Interacting Quantum Observables: Categorical Algebra and Diagrammatics

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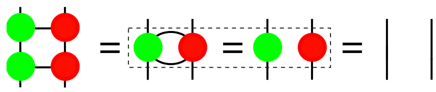
Abstract: Within an intuitive diagrammatic calculus and corresponding high-level category-theoretic algebraic description we axiomatise *complementary observables* for quantum systems described in finite dimensional Hilbert spaces, and study their interaction. We also axiomatise the *phase shifts* relative to an observable. The resulting graphical language is expressive enough to denote any quantum physical state of an arbitrary number of qubits, and any processes thereof. The rules for manipulating these result in very concise and straightforward computations with elementary quantum gates, translations between distinct quantum computational models, and simulations of quantum algorithms such as the quantum Fourier transform. They enable the description of the interaction between classical and quantum data in quantum informatic protocols.

More specifically, we rely on the previously established fact that in the symmetric monoidal category of Hilbert spaces and linear maps non-degenerate observables correspond to special commutative †-Frobenius algebras. This leads to a generalisation of the notion of observable that extends to arbitrary †-symmetric monoidal categories (†-SMC). We show that any observable in a †-SMC comes with an abelian group of phases. We define complementarity of observables in arbitrary †-SMCs and prove an elegant diagrammatic characterisation thereof. We show that an important class of complementary observables give rise to a Hopf-algebraic structure, and provide equivalent characterisations thereof.

Contents

1. Introduction

In classical physics all observables are *compatible*, that is, they all admit sharp values at the same time. In contrast, quantum observables are typically incompatible. Maximally incompatible observables are called *complementary* or *unbiased*. Incompatible observables are usually identified by properties which they they *fail* to obey, for example, in algebraic terms they do *not commute* and in order-theoretic terms they yield *non-distributivity*. In this paper we will take a more constructive stance and treat complementarity as a *resource*. We study the *capabilities* a pair of complementary observables, and show how these capabilities are exploited in quantum information and computation. Indeed, many computations with elementary quantum gates turn out to crucially rely on the capabilities of complementarity, and so do algorithms and protocols. For example, we shall see in Section 6.1 that the fact that the composite of two $\wedge$ Z-gates is the identity boils down to the graphical derivation:

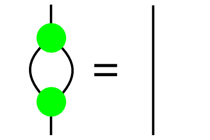
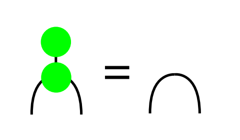
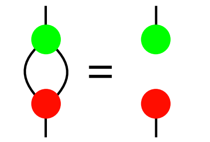
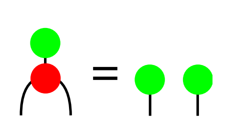


where the dotted area is a diagrammatic characterisation of complementarity.

We perform this study within the framework of †*-symmetric monoidal categories*. In a recent series of papers it was shown that many important features of quantum theory are already present at this highly abstract level, and that this abstract level suffices for important fragments of quantum informatic reasoning [1,8,37,11,13,14]. The main difference between this program and earlier axiomatic accounts of quantum theory is that the primitive concept here is the description of compound quantum systems, which we model by the monoidal tensor, while most other approaches take the notion of observable as primitive. In our approach observables are a definable concept, namely special commutative †-Frobenius algebras [15,16], or in short, observable structures. Hence complementarity will be a special relation between two observable structures.

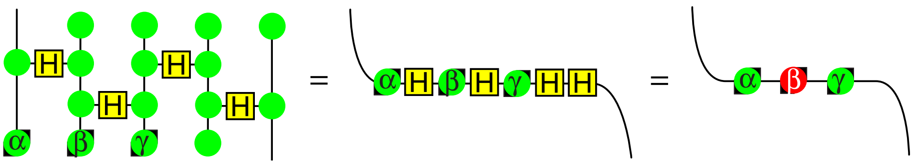
An important feature of †-symmetric monoidal categories is that they admit an intuitive diagrammatic calculus [33,23,37,38]. Within this graphical calculus, each observable induces a normal form which makes them particularly easy to reason with [29,11]. One typically refers to this result as the 'spider' theorem.

Different kinds of observables induce different equations over diagrams: compatible observables are characterised by homotopic transformations of diagrams, reducing connected components to points, while complementary observables, as we shall see in this paper, introduce changes in topology, characterised by disconnecting components. The following table shows the key examples of this behaviour, where the green components are defined in terms of one observable, and the red components in terms of a complementary one:

compatible (self-)interaction:		
complementary interaction:		

For both of the depicted interactions, complementarity yields two disconnected components, while for compatible observables connectedness is preserved. This topological distinction has very important implications for the capabilities of complementary observables in quantum informatics. In particular, the disconnectedness in the case of complementary observables stands for the fact that there is no flow of information from one component to the other, a *dynamic* counterpart to the fact that knowledge of one observable in a pair of complementary observables doesn't result in any knowledge of the other observable. The definition and derivation of the above graphical laws is in Section 6.1.

Before arriving at the definition of complementarity, in Section 5 we provide a category-theoretic account of an important related concept, namely the phase relative to an observable. We show that each observable structure always comes with an abelian group of phases. As this group of phases is defined in terms of observable structure, it behave particularly well with respect to the normal form theorem for diagrams involving observables. We refer to this result as the 'decorated spider' theorem. Together, the algebra of complementary observables and the calculus of phases radically simplifies certain computations. For example,



is an important computation in the context of measurement based quantum computing [35], which in Hilbert space terms would involve computations with 32×32 matrices. This example provides a straightforward translation between quantum computational models, transforming a measurement-based configuration into a circuit. We provide several such quantum computational examples throughout the paper and Section 9 is entirely devoted to them.

In Section 7 we identify a special kind of complementary observables, which we refer to as *closed*. These include the complementary observables that are relevant to quantum computing. We show that any pair of closed complementary observables defines a Hopf-algebra, hence exposing an unexpected connection between quantum computation and the area of Hopf algebras and quantum groups [7, 24, 41]. We moreover provide further, equivalent, characterisations of these closed complementary observables. All of these equivalent characterisations take the form of some sort of commutativity, be it either commutativity of multiplication and a comultiplication, commutativity of a multiplication and an operation, or commutativity of operations. These commutation properties present a remarkable contrast with the usual characterisation of incompatibility as non-commutativity.

The observable structures were introduced under the name *classical objects* [15] and then used under the name *classical structures* [13]. In those papers their main use was somewhat different than in this paper. They were used to describe classical data flows in quantum informatic protocols such as quantum teleportation [5]. This unified treatment of classical and quantum data within a single diagrammatic calculus is indeed an important achievement of our categorical approach. In Section 9.6 we present a diagrammatic presentation of the quantum teleportation protocol which combines the use of observable structures to describe complementarity with their use as a representation of classical data.

Section 2 reviews the necessary category theoretic background, symmetric monoidal †-categories and generalised scalars in particular, and diagrammatic calculus for symmetric monoidal †-categories. The category-theoretic structures employed in this paper have a long history, tracing back to Carboni and Walters' Frobenius law [6], Kelly and Laplaza's compact (closed) categories [25], and Joyal and Street's and Lack's work on diagrammatic calculus [23,29]. Appendix A provides a more detailed account on internal comonoids in a †-symmetric monoidal category. Section 3 reviews the quantum mechanical background, states, logic gates, observables and unbiased observables in particular, as well as the basics of categorical quantum mechanics, and the standard model of categorical quantum mechanics, i.e. **FdHilb**. We also define the notions of *state basis* and *coherent unbiased basis* for Hilbert spaces, and study these. These concepts play a key role in this paper and to our knowledge have not appeared in the literature yet.

2. †-symmetric monoidal categories

Some tutorials with a focus on symmetric monoidal categories and corresponding diagrammatic calculi intended for physicists are now available [3,4,12,38]. Specialised papers are [21,23,37]. A standard textbook is [30].

2.1. Symmetric monoidal categories.

A *monoidal category* $(\mathbf{C}, \otimes, \mathrm{I})$, for details of which we refer the reader to [30,12], is a category which comes with a bifunctor

$$- \otimes - : \mathbf{C} \times \mathbf{C} \to \mathbf{C}\,,$$

a unit object I, natural unit isomorphisms

$$\lambda_A : A \simeq \mathrm{I} \otimes A \qquad \text{and} \qquad \rho_A : A \simeq A \otimes \mathrm{I}\,,$$

and a natural associativity isomorphism

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C\,,$$

which are subject to certain coherence conditions. It is a *symmetric monoidal category* if it also comes with a natural symmetry isomorphism

$$\sigma_{A,B} : A \otimes B \simeq B \otimes A$$

again subject to some coherences. We refer to these four morphisms as the *structure maps* of a monoidal category.

Definition 1. The morphisms of type $\mathrm{I} \to \mathrm{I}$ in a monoidal category $\mathbf{C}$ are called *scalars*. For $c : \mathrm{I} \to \mathrm{I}$ a scalar we call the natural transformation

$$\left\{ c \cdot 1_A := \lambda_A^{-1} \circ (c \otimes 1_A) \circ \lambda_A : A \to A \;\middle|\; A \in |\mathbf{C}| \right\}$$

the *scalar multiplication with scalar c*. More explicitly, we can define

$$c \cdot f := \lambda_B^{-1} \circ (c \otimes f) \circ \lambda_A = f \circ (c \cdot 1_A) = (c \cdot 1_B) \circ f$$

to be the *scalar multiplication of morphism $f : A \to B$ with scalar c.*

The scalars, in any monoidal category, form a commutative monoid [25]. From the definition of scalar multiplication it follows that

$$(c \cdot f) \circ (c' \cdot g) = (c \circ c') \cdot (f \circ g) \quad \text{and} \quad (c \cdot f) \otimes (c' \cdot g) = (c \circ c') \cdot (f \otimes g)\,.$$

Intuitively, in the language of symmetric monoidal categories, if a scalar appears in the description of a morphism, it does not matter where it appears: its effect is that of a global multiplier for the entire morphism.

2.2. The † functor.

Following [1,2,37] we augment symmetric monoidal categories with additional structure that plays an essential role in the quantum mechanical formalism.

Definition 2. A †-*symmetric monoidal category* (†-SMC) is a symmetric monoidal category equipped with an identity-on-objects contravariant endofunctor

$$(-)^\dagger : \mathbf{C}^{op} \to \mathbf{C}$$

which coherently preserves the monoidal structure, that is,

$$(f \circ g)^\dagger = g^\dagger \circ f^\dagger \qquad 1_A^\dagger = 1_A \qquad f^{\dagger\dagger} = f \qquad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$$

together with the fact that the natural isomorphisms θ of the symmetric monoidal structure are all *unitary*, that is, $\theta^{-1} = \theta^\dagger$.

The † functor provides an involution for the monoid of scalars.

Example 1. The category **FdHilb**, has finite dimensional Hilbert spaces as objects and linear maps as its morphisms. It is a †-SMC with the usual tensor product as the monoidal structure, and the † functor is given by the adjoints of linear algebra. The unit object for the monoidal structure is the field of complex numbers $\mathbb{C}$ —which is a one-dimensional Hilbert space over itself— since

$$\mathcal{H} \otimes \mathbb{C} \simeq \mathcal{H}\,,$$

and the monoid of scalars is isomorphic to the monoid of the complex numbers $(\mathbb{C}, \cdot, 1)$. Indeed, a linear map $s : \mathbb{C} \to \mathbb{C}$ is completely determined by $s(1)$, due to linearity, so there is a bijection

$$\mathbf{FdHilb}(\mathbb{C}, \mathbb{C}) \to \mathbb{C} :: s \mapsto s(1)\,.$$

The involution for the monoid of scalars is complex conjugation.

The internal structure of the state spaces is hidden—abstracted away—in this categorical setting; the state spaces are effectively reduced to labels which determine when morphisms may be composed. However, in **FdHilb** and many other important examples, the internal structure of the spaces may be reconstructed via the structure of the morphisms into that space. In particular, any linear map $\psi : \mathbb{C} \to \mathcal{H}$ is completely determined by $\psi(1)$, due to linearity, hence there is a bijection,

$$\mathbf{FdHilb}(\mathbb{C}, \mathcal{H}) \to \mathcal{H} :: \psi \mapsto \psi(1).$$

We call the elements of $\mathbf{FdHilb}(\mathbb{C}, \mathcal{H})$ the *points* of object $\mathcal{H}$. To distinguish between the linear map ψ and the vector $\psi(1)$ we will denote the latter by $|\psi\rangle$.

Example 2. The category **Rel** which has sets as objects and relations as morphisms is a †-SMC with the Cartesian product of sets as monoidal structure, and the relational converse as the † functor. The unit object for the monoidal structure is the singleton set $\{*\}$, since

$$X \times \{*\} \simeq X$$

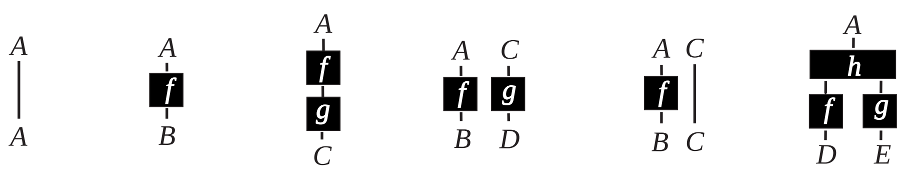
for any set X, and the monoid of scalars is now isomorphic to the Boolean monoid $(\mathbb{B}, \wedge, 1)$, since there are only two relations $r : \{*\} \to \{*\}$, namely the empty relation and the identity relation. The involution for the monoid of scalars is now trivial. We write **FRel** when restricting to finite sets. In this case the points of an object X are not its elements but its *subsets* [12]. While at first sight (**F**)**Rel** seems to have little to do with physics, it enables to encode a surprising amount of quantum phenomena, e.g. Spekkens' toy quantum theory [40] can be embedded within it as a sub-†-SMC **Spek** [9].

2.3. Diagrammatic calculus.

Diagrammatic calculus for tensor categories traces back to Penrose [33]. The first formal treatments are in [21,23]. We refer the reader in particular to [37] and to the tutorials [4,12,38]. Also [27] is helpful.

Morphisms in †-SMCs are represented by *boxes*, domain types by *input wires*, and codomain types by *output wires*. Composition is depicted by connecting matching outputs and inputs by wires, and tensor by juxtaposing wires or boxes side by side. Pictures are to be read from top to bottom. E.g.

1_A $\quad$ f $\quad$ $g \circ f$ $\quad$ $f \otimes g$ $\quad$ $f \otimes 1_C$ $\quad$ $(f \otimes g) \circ h$



We translate axioms of symmetric monoidal structure such as

$$\begin{array}{ccc} A \otimes C & \xrightarrow{1_A \otimes g} & A \otimes D \\ {\scriptstyle f \otimes 1_C}\downarrow & & \downarrow{\scriptstyle f \otimes 1_D} \\ B \otimes C & \xrightarrow[1_B \otimes g]{} & B \otimes D \end{array} \qquad\qquad \begin{array}{ccc} A \otimes C & \xrightarrow{\sigma_{A,C}} & C \otimes A \\ {\scriptstyle f \otimes g}\downarrow & & \downarrow{\scriptstyle g \otimes f} \\ B \otimes D & \xrightarrow[\sigma_{B,D}]{} & D \otimes B \end{array}$$

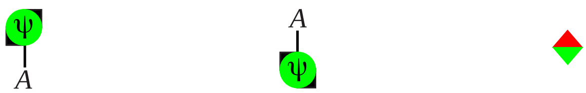
into the following diagrammatic equations:

We are allowed to 'slide' boxes along the wires, and also along the crossings which represent the symmetry natural isomorphism. The adjoint corresponds to reflection of the picture in the horizontal axis:

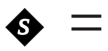
$f : A \to B$ $\qquad$ $f^\dagger : B \to A$

where we made the box asymmetric, by introducing the white square, to expose this reversal. Reversing twice leaves the box invariant, exposing the involutive nature of the adjoint. The monoidal unit I is represented by 'no wire'. For example, in this paper you will encounter the following symbols:

$\psi : \mathrm{I} \to A$ $\qquad$ $\psi^\dagger : A \to \mathrm{I}$ $\qquad$ $\sqrt{D} : \mathrm{I} \to \mathrm{I}$



The identity on the monoidal unit $1_\mathrm{I} : \mathrm{I} \to \mathrm{I}$ is represented by an 'empty wire' and hence an equation of the form $s = 1_\mathrm{I}$ is depicted as:



leaving the right-hand side empty.

3. Some concrete and generalised quantum theory

We restrict ourselves here to ingredients that play a role in this paper. In particular, all Hilbert spaces involved will be finite-dimensional.

3.1. Some Quantum Mechanics, von Neumann style.

Quantum mechanics postulates that a quantum system has *states* which live within a complex Hilbert space. More precisely, states are unit vectors within this space, and for reasons discussed below, two unit vectors which differ only in terms of a scalar $e^{i\theta}$ describe the same state. States correspond, therefore, to one-dimensional subspaces, or *rays*. Somewhat abusively, we will denote the set of rays in a Hilbert space $\mathcal{H}$, which we refer to as the *state space*, by $\mathcal{H}$ itself. To distinguish between states and vectors, we write $\|\psi\rangle\rangle$ to denote the unique state containing the (non-zero) vector $|\psi\rangle$. Similarly, $|\sum_i c_i|\, v_i \,\rangle\rangle$ is the state spanned by vector $\sum_i c_i| v_i \rangle$. In quantum computation the state space of the elementary computational unit is the *qubit*, that is, $\mathcal{Q} = \mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$; the vectors of the *computational basis* for the underlying vector space are written $|0\rangle, |1\rangle$.

