT00], i.e. $[q \vee q']m \vdash [q]m \wedge [q']m$ is as follows:

$$\dfrac{\dfrac{\dfrac{}{q \vdash q \vee q'}\ Ax.}{[q \vee q']m \vdash [q]m}\ anti \qquad \dfrac{\dfrac{}{q' \vdash q \vee q'}\ Ax.}{[q \vee q']m \vdash [q']m}\ anti}{[q \vee q']m \vdash [q]m \wedge [q']m}\ \wedge$$

The other direction, i.e. $[q]m \wedge [q']m \vdash [q \vee q']m$ is also derivable, but the proof tree was so big that we could not fit it in the page.

4 Second Order di-Systems

For our application purposes, we must go higher order by defining a di-System that acts on another di-System (rather than on just a module).

Definition 4.1 *A second order di-System $(\mathcal{D}, Q', \odot, \otimes)$ is a di-System whose module is itself a di-System, i.e. $\mathcal{D} = (M, Q, \cdot, \times)$, i.e. we have $- \odot -: \mathcal{D} \times Q' \longrightarrow \mathcal{D}$ and $- \otimes -: \mathcal{D} \times Q' \longrightarrow \mathcal{D}$.*

Example 4.2. An example of a second order di-System would be to take $\mathcal{D}$ as in example 2.15, that $\mathcal{D} = (\mathcal{P}(S), \mathcal{P}(S \times S), \cdot, \cdot^{-1})$, set $Q = Q' = \mathcal{P}(S \times S)$, take $- \otimes -$ to be as defined as in example 2.17. For $R \subseteq S \times S, W \subseteq S$, examples of $- \odot -$ can be the action that deletes the effects of the action, just performed, from the module, i.e. $W \odot R := \emptyset$ or set it to the diagonal, i.e. $W \odot R := \triangle_W[W] = W$. In either case the action acts as identity on the quantale, i.e. for $\Sigma \subseteq Q$ we set $\Sigma \odot R = \Sigma$.

For our applications, we need extra data, encoded in endomorphisms of the setting[4], defined as

[4]Note that imitating the definition of [JT84] provides us with a definition for a di-System endomorphism $f \colon \mathcal{D} \longrightarrow \mathcal{D}$ given by $f(m \cdot q) = f(m) \cdot q$ and $f(m \cdot^{-1} q) = f(m) \cdot^{-1} q$. Our endomaps

Definition 4.3 *A lax endomorphism* u *of a 2nd order di-System* $((M, Q, - \cdot -, \times), Q', \odot, \otimes)$ *consists of a pair of endomorphisms* $u = (u^M \colon M \longrightarrow M, u^Q \colon Q \longrightarrow Q)$*, where* u^M *preserves joins of* M *and* u^Q *preserves joins of* Q*, and moreover the following three axioms hold*

$$\begin{aligned} u^M(m \odot q) &\leq \bigvee \left\{ m' \in M \mid m' \leq u^M(m \cdot q), \quad m' \times u^Q(q) \neq \bot \right\} & (1) \\ u^Q(q \bullet q') &\leq u^Q(q) \bullet u^Q(q') & (2) \\ 1 &\leq u^Q(1) & (3) \end{aligned}$$

Example 4.4. Consider the di-System of example 4.2. The lax di-System endomorphism on the module u^M can be determined by indistinguishability of the points under one of the actions, e.g. $- \cdot -$, i.e. two propositions $W, W' \subseteq S$ are indistinguishable iff the same relation R can be performed on them. In formal terms

$$u^M(W) = \bigcup \{W' \subseteq S \mid \forall R \subseteq S \times S, \quad W \cdot R \neq \emptyset \quad \text{iff} \quad W' \cdot R \neq \emptyset\}$$

The lax di-System endomorphism on the quantale can be defined similarly, i.e. two relations $R, R' \subseteq S \times S$ are indistinguishable iff they can be performed via $- \cdot -$ on the same propositions. In formal terms,

$$u^Q(R) = \bigcup \{R' \subseteq S \times S \mid \forall W \subseteq S, \quad W \cdot R \neq \emptyset \quad \text{iff} \quad W \cdot R' \neq \emptyset\}$$

Since each projection of u is join preserving, it has a Galois right adjoint, we denote it by $\Box = (\Box^M, \Box^Q)$. These are canonically defined as follows

$$\Box^M l := \bigvee \{m' \in M \mid u^M(m') \leq m\} \qquad\qquad \Box^Q q := \bigvee \{q' \in Q \mid u^Q(q') \leq q\}$$

Definition 4.5 *A 2nd order Informative di-System* $((M, Q, - \cdot -, \times), Q', \odot, \otimes, u)$ *is a second order di-System endowed with a di-System lax endomorphism* $u = (u^M \colon M \longrightarrow M, u^Q \colon Q \longrightarrow Q)$*.*

The 2nd order Informative di-System that we need for applications are 2nd order converse di-systems. As we shall see in more details later, these will be a generalization of Epistemic Systems of [BCS07] to also include actions that are not necessarily epistemic. We refer to them as *Navigation Di-Systems*, defined as follows

come in pairs and are quite different from these.