Quantum mechanics postulates that the joint state space of two systems with state spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is the tensor product state space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Quantum mechanics postulates that the reversible operations that we can apply to a quantum system with states $\mathcal{H}$ are the unitary maps $U : \mathcal{H} \to \mathcal{H}$, i.e. linear maps that preserve the inner product, or equivalently, in finite dimensions, linear maps for which the adjoint is equal to the inverse. Typical unitary operations, known as *quantum logic gates* in quantum computation, include the Pauli X-gate, the Pauli Z-gate, the Hadamard gate, the controlled-X gate and the controlled-Z gate, whose matrix forms respectively are:

$$\mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\wedge\mathsf{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \wedge\mathsf{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Quantum mechanics postulates that each *observable quantity* O for a quantum system with states $\mathcal{H}$ corresponds to a self-adjoint operator $\hat{O} : \mathcal{H} \to \mathcal{H}$. Here we will assume that $\hat{O}$ has a non-degenerate spectrum, that is, for each eigenvalue λ_i the space of eigenvectors is one-dimensional, hence

$$\hat{O} = \sum_i \lambda_i \, |v_i\rangle \, \langle v_i|$$

where we assume eigenvectors v_i to be of unit length, and we can speak of *the* eigen*state* $\|v_i\rangle\!\rangle$ for an eigenvalue λ_i. If a quantum system is in state $\|\psi\rangle\!\rangle$ then measuring O will result with probability

$$p_i = |\langle v_i \mid \psi\rangle|^2$$

in observing λ_i. Note here that for angle α, we have

$$\left|\langle v_i \mid e^{i\alpha}\psi\rangle\right|^2 = \left|e^{i\alpha}\langle v_i \mid \psi\rangle\right|^2 = |\langle v_i \mid \psi\rangle|^2 \ ;$$

hence the vectors $\left|e^{i\alpha}\psi\right\rangle$ and $|\psi\rangle$ yield the same outcomes with the same probability for every possible measurement. Since there is no observable distinction between $\left|e^{i\alpha}\psi\right\rangle$ and $|\psi\rangle$, they define the same state $\|e^{i\alpha}\psi\rangle\!\rangle = \|\psi\rangle\!\rangle$.

Observation of λ_i is accompanied by a change of state,

$$\|\psi\rangle\!\rangle \mapsto \|v_i\rangle\!\rangle \, .$$

Note that the actual value of λ_i does not affect this change of state, nor does it affect the probability p_i. Therefore it makes sense to abstract over the values λ_i and identify an observable by the set

$$\{\|v_1\rangle\!\rangle, \ldots, \|v_n\rangle\!\rangle\}$$

of eigenstates of $\hat{O}$. Below, by *observable* we intend such a set. We refer to a corresponding orthonormal basis

$$\{|v_1\rangle, \ldots, |v_n\rangle\}$$

as a *vector basis*; cf. Definition 4 below. To summarise:

vector	state	vector basis	observable
$\|\psi\rangle$	$\|\psi\rangle\rangle$	$\{\|v_1\rangle, \ldots, \|v_n\rangle\}$	$\{\|v_1\rangle\rangle, \ldots, \|v_n\rangle\rangle\}$

Definition 3. A vector $|\psi\rangle$ (or state $\|\psi\rangle\rangle$) is *unbiased* relative to vector basis $\{|v_1\rangle, \ldots, |v_n\rangle\}$ (or observable $\{\|v_1\rangle\rangle, \ldots, \|v_n\rangle\rangle\}$) if for all i, j we have

$$|\langle v_i \mid \psi\rangle| = |\langle v_j \mid \psi\rangle| \ ,$$

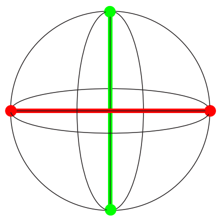
and two vector bases (or two observables) are *mutually unbiased* if each vector (or state) in one of these vector bases (or observables) is unbiased relative to the other vector basis (or observable).

When $\mathcal{H} = \mathbb{C}^D$ then we have $|\langle v_i \mid \psi\rangle| = |\langle v_j \mid \psi\rangle| = 1/\sqrt{D}$.

Example 3. A typical example of such observables are the spin observables of the electron measured along the X and Z axes. The state space is $\mathbb{C}^2$ and the corresponding self-adjoint operators are the Pauli X and Z matrices mentioned above. The Z operator has $|0\rangle$ and $|1\rangle$ as eigenvectors and hence can be identified by the observable $\{\|0\rangle\rangle, \|1\rangle\rangle\}$ while the X operator has

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad \text{and} \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

as eigenvectors and hence can be identified by the observable $\{\|+\rangle\rangle, \|-\rangle\rangle\}$. One easily sees that the Hadamard gate transforms the X-observable in the Z-observable and vice versa. On the Bloch sphere an observable is a pair of antipodal points. The X- and Z-observables can be represented:



where the green dots represent the eigenstates of the Z-observable and the red dots represent the eigenstates of the X-observable.

3.2. Bases for vectors vs. bases for states.

A vector basis of a Hilbert space is characterised by the following property:

Proposition 1. *Let $\{|v_1\rangle, \ldots, |v_n\rangle\}$ be any vector basis of a Hilbert space $\mathcal{H}$. Then, any linear map $f : \mathcal{H} \to \mathcal{H}'$ is completely determined by the values f takes on $|v_1\rangle, \ldots, |v_n\rangle$. Further, no proper subset $\{|v_1\rangle, \ldots, |v_n\rangle\}$ suffices to determine f.*

Now, any regular linear map induces a map from states to states, namely

$$\hat{f} :: \|\psi\rangle\!\rangle \mapsto |f(|\psi\rangle)\rangle$$

However, knowing the values that $\hat{f}$ takes on an observable $\{\|v_1\rangle\!\rangle, \ldots, \|v_n\rangle\!\rangle\}$ does not suffice to fix $\hat{f}$ itself. For example, define a family of linear maps

$$f_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix},$$

where the matrix is expressed in the vector basis $\{|0\rangle, |1\rangle\}$. Every $\hat{f}_\theta$ leaves both $\|0\rangle\!\rangle$ and $\|1\rangle\!\rangle$ invariant, while

$$\hat{f}(\|+\rangle\!\rangle) = \|0\rangle + e^{i\theta}|1\rangle\!\rangle \neq \|0\rangle + e^{i\theta'}|1\rangle\!\rangle = \hat{f_{\theta'}}(\|+\rangle\!\rangle)$$

whenever $\theta \neq \theta'$ for $\theta, \theta' \in [0, 2\pi)$. Is there an analogue to Proposition 1? Can $\hat{f}$ be characterised by its image on some minimal set of states? The answer is yes:

Definition 4. We call a set of states of the form

$$\text{observable} \cup \{\text{unbiased state for that observable}\} \qquad (1)$$

a *state basis*. We call the unbiased state the *deleting point*.

Proposition 2. *Any map on states $\hat{f}$ induced by a regular linear map $f : \mathcal{H} \to \mathcal{H}'$ is completely determined by the values it takes on a state basis for some arbitrary observable. Moreover, no proper subset of such a set of states suffices to determine $\hat{f}$.*

Proof. Let f be a regular linear map, and let $\hat{f}$ be the corresponding map of states. Let $\{\|v_1\rangle\!\rangle, \ldots, \|v_n\rangle\!\rangle\} \cup \{\|s\rangle\!\rangle\}$ form a state basis, and suppose that $\hat{f}$ takes known values upon these states. We will show this determines f on a vector basis $\{|\eta_1\rangle, \ldots, |\eta_n\rangle\}$ of $\mathcal{H}$, up to a common, overall phase.

Set $|\eta_i\rangle = \langle v_i \mid s\rangle\, |v_i\rangle$ and let $\mathcal{H}''$ be the subspace spanned by $f(|\eta_1\rangle), \ldots, f(|\eta_n\rangle)$. Since f is regular $\{f(|\eta_1\rangle), \ldots, f(|\eta_n\rangle)\}$ is a basis for $\mathcal{H}''$, and let $\langle - \mid - \rangle_\diamond$ denote the inner-product on $\mathcal{H}''$ for which $\{f(|\eta_1\rangle), \ldots, f(|\eta_n\rangle)\}$ is orthonormal. Then the codomain restriction of f to $(\mathcal{H}'', \langle - \mid - \rangle_\diamond)$ is unitary. Relying on this we have

$$f(|\eta_i\rangle) = f(\langle v_i \mid s\rangle\, |v_i\rangle) = \langle v_i \mid s\rangle f(|v_i\rangle) = \langle f(|v_i\rangle) \mid f(|s\rangle)\rangle_\diamond\, f(|v_i\rangle)\,.$$

This expression is completely determined by $\hat{f}(\|v_i\rangle\!\rangle)$ and $\hat{f}(\|s\rangle\!\rangle)$ up to a phase factor contributed by $f(|s\rangle)$, but this phase factor is the same for all $f(|\eta_i\rangle)$.

It now follows that, given an arbitary state $\|\psi\rangle\rangle = |\sum_i c_i |\eta_i\rangle\rangle$, we have

$$\hat{f}(\|\psi\rangle\rangle) = |f(|\psi\rangle)\rangle = |f(\sum_i c_i |\eta_i\rangle)\rangle = |\sum_i c_i f(|\eta_i\rangle)\rangle ,$$

where each $f(|\eta_i\rangle)$ is determined upto the common phase. This phase is therefore a global phase for the vector $\sum_i c_i f(|\eta_i\rangle)$, hence $\hat{f}(\|\psi\rangle\rangle)$ yields a unique state, and so $\hat{f}$ is well-defined on all states.

It is easily seen that no proper subset of $\{\|v_1\rangle\rangle, \ldots, \|v_n\rangle\rangle, \|s\rangle\rangle\}$ is sufficient to completely determine $\hat{f}$. □

State bases and vector bases are related by the following proposition:

Proposition 3. *Let $\mathcal{S}$ be the set of all state bases for $\mathcal{H}$, let $\mathcal{V}$ be the set of all vector bases for $\mathcal{H}$, and let $\mathcal{V}/\sim$ be the set of equivalence classes $[-]_\sim$ in $\mathcal{V}$ for the equivalence relation 'equal up to an overall phase' i.e.*

$$\{|v_1\rangle, \ldots, |v_n\rangle\} \sim \{|w_1\rangle, \ldots, |w_n\rangle\} \;\Leftrightarrow\; \exists\theta \textit{ such that } \forall j : |v_j\rangle = e^{i\theta} \cdot |w_j\rangle .$$

There is a bijective correspondence

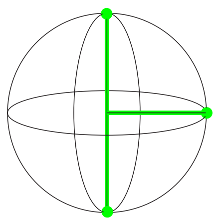
$$\mathcal{S} \underset{vs}{\overset{sv}{\rightleftarrows}} \mathcal{V}/\sim$$

where

- $sv : \{\|v_1\rangle\rangle, \ldots, \|v_n\rangle\rangle\} \cup \{\|s\rangle\rangle\} \mapsto [\{\langle v_1 \mid s\rangle |v_1\rangle, \ldots, \langle v_n \mid s\rangle |v_n\rangle\}]_\sim$
- $vs : [\{|v_1\rangle, \ldots, |v_n\rangle\}]_\sim \mapsto \{\|v_1\rangle\rangle, \ldots, \|v_n\rangle\rangle\} \cup \{|\sum_i |v_i\rangle\rangle\}$.

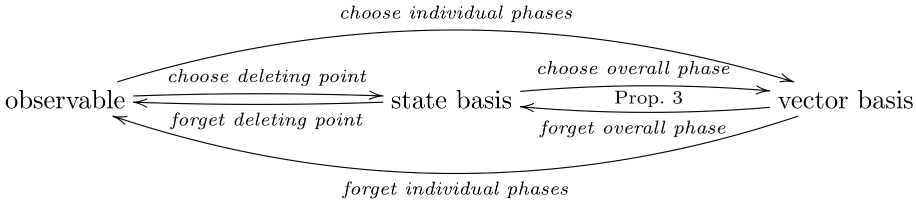
Proof. Note that sv is indeed well-defined in the sense that its prescription does not depend on the choice of the respective vectors $|v_1\rangle, \ldots, |v_n\rangle, |s\rangle$ in the states $\|v_1\rangle\rangle, \ldots, \|v_n\rangle\rangle, \|s\rangle\rangle$. Also vs is easily seen to be well-defined. Verifying that these maps are each other's inverse is straightforward. □

Example 4. On the Bloch sphere the deleting point lies on the equator for the antipodal points that represent the observable. E.g. for the Z-observable $\{\|\psi_0\rangle\rangle, \|\psi_1\rangle\rangle\}$ and the deleting point $\|+\rangle\rangle := \|0\rangle + |1\rangle\rangle$ we have:



so the observable and the deleting point together make up a T-shape. The choice of the name 'deleting point' will become clear below.

Given a vector basis, we can turn it into an observable by forgetting the phases of each of the basis vectors, which we formalise by passing to equivalence classes. To construct a vector basis from an observable, we have to choose a phase for each basis element. This construction factors over the construction of state bases as follows:



3.3. Coherent bases.

Definition 5. We call two mutually unbiased vector bases (= MUVBs), $\mathcal{V} = \{|v_1\rangle, \ldots, |v_n\rangle\}$ and $\mathcal{W} = \{|w_1\rangle, \ldots, |w_n\rangle\}$, *coherent* iff

$$\frac{1}{\sqrt{n}} \sum_i v_i \in \mathcal{W} \qquad \text{and} \qquad \frac{1}{\sqrt{n}} \sum_i w_i \in \mathcal{V}\,.$$

We call two mutually unbiased state bases (= MUSBs) *coherent* iff the deleting point of each is contained in the observable of the other.

These notions of coherence coincide along the bijection of Proposition 3:

Proposition 4. *If $\mathcal{V}$ and $\mathcal{W}$ are coherent MUVBs then $vs([\mathcal{V}]_\sim)$ and $vs([\mathcal{W}]_\sim)$ are coherent MUSBs, and if $\mathcal{S}$ and $\mathcal{T}$ are coherent MUSBs then there exist $\mathcal{V} \in sv(\mathcal{S})$ and $\mathcal{W} \in sv(\mathcal{T})$ such that $\mathcal{V}$ and $\mathcal{W}$ are coherent MUVBs.*

Proof. The first statement follows directly from the definition of vs. For coherent MUSBs $\mathcal{S} = \{\|v_1\rangle\!\rangle, \ldots, \|v_n\rangle\!\rangle, \|w_1\rangle\!\rangle\}$ and $\mathcal{T} = \{\|w_1\rangle\!\rangle, \ldots, \|w_n\rangle\!\rangle, \|v_1\rangle\!\rangle\}$ for $\mathcal{V} \in sv(\mathcal{S})$ and $\mathcal{W} \in sv(\mathcal{T})$ we have $\sum_i \mathcal{V} = \sum_i \langle v_i \mid w_1\rangle |v_i\rangle = |w_1\rangle$ and $\sum_i \mathcal{W} = |v_1\rangle$. Hence we obtain coherence if $\langle w_1 \mid v_1\rangle |w_1\rangle = |w_1\rangle$ and $\langle v_1 \mid w_1\rangle |v_1\rangle = |v_1\rangle$, that is, $\langle v_1 \mid w_1\rangle = 1$. This can be realised by choosing an appropriate overall phase for $\mathcal{V}$ relative to $\mathcal{W}$. □

Pairs of observables arise from a coherent bases:

Theorem 1. *For each pair of MUVBs $\{|v_1\rangle, \ldots, |v_n\rangle\}$ and $\{|w_1\rangle, \ldots, |w_n\rangle\}$ there exists a pair of coherent MUVBs $\{|v_1'\rangle, \ldots, |v_n'\rangle\}$ and $\{|w_1'\rangle, \ldots, |w_n'\rangle\}$ that induces the same observables i.e. $\|v_i\rangle\!\rangle = \|v_i'\rangle\!\rangle$ and $\|w_i\rangle\!\rangle = \|w_i'\rangle\!\rangle$ for all i.*

Proof. Given a pair of MUVBs forget all phases to obtain the corresponding pair of induced observables. Then adjoin as a deleting point to each of these a state of the other, in order to obtain coherent MUSBs. Now we can rely on Proposition 4 to obtain coherent MUVBs that induce the same observable as the initial one. □

3.4. Some quantum mechanics, categorically.

The categorical version of quantum mechanics has become an active area of research, some of the key papers being [1,2,37,15]. Particularly relevant here are [8,15,16,14,18,9]. This approach seeks to axiomatise the structural features of the concrete physical theory and to extract the most general mathematical environment in which quantum mechanics could be carried out. The required environment is generally a †-SMC.

The objects of the category represent the *systems* (or *entities*) of the theory; for example, in **FdHilb** the Hilbert spaces represent quantum systems, and the two-dimensional Hilbert space represents the qubit. Tensors $A \otimes B$ represent the joint system consisting of A and B. Morphisms are interpreted as *operations* (or *processes*). In this paper all processes will be *pure* (or *closed*) – *mixed* (or *open*) processes can be derived from these via Selinger's CPM-construction [37].

The monoidal unit I, given that it satisfies $A \otimes \mathrm{I} \cong \mathrm{I} \otimes A \cong A$, is interpreted as 'no system'. The *points*, that is, the morphisms of type $\mathrm{I} \to A$, are the *states* of system A. We can think of these also as *preparation procedures.* As demonstrated in Example 1, in **FdHilb** these are the vectors of the corresponding Hilbert space. But as discussed above, states of quantum systems are not vectors, but equivalence classes of vectors, that is, rays. This problem can be overcome by passing to the category $\mathbf{FdHilb}_p$ which has the same objects as **FdHilb** but whose morphisms are equivalence classes of **FdHilb**-morphisms, given by the equivalence relation

$$f \sim g \;\Leftrightarrow\; \exists c \in \mathbb{C} \setminus \{0\} \; s.t. \; f = c \cdot g.$$

In $\mathbf{FdHilb}_p$ the states are now indeed the rays of the Hilbert space, together with one point representing the zero vector. The points for the two-dimensional Hilbert space in $\mathbf{FdHilb}_p$, that is, the set $\mathbf{FdHilb}_p(\mathbb{C}, \mathbb{C}^2)$, exactly correspond with the points of the Bloch sphere. General morphisms correspond to the partial maps from states to states induced by linear maps – cf. Section 3.2.

A point $\Psi : \mathrm{I} \to A \otimes B$ is a state of the compound system $A \otimes B$, and this state may or may not be entangled. If it is not entangled, then we have

$$\Psi = (\psi_A \otimes \psi_B) \circ \lambda_{\mathrm{I}}\,,$$

that is, the state Ψ factors in state ψ_A of system A and state ψ_B of system B. It is entangled if such a factorisation does not exist. Additional structure of the †-SMC can guarantee the existence of entangled states, and, for example, enable derivation of teleportation-like protocols [1].

The morphisms of type $\mathrm{I} \to \mathrm{I}$ can be interpreted as probability amplitudes. In **FdHilb** they are the complex numbers, hence too many, and in $\mathbf{FdHilb}_p$ there are only two, hence too few. The solution consists of enriching $\mathbf{FdHilb}_p$ with *probabilistic weights*, i.e. to consider morphisms of the form $r \cdot f$ where $r \in \mathbb{R}^+$ and f a morphism in $\mathbf{FdHilb}_p$. Therefore, let $\mathbf{FdHilb}_{wp}$ be the category whose objects are those of **FdHilb** and whose morphisms are equivalence classes of **FdHilb**-morphisms for the congruence

$$f \sim g \;\Leftrightarrow\; \exists \alpha \in [0, 2\pi) \text{ s.t. } f = e^{i\alpha} \cdot g.$$

A detailed categorical account on $\mathbf{FdHilb}_{wp}$ is in [8].

The three categories considered above are related via inclusions:

$$\mathbf{FdHilb}_p \twoheadleftarrow\hookrightarrow \mathbf{FdHilb}_{wp} \twoheadleftarrow\hookrightarrow \mathbf{FdHilb}$$

The theorems proven in this paper apply to each of these.

Further, in any †-category, given any two points ψ, ϕ, the composition

$$\psi^\dagger \circ \phi : \mathrm{I} \to A \to \mathrm{I}$$

is a scalar. In **FdHilb**, $(\cdot)^\dagger$ is the usual linear algebraic adjoint, so $\psi^\dagger \circ \phi$ is the inner product $\langle \psi \mid \phi \rangle$. On the other hand, in $\mathbf{FdHilb}_{wp}$ it provides the absolute value of the inner-product, of which the square accounts for transition probabilities. The notion of unitarity in Definition 2 guarantees that this inner-product is preserved. In the case of **FdHilb** this notion of unitarity coincides with the usual one.

Example 5. The *unitaries* in $\mathbf{FdHilb}_p(\mathbb{C}, \mathbb{C}^2)$, i.e. those maps U satisfying

$$U \circ U^\dagger = U^\dagger \circ U = 1_{\mathbb{C}^2} ,$$

correspond to rotations of the Bloch sphere.

It is standard to interpret the *eigenstates* $\|\psi_i\rangle\!\rangle$ for an observable $\{\|\psi_i\rangle\!\rangle\}_i$ as classical data. Hence, in $\mathbf{FdHilb}_{wp}$, the operation δ of an observable structure copies the eigenstates $\|\psi_i\rangle\!\rangle$ of the observable it is associated with. We can interpret the deleting point $\|\varepsilon\rangle\!\rangle$ as the state which uniformly deletes these eigenstates: by unbiasedness the probabilistic distance of each eigenstate $\|\psi_i\rangle\!\rangle$ to $\|\varepsilon\rangle\!\rangle$ is equal. Therefore we will refer to δ as (classical) copying and to ϵ as (classical) erasing. A crucial point here is that given an observable there is a choice involved in picking ϵ.

4. Observable structures and their classical points

4.1. Observable structures.

An *internal commutative monoid* in an SMC is a triple

$$(A \ , \ m : A \otimes A \to A \ , \ e : \mathrm{I} \to A)$$

for which the *multiplication* $m : A \otimes A \to A$ is both *associative* and *commutative* and it has e as its *unit*, that is, respectively,

$$m \circ (m \otimes 1_X) = m \circ (1_X \otimes m) \qquad m \circ \sigma_{A,A} = m \qquad m \circ (1_X \otimes \epsilon) = 1_X .$$

Dually, an *internal commutative comonoid* in an SMC is a triple

$$(A \ , \ \delta : A \to A \otimes A \ , \ \epsilon : A \to \mathrm{I})$$

for which the *comultiplication* $\delta : A \to A \otimes A$ is both *coassociative* and *cocommutative* and it has ϵ as its *unit*, that is, respectively,

$$(\delta \otimes 1_X) \circ \delta = (1_X \otimes \delta) \circ \delta \qquad \sigma_{A,A} \circ \delta = \delta \qquad (1_X \otimes \epsilon) \circ \delta = 1_X .$$

Appendix A provides some additional background and intuitions on these concepts. In a †-SMC each internal commutative monoid is also an internal commutative comonoid, for $\delta := m^\dagger$ and $\epsilon := e^\dagger$, and vice versa.

Definition 6. [15] An *observable structure* in a †-SMC is an internal cocommutative comonoid

$$(A \, , \ \delta : A \to A \otimes A \, , \ \epsilon : A \to \mathrm{I} \,)$$

for which δ is *special* and satisfies the *Frobenius law*, that is, respectively,

$$\delta^\dagger \circ \delta = 1_A \qquad \text{and} \qquad \delta \circ \delta^\dagger = (\delta^\dagger \otimes 1_A) \circ (1_A \otimes \delta).$$

Example 6. The unit object I canonically comes with observable structure

$$\delta_{\mathrm{I}} := \lambda_{\mathrm{I}} : \mathrm{I} \simeq \mathrm{I} \otimes \mathrm{I} \qquad \text{and} \qquad \epsilon_{\mathrm{I}} := 1_{\mathrm{I}} \, .$$

Example 7. In [15] it was observed that any orthonormal basis $\{|\psi_i\rangle\}_i$ for a Hilbert space $\mathcal{H}$ induces an observable structure by considering the linear maps that respectively 'copy' and 'uniformly delete' the basis vectors:

$$\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |\psi_i\rangle \mapsto |\psi_i\rangle \otimes |\psi_i\rangle \quad \text{and} \quad \epsilon : \mathcal{H} \to \mathbb{C} :: |\psi_i\rangle \mapsto 1 \, . \tag{2}$$

This observable structure is moreover 'basis capturing': we can recover the basis vectors from which we constructed δ as the solutions to the equation

$$\delta(|\psi\rangle) = |\psi\rangle \otimes |\psi\rangle \ .$$

This also shows that the basis $\{|\psi_i\rangle\}_i$ is faithfully encoded in the linear map δ, and that ϵ does not carry any additional data.

The following theorem characterises all observable structures in **FdHilb**.

Theorem 2. [16] *All observable structures in* **FdHilb** *are of the form* (2)*.*

Hence, observable structures precisely *axiomatise* orthonormal bases; more specifically, they characterise bases in terms of the tensor product only, without any reference to the additive linear structure of the underlying vector spaces.

Example 8. Each observable structure in **FdHilb** induces one in $\mathbf{FdHilb}_{wp}$ in the obvious manner. But, in the light of Proposition 3, the correspondence in **FdHilb** between observable structures and vector bases, becomes one between observable structures and state basis in $\mathbf{FdHilb}_{wp}$. Note that

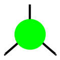
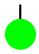
$$\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: \| \psi_i \rangle\rangle \mapsto \| \psi_i \rangle \otimes | \psi_i \rangle\rangle$$

does *not* define a unique δ anymore. In addition we need to specify that $| \sum_i |\psi_i\rangle\rangle$, the deleting point of the state basis, provides the unit for the comultiplication:

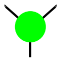
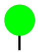
$$\delta^\dagger(- \otimes | \textstyle\sum_i |\psi_i\rangle\rangle) = 1_{\mathcal{H}} \, .$$

Corollary 1. *Observable structures in* $\mathbf{FdHilb}_{wp}$ *are in bijective correspondence with state bases via the correspondence outlined in example 8.*

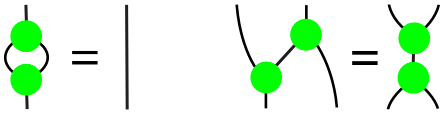
Notation. Within graphical calculus for †-SMCs we depict δ and ϵ by

 and  ,

and their adjoints, $\delta^\dagger$ and $\epsilon^\dagger$ by

 and  .

In the graphical language the speciality equation and the Frobenius law are:



Example 9. The notion of observable structure also applies to non-standard quantum-like theories. For example, it provides a generalised notion of basis for Spekkens' toy theory [9], despite the lack of an underlying vector space structure.

4.2. Spider theorem.

When the monoidal structure is strict, something which is of course always the case in graphical representation, observable structures obey the following remarkable theorem [29, 11] – similar results are known for concrete †-Frobenius algebras, for example, for 2D topological quantum field theories, as well as in more general categorical settings [27].

Theorem 3. *Let $\delta_n : A \to A^{\otimes n}$ be defined by the recursion*

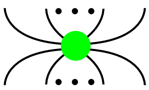
$$\delta_0 = \epsilon\,, \qquad\qquad \delta_n = \delta \circ (\delta_{n-1} \otimes 1_A)\,.$$

If $f : A^{\otimes n} \to A^{\otimes m}$ is a morphism generated from the observable structure (A, δ, ϵ), the symmetric monoidal structure maps, and the adjoints of all of these, and if the graphical representation of f is connected, then we have

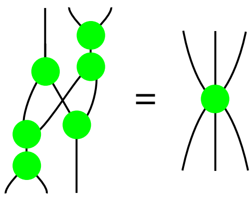
$$f = \delta_m \circ \delta_n^\dagger.$$

Hence, f only depends on the object A and the number of inputs and outputs.

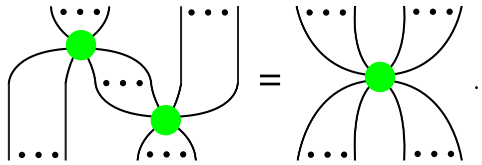
We represent this morphism as an $n + m$*-legged spider*

 .

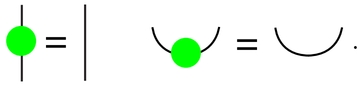
Theorem 3 provides a formal justification for 'fusing' connected dots representing δ, ϵ, $\delta^\dagger$ and $\epsilon^\dagger$ into a single dot:

 .

Otherwise put, 'spiders' admit the following composition rule:



i.e. when two spiders 'shake hands' they fuse into a single spider. There are some special cases of spiders were we simplify notation:



Note also that, conversely, the axioms of observable structure are consequences of this fusing principle, since the defining equations only involve connected pictures.

4.3. Induced compact structure.

Observable structures refine the †-compact structure which was used in [1,37], provided the latter is self-dual. Such †-compact categories are the †-augmented variant of Kelly's compact categories [25].

Proposition 5. [13] *Every observable structure is a self-dual †-compact structure, that is, a tuple $(A, \eta : \mathrm{I} \to A \otimes A)$ for which we have*

$$\lambda_A^\dagger \circ (\eta^\dagger \otimes 1_A) \circ (1_A \otimes \eta) \circ \rho_A = 1_A \qquad \text{and} \qquad \sigma_{A,A} \circ \eta = \eta\,.$$

Proof. It suffices to set $\eta := \delta \circ \epsilon^\dagger$. The first equation then follows from the Frobenius law together with the unit law for the comonoid. The second equation follows by commutativity of the comonoid. □

Graphically, the defining equations of self-dual compactness are

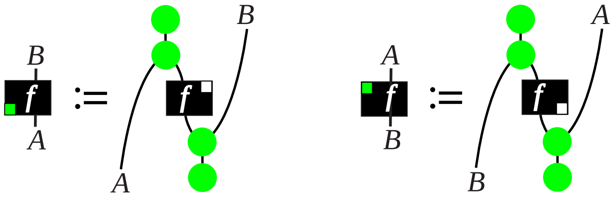
and these indeed straightforwardly follow from Theorem 3. We can now define the *transpose* $f^* : B \to A$ of a morphism $f : A \to B$ *relative to observable structures* $(A, \delta_A, \epsilon_A)$ and $(B, \delta_B, \epsilon_B)$ to be the morphism

$$f^* := (1_A \otimes \eta_B^\dagger) \circ (1_A \otimes f \otimes 1_B) \circ (\eta_A \otimes 1_B)\,,$$

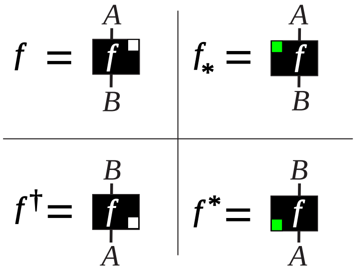
where $\eta_X := \delta_X \circ \epsilon_X^\dagger$, and the *conjugate* $f_* : A \to B$ of a morphism $f : A \to B$ *relative to observable structures* $(A, \delta_A, \epsilon_A)$ and $(B, \delta_B, \epsilon_B)$ to be the morphism

$$f_* := (1_B \otimes \eta_A^\dagger) \circ (1_B \otimes f^\dagger \otimes 1_A) \circ (\eta_B \otimes 1_A)\,.$$

Diagrammatically, f^* and f_* respectively are:

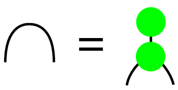


The green squares indicated the dependency of f^* and f_* on the observable structures $(A, \delta_A, \epsilon_A)$ and $(B, \delta_B, \epsilon_B)$. We are following [37] when representing the conjugate by reflection of the picture relative to a vertical mirror and the transpose by a 180° rotation:



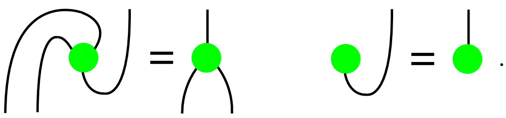
which is consistent with the fact that $f_* = (f^\dagger)^* = (f^*)^\dagger$, or equivalently, $f^\dagger = (f_*)^* = (f^*)_*$. In **FdHilb** the linear function f_* is obtained by conjugating the entries of the matrix of f when expressed in the observable structure bases.

Remark 1. The compact structures induced by different observable structures may or may not coincide [14]. For example, while compact structures induced by the basis structures that copy the $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ coincide, this is not the case anymore for the bases $\{|0\rangle, |1\rangle\}$ and $\{|0\rangle + i|1\rangle, |0\rangle - i|1\rangle\}$. When no confusion is possible we denote compact structure by:



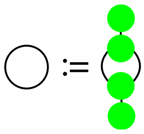
Corollary 2. [15] *If (A, δ, ϵ) is an observable structure then δ and ϵ are self-conjugate, that is, $\delta_* = \delta$ and $\epsilon_* = \epsilon$.*

Proof. By Theorem 3 we have:



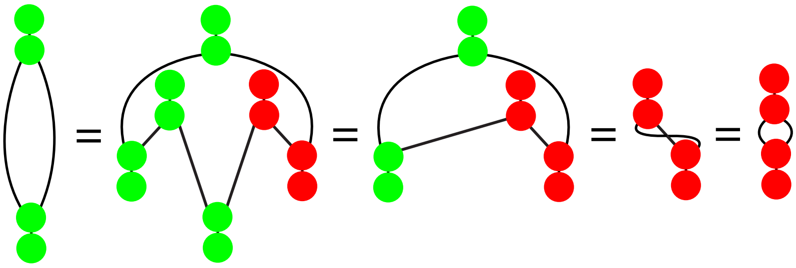
The first equality states that δ is self-conjugate and the second equality states that ϵ is self-conjugate. □

Definition 7. Let A be an object in a †-SMC which comes with a observable structure, and hence an induced compact structure (A, η). The *dimension* of A is $\dim(A) := \eta^\dagger \circ \eta$, represented graphically by a circle:



Lemma 1. *Dimension is independent of the choice of observable structure.*

Proof. We will depict two distinct observable structures in green and red respectively. Then, repeatedly relying on compactness we have



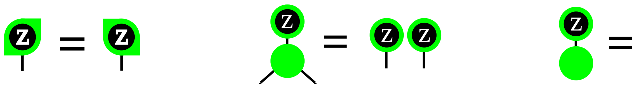
so the circles induced by the two observable structures coincide. □

4.4. Classical points and generalised bases.

We now provide a category-theoretic counterpart to the role played by basis vectors/states in **FdHilb** and $\mathbf{FdHilb}_{wp}$.

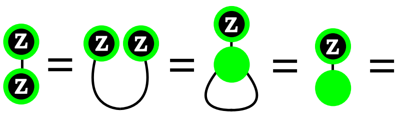
Definition 8. Given an observable structure (A, δ, ϵ), a morphism $z : \mathrm{I} \to A$ is a *classical point* iff it is a self-conjugate comonoid homomorphism.

Graphically, the defining equations for classical points become:



Proposition 6. *Classical points are normalised.*

Proof. Since each classical point $z : \mathrm{I} \to A$ is self-conjugate its adjoint $z^\dagger : A \to \mathrm{I}$ and its transpose $z^* : A \to \mathrm{I}$ coincide. Hence we have:



that is, $z^\dagger \circ z = 1_\mathrm{I}$. □

Example 10. In **FdHilb** the classical points for an observable structure are exactly the basis vectors $\{|v_1\rangle, \ldots, |v_n\rangle\}$ and in $\mathbf{FdHilb}_{wp}$ they constitute the corresponding observable $\{\|v_1\rangle\!\rangle, \ldots, \|v_n\rangle\!\rangle\}$. Hence, while in **FdHilb** the classical points completely determine an observable structure, this is not the case in $\mathbf{FdHilb}_{wp}$, where it is the classical points together with a deleting point that determine an observable structure.

The following is a category-theoretic generalisation of a notion of basis, either in the sense of Proposition 1 or in the sense of Proposition 2, which respectively applies to the categories **FdHilb** and $\mathbf{FdHilb}_{wp}$.

Definition 9. An observable structure (A, δ, ϵ) with classical points $\mathcal{B}$ is called a *vector basis* iff for all objects B and all morphisms $f, g : A \to B$ we have

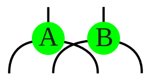
$$\big[\forall z \in \mathcal{B} : f \circ z = g \circ z\,\big] \Rightarrow\; f = g\,.$$

It is called a *state basis* iff for all B and all $f, g : A \to B$ we have

$$\big[\forall z \in \mathcal{B} \cup \{\epsilon^\dagger\} : f \circ z = g \circ z\,\big] \Rightarrow\; f = g\,.$$

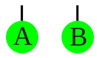
Proposition 7. *Two observable structures $(A, \delta_A, \epsilon_A)$ and $(B, \delta_B, \epsilon_B)$ canonically induce an observable structure on $A \otimes B$ with*

$$\delta_{A\otimes B} = (1_A \otimes \sigma_{A,B} \otimes 1_B) \circ (\delta_A \otimes \delta_B) \qquad \textit{i.e.}$$



and

$$\epsilon_{A\otimes B} = \epsilon_A \otimes \epsilon_B \qquad \textit{i.e.}$$



.

Moreover, if a is a classical point for $(A, \delta_A, \epsilon_A)$ and b is a classical point for $(B, \delta_B, \epsilon_B)$ then $a \otimes b$ is a classical point for $(A \otimes B, \delta_{A\otimes B}, \epsilon_{A\otimes B})$.

Proof. Straightforward verification of Definitions 6 and 8. □

Definition 10. We say that the monoidal tensor *lifts vector bases* iff for all vector bases $(A, \delta_A, \epsilon_A)$ with classical points $\mathcal{B}$ and $(B, \delta_B, \epsilon_B)$ with classical points $\mathcal{B}'$, all objects C, and all morphisms $f, g : A \otimes B \to C$, we have that

$$\big[\forall (z, z') \in \mathcal{B} \times \mathcal{B}' : f \circ (z \otimes z') = g \circ (z \otimes z')\big] \;\Rightarrow\; f = g$$

– hence it follows that the observable structure $(A\otimes B, \delta_{A\otimes B}, \epsilon_{A\otimes B})$ is also vector basis. Similarly, the monoidal tensor *lifts state bases* iff

$$\big[\forall (z, z') \in (\mathcal{B} \times \mathcal{B}') \cup \{(\epsilon_A^\dagger, \epsilon_B^\dagger)\} : f \circ (z \otimes z') = g \circ (z \otimes z')\,\big] \Rightarrow\; f = g\,.$$

– hence it follows that $(A \otimes B, \delta_{A\otimes B}, \epsilon_{A\otimes B})$ is also state basis.