Definition 4.6 *A Navigation di-System (Nav-diSys)* $\big((M, Q, \cdot, \cdot^{-1}), Q, \odot, \odot^{-1}, u\big)$ *is a 2nd order Informative di-System, where* $(M, Q, \cdot, \cdot^{-1})$ *is a converse di-System,* u *is a 2nd order lax di-System endormorphism, and* $- \odot -$ *and* $-\odot^{-1}$ *are given by*

$$\begin{array}{rcll} - \odot - & : & (M, Q, \cdot, \cdot^{-1}) \times Q \longrightarrow (M, Q, \cdot, \cdot^{-1}) & :: M \odot Q \longrightarrow M \\ - \odot^{-1} - & : & (M, Q, \cdot, \cdot^{-1}) \times Q \longrightarrow (M, Q, \cdot, \cdot^{-1}) & :: M \odot^{-1} Q \longrightarrow M \end{array}$$

The 2nd order action $- \odot -$ does not change the base di-System, but changes the lax endomorpshisms via inequality (1) of definition 4.3. It can be read in an operational way as *restricting* $u^M(m \odot q)$ to elements in the uncertainty of its corresponding lower action, i.e. $u^M(m \cdot q)$, to the ones that can be reached via a $u^Q(q)$ action.

Definition 4.7 *An atomic Nav-diSys is one that has an atomic module with set of atoms* $At(M)$ *and an atomic quantale with a set of atoms* $At(Q)$.

May be it would be worth working on obtaining a logic which is complete with regard to these, this is obtained from our previous PDL^{+-} by adding one or two more modalities to the propositional level, corresponding to uncertainty and its right adjoint. Challenge would be to come up with axioms and rules for conditions of definition 4.3, esp condition (1).

4.1 Interpretation

Given a Navigation di-System (Nav-diSys)$\big((M, Q, \cdot, \cdot^{-1}), Q, \odot, \odot^{-1}, u\big)$ we interpret elements of the module as *propositions* and the order as entailment, thus $m \vee m'$ is the logical disjunction and $\bot$ is the falsum. The elements of the quantale are interpreted as actions and the order is the order of non-determinism, thus $q \vee q'$ is the non-deterministic choice and $\bot$ is crash, monoid multiplication $q \bullet q'$ is sequential composition, and its unit 1 is the action that does nothing. The actions of the base converse di-System are to be thought of as *potential*, e.g. descriptions of actions on a map, the actions of the second order di-System are the realizations of their potential counterparts. For simplicity, we make the two di-Systems share the same action labels that live in the quantale Q and the higher order action to mimicc their base counterparts.

As is usual, the real world comes with some uncertainties and the effect of performing the real actions is to remove some of these uncertainties. The uncertainties are modeled by the endomorphisms of the module and quantale. Following previous work [BCS07], we read them as follows

- $u^M(m)$ is the uncertainty about proposition m, it is the join of all propositions that might be true, that are possibly true, when in reality m is true. For example $u^M(m) = m \vee m'$, says that in reality m is true, but the agent considers it possible that either m or m' might be true, it encodes the uncertainties about the truth of proposition m.

- $u^Q(q)$ is the uncertainty about action q, it is the join of all actions that might be happening, that are possibly happening, when in reality action q is happening. For example $u^Q(q) = q \vee q'$ says that in reality action q is happening but the agent considers it possible that either q or q' is. We can have $u^Q(q) = 1$, which says in reality action q is happening, where as it might be possible that no action is. But we assume that $1 \leq u^Q(1)$, that is if nothing is happening in reality, it is considered possible that this is the case, so we cannot have $u^Q(1) = q \vee q'$, it has to be $1 \vee q$.

We interpret the right adjoints to uncertainty, i.e. the $\Box$ operators, as information modalities, standing for the common part of all the uncertainties, on the module about propositions, and on the quantale about actions. We read them in an operational manner as follows

- $\Box^M m$ reads as 'according to the information available m holds in reality'. Alternatively, one can use the belief modality of doxastic logic and read it as 'it is believed that m holds in reality'.

- $\Box^Q q$ reads 'according to the information available q is happening in reaity', or using the belief modality as 'it is believed that action q is happening in reality'.

These can easily be extended to an agent-indexed family to encode information of multiple agents.

We refer to axiom (1) of definition 4.3 as the *uncertainty reduction* axiom. The intuition behind it is as follows: when one does actions in reality, they change our uncertainty. In navigation systems this change is as follows: the uncertainty after performing an action in reality $u^M(m \odot q)$ is the uncertainty of performing a potential action according to the description of the system, i.e. $u^M(m \cdot q)$ minus the choices to which one could not have reached via a q action (according to the description). For example, $u^M(m \cdot q)$ can be a choice of $m' \vee m''$ and it is not possible to reach m' via a q action, i.e. $m' \cdot^{-1} q = \bot$. Hence m' is removed from the choices in $u^M(m \odot q)$, hence $u^M(m \odot q) = m''$.

The other two inequalities are for coherence of uncertainty with regard to composition. The first one allows us to decompose the uncertainty of the sequential composition of an action according to the uncertainty of each individual; it can be read as stating that the uncertainty of a composition of actions is more uncertain than the composition of their uncertainties. The second says when nothing is happening "in reality" then the uncertainty includes this option, i.e. the agent considers it possible that nothing is happening. So it rules out a totally paranoid account of the world: when nothing is happening, people think either nothing happening or something terrible is happening, but they are just not sure which.

5 Application to Navigation

The systems we are interested in are described by a set of locations S, a set of atomic actions Ac, and a set of possible movements $Ac \subseteq S \times S$. The purpose of the scenarios we consider is that an agent moves between the locations by taking the actions available to it at each location. It is uncertain about being in each location: it does not know where in the system it is located at present. It has no uncertainty about the actions it can take. It moves in order to find out where it is. We model each system in an Nav-diSys and use the axiomatics available to prove that the desired information is acquired by the agent after certain navigation actions.

From the data available, a concrete Nav-diSys with *internal uncertainties* is constructed as follows:

Definition 5.1 *On a set of locations S and a set of action labels $Ac \subseteq S \times S$, a concrete Nav-diSys is built as follows*

$$\mathcal{N} \quad = \quad \big((\mathcal{P}(S), \mathcal{P}(S \times S), - \cdot -, - \cdot^{-1} -), \mathcal{P}(S \times S), - \odot -, - \odot^{-1} -, (u^M, id^Q)\big)$$

where the actions $- \cdot -$ and $- \cdot^{-1} -$ are as defined as in example 2.15, the uncertainty of locations is as defined in example 4.4

For some applications, it might be useful to quotient the di-System over the following property

$$\forall l \in S, \quad \text{if} \quad \exists a \in Ac, \quad \text{s.t.} \quad R_a[l] \neq \emptyset \qquad \text{then} \qquad u^M(s_i) \not\leq l\,. \qquad (4)$$

This property means that we assume no location in which some action can be done can have no uncertainty. By adjunction this is equivalent to

$$\forall l \in S, \quad \text{if} \quad \exists a \in Ac, \quad \text{s.t.} \quad R_a[l] \neq \emptyset \qquad \text{then} \qquad l \not\leq \Box l\,.$$

That is, if the agent can do some action at a location then it cannot know where he is at that location. The intuition behind this property is that, agents move to be able to find out where they are, if they already know where they are, then there would be no point in moving and following a scenario.

We will also work with concrete Nav-diSys that have *external uncertainties*, defined in the same way as above, with the difference that u^M is not set as in example 4.4, but is provided as external data.

5.1 Robot Navigation

Consider given to a robot, the map of a small computing laboratory as five locations $S = \{s_1, s_2, s_3, s_4, s_5\}$, the set of actions $Ac = \{a, b, c\}$, and the following diagram

$$\begin{tikzcd}
s_2 \arrow[r, "b"] \arrow[d, "a"'] & s_3 \arrow[r, "c"] & s_5 \\
s_1 \arrow[ur, "b"] \arrow[r, "a"'] & s_4 \arrow[ur, "c"] &
\end{tikzcd}$$

The uncertainties are set internally as follows

Location	Actions Available	Uncertainty
s_1, s_2	a, b	s_1, s_2
s_3, s_4	c	s_3, s_4