Since observable structures induce compact structures we have the following.

Proposition 8. *Monoidal tensors always lift vector bases and state bases.*

Proof. If $f \circ (z \otimes z') = g \circ (z \otimes z')$ then $(f \circ (1_A \otimes z') \circ \rho_A) \circ z = (g \circ (1_A \otimes z') \circ \rho_A) \circ z$ so $f \circ (1_A \otimes z') \circ \rho_A = g \circ (1_A \otimes z') \circ \rho_A$ since $(A, \delta_A, \epsilon_A)$ is a vector/state basis, hence $f \circ (1_A \otimes z') = g \circ (1_A \otimes z')$, and hence $(1_A \otimes f) \circ (\eta_A \otimes 1_B) \circ \lambda_B \circ z' = (1_A \otimes g) \circ (\eta_A \otimes 1_B) \circ \lambda_B \circ z'$ with $\eta_A = \delta_A \circ \epsilon_A^\dagger$. So $(1_A \otimes f) \circ (\eta_A \otimes 1_B) \circ \lambda_B = (1_A \otimes g) \circ (\eta_A \otimes 1_B) \circ \lambda_B$ since $(B, \delta_B, \epsilon_B)$ is a vector/state basis, and hence $f = g$ follows by compactness. □

5. Phase shifts and a generalised spider theorem

In the preceding section we saw that observable structures correspond to bases of our state space; now we introduce an abstract notion of *phase* relative to a given basis, and generalise Theorem 3 to incorporate such phase shifts.

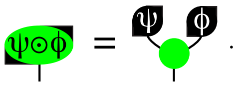
5.1. A monoid structure on points.

Definition 11. Let (A, δ, ϵ) be an observable structure in a †-SMC $\mathbf{C}$ and let $\mathbf{C}(\mathrm{I}, A)$ be the underlying set of points. We define a *multiplication on points*

$$- \odot - : \mathbf{C}(\mathrm{I}, A) \times \mathbf{C}(\mathrm{I}, A) \to \mathbf{C}(\mathrm{I}, A)$$

by setting, for all points $\psi, \phi \in \mathbf{C}(\mathrm{I}, A)$,

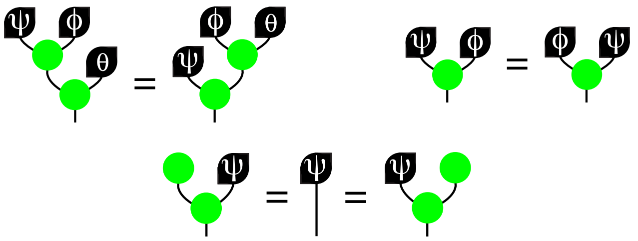
$$\psi \odot \phi := \delta^\dagger \circ (\psi \otimes \phi) \qquad \text{i.e.}$$



Notation. We use an asymmetric depiction of these points to indicate that they need not be self-conjugate.

Proposition 9. $(\mathbf{C}(\mathrm{I}, A), \odot, \epsilon^\dagger, (-)_*)$ *is an involutive commutative monoid.*

Proof. Associativity, commutativity, and that $\epsilon^\dagger$ is the monoid's unit, i.e.:



follow immediately from internal monoid laws for $(A, \delta^\dagger, \epsilon^\dagger)$. □

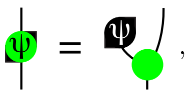
As well as giving an involutive commutative monoid on the points of A, we can use $\delta^\dagger$ to lift this monoid up to the endomorphisms of A.

Definition 12. For (A, δ, ϵ) an observable structure in a †-SMC $\mathbf{C}$ let

$$\Lambda : \mathbf{C}(I, A) \to \mathbf{C}(A, A)$$

be defined by setting, for each point $\psi \in \mathbf{C}(\mathrm{I}, A)$,

$$\Lambda(\psi) = \delta^\dagger \circ (\psi \otimes 1_A) \qquad \text{i.e.}$$



and we denote the range of Λ by $\Lambda(A, A)$.

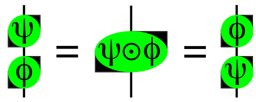
Proposition 10. *The map Λ is an isomorphism of involutive commutative monoids:*

$$\left(\mathbf{C}(\mathrm{I},A),\odot,\epsilon^\dagger,(-)_*\right)\simeq\left(\Lambda(A,A),\circ,1_A,(-)^\dagger\right)\ .$$

that is, explicitly:

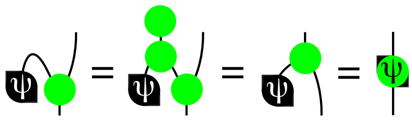
$$\Lambda(\psi\odot\phi)=\Lambda(\psi)\circ\Lambda(\phi)\qquad\Lambda(\epsilon^\dagger)=1_A\qquad\Lambda(\psi_*)=\Lambda(\psi)^\dagger,$$

and hence commutativity is inherited:

$$\Lambda(\psi)\circ\Lambda(\phi)=\Lambda(\psi\odot\phi)=\Lambda(\phi)\circ\Lambda(\psi)\quad \textit{i.e.}$$ 

.

Proof. Preservation of monoid multiplication and unit follow directly from the unit and commutativity law of the internal monoid. By Theorem 3 we have:



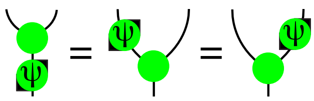
that is, conjugation of points is mapped to the adjoint of endomorphisms. □

Note that the notation of endomorphisms in $\Lambda(A,A)$ is invariant under 180° rotations. This is justified by the following proposition.

Proposition 11. *Each $\Lambda(\psi)\in\Lambda(A,A)$ is equal to its transpose.*

Proof. By Corollary 2 we have that δ is self-conjugate so it follows that $\Lambda(\psi)^*=\Lambda(\psi_*)$, which is equal to $\Lambda(\psi)^\dagger$ by Proposition 10. Conjugating $\Lambda(\psi)^*=\Lambda(\psi)^\dagger$ yields the desired result. This again could also be directly seen in graphical terms by relying on Theorem 3. □

Proposition 12. *The endomorphisms in $\Lambda(A,A)$ obey*

$$\Lambda(\psi)\circ\delta^\dagger=\delta^\dagger\circ(1_A\otimes\Lambda(\psi))=\delta^\dagger\circ(\Lambda(\psi)\otimes 1_A)\quad \textit{i.e.}$$ 

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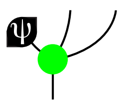
Proof. As a consequence of Theorem 3 all three diagrams normalise to:



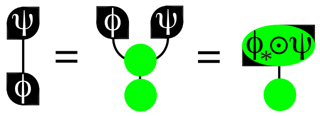
so they are indeed all equal. □

The following proposition shows that the inner-product structure on points is subsumed by the commutative involutive monoid structure on points.

Proposition 13. *For points $\psi, \phi : I \to A$ in* **C** *we have:*

$$\langle \phi | \psi \rangle := \phi^\dagger \circ \psi = \epsilon \circ (\phi_* \odot \psi)\,.$$

Proof. Unfolding the definition of the conjugate yields:

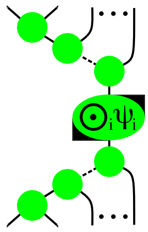


which completes the proof. □

5.2. The generalised spider theorem.

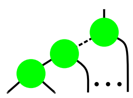
Theorem 4. *Let $f : A^{\otimes n} \to A^{\otimes m}$ be a morphism generated from the observable structure (A, δ, ϵ), the symmetric monoidal structure maps, their adjoints, and points $\psi_i : I \to A$ (not necessarily all distinct). If the graphical representation of f is connected then*

$$f = \delta_m \circ \Lambda\Big(\bigodot_i \psi_i\Big) \circ \delta_n^\dagger \qquad \textit{i.e.}$$



where

$$\delta_n := (\delta \otimes 1_{X^{\otimes n-2}}) \circ (\delta \otimes 1_{X^{\otimes n-3}}) \circ \ldots \circ (\delta \otimes 1_X) \circ \delta \quad \textit{i.e.}$$



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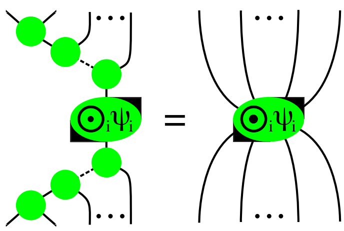
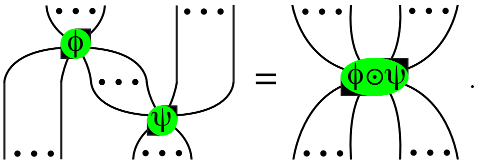
Proof. If neither δ nor $\delta^\dagger$ occurs in f, then result is trivial. Otherwise, all points ψ occurring in f may be lifted to $\Lambda(\psi)$; by virtue of Proposition 12, these commute freely with the observable structure, hence can all be collected together. The result now follows by applying Theorem 3. □

Theorem 4 is a strict generalisation of Theorem 3: diagrams with equal numbers of inputs and outputs are equal whenever the product of points occurring in them is equal. The theorem gives a specific normal form to which this entire class of diagrams is equal; this justifies the use of a single simplified diagram to represent the whole class: the *decorated spider*.



In graphical terms, Theorem 4 allows arbitrary diagrams constructed from the same observable structure to 'fuse' together provided we 'multiply decorations'. Otherwise put, 'decorated spiders' admit the following composition rule:



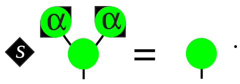
i.e. when two decorated spiders 'shake hands' they fuse into a single spider and their 'decorations' are multiplied.

5.3. Unbiased points.

Example 11. Let (δ, ϵ) be the observable structure corresponding to the standard basis of $\mathbb{C}^n$, and consider $|\psi\rangle = \sum_i c_i |\, i\,\rangle$. When written in the basis fixed by (δ, ϵ), $\Lambda(\psi)$ consists of the diagonal $n \times n$ matrix with $c_1, \ldots, c_n$ on the diagonal. Hence, $\Lambda(\psi)$ is unitary, upto a normalisation factor, if and only if $\|\psi\rangle\rangle$ is unbiased for $\{\|1\rangle\rangle, \ldots, \|n\rangle\rangle\}$. This fact admits generalisation to arbitrary †-SMCs.

Definition 13. We call a point $\alpha : \mathrm{I} \to A$ *unbiased* relative to (A, δ, ϵ) if there exists a scalar $s : \mathrm{I} \to \mathrm{I}$ such that:

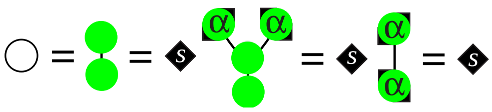
$$s \cdot \alpha \odot \alpha_* = \epsilon^\dagger \qquad \text{i.e.}$$



Example 12. Since the point $\epsilon^\dagger$ satisfies this definition due to the spider theorem, every observable structure has at least one unbiased point.

Lemma 2. *If an unbiased point α is normalised, i.e. $\alpha^\dagger \circ \alpha = 1_I$, then the scalar s in the above definition is equal to $D = \dim A$. Hence, if on the other hand $|\alpha|^2 := \alpha^\dagger \circ \alpha = D$ then this scalar is 1_I.*

Proof. We have:

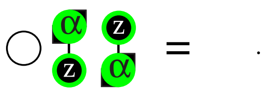


where we relied on Proposition 13. □

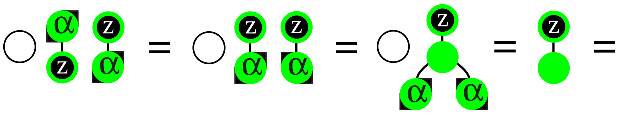
The expression $\alpha \odot \alpha_*$ denotes 'convolution' of α with itself; since the point $\epsilon^\dagger$ represents the uniform distribution over the basis defining δ, this definition indeed captures the usual understanding of what it means for a vector to be unbiased with respect to a basis. The following shows that this is exactly correct.

Lemma 3. *Let $\alpha, z : \mathrm{I} \to A$ be points of A such that α is unbiased and normalised, and z is classical, for (A, δ, ϵ); then*

$$D \cdot |\langle z|\alpha\rangle|^2 := D \cdot (z^\dagger \circ \alpha) \cdot (\alpha^\dagger \circ z) = 1_\mathrm{I} \qquad \textit{i.e.}$$



Proof. We have:



where we used the adjoint to the unbiasedness law. □

Since we operate in an arbitrary †-SMC, the scalars form a commutative monoid rather than a group; to retrieve the usual definition of unbiasedness,

$$|\langle z \mid \alpha \rangle| = \frac{1}{\sqrt{\dim A}}\,,$$

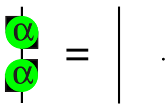
we simply divide.

5.4. The phase group.

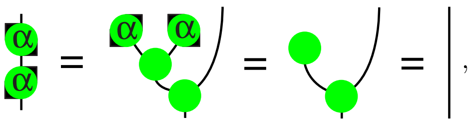
Proposition 14. *A point α of length $|\alpha|^2 = D$ is unbiased iff $\Lambda(\alpha)$ is unitary.*

Proof. Due the commutativity property of Λ in Proposition 10 we need check only one equation to show that $\Lambda(\alpha)$ is unitary, namely,

$$\Lambda(\alpha) \circ \Lambda(\alpha)^\dagger = 1_A \qquad \text{i.e.}$$



Suppose that α is unbiased; then by the spider theorem we have



as required. The other direction of the proof is essentially the same. □

Since unitary maps are invertible, they form a group, and this group structure transfers back onto the unbiased points.

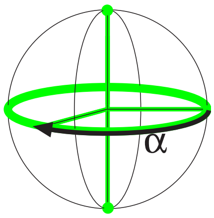
Theorem 5. *If in the isomorphism of involutive commutative monoids of Proposition 10 we restrict ourselves to unbiased points relative to the observable structure of length $|\alpha|^2 = D$, and unitary endomorphisms, then we obtain an isomorphism of abelian groups, with the involution as the inverse.*

Proof. This immediately follows from Proposition 14 and the fact that for a unitary morphism the adjoint is the inverse. □

Definition 14. The abelian group of endomorphisms of Theorem 5 is called the *phase group*, and its elements are called *phase shifts*.

Remark 2. We chose length $|\alpha|^2 = D$ since this results in the inverse taking a simple form. However, fixing another length would also have given us an abelian group structure.

Example 13. In $\mathbb{C}^2$, the phase shifts of the observable structure $\{\|0\rangle\rangle, \|1\rangle\rangle, \|+\rangle\rangle\}$ (cf. Proposition 1) are obtained by rotating along the equator of the Bloch sphere:



.

The states which are unbiased relative to $\{\|0\rangle\rangle, \|1\rangle\rangle\}$ are of the form $\|+_\theta\rangle\rangle := \|0\rangle + e^{i\theta}|1\rangle\rangle$, and so form a family parameterised by a phase θ. In particular, we have $\|+_{\theta_1}\rangle\rangle \odot \|+_{\theta_2}\rangle\rangle = \|+_{\theta_1+\theta_2}\rangle\rangle$, that is, the operation $\odot$ is simply addition modulo 2π, which is an abelian group with minus as inverse.

Example 14. Phase groups can provide an algebraic witness for physical differences between theories. For example, as shown in [10], the toy model category **Spek** (cf. Example 2) and the category **Stab** (a restriction of **FdHilb** to the qubit stabilizer states) are essentially the same except for the phase groups of their respective qubits. In the case of **Spek** the phase group is the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$, while for **Stab** the phase group is the cyclic four group $\mathbb{Z}_4$. This difference in phase groups is closely connected to the fact that while states in **Spek** always admit a local hidden variable representation, in **Stab** there are states which don't, namely the GHZ state [31].

Using the 'decorated spider notation' justified by Theorem 4, we can set

$:= \big(\Lambda(\alpha) \otimes \Lambda(\alpha)\big) \circ \delta \circ \Lambda(\alpha)^\dagger$ and


$:= \epsilon \circ \Lambda(\alpha)^\dagger$,

and for α unbiased relative to (A, δ, ϵ), again by the generalised spider theorem, it follows that these morphisms define an observable structure. Hence elements of the phase group define an observable structure.

5.5. Example: state transfer.

Referring to eq.(2), we define a particular observable structure $(\mathbb{C}^2, \delta_Z, \epsilon_Z)$, for the specific case of a qubit $\mathbb{C}^2$ in **FdHilb**, namely

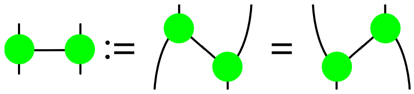
$$\delta_Z : |i\rangle \mapsto |ii\rangle\,, \qquad\qquad \epsilon_Z : |0\rangle + |1\rangle \mapsto 1\,.$$

Its classical points are $\{|0\rangle, |1\rangle\}$. As mentioned above, the unbiased points for $(\mathbb{C}^2, \delta_Z, \epsilon_Z)$ are of the form $|\alpha_Z\rangle = |0\rangle + e^{i\alpha}|1\rangle$, and $|\alpha_Z\rangle \odot_Z |\beta_Z\rangle = |(\alpha+\beta)_Z\rangle$, hence the group of unbiased points is isomorphic to the interval $[0, 2\pi)$ under addition modulo 2π. Direct calculation verifies that

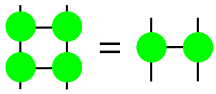
$$\Lambda^Z(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix},$$

and in particular, $\Lambda^Z(\pi) = \mathsf{Z}$. We will use $(\mathbb{C}^2, \delta_Z, \epsilon_Z)$ to study some protocols.