When the robot is at location s_1, it cannot tell if it is at s_1 or s_2, since it can perform the same set of actions in either, similarly for s_2, s_3 and s_4. But if it is at s_1 it performs an a action then he reaches s_4, now it learns where it is. This is because the only way it could have reached s_4 would be that it was originally in s_1. It rules out s_3 from its uncertainty set about s_4, because, according to its description, it could not have reached s_3 via an a action. So by moving from s_1 to s_4, the robot has removed its uncertainty, has acquired information, and has become sure where it is located. Furthermore, it even learns where it was before taking the action a, i.e. in s_1. This is exactly the manner in which the *uncertainty reduction* inequality formalized the elimination of past uncertainties: after performing a certain move in the real world, the robot consults its description, considers its possibilities and eliminates the ones that could not have been reached as a result of the action it just performed.

We encode this navigation scenario in a concrete Nav-diSys as in definition 5.1 quotiented over property 4, denote it by Θ, and prove the above two statements.

Proposition 5.2 *The following inequalities hold in a concrete $\mathcal{N}/\Theta$ based on the above data.*

$$s_1 \leq [a]\Box^M s_4 \qquad\qquad s_1 \leq [a]\Box^M [a]^{-1} s_1$$

Proof: Consider the first one: by the adjunction $- \odot a \dashv [a]-$, it is equivalent to $s_1 \odot a \leq \Box^M s_4$. By the adjunction $u^M \dashv \Box^M$, this is equivalent to $u^M(s_1 \odot a) \leq s_4$. Now by the uncertainty reduction inequality, it is enough to show that

$$\bigvee \{s_i \in S \mid s_i \leq u^M(s_1 \cdot a),\ s_i \cdot^{-1} a \neq \bot\} \ \leq\ s_4$$

Since $s_1 \cdot a = s_4$, and $u^M(s_4) = s_3 \vee s_4$, but $s_3 \cdot^{-1} a = \bot$ where as $s_4 \cdot^{-1} a \neq \bot$, hence the lhs of the above is equal to s_4, which is $\leq s_4$. Consider the second inequality, it becomes equivalent to $u^M(s_1 \odot a) \odot^{-1} a \leq s_1$, by a series of 3 unfoldings of adjunctions. We have shown that $u^M(s_1 \odot a) \leq s_4$, so it suffices to show $s_4 \odot^{-1} a \leq s_1$, which is true since $s_4 \odot^{-1} a = s_4 \cdot^{-1} a = s_1 \leq s_1$. ∎

5.2 Counting on the Staircase

We model the staircase scenario of the introduction. There are $n \in \mathbb{N}$ locations $S = \{f_n \mid n \in \mathbb{N}\}$, one for each floor. The atomic actions available to the robot are $Ac = \{up, down\}$. The description of the staircase is depicted in the following diagram

$$f_1 \underset{\text{down}}{\overset{\text{up}}{\rightleftarrows}} f_2 \underset{\text{down}}{\overset{\text{up}}{\rightleftarrows}} \cdots \underset{\text{down}}{\overset{\text{up}}{\rightleftarrows}} f_{n-1} \underset{\text{down}}{\overset{\text{up}}{\rightleftarrows}} f_n$$

The uncertainties are set internally, so the floors f_2 to f_{n-1} are indistinguishable from one another, i.e. for $1 < i < n$, we have $u^M(f_i) = \bigvee_{1<i<n} f_i$, the first and last floor and the actions have no uncertainty.

Anywhere except for the first and last floor, the robot does not know where it is, but from there if it goes up until it reach the last floor, or goes down until it reaches the first floor, it will find out where it is, and he will find out where it was before starting to move.

Proposition 5.3 *In a concrete Nav-diSys with internal uncertainties built from the above data, the following inequalities hold for* $1 < k < n$

$$f_k \le [\underbrace{up \bullet \cdots \bullet up}_{n-k}]\Box^M f_n \qquad f_k \le [\underbrace{up \bullet \cdots \bullet up}_{n-k}]\Box^M [\underbrace{up \bullet \cdots \bullet up}_{n-k}]^{-1} f_k$$

$$f_k \le [\underbrace{down \bullet \cdots \bullet down}_{k-1}]\Box^M f_1 \qquad f_k \le [\underbrace{down \bullet \cdots \bullet down}_{k-1}]\Box^M [\underbrace{down \bullet \cdots \bullet down}_{k-1}]^{-1} f_1$$

$$f_k \le [\underbrace{up \bullet \cdots \bullet up}_{n-k}]\Box^M [\underbrace{down \bullet \cdots \bullet down}_{n-k}] f_k \qquad f_k \le [\underbrace{down \bullet \cdots \bullet down}_{k-1}]\Box^M [\underbrace{up \bullet \cdots \bullet up}_{k-1}] f_k$$

Proof: Consider the first inequality, it is equivalent to the following by applying the adjunction $- \odot q \dashv [q]-$

$$f_k \odot (\underbrace{up \bullet \cdots \bullet up}_{n-k}) \le \Box^M f_n$$

By the associativity of the quantale action $\odot$ and multiplication $\bullet$, the above is equivalent to

$$f_k \odot \underbrace{up \odot \cdots \odot up}_{n-k} \le \Box^M f_n$$

By the adjunction $u^M \dashv \Box^M$, the above is equivalent to

$$u^M(f_k \odot \underbrace{up \odot \cdots \odot up}_{n-k}) \le f_n$$

By the uncertainty reduction axiom it suffices to show

$$\bigvee \left\{ m' \mid m' \le u^M(f_k \cdot \underbrace{up \cdot \cdots \cdot up}_{n-k}), \quad m' \cdot^{-1} \underbrace{up \cdot^{-1} \cdots \cdot^{-1} up}_{n-k} \neq \bot \right\} \le f_n$$

The lhs above is equal to f_n, since

$$f_k \cdot \underbrace{up \cdot \cdots \cdot up}_{n-k} = f_n, \quad u^M(f_n) = f_n, \quad f_n \cdot^{-1} \underbrace{up \cdot^{-1} \cdots \cdot^{-1} up}_{n-k} \neq \bot$$

Consider the second inequality, by adjunction and associativity it is equivalent to

$$u^M(f_k \odot \underbrace{up \odot \cdots \odot up}_{n-k}) \odot^{-1} (\underbrace{up \odot^{-1} \cdots \odot^{-1} up}_{n-k}) \le f_k$$

Since in the first inequality we have shown that $u^M(f_k \odot \underbrace{up \odot \cdots \odot up}_{n-k}) \le f_n$, it suffices to show that $f_n \odot^{-1} (\underbrace{up \odot^{-1} \cdots \odot^{-1} up}_{n-k}) \le f_k$, which is since $f_n \odot^{-1} (\underbrace{up \odot^{-1} \cdots \odot^{-1} up}_{n-k}) = f_k$. The other inequalities are proven similarly. ∎

5.3 Navigating in a Grid

Grids are generalizations of our staircase scenario above to two dimensions. Like a staircase, there is always a way to find out where you are and were. A typical grid navigation protocol is as follows, a robot is in a grid with n rows and m columns, it can go up, down, left, and right and is supposed to move about and find out where it is. The grid cells look alike to it as long as he can do the same movements in them, hence it knows where it is iff he ends up in one of the four corner cell. We model this protocol in a Nav-diSys and show that no matter where the robot is, there is always some sequences of movements that it can do to get it to one of the corners. After doing either of these it learns where it is and where it was beforehand.

Each grid cell is modeled by a state s_{ij} in the i'th row and j'th column. Uncertainty of corner states is identity

$$u^M(s_{11}) = s_{11} \quad u^M(s_{1m}) = s_{1m} \quad u^M(s_{n1}) = s_{n1} \quad u^M(s_{nm}) = s_{nm}$$

For the rest of the cells we have

$$u^M(s_{ij}) = \bigvee_{1<x<n} \quad \bigvee_{1<y<m} s_{xy} \qquad u^M(s_{1j}) = \bigvee_{1<y<m} s_{1y} \qquad u^M(s_{i1}) = \bigvee_{1<x<n} s_{x1}$$

The set of actions is $Ac = \{u, d, l, r\}$, their non-applicability is as follows

$$s_{1j} \cdot u = s_{1j} \cdot^{-1} d = s_{i1} \cdot l = s_{i1} \cdot^{-1} r = s_{nj} \cdot d = s_{nj} \cdot^{-1} u = s_{im} \cdot r = s_{im} \cdot^{-1} l = \bot$$

All the other actions are applicable in all the other states.