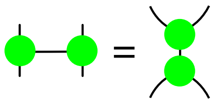
Assume that we have two qubits, one in an unknown state and one in the $|+\rangle$ state. We want to transfer the unknown state from the first qubit to the second. To simplify the discussion we condition on measurements outcomes, hence we can consider arbitrary projections. Our claim is that this can be done by means of two projections. Consider:



We claim that this is a projector. Idempotence, that is

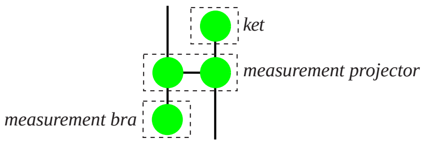
 ,

follows by the spider theorem, and self-adjointness follows by the fact that its picture is invariant under reflection. The reader may check these graphical computations by numerical ones, given that



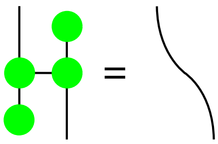
$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis. We propose the following protocol:



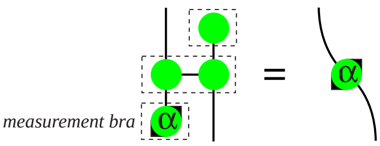
which, by the spider theorem, indeed provides the desired transfer:

$$\left(\begin{pmatrix} 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) =$$



$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So in the graphical calculus the whole matrix computation becomes a trivial, one-step application of the spider theorem. Now, in addition to the transfer, we also would like to at the same time apply a phase gate to the qubit we start from. From the decorated spider theorem we can conclude that



$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

indeed does the job. This protocol was introduced in [34] in order to reduce the required resources in measurement based quantum computing.

6. Complementarity ≡ Hopf law

6.1. The case of coinciding compact structures.

Definition 15. Two observable structures $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ in a †-SMC are called *complementary* if they obey the following rules:

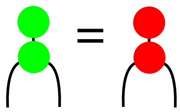
comp1 whenever $z : I \to A$ is classical for (δ_Z, ϵ_Z) it is unbiased for (δ_X, ϵ_X);
comp2 whenever $x : I \to A$ is classical for (δ_X, ϵ_X) it is unbiased for (δ_Z, ϵ_Z);

We abbreviate *complementary observable structure* as COS.

Notation. Graphically we distinguish two distinct observable structures in terms of their colour. To emphasise that classical points are copied by an observable structure of one colour, say green, while unbiased with respect to an observable structure of another colour, say red, we denote them by:

that is, the outside colour indicates which observable structure copies this point, while the inner colour shows to which observable structure the point is unbiased. The fact that we denote these points in a manner which is invariant under conjugation is a consequence of the fact that in this section we will assume that the compact structures induced by the two COSs coincide, i.e.:

$$\delta_Z \circ \epsilon_Z^\dagger = \delta_X \circ \epsilon_X^\dagger \qquad \text{i.e.}$$



It then follows that classical points of one colour are not only self-conjugate for 'their own colour' (cf. Corollary 2), but also self-conjugate for 'the other colour':

Proposition 15. *If $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ are two observable structures for which the induced compact structures $\eta_Z := \delta_Z \circ \epsilon_Z^\dagger$ and $\eta_X := \delta_X \circ \epsilon_X^\dagger$ coincide, then for classical points $z : I \to A$ of (δ_Z, ϵ_Z) and $x : I \to A$ of (δ_X, ϵ_X) we have*

$$z = (z^\dagger \otimes 1_A) \circ \eta_X \qquad \textit{and} \qquad x = (x^\dagger \otimes 1_A) \circ \eta_Z \,,$$

which we can depict graphically as:

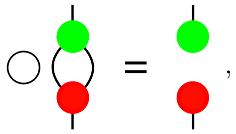
As mentioned in Remark 1 this is not the case in general, but all our results can be easily adjusted; we discuss the general case in Section 6.3.

The comonoid homomorphism laws governing classical points become:

and the mutual unbiasedness conditions become:

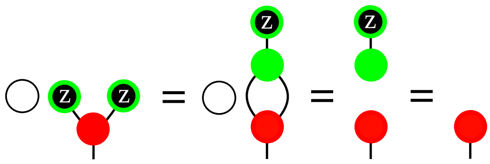
Theorem 6. *Let $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ be two observable structures of which the induced compact structures coincide. If these observable structures obey a 'scaled Hopf law with trivial antipode', namely*

$$D \cdot \delta_X \circ \delta_Z^\dagger = \epsilon_X^\dagger \circ \epsilon_Z \qquad \text{i.e.}$$



then they are complementary observable structures.

Proof. We have to show that **comp1** and **comp2** of Definition 15 hold. We do this graphically for **comp1**:

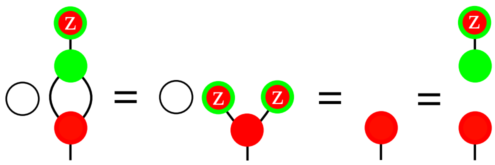


and the same argument holds for **comp2** subject to exchanging the colours. □

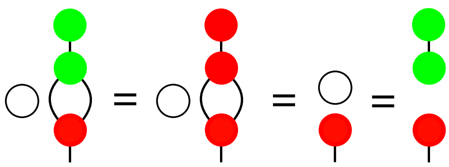
The converse to Theorem 6 also holds if one of the observable structures involved is either a vector or state basis – cf. Definition 9.

Theorem 7. *If $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ are COSs, and if at least one of these is either a vector basis or a state basis, then the Hopf law of Theorem 6 holds.*

Proof. We need to show that when applying the left-hand side and the right-hand side of the Hopf law to an element of the basis that both sides are equal, for all the elements of the basis. For the case of a vector basis we have:



and for the case of a state basis we moreover have:



where the first equation relies on coinciding compact structures. □

6.2. Example: the controlled not gate.

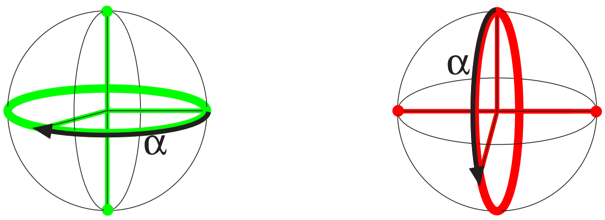
We take the ‘green’ observable structure (δ_Z, ϵ_Z) as in the example in Section 5.5, and define a complementary ‘red’ observable structure (δ_X, ϵ_X) setting

$$\delta_X : \begin{cases} |+\rangle \mapsto |{+}{+}\rangle \\ |-\rangle \mapsto |{-}{-}\rangle \end{cases} \qquad\qquad \epsilon_X : \sqrt{2}\,|0\rangle \mapsto 1\,.$$

Its classical points are $\{|+\rangle, |-\rangle\}$ and its unbiased points have the form

$$|\alpha_X\rangle = \sqrt{2}(\cos\frac{\alpha}{2}\,|0\rangle + \sin\frac{\alpha}{2}\,|1\rangle)\,, \qquad \text{and} \qquad \Lambda^X(\alpha) = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}\,,$$

in particular, $\Lambda^X(\pi) = \mathsf{X}$. On the Bloch sphere we have:



.

We have $\mathsf{Z} \circ |\alpha_X\rangle = |{-\alpha_X}\rangle$, and, upto a global phase, $\mathsf{X} \circ |\alpha_Z\rangle = |{-\alpha_Z}\rangle$.

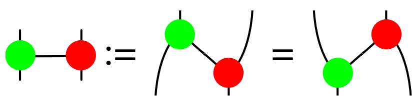
We claim that we now have enough expressive power to write down any arbitrary linear map from n qubits to m qubits. In the example in Section 5.5 we showed that we can express arbitrary one-qubit phase gates by relying on the phase group structure. Combining both the ‘green’ and the ‘red’ phases allows us to write down any arbitrary one-qubit unitary in terms of its Euler-angle decompositions on the Bloch sphere:

$$\Lambda^Z(\gamma) \circ \Lambda^X(\beta) \circ \Lambda^Z(\alpha) =$$

.

Moreover, the pair of complementary observables also allows us to graphically denote the controlled not gate $\wedge\mathsf{X}$. One verifies by concrete calculation that:

$$\begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,1\,0 \end{pmatrix} =$$

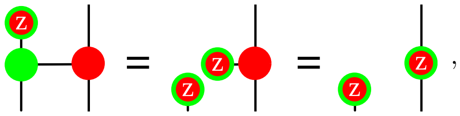


.

But we can also give a purely abstract proof. Let $|z\rangle$ be a classical point for the green observable structure (δ_Z, ϵ_Z). Since (δ_Z, ϵ_Z) and the red observable structure (δ_X, ϵ_X) are complementary, $|z\rangle$ is an unbiased point for (δ_X, ϵ_X). So z can be represented by certain special ‘angles’ of the phase group of (δ_X, ϵ_X). In the case of the qubit these are the angles 0 and π respectively, since $|0\rangle = |0_X\rangle$ and $|1\rangle = |\pi_X\rangle$. Graphically we denote $|0_X\rangle$ and $|\pi_X\rangle$ respectively as:

.

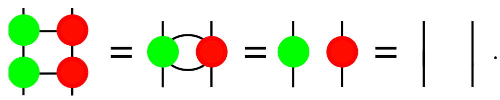
When evaluating $\wedge\mathsf{X}$ with an input to its control qubit (the green end), by first relying on the fact that $|z\rangle$ is copied by the green observable structure and then that it is unbiased for the red one we obtain:



where for $z = 0$ we have $\Lambda^X(0) = \text{identity}$, while for $z = \pi$ we have $\Lambda^X(\pi) = \mathsf{X}$.

We now compose this gate with itself:

$$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix} \circ \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix} =$$



Hence the Hopf law, which stands for complementarity, plays a key role in this computation, from which it also follows that $\wedge\mathsf{X}$ is indeed unitary.

With a $\wedge\mathsf{X}$ gate and arbitrary one qubit unitaries at our disposal we can now depict arbitrary n-qubit unitaries. Hence, we can also depict arbitrary n-qubit states as the image of any n-qubit state—for example $\epsilon_X^\dagger \otimes \ldots \otimes \epsilon_X^\dagger$—by a well-chosen unitary. Finally, compact structure allows us to obtain any arbitrary linear map from n qubits to m qubits from some $n+m$ qubit state by relying on the compact structure, which realises map-state duality.

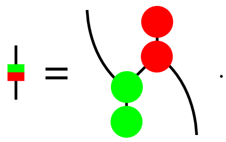
The quantum teleportation protocol, including classical communication, also crucially relies on the Hopf law. We defer this example until in Section 9.6.

6.3. The general case: dualisers as antipodes.

The results of this section still hold when the compact structures induced by the two COSs do coincide, provided we extend the observable structures formalism with dualisers, as described in [14] by Paquette, Perdrix and one of the authors.

Definition 16. The *dualiser* of two distinct observable structures $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ on the same object A is

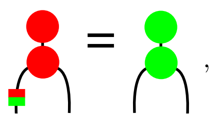
$$d_{ZX} = ((\epsilon_Z^\dagger \circ \delta_Z) \otimes 1_A) \circ (1_A \otimes (\epsilon_X \circ \delta_X^\dagger)) \quad \text{i.e.}$$



Remark 3. If the induced compact structures of the two observable structures on A happen to coincide then their dualiser is 1_A, hence trivial. More generally, the dualiser is easily seen to always be unitary, by compactness.

Lemma 4. *For observable structures $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ we have*

$$(d_{XZ} \otimes 1_A) \circ \delta_X \circ \epsilon_X^\dagger = \delta_Z \circ \epsilon_Z^\dagger \qquad \text{i.e.}$$



and the equation obtained by exchanging the colours also holds.

Proof. Straightforward by compactness. □

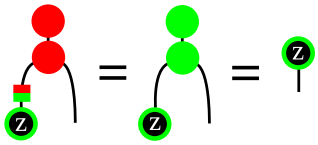
Remark 4. Lemma 4 together with unitarity of the dualiser provides a more concise proof of the fact that dimension does not depend on observable structure.

Lemma 5. *Let $z : \mathrm{I} \to A$ be a classical point for observable structure $(A, \delta_Z, \epsilon_Z)$ and let $(A, \delta_X, \epsilon_X)$ be another observable structure. Then the point*

$$d_{ZX} \circ z : \mathrm{I} \to A \qquad \text{i.e.}$$

is the conjugate to z for the compact structure induced by $(A, \delta_X, \epsilon_X)$.

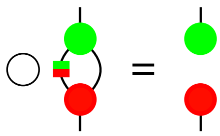
Proof. The (A, η_X)-conjugate to $d_{ZX} \circ z$ is, using Lemma 4,



since the (A, η_Z)-transpose to z is also its adjoint. □

Theorem 8. *If two observable structures obey*

$$D \cdot \delta_X \circ (d_{ZX} \otimes 1_A) \circ \delta_Z^\dagger = \epsilon_X^\dagger \circ \epsilon_Z \qquad \text{i.e.}$$



,

to which we refer as the 'scaled Hopf law with the dualiser as the antipode', then they are complementary observable structures. Conversely, if $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ are COSs, and if at least one of these is either a vector basis or a state basis, then the Hopf law depicted above holds.

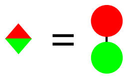
Proof. Lemma 4 and Lemma 5 straightforwardly allow to adjust the proofs of Theorem 6 and Theorem 7 to this more general situation. □

7. Closed complementary observable structures

In this section we study a special case of complementary observables, to which we refer as *closed*. The main theorem of this section provides a number of equivalent characterisations of these. Again, for the sake of conciseness and clarity, we assume that the compact structures induced by the observable structures coincide. In Subsection 7.5 we briefly discuss the general case.

7.1. Coherence for observable structures.

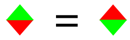
In this section we provide a category-theoretic generalisation of coherence for mutually unbiased vector and state bases of Definition 5 above. First we set:



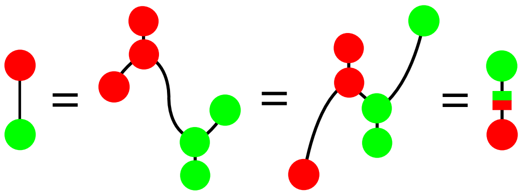
and denote this scalar by $\sqrt{D}$ for reasons we explain below.

Lemma 6. *If $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ are two observable structures for which the induced compact structures coincide then we have:*

$$\sqrt{D}^{\dagger} = \sqrt{D} \qquad \text{i.e.}$$

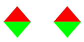
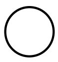
 .

Proof. By Corollary 2 we know that ϵ_Z are ϵ_X are both self-conjugate, that is, ϵ_Z is self-conjugate for the compact structure induced by $(A, \delta_Z, \epsilon_Z)$ and ϵ_X is self-conjugate for the compact structure induced by $(A, \delta_X, \epsilon_X)$. Hence:

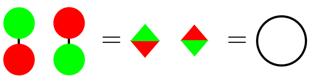


so the result follows by the fact that the dualiser is trivial when the induced compact structures coincide. □

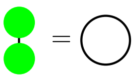
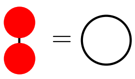
We now explain our choice of the notation $\sqrt{D}$ for  , which translates as

 $= \sqrt{D} \cdot \sqrt{D} = D =$  ,

so by Lemma 6 we have

 . (3)

Since the 'length' of both $\epsilon_Z^{\dagger}$ and $\epsilon_X^{\dagger}$ is $\sqrt{D}$—cf. Lemma 3—i.e.

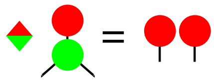
  ,

eq.(3) states that $\epsilon_Z^{\dagger}$ and $\epsilon_X^{\dagger}$ are unbiased, which is a natural requirement for a pair of COSs. It moreover follows from coherence of complementary observables as we show below in Proposition 16.

Definition 17. Two observable structures $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ in a †-SMC are called *coherent* if they obey the following two rules:

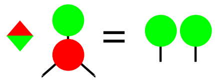
coher1 ϵ_X satisfies:

$$\sqrt{D} \cdot \delta_Z \circ \epsilon_X^\dagger = \epsilon_X^\dagger \otimes \epsilon_X^\dagger \qquad \text{i.e.}$$

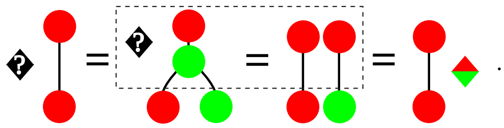


coher2 ϵ_Z satisfies:

$$\sqrt{D} \cdot \delta_X \circ \epsilon_Z^\dagger = \epsilon_Z^\dagger \otimes \epsilon_Z^\dagger \qquad \text{i.e.}$$



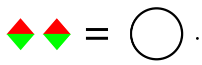
Remark 5. The condition **coher1** in Definition 17 states that $\epsilon_X^\dagger$ differs from a classical point of $(A, \delta_Z, \epsilon_Z)$ by a scalar factor of $\sqrt{D}$. The choice of $\sqrt{D} = (\epsilon_Z^\dagger \circ \epsilon_X^\dagger)$ for this scalar is not arbitrary but is imposed by the fact that ϵ_Z is the unit for the comultiplication δ_Z. To see this, it suffices to post-compose both sides of **coher1** with $\epsilon_Z \otimes \epsilon_X$, which results in:



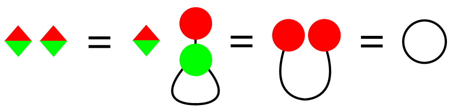
Dually, condition **coher2** asserts the same relationship between $\epsilon_Z^\dagger$ and $(A, \delta_X, \epsilon_X)$.

Proposition 16. *For coherent observable structures on A we have:*

$$\sqrt{D} \cdot \sqrt{D} = D = \dim(A) \qquad \textit{i.e.}$$



Proof. We have:



where in the last step we relied on coinciding compact structures. □

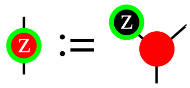
Example 15. For the case of **FdHilb** and **FdHilb**$_{wp}$ this category-theoretic notion of coherence coincides with the one defined in Definition 5. For these cases Theorem 1 tells us that we don't loose generality by requiring that COSs are coherent.

7.2. Commutation for observable structures.

Several notions of 'commutation' may apply to observable structures. In this section we consider three of these.

Remark 6. One should clearly distinguish the notions of commutation that we consider in this paper from that of *commuting observables* as found in most of the quantum theory literature. The kind of observables considered here are complementary, and are thus non-commuting in the usual sense. What we wish to expose here is that certain alternative notions of commutation, which are useful in computations, do apply to the specific case of complementary observables.

Notation. For an observable structure $(A, \delta_Z, \epsilon_Z)$, with classical points $\mathcal{B}_Z$ depicted in green, and an observable structure $(A, \delta_X, \epsilon_X)$ depicted in red, we set for all $z \in \mathcal{B}_Z$:



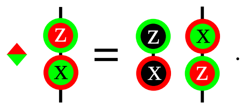
The use of two colours in this graphical representation reflects its dependence on two observable structures. We denote this morphism by $\Lambda^X(z)$. By Lemma 2 we know that, when z is unbiased for $(A, \delta_X, \epsilon_X)$, this morphism is unitary if and only if $z^\dagger \circ z = D$. Therefore it is more convenient to consider classical points to have length $\sqrt{D}$ rather than being normalised. The comonoid homomorphism laws governing classical points then become:

where we somewhat abusively depict $\sqrt{D}$ by ◆ as in the case of coherent observable structures. Complementarity, in the case it holds, becomes:

Remark 7. The similarity between the graphical notation for $\Lambda^X(z)$ and that of the classical points for COSs in Section 6.1 anticipates Theorem 10.