Proposition 5.4 *The following hold in a concrete $\mathcal{N}$ based on the above data.*

$$s_{ij} \leq [\alpha]\Box^M(s_{11} \vee s_{1m} \vee s_{n1} \vee s_{nm}) \qquad\qquad s_{ij} \leq [\alpha]\Box^M[\alpha]^{-1}s_{ij}$$

for $1 < i < n, 1 < j < m$ and α the following choices of sequences of movements

$$(u^{i-1} \vee d^{n-i}) \bullet (l^{j-1} \vee r^{m-j}) \vee (l^{j-1} \vee r^{m-j}) \bullet (u^{i-1} \vee d^{n-i})$$

Proof: Consider the first property, by the adjunctions $- \odot q \dashv [q]-$ and $u^M \dashv \Box^M$, it is equivalent to

$$u^M(s_{ij} \odot \alpha) \leq s_{11} \vee s_{1m} \vee s_{n1} \vee s_{nm}$$

By join preservation of $\odot$ and u^M, the above becomes equivalent to showing a join of 8 terms on the left to be less than or equal to the a join of 4 locations on the right. So by definition of join, we msut show that all of the following 8 cases hold

$$
\begin{array}{ll}
u^M(s_{ij} \odot (u^{i-1} \bullet l^{j-1})) \leq s_{11} & u^M(s_{ij} \odot (u^{i-1} \bullet r^{m-j})) \leq s_{1m} \\
u^M(s_{ij} \odot (d^{n-i} \bullet l^{j-1})) \leq s_{n1} & u^M(s_{ij} \odot (d^{n-i} \bullet r^{m-j})) \leq s_{nm} \\
u^M(s_{ij} \odot (l^{j-1} \bullet u^{i-1})) \leq s_{11} & u^M(s_{ij} \odot (l^{j-1} \bullet d^{n-i})) \leq s_{n1} \\
u^M(s_{ij} \odot (r^{m-j} \bullet u^{i-1})) \leq s_{1m} & u^M(s_{ij} \odot (r^{m-j} \bullet d^{n-i})) \leq s_{nm}
\end{array}
$$

Consider the first one, by uncertainty reduction it suffices to show that

$$\bigvee\{m' \mid m' \leq u^M(s_{ij} \cdot (u^{i-1} \bullet l^{j-1})), m' \cdot^{-1} (u^{i-1} \bullet l^{j-1}) \neq \bot\} \leq s_{11}$$

By associativity of $\odot$ over $\bullet$ and the grid assumptions we have that $s_{ij} \cdot (u^{i-1} \bullet l^{j-1}) = s_{11}$ and that $u^M(s_{11}) = s_{11}$, also that $s_{11} \cdot^{-1} (u^{i-1} \bullet l^{j-1}) \neq \bot$, hence the left hand side is qual to s_{11}, and trivially we have that $s_{11} \leq s_{11}$. Proofs of the other 7 inequalities are similar.

Now consider the second property $s_{ij} \leq [\alpha]\Box^M[\alpha]^{-1}s_{ij}$, which is equivalent to the following by adjunction

$$u^M(s_{ij} \odot \alpha) \odot^{-1} \alpha \leq s_{ij}$$

Like above $u^M(s_{ij} \odot \alpha)$ breaks down to 8 terms and since $\odot^{-1}$ is also join preserving, one has to show 8×8 inequalities similar to those in the above, but updated with the $\odot^{-1}$ of the 8 combinations of composition of actions on the left and s_{ij} on the right. That is, we have 8 inequalities of the form

$$u^M(s_{ij} \odot (u^{i-1} \bullet l^{j-1})) \odot^{-1} ((u^{i-1} \vee d^{n-i}) \bullet (l^{j-1} \vee r^{m-j}) \vee (l^{j-1} \vee r^{m-j}) \bullet (u^{i-1} \vee d^{n-i})) \leq s_{ij}$$

We have shown that $u^M(s_{ij} \odot (u^{i-1} \bullet l^{j-1})) = s_{11}$, so the above is equivalent to

$$s_{11} \odot^{-1} ((u^{i-1} \vee d^{n-i}) \bullet (l^{j-1} \vee r^{m-j}) \vee (l^{j-1} \vee r^{m-j}) \bullet (u^{i-1} \vee d^{n-i})) \leq s_{ij}$$

To show the above, one must show all of the following 8 inequalities

$$
\begin{array}{ll}
s_{11} \odot^{-1} (u^{i-1} \bullet l^{j-1}) \leq s_{ij} & s_{11} \odot^{-1} (u^{i-1} \bullet r^{m-j}) \leq s_{ij} \\
s_{11} \odot^{-1} (d^{n-i} \bullet l^{j-1}) \leq s_{ij} & s_{11} \odot^{-1} (d^{n-i} \bullet r^{m-j}) \leq s_{ij} \\
s_{11} \odot^{-1} (l^{j-1} \bullet u^{i-1}) \leq s_{ij} & s_{11} \odot^{-1} (l^{j-1} \bullet d^{n-i}) \leq s_{ij} \\
s_{11} \odot^{-1} (r^{m-j} \bullet u^{i-1}) \leq s_{ij} & s_{11} \odot^{-1} (r^{m-j} \bullet d^{n-i}) \leq s_{ij}
\end{array}
$$