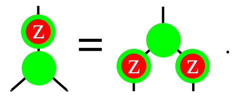
Definition 18. An observable structure $(A, \delta_Z, \epsilon_Z)$ with classical points $\mathcal{B}_Z$, and an observable structure $(A, \delta_X, \epsilon_X)$ with classical points $\mathcal{B}_X$, satisfy *operator commutation* iff for all $z \in \mathcal{B}_Z$ and all $x \in \mathcal{B}_X$:

$$\Lambda^Z(x) \circ \Lambda^X(z) = (x^\dagger \circ z) \cdot (\Lambda^X(z) \circ \Lambda^Z(x)) \qquad \text{i.e.}$$



Definition 19. An observable structure $(A, \delta_Z, \epsilon_Z)$ with classical points $\mathcal{B}_Z$, and an observable structure $(A, \delta_X, \epsilon_X)$, satisfy *comultiplicative commutation* iff for all $z \in \mathcal{B}_Z$:

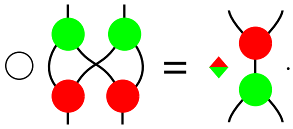
$$\delta_Z \circ \Lambda^X(z) = (\Lambda^X(z) \otimes \Lambda^X(z)) \circ \delta_Z \qquad \text{i.e.}$$



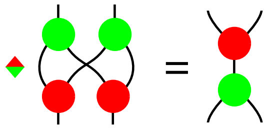
Remark 8. While this equation seems akin to the defining equation for classical points it carries a lot more structure. The reason for this is the involvement of two observable structures, which is exposed by the colouring.

Definition 20. Observable structures $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ satisfy *bialgebraic commutation* iff:

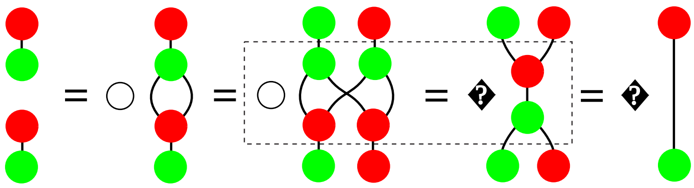
$$D \cdot (\delta_X^\dagger \otimes \delta_X^\dagger) \circ (1 \otimes \sigma \otimes 1) \circ (\delta_Z \otimes \delta_Z) = \sqrt{D} \cdot \delta_Z \circ \delta_X^\dagger \quad \text{i.e.}$$



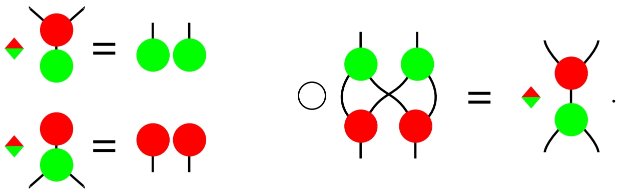
Remark 9. For coherent observable structures, by Proposition 16, when $\sqrt{D}$ admits an inverse we can simplify the bialgebraic commutation equation to:



To see that the choice of scalars is not arbitrary, we can, for example, either assume coherence or the Hopf law for the observable structures, both resulting in:



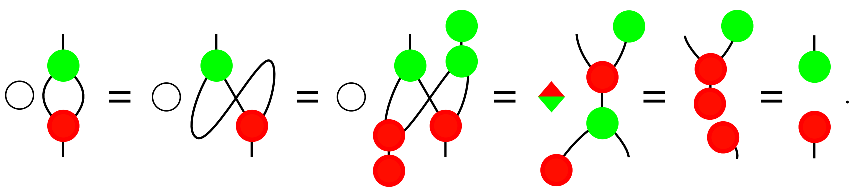
Definition 21. A *scaled bialgebra* is a pair of coherent observable structures which satisfy bialgebraic commutation, that is, all together:



Remark 10. If we remove the scalars from the definition of a scaled bialgebra and adjoin the equation $\epsilon_Z \circ \epsilon_X^\dagger = 1_{\mathrm{I}}$ – which is trivial anyway when taken 'up to a scalar' – then we obtain the usual notion of a *bialgebra* [7, 41].

Theorem 9. *Each scaled bialgebra satisfies the Hopf law.*

Proof. We have:



where the 1st step uses compactness, the 2nd step the spider theorem, the 3rd step uses the bialgebra law, the 4th step uses coherence i.e. the green comultiplication copies the red unit, and the 5th step uses the spider theorem. □

Corollary 3. *If a pair of observable structures constitutes a scaled bialgebra then these are complementary observable structures.*

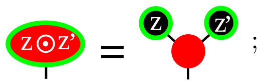
While at first sight the three notions of commutation we have introduced in this section look very different, in fact, they boil down to the same thing in all of our example categories, as we shall see in Theorem 10 below.

7.3. Closedness for observable structures.

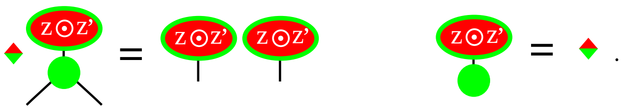
Definition 22. The classical points $\mathcal{B}_Z$ of an observable structure $(A, \delta_Z, \epsilon_Z)$ are *closed* for another observable structure $(A, \delta_X, \epsilon_X)$ iff for all $z, z' \in \mathcal{B}_Z$ we have

$$z \odot_X z' \in \mathcal{B}_Z .$$

From the assumption that the induced compact structures coincide, since by Corollary 2 and Definition 8 we have that δ_X, z and z' are all self-conjugate, it follows that the composite $z \odot_X z'$ is also self-conjugate. Hence, setting:



The closedness requirement is depicted graphically as:



Remark 11. If the observable structures are coherent, then the normalisation condition is also trivially satisfied. If classical points were taken to be normalised then we would take $\sqrt{D} \cdot z \odot_X z'$ rather than $z \odot_X z'$ in Definition 22.

Remark 12. Again, similarly to Remark 7, this notation which seems to indicate that $z \odot_X z'$ is unbiased to $(A, \delta_X, \epsilon_X)$ anticipates Theorem 10 below.

We now show that on every Hilbert space we can find a pair of closed COSs, and hence by Theorem 1 we can find a pair of closed coherent COSs.

Proposition 17. *In* **FdHilb** *there exist pairs of closed coherent COSs on Hilbert spaces $\mathbb{C}^n$ for any dimension $n \in \mathbb{N}$.*

Proof. Without loss of generality we take the first observable structure as being defined by the standard basis on $\mathbb{C}^n$, i.e. $\delta : |i\rangle \mapsto |i\rangle \otimes |i\rangle$ with the deleting point $\epsilon^\dagger = \sum_i |i\rangle$. Notice that the multiplication induced by this observable structure is simply point-wise:

$$(\sum_i a_i\,|i\rangle) \odot (\sum_i b_i\,|i\rangle) = \sum_i a_i b_i\,|i\rangle .$$

We need to find a basis which contains $\epsilon^\dagger$, is unbiased with respect to the standard basis, and is closed under $\odot$. It is routine to check that the family

$$|f_j\rangle = \frac{1}{\sqrt{N}} \sum_k \omega_n^{jk} |k\rangle \ ,$$

where j and k range from 0 to $n-1$, and $\omega_n = e^{2\pi i/n}$, provides an orthonormal basis satisfying these conditions. To complete the proof we simply choose $|0\rangle$ as the deleting point. □

Corollary 4. *There exist pairs of closed coherent complementary observable structures for any dimension* $\mathbf{FdHilb}_{wp}$.

Remark 13. Thanks to Theorem 1, to find a pair of coherent COSs on $\mathbb{C}^d$ it suffices to find any *dephased complex Hadamard matrix*, that is, an orthogonal matrix whose entries are all complex units, and whose first row and column are all ones. The columns of the matrix will provide the required basis. The family $|f_j\rangle$ used above are a particular example: they form the columns of the *d-dimensional Fourier matrix.* If $d = 2$, 3 or 5 the *only* dephased Hadamards are Fourier matrices [42], hence we can conclude that every pair of coherent COSs in these dimensions is closed. However this does not hold in general. If $d = 4$, for example,

$$F_4(x) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & ie^{ix} & -1 & -ie^{ix} \\ 1 & -1 & 1 & -1 \\ 1 & -ie^{ix} & -1 & ie^{ix} \end{pmatrix}$$

is not closed when x is irrational. Similar counterexamples can be constructed for dimensions $d \geq 6$. This shows that the notion of a closed COSs is strictly stronger than that of a COS.

Since closed COSs exist for all dimensions, for most practical purposes we can assume that COSs are both coherent and closed.

Closed COSs moreover behave well with respect to the monoidal structure, in that the canonical induced observable structures of Proposition 7, which are defined on the tensor space, inherit both complementarity and closedness.

Proposition 18. *Let* $(A, \delta_Z, \epsilon_Z)$ *and* $(A, \delta_X, \epsilon_X)$ *be coherent COSs such that* (δ_Z, ϵ_Z) *is closed with respect to* (δ_X, ϵ_X)*, and let* $(B, \delta_{Z'}, \epsilon_{Z'})$ *and* $(B, \delta_{X'}, \epsilon_{X'})$ *be coherent COSs such that* $(\delta_{Z'}, \epsilon_{Z'})$ *is closed with respect to* $(\delta_{X'}, \epsilon_{X'})$*; then the canonical observable structure on the joint space* $(A \otimes B, \delta_Z \otimes \delta_{Z'}, \epsilon_Z \otimes \epsilon_{Z'})$ *is both complementary and closed with respect to* $(A \otimes B, \delta_X \otimes \delta_{X'}, \epsilon_X \otimes \epsilon_{X'})$.

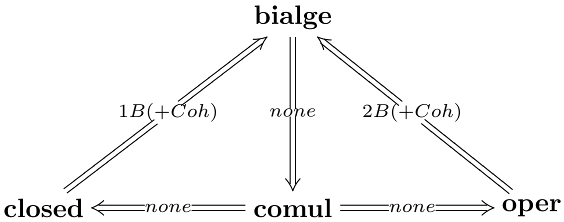
7.4. Our main theorem on pairs of closed observable structures.

Theorem 10. *The following are equivalent for two observable structures:*

closed *They are closed.*
oper *They obey operator commutation.*
comul *They obey comultiplicative commutation.*

bialge *They obey bialgebraic commutation.*

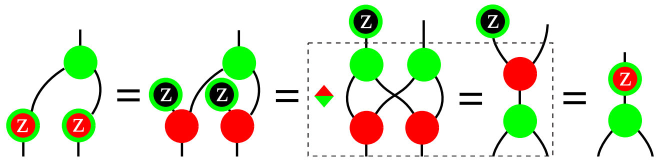
subject to the following requirements:



where 'none' stands for no additional requirements, except for the ones explicitly stated in the proof; where '1B' means that at least one of the observable structures has either a vector basis or a state basis, where '2B' means that this is the case for both observable structures, and '(+ Coh)' means that in the case of state bases we also require coherence. We indicate in the proof where we assume that $\sqrt{D}$ has an inverse and where we use the fact that compact structures coincide.

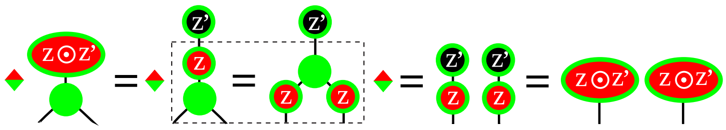
Proof. We show all required implications graphically:

– **bialge** ⇒ **comul**:

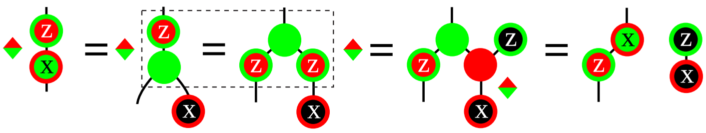


Here we assumed that $\sqrt{D}$ has an inverse.

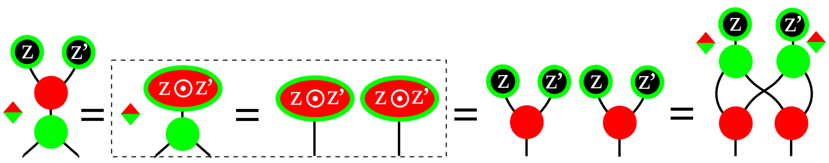
– **comul** ⇒ **closed**:



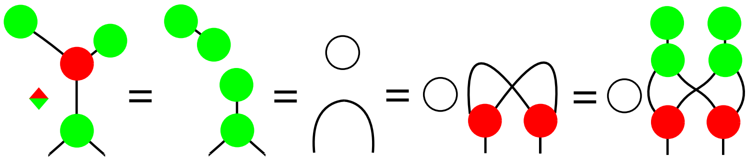
– **comul** ⇒ **oper**:



– **closed** ⇒ **bialge**:

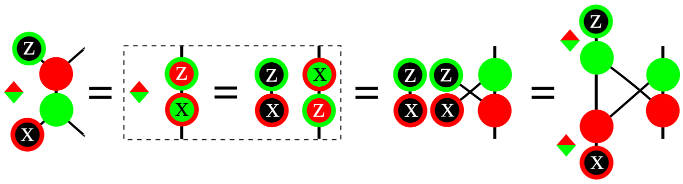


The assumption that the classical points for the green observable structure constitute a vector basis, together with the fact that the monoidal tensor lifts to vector basis, imply bialgebraic commutation. By coherence we have:



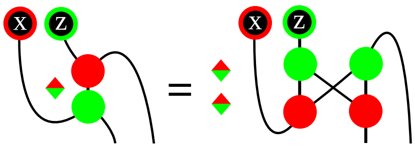
so the result holds when there is a state basis for the green observable structure. Steps 2–4 assume that the induced compact structures coincide.

– **oper** ⇒ **bialge**:

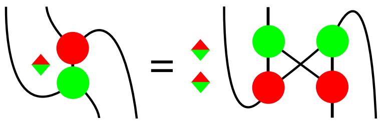


(4)

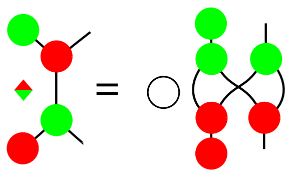
so by compactness we have:



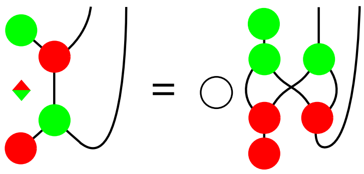
Under the assumption that the classical points both for the green and the red observable structure constitute a vector basis, together with the fact that the monoidal tensor lifts vector bases, we have:



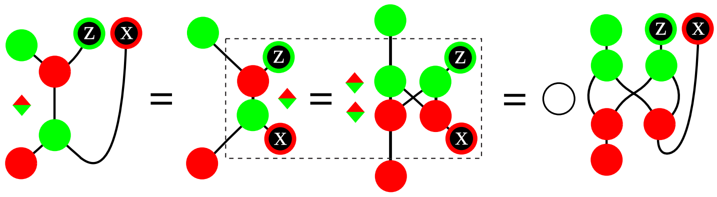
from which the bialgebra follows by compactness. The two diamonds are equal to a circle given that the compact structures coincide. For the case that both observable structures have a state basis it remains to be shown that:



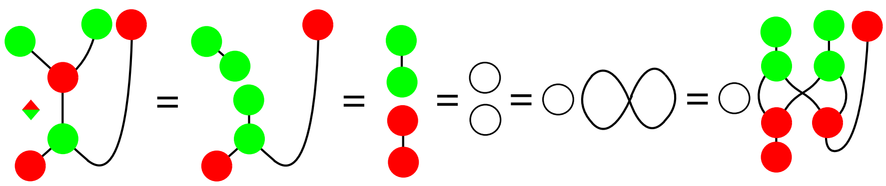
To do so, we now show that the equation



holds, by relying on the fact that we have a state basis for both observable structures, and as above, by again relying on compactness. We have:



where in the dotted area we used derivation (4) above. Finally:



where the last step assumes that induced compact structures coincide.

This concludes this proof. □

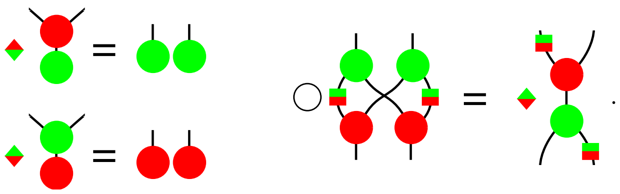
Remark 14. We leave it to the reader to see how '2B (+ Coh)' factors into requirements for **oper** ⇒ **comul** and **comul** ⇒ **bialge**.

The examples of COSs discussed in Proposition 17 satisfy all the equations stated in Theorem 10. In particular, they constitute scaled bialgebras. These equations are strictly stronger than the Hopf law by Theorem 11, and hence all pairs of observable structures that satisfy them are COSs.

7.5. The case of non-coinciding induced compact structures.

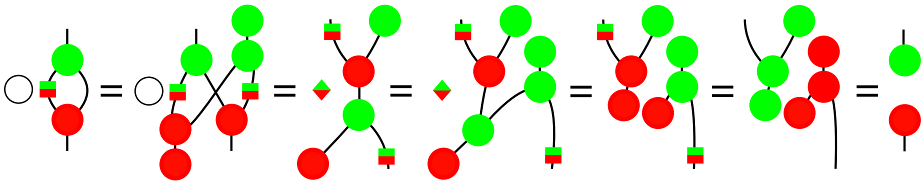
Again, we can adjust Theorem 10 to the case that compact structures do not coincide. We do not present this in detail but merely indicate how one needs to adjust the scaled bialgebra structure such that it still implies complementarity, that is, the *dualised Hopf law* of Theorem 8.

Definition 23. A *scaled bialgebra with dualisers* is a pair of coherent observable structures which satisfy a modified form of bialgebraic commutation:



Theorem 11. *Any scaled bialgebra with dualisers satisfies the scaled Hopf law with the dualiser as the antipode, so its observable structures are complementary.*

Proof. We have:



where we used the properties of the dualiser from Section 6.3. □

8. Further group structure and the classical automorphisms

In Section 5.4 we saw how the abelian group of phase shifts arose naturally from the presence of unbiased points for a given observable structure. When we have a pair of COSs, the two phase groups can interfere with each other, an interaction which arises from the special role of the classical points within each phase group.

In the following, suppose $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ are coherent COSs which jointly form a scaled bialgebra. Let U_Z denote all the unbiased points for (δ_Z, ϵ_Z), and let C_Z denote its classical points; define U_X and C_X similarly. By virtue of complementarity we have $C_X \subseteq U_Z$ and $C_Z \subseteq U_X$. Recall that by Proposition 5, $(U_Z, \odot_Z)$ is an abelian group, isomorphic to the phase group of (δ_Z, ϵ_Z).

Proposition 19. *C_X is a subgroup of $(U_Z, \odot_Z)$ if either: C_X is finite; or, if the two observable structures give rise to the same compact structure.*

Proof. C_X is always a submonoid of U_Z because of the closure and coherence of the two observable structures; any finite submonoid is a subgroup. Alternatively, given a point $x \in C_X$, its inverse in U_Z is given by its conjugate with respect to the compact structure of (δ_Z, ϵ_Z); by the definition of classicality, x is self conjugate with respect to the compact structure of (δ_X, ϵ_X). Hence, if these compact structures agree (cf. Proposition 15), $x^{-1} = x$ in U_Z, so C_X is a subgroup. □

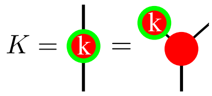
Remark 15. Shared compact structure is a powerful assumption. The proof above indicates that in the case of coinciding compact structures, not only do the classical points within U_Z form a subgroup, but the resulting group is a product of copies of S_2. In the case of qubits described by X and Z spins, the compact structure is shared, and the resulting classical subgroup is just S_2.

Proposition 20. *For all $x \in C_X$, $\Lambda^Z(x)$ is a left action on U_Z and, in particular, a permutation on C_X.*

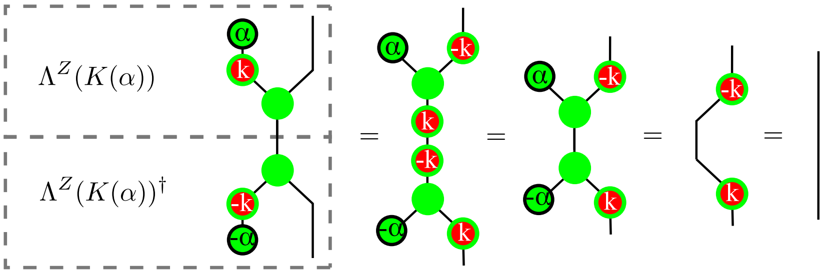
Proof. For any $\psi : I \to A$, we have $\Lambda^Z(x) \circ \psi = x \odot_Z \psi$ by definition; that this is a permutation on C_X follows from the closure of C_X. □

Theorem 12. *Suppose $k \in C_Z$, and define $K = \Lambda^X(k)$; then K is a group automorphism of U_Z.*

Proof. Graphically we depict K as:

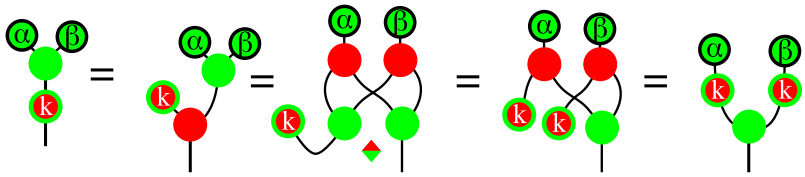


Since $k \in U_X$, K is unitary, and so is invertible. We must show that if $\alpha \in U_Z$ then also $K \circ \alpha \in U_Z$. This holds if and only if $\Lambda^Z(K \circ \alpha)$ is unitary; we show this directly:



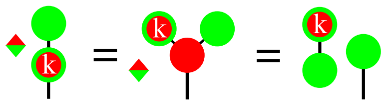
where the equations are by the comultiplication property, the unitarity of K, the unbiasedness of α, and the unitarity of K again. It remains to show that K is a homomorphism of the group structure.

1. $K \circ (\alpha \odot_Z \beta) = (K \circ \alpha) \odot_Z (K \circ \beta)$:



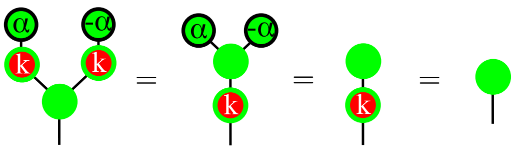
   The equations are: the definition of K; the bialgebra law; the classical property of k; and the definition of K.

2. $K \circ \epsilon_Z^\dagger = \epsilon_Z^\dagger$ (upto global phase):



   The equation simply uses coherence of δ_X and ϵ_Z; the result follows by dividing by the scalar factor as per Lemma 3.

3. $(K \circ \alpha)^{-1} = K \circ \alpha^{-1}$:



   where we relied upon the comultiplication property of K and the unbiasedness of α, showing that the inverse $K \circ \alpha$ in U_Z is $K \circ \alpha^{-1}$ as required.

Hence K is an automorphism of U_Z. □

Corollary 5. *$(C_Z, \odot_X)$ is an abelian group of automorphisms on U_Z whose action is defined by $(x, z) \mapsto \Lambda^X(x) \circ z$.*

The possibility for the classical points to act as automorphisms on the corresponding phase group gives rise to "interference" phenomena; this will be illustrated by the example of the quantum Fourier transform in Section 9.

9. Application to quantum computation

The examples in Sections 5.5 and 6.2 indicated that the structures introduced in this paper provide a useful tool for reasoning about quantum computation. This section provides some more examples to further substantiate this claim.

9.1. Simplifying the notation.

The diagrammatic notation we have used up till this point is well adapted to deriving the properties of the theory. For practical uses, such as computing the examples of this section, some simplifications can be made.

In the examples we consider a pair of COSs, which are coherent, and form a scaled bialgebra. Furthermore we assume that all the classical points have length $\sqrt{D}$. We restrict our attention to points which are unbiased for one observable structure or the other, and we represent each dot with a single colour, that of the observable structure to which it is unbiased. Points which are classical are indicated by distinguished indices.

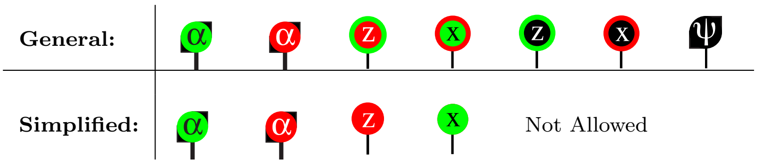
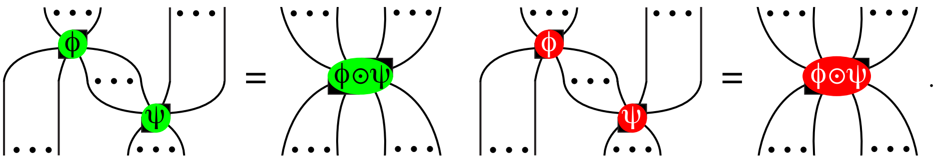


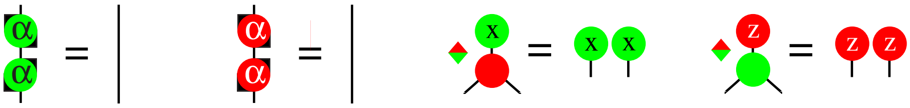
Table 1. Translation between general and simplified graphical notation

The rules of the simplified calculus are summarised in these slogans:

1. The spider law applies to points of the same colour.
2. Within one colour conjugate points are mutually inverse.
3. Red copies green, green copies red, and copying consumes a diamond.

These respectively capture observable structure, unbiasedness, and classicality of points, and are explicitly depicted as:



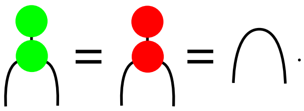


9.2. Qubit Calculus.

As in the earlier examples we consider a 'green' observable structure (δ_Z, ϵ_Z) corresponding to the Z-spin observable and a 'red' observable structure (δ_X, ϵ_X), corresponding to the X-spin.

$\delta_Z : |i\rangle \mapsto |ii\rangle$ $\qquad \epsilon_Z : \sqrt{2}\,|+\rangle \mapsto 1$ $\qquad \delta_X : |\pm\rangle \mapsto |\pm\pm\rangle$ $\qquad \epsilon_X : \sqrt{2}\,|0\rangle \mapsto 1$

We first observe that $\delta_Z \circ \epsilon_Z^\dagger = \delta_X \circ \epsilon_X^\dagger = |00\rangle + |11\rangle$ so both observables induce the same compact structure. Pictorially, we have:



Concentrating first on the Z observable, from the definition of δ_Z we immediately read that its classical points are $|0\rangle$ and $|1\rangle$; these points, together with $\epsilon_Z^\dagger = |+\rangle$ (ignoring the normalising factor), form a state basis for the space. Recall that the points which are unbiased for Z have the form $|\alpha_Z\rangle = |0\rangle + e^{i\alpha}|1\rangle$ where $0 \leq \alpha < 2\pi$, and hence the phase group consists of matrices of the form:

$$\Lambda^Z(|\alpha_Z\rangle) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} =$$  .

The group $(U_Z, \odot_Z)$ is therefore isomorphic to the circle. Notice that $|0_Z\rangle = |+\rangle$ and $|\pi_Z\rangle = |-\rangle$; these points form a 2-element subgroup of U_Z, corresponding to the identity and the Pauli Z matrices. Since $|+\rangle$ and $|-\rangle$ are the classical points C_X of the X observable, this shows that the observable structure for X is closed.

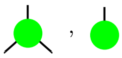
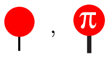
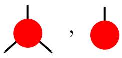
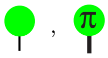
The structure of the X observable is essentially the same. The state basis is made up of the classical points $|+\rangle$ and $|-\rangle$, and the unit element $|0\rangle$. Its phase group consists of rotations around X, that is matrices of the form

$$\Lambda^X(|\alpha_X\rangle) = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} =$$  ,

generated by the unbiased points $|\alpha_X\rangle = \cos\frac{\alpha}{2}|0\rangle + i\sin\frac{\alpha}{2}|1\rangle$. We have $C_Z = \{|0_X\rangle, |\pi_X\rangle\}$ so the C_Z forms a 2-element subgroup of U_X as before; its matrix form consists of the identity and the Pauli X.

Notation. In following examples, we will not apply Λ^Z and Λ^X to any vectors other than the $|\alpha_Z\rangle$ and $|\alpha_X\rangle$ respectively, so we will simply write Z_α and X_α to denote these unitary matrices.

The remaining piece of the structure to be described is the action of C_Z on U_Z. Since $X_\pi = \mathsf{X}$ we see that the non-trivial element of C_Z sends $|\alpha_Z\rangle \mapsto |-\alpha_Z\rangle$; i.e. X assigns elements of U_Z to their inverses. The action of C_X on U_X is exactly dual. The group structure is summarised in Table 2.

Observable	Classical Points	Unbiased Points	Phase Group
$Z = (\delta_Z, \epsilon_Z)$ 	$\|0\rangle, \|1\rangle$ 	$\|0\rangle + e^{i\alpha}\|1\rangle$ 	$Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$ 
$X = (\delta_X, \epsilon_X)$ 	$\|+\rangle, \|-\rangle$ 	$\cos\frac{\alpha}{2}\|0\rangle + i\sin\frac{\alpha}{2}\|1\rangle$ 	$X_\alpha = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$ 

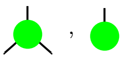
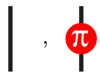
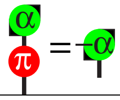
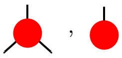
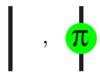
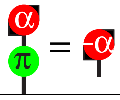
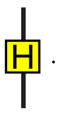
Observable	Classical Subgroup	Automorphism Action
$Z = (\delta_Z, \epsilon_Z)$ 	$1, \mathsf{X}$ 	$\mathsf{X} : Z_\alpha \mapsto Z_{-\alpha}$ 
$X = (\delta_X, \epsilon_X)$ 	$1, \mathsf{Z}$ 	$\mathsf{Z} : X_\alpha \mapsto X_{-\alpha}$ 

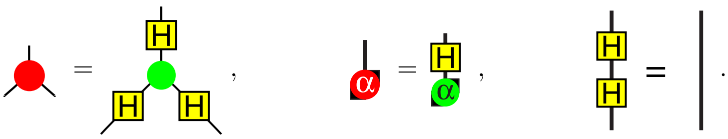
Table 2. Summary of the group structure for qubits

The presentation of the examples will be greatly simplified by adding one further operation to the language, namely the unitary map which exchanges the X and Z bases. From the graphical perspective, this map is an explicit colour changing operation which transforms "green structures" into "red structures" and vice versa. For the Z and X observables described above, the desired map is the familiar Hadamard gate:

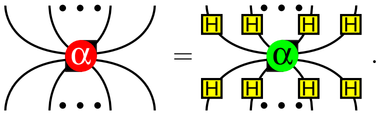
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$



We will introduce H into the graphical language with the following equations:



As well as being unitary, H is self-adjoint, so we denote it with a square; its symmetrical form indicates that is not changed by transposition, nor by conjugation. The three equations above are equivalent to the following equation between spiders:



The spider form of the equations is frequently more useful.

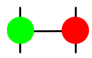
Remark 16. As discussed in Section 6.2, since the calculus contains symbols for X and Z rotations, every 1-qubit unitary can be represented (upto a global phase) by its Euler decomposition, $Z_\alpha X_\beta Z_\gamma$. In particular, $H = Z_{-\frac{\pi}{2}} X_{-\frac{\pi}{2}} Z_{-\frac{\pi}{2}}$. This equation cannot be derived in the calculus as it stands. In fact this equation is equivalent to Van den Nest's theorem on local complementation of graph states [43]; see [19] for details.

Now we move on to some examples of the kinds of calculations which the graphical calculus can perform. In these examples we disregard scalar factors as these only distract from the essential; the reader will find it easy to restore the scalars if desired.

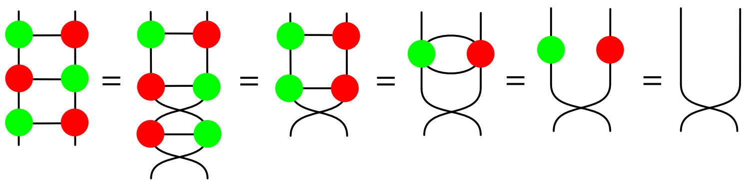
9.3. Quantum gates, circuits, and algorithms.

The 1-qubit unitaries Z_α and X_β correspond to rotations in the X-Y and the Y-Z planes respectively; these suffice to represent all 1-qubit unitaries, and their basic equational properties follow from the various lemmas introduced in the preceding sections. We demonstrate how to define the $\wedge\mathsf{X}$ and $\wedge\mathsf{Z}$ gates, and prove two elementary equations involving them. The addition of these operations will provide a computationally universal set of gates.

Example 16 ($\wedge\mathsf{X}$ gate). In Section 6.2 we saw that the $\wedge\mathsf{X}$ gate can be defined as

$$\wedge\mathsf{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$  ;

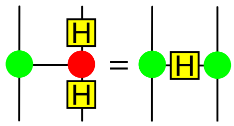
by appealing to the Hopf law, we showed that this gate is involutive. We now show how the bialgebra law provides additional computational power. By applying a $\wedge\mathsf{X}$ gate three times, alternating the target and control input, we obtain a swap—i.e. the symmetry map of the monoidal structure.



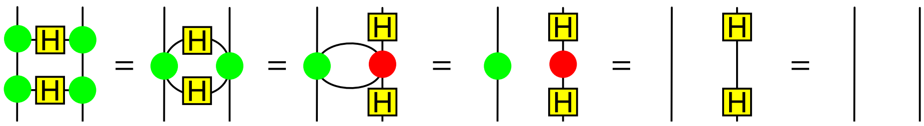
While this is a well-known property of $\wedge\mathsf{X}$, our proof uses only the bialgebra structure, hence it will hold in much greater generality than for qubits.

Example 17 ($\wedge$Z gate). Since $\mathsf{Z} = H\mathsf{X}H$ we can obtain the $\wedge$Z gate from the $\wedge$X gate by conjugating the target qubit with H gates, as shown below:

$$\wedge\mathsf{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$



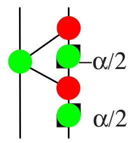
Notice that we can immediately read off a property of this gate from its graphical representation: it is symmetric in its inputs. We can use the colour change equation and the Hopf law to prove that $\wedge\mathsf{Z} \circ \wedge\mathsf{Z} = 1$, as shown below.



Example 18 (An algorithm: the quantum Fourier transform). The quantum Fourier transform is one of the most important quantum algorithms, lying at the centre of Shor's famous factoring algorithm [39]. The equations of the diagrammatic calculus are strong enough to simulate this algorithm. In this example the interaction between the two phase groups—as described in Theorem 12—is crucial.

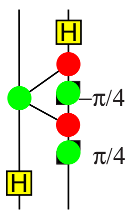
To write down the required circuit, we must realise a controlled phase gate, where the phase is an arbitrary angle α; this is shown below—the control qubit is on the left. (Recall that the inputs are at the top of the diagram, and the outputs at the bottom.)

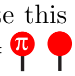
$$\wedge\mathsf{Z}_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{pmatrix} =$$

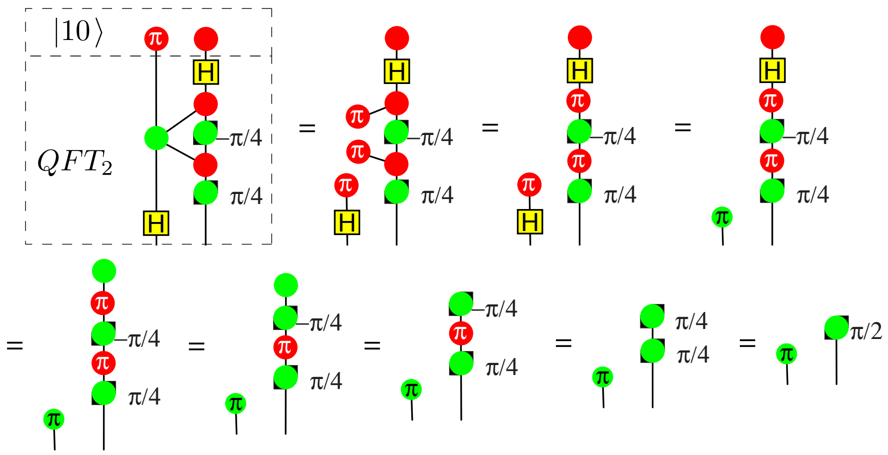


The only gates which are required to construct the circuit implementing the quantum Fourier transform are the Hadamard and the $\wedge\mathsf{Z}_\alpha$—see for example [32]. The circuit for the 2-qubit QFT is shown below.

$$QFT_2 =$$



How can we simulate this circuit? First, we choose an input state, in this case $|10\rangle = |\pi_X\rangle \otimes |0_X\rangle =$  ; then we simply concatenate the input to the circuit, and begin rewriting according to the equations of the theory, as shown below.