For two of these we have $s_{11} \odot^{-1} (d^{n-i} \bullet r^{m-j}) = s_{11} \odot^{-1} (r^{m-j} \bullet d^{n-i}) = s_{ij}$, and the rest are equal to $\bot$ (by the inapplicability assumptions of the grid), which is fine since $\bot$ is less than or equal anything, in particular s_{ij}. ∎

5.4 Map Reading

Consider the map reading example of the introduction. Take the location set S to be the set of all street names on the map of a city, the set of action labels to be one label indexed by the street name $Ac = \{*_l \mid s \in L\}$, to encode walking on a street in a fixed direction. Assume that the uncertainty about each location is given externally as a relation $R \subseteq S \times S$, where $(l, l') \in R$ means, while at location l the agent considers it possible that it is at location l'. For example, $R = \{(l, l), (l, l')\}$ means while at l the agent thinks it might be at l or at l'. Now consider the map of Montreal and five streets therein: "Milton, St. Laurent, St. Denis, University and Peel"; fix the directions to be east-west and north-south. The given uncertainties are (e.g. given by an agent's previous memories about Montreal) as follows: it cannot distinguish "St. Laurent" from "St. Denis" and cannot distinguish "University" from "Peel". Assume it recognizes the McGill University campus, thus if it is on "St. Laurent", it is not sure where it is, but if walks on "Milton" to the end, then it becomes certain that it was on "St. Laurent" and that now it is on "University".

We build a concrete Nav-diSys with external uncertainties consisting of

$$\Big\{St.Lrnt, St.Denis, Univ, Peel, Milton\Big\} \subseteq L(S)$$

$$\Big\{ *_{Milton} \Big\} \subseteq Ac, \quad \Big\{(St.Lrnt, Univ)\Big\} \subseteq *_{Milton}$$

$$\Big\{(St.Lrnt, St.Lrnt), (St.Lrnt, St.Denis), (Univ, Univ), (Univ, Peel)\Big\} \subseteq R$$

It is easy to see that given the data above the following inequalities hold in this setting

$$St.Lrnt \leq [*_{Milton}]\Box Univ \qquad\qquad St.Lrnt \leq [*_{Milton}]\Box[*_{Milton}]^{-1} St.Lrnt$$

6 Embedding Epistemic Systems in Navigation di-Systems

When lost in the street, one can also acquire information about where one is by asking people. The information thus acquired is from communication and has been formalized in previous work [BCS07] in the modal quantale module setting of *Epistemic Systems*. An example of a *token game* where public announcement and communication actions are mixed with special sort of fact changing actions has been modeled in [Phi09].

In this section we make the connection between Epistemic Systems and Nav-diSys formal. To do so, we work with mono-modal version of the former. We had to assume atomic modules and quantales whose uncertainty maps satisfy a weak reflexivity condition. In fact, this condition does hold in concrete systems that arise from applications so far exemplified in both Epistemic and Navigation situations.

Definition 6.1 *An atomic mono-modal Epistemic System* $(M, Q, - \otimes -, f)$ *as defined in [BCS07] is a quantale* Q *acting on its right module* M *via the action* $- \otimes -\colon M \times Q \longrightarrow M$*, where* $f = (f^M\colon M \longrightarrow M, f^Q\colon Q \longrightarrow Q)$ *is a lax system endormosphism of the setting satisfying*

$$\begin{array}{rcl} f^M(m \otimes q) & \leq & f^M(m) \otimes f^Q(q) \\ f^Q(q \bullet q') & \leq & f^Q(q) \bullet f^Q(q') \\ 1 & \leq & f^Q(1) \end{array}$$

Moreover every element of the quantale $q \in Q$ *has a kernel,* $ker(q) = \bigvee\{m \in M \mid m \otimes q = \bot\}$ *and the module has a special subset* $Fact \subseteq M$*, defined as* $\Phi = \{p \in M \mid \forall q \in Q, p \otimes q \leq p\}$*. The module and quatale have a set of atoms* $At(M)$ *and* $At(Q)$ *and we have that* $At(M) \subseteq \Phi$*.*

The first inequality in the above is referred to as the *Appearance-update* axiom.

Definition 6.2 *A weak reflexive Nav-diSys is an atomic one in which for* $s \in At(M), \pi \in At(Q)$ *we have* $s \leq u^M(s)$ *and* $\pi \leq u^Q(\pi)$*.*

It might become handy later to also define a *weak transitivity* condition as $u^M u^M(s) \leq u^M(s)$ and $u^Q u^Q(\pi) \leq u^Q(\pi)$ and call the di-Systems (or systems) that satisfy both of these *clustered* di-Systems (or systems). It is easy to see that these conditions lift to all the elements of the module and quantale.

Theorem 6.3 *For* $\mathcal{N}$ *a weak reflexive Nav-diSys,* $\mathcal{N}^\sigma = (M^\sigma, Q^\sigma, - \otimes -, f)_\Phi$ *given by* $M^\sigma = M, Q^\sigma = Q, f = u, \Phi = At(M), m \otimes q = m \odot q$*, is an atomic mono-modal Epistemic System.*

Proof: We need to show that $\mathcal{N}^\sigma$ satisfies the *appearance-update* axiom. We do so by deriving it from the *uncertainty reduction* axiom of $\mathcal{N}$. In an atomic setting the *uncertainty reduction* axiom becomes equivalent to the following

$$(I) \quad u^M(m \odot q) \leq \bigvee\{s_i \in At(M) \mid s_i \leq u^M(m \cdot q), s_i \cdot^{-1} u^Q(q) \neq \bot\}$$

In the atomic Epistemic System $\mathcal{N}^\sigma$, the *appearance-update* axiom becomes equivalent to the following

$$(II) \quad u^M(m \odot q) \leq \bigvee\{t_j \in At(M) \mid t_j \leq u^M(m), t_j \odot u^Q(q) \neq \bot\}$$

This is a result of atoms becoming facts, that is since $t_j \in \Phi$ we obtain $t_j \otimes u^Q(q) \leq t_j$. We show $(I) \leq (II)$. Take $s_i \leq (I)$, that is $s_i \leq u^M(m \cdot q)$ where $s_i \cdot^{-1} u^Q(q) \neq \perp$. We analyze $u^M(m \cdot q)$ by analyzing $m \cdot q$, which is the same as $m \odot q$ in $\mathcal{N}^\sigma$, and is thus equivalent to

$$m \odot q = \bigvee \{w_k \in At(M) \mid w_k \leq m, w_k \odot q \neq \perp\}$$

Form this by monotonicity of u^M, we obtain

$$u^M(m \odot q) = \bigvee \{u^M(w_k) \in At(M) \mid w_k \leq m, w_k \odot q \neq \perp\}$$

From the above and $s_i \leq u^M(m \cdot q) = u^M(m \odot q)$ in $\mathcal{N}^\sigma$ we obtain that $s_i \leq u^M(w_k)$ where $w_k \odot q \neq \perp$. Since $w_k \leq m$ then $u^M(w_k) \leq u^M(m)$, thus $s_i \leq u^M(m)$. Since $w_k \odot q \neq \perp$ and by weak reflexivity from $w_k \leq u^M(w_k)$ and $q \leq u^Q(w_k)$, we have $w_k \odot q \leq u^M(w_k) \odot u^Q(q)$, we obtain that $u^M(w_k) \odot u^Q(q) \neq \perp$, hence $s_i \leq (II)$.

∎

It is possible, but tedious and not very illuminating, to show that weak reflexive (or clustered) Nav-diSys's and Epistemic Systems form a pair of categories where the morphisms of each is its corresponding lax endomorphisms. In that setting the above construction lends itself to a forgetful functor from the latter to the former, it seems possible to find a right adjoint to it. It would be interesting to show that the uncertainty reduction axiom is based on or gives rise to some meaningful categorical operation.

7 Conclusions and future work

We have developed an algebraic framework for dynamic epistemic logic in which the dynamic and epistemic modalities appear as right adjoints. The key new feature in the present work relative to previous work [Sad06, BCS07] is the presence of converse actions and the algebraic laws that govern uncertainty reduction. Robot navigation protocols, as well as the three-player game in Phillips' thesis, give examples in which the old learning inequality was violated showing that there were new subtleties that arise when there are actions that really change the state of the world.

A number of directions for future work naturally suggest themselves. Purely theoretical, we would like to relate Boolean converse di-Systems to Kleene Algebras with test and converse. We would like to develop a cut-free proof system for NavdiSys and a categorical semantics and representation theorems for it. We are also particularly interested in extending this work to apply to examples that involve security protocols where "knowledge" and "learning" play evident roles. A fundamental extension, and one in which we have begun preliminary investigations, is the extension to the probabilistic case. Here knowledge and information theory may well merge in an interesting and not obvious way.

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