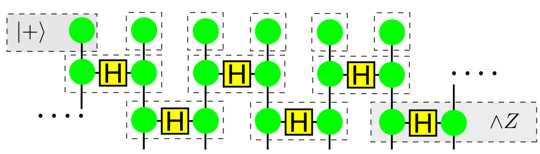


The final diagram in the sequence is simply the tensor product of $|\pi_Z\rangle$ and $|\pi/2_Z\rangle$, or in the standard notation $(|0\rangle - |1\rangle) \otimes (|0\rangle + i\,|1\rangle)$, which is indeed the desired result. Notice how this example makes use of classical values coded as quantum states to control the interference of phases (the second last equation). In passing we remark upon another feature of the graphical language: since the last diagram is a disconnected graph, it represents a separable quantum state.

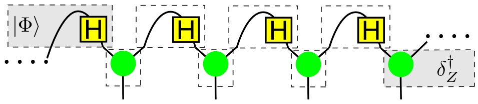
9.4. Multi-partite entanglement.

In our graphical language, a quantum state is nothing more than a circuit with no inputs; output edges correspond to the individual qubits making up the state. The interior of the diagram—i.e. its graph structure—describes how these qubits are related. Hence this notation is ideal for representing large entangled states.

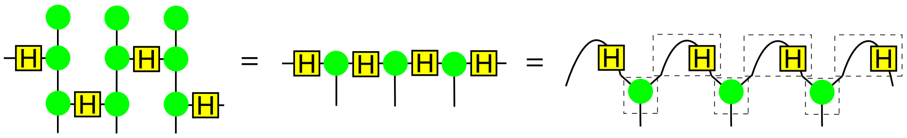
Example 19. The cluster states used in measurement-based quantum computing [35], can be prepared in several ways; the graphical calculus provides short proofs of their equivalence. For example, the original scheme describes a $\wedge$Z interaction between qubits initially prepared in the state $|+\rangle$; in our notation this is $|0_Z\rangle$, or  . A one-dimensional cluster state can be presented diagrammatically as:



where the boxes delineate the individual $|+\rangle$ preparations and $\wedge$Z operations. Alternatively, the cluster state can be prepared by fusion of states of the form $|\varPhi\rangle = |0{+}\rangle + |1{-}\rangle$ which are obtained by applying a Hadamard gate to one part of a Bell pair [22]. The fusion operation is exactly the multiplication $\delta_Z^\dagger$, so a 1D cluster prepared with this method looks like:



Again we use dashed lines to indicate the individual components. While conventional methods require some calculation to show that these methods of preparation produce the same state, using the spider theorem, the two diagrammatic forms are immediately equivalent:



From the example of the 1D cluster, it's easy to see how to construct diagrams corresponding to arbitrary graph states. Indeed given a graph state $|G\rangle$, with underlying graph G, we represent $|G\rangle$ by the same graph G, with green dots at each vertex, and H gates on each edge; to complete the construction we must add one output edge to each at each vertex.

Ongoing work seeks to classify multipartite entangled states in terms of their graphical representatives, for example, the GHZ and the W state [20], which respectively witness each of the two non-comparable SLOCC classes of genuine three qubit entangled states, can be respectively depicted as:

The behaviors of these states follow from the algebraical laws in this paper.

9.5. Properties of quantum computational models.

Our formalism axiomatises two key features of quantum mechanics: the underlying monoidal structure and the interaction of complementary observables. Furthermore it is a semantic, which is to say *extensional*, framework which makes it ideal for unifying various approaches to quantum computation. For example, we can demonstrate equivalence between different quantum computational models.

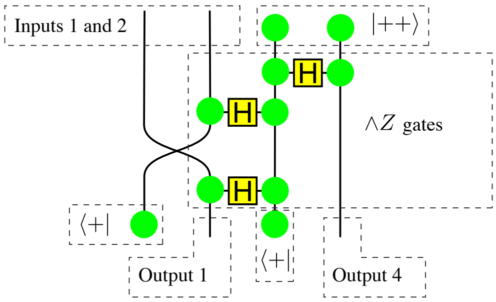
Example 20 (Verifying one-way quantum computations). We show how to verify some example programs for the one-way model, taken from [17], by translation to equivalent quantum circuits. In these examples we replace measurements with projections onto a particular output state. These projections are represented by copoints such as  .

Remark 17. The use of post-selection is not essential here, although it results in a radical simplification of the diagrams. Observable structures were initially introduced in [15] to represent classical control structure; the branching behaviour of quantum measurements, and the unitary corrections that are thus required, can easily be added to these examples. The last example demonstrates this structure in the simpler setting of quantum teleportation.

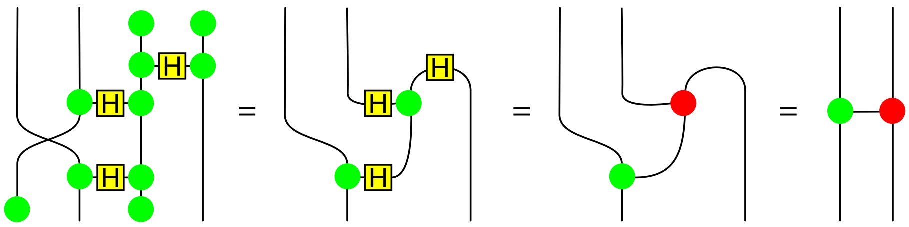
First we consider a measurement-based program involving 4 qubits, which computes a ∧X gate upon its inputs. In the syntax of the measurement calculus [17] this pattern[1] is written

$$M_2^0 M_4^0 E_{13} E_{23} E_{34} N_3 N_4.$$

Reading from right to left, this specifies that qubits 3 and 4 should be prepared in a $|+\rangle$ state, then ∧Z operations should be applied pairwise between qubits 1 and 3, 2 and 3, and 3 and 4; finally X basis measurements should be performed upon qubits 2 and 4. Implicitly, qubits 1 and 2 are the inputs and qubits 1 and 4 are the outputs. We represent this pattern diagrammatically as:



Note that the measurements have been replaced with projections onto the $|+\rangle$ state. The spider theorem allows this one-way program to be rewritten to a ∧X gate in no more than two steps:

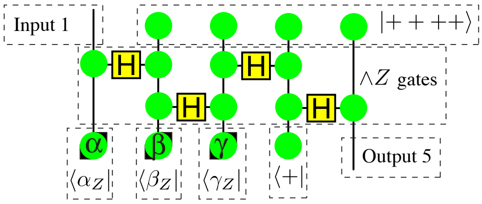


Our next example is also a one-way program, this time implementing an arbitrary 1-qubit unitary. Recall that any single qubit unitary map U has an Euler decomposition as such that $U = Z_\gamma X_\beta Z_\alpha$. Such a unitary can be implemented by the following measurement pattern:

$$M_3^\gamma M_2^\beta M_1^\alpha E_{12} E_{23} E_{34} E_{45} N_2 N_3 N_4 N_5$$

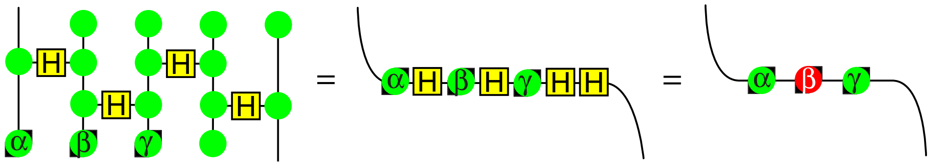
[1] Since we are post-selecting the measurements, we have omitted from this pattern, and the next, the corrections which are needed to ensure that the program behaves deterministically; see [17] for details.

where M_i^α denotes a measurement of qubit i in the $X-Y$ plane at angle α. Again, we will replace this measurement by a projection onto the state $|0\rangle + e^{i\alpha}|1\rangle$. The graphical form of this pattern is shown below:



.

Again a sequence of simple rewrites shows that the one-way program intended to compute such a unitary does indeed produce the desired map.



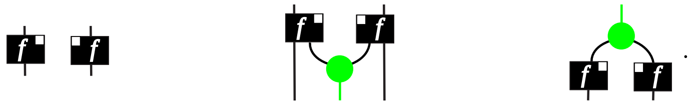
9.6. Observable structure as classical control structure.

As already mentioned in the introduction, observable structures were initially introduced under the name *classical structures* in [15] to model classical data in quantum informatic protocols. One of the initial goals of categorical quantum axiomatics in [1] was indeed to provide a fully comprehensive description of both classical and quantum data in quantum informatic protocols. In [1] this relied on an explicit syntactical distinction of how objects are denoted. In [8, 37] the classical-quantum distinction was achieved in category-theoretic terms, but relied on a second 'additive' monoidal structure. In [15, 13] a description of classical data in terms of observable structure was proposed. Hence with a single concept we can account both for quantum observables, complementarity, phases, as well as classical information flow. To illustrate this we provide a description of the quantum teleportation protocol.

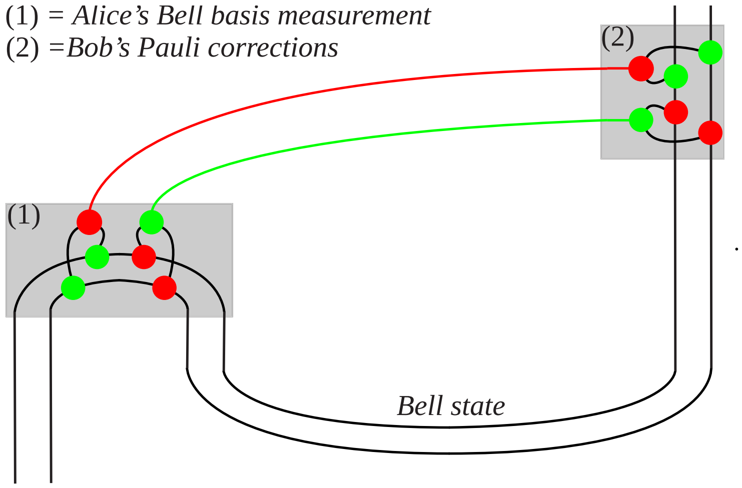
Notation. (!) In keeping with how the quantum teleportation protocol is usually depicted, we reverse our earlier convention, and read this example from bottom to top.

We will be very brief on the use of observable structures in order to describe classical data flow; we refer the reader to [13] for more details. Graphically, classical data and classical operation are represented by a single wire, while a quantum data and quantum operations are represented by double wires. These double wires arise via the CPM-construction of [37]. This '1 vs. 2' as 'classical vs. quantum' is also present in Dirac notation; for a mixed state $\sum_i \omega_i |\psi_i\rangle\langle\psi_i|$ the clasical probabilistic state $(\omega_1, \ldots, \omega_n)$ occurs only once while the quantum states occur both as a ket $|\psi_i\rangle$ and as a bra $\langle\psi_i|$.

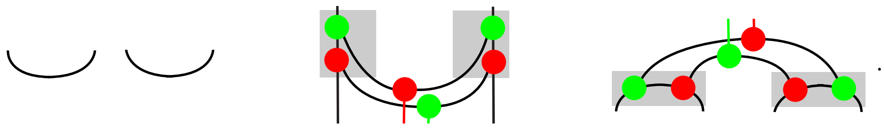
Pure quantum operations, controlled pure quantum operations, and destructive measurements are of the forms:



respectively, where for clarity we choose to colour the classical wires in the colour of the observable structure which encodes that classical data, that is, for which the classical data are classical points. We claim that the following constitutes the quantum teleportation protocol, including the classical correction:

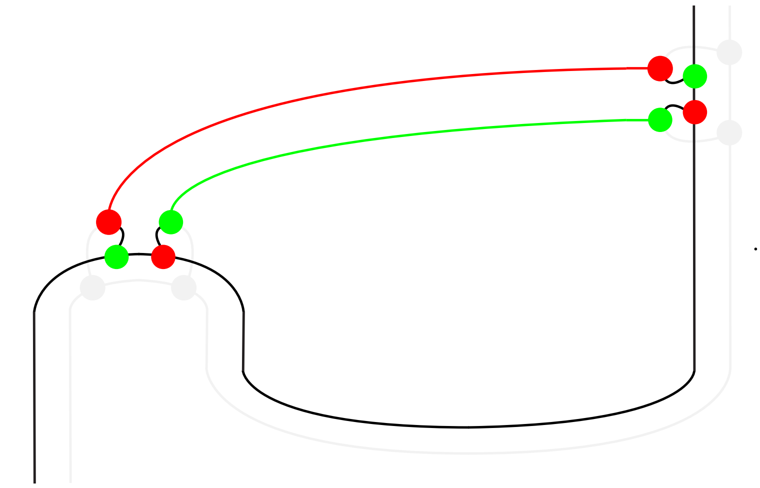


The green and the red wire represent the two qubits Alice has to send to Bob to inform him of the measurement outcome. The Bell state, Bob's Pauli corrections and Alice's Bell basis measurement can be rewritten respectively as:

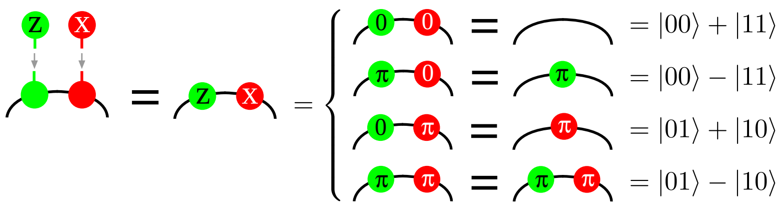


hence, they are of the forms shown above.

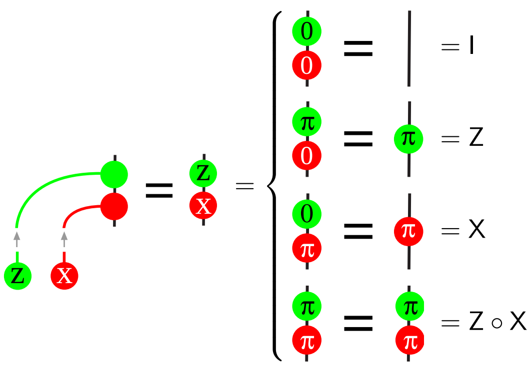
The picture might seem somewhat complicated; the reason for this is that in order to display the quantum-classical distinction graphically, all the quantum operations are doubled. If we hide one of the two copies it becomes much clearer what is going on:



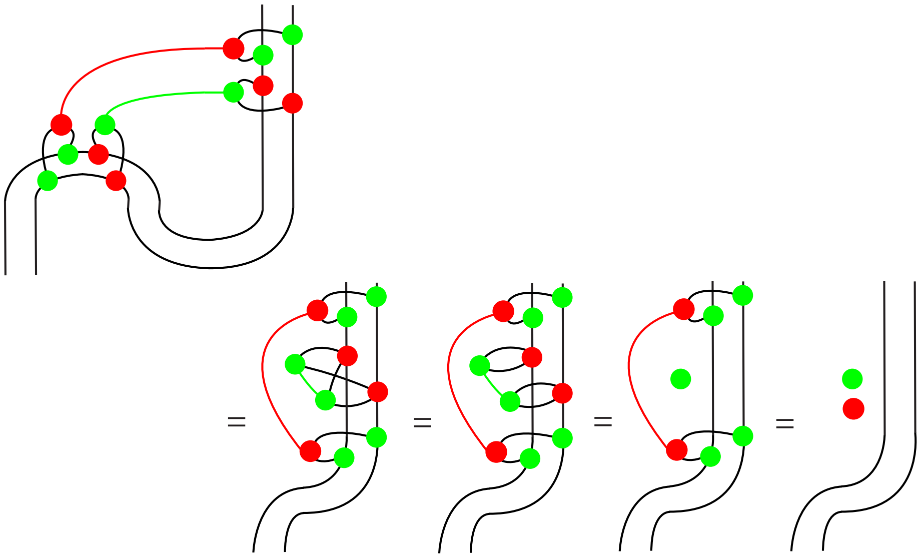
We show that the measurement and corrections in this picture are indeed the ones we claim them to be. Post-selecting the Bell-basis measurement we obtain:



Similarly, selecting the Pauli corrections we obtain:



We now rely on the Hopf law to compute the overall result of the teleportation protocol diagrammatically:



The first step uses compactness for the black wires and the spider theorem for the red dots, the second step uses the spider theorem for the green dots, the third step uses complementarity of green and red dots, and the fourth step does again all of the previous but now for swapped colours. The scalars that remain at the end are a consequence of the fact that we didn't normalise the Bell state nor the Bell basis measurement.

10. Acknowledgements

Work supported by the authors' respective EPSRC Advanced and Postdoctoral Research Fellowships, by EC FP6 STREP QICS, by FQXi and by ONR, and we also acknowledge support of Perimeter Institute for Theoretical Physics which hosted both of the authors. We thank Samson Abramsky, Bertfried Fauser, Prakash Panangaden, Simon Perdrix and Marni Dee Sheppeard for valuable feedback on an earlier version of this paper, and to Stephen Brierly and Stefan Weigert for providing the counter-example of Remark 13.

A. Internal comonoids in a symmetric monoidal category

Recall that a *monoid* is a triple $(M, \bullet, 1_\bullet)$ where $\bullet$ is the associative *multiplication* of the monoid and $1_\bullet \in M$ is its unit. The multiplication is of course a map

$$m_\bullet : M \times M \to M :: (x, y) \mapsto x \bullet y$$

and we can also represent the unit as a map

$$e_\bullet : \mathbb{I} \to M :: \star \mapsto 1_\bullet$$

where $\mathbb{I} := \{\star\}$ is a singleton set. The associativity and unit laws of the monoid can now be re-written in terms of composition of maps and cartesian product:

$$m_\bullet \circ (m_\bullet \times 1_M) = m_\bullet \circ (1_M \times m_\bullet) \qquad m_\bullet \circ (e_\bullet \times 1_M) \simeq m_\bullet \circ (1_M \times e_\bullet) \simeq 1_M$$

where $1_M : M \to M$ is the identity on the set M.

An *(internal) monoid* in a symmetric monoidal category is a triple

$$(M, m : M \otimes M \to M, e : \mathrm{I} \to M)$$

satisfying

$$m \circ (m \otimes 1_M) = m \circ (1_M \otimes m) \qquad m \circ (e \otimes 1_M) \circ \lambda_M = m \circ (1_M \otimes e) \circ \rho_M = 1_M \,,$$

and it is moreover *commutative* if we have that $m \circ \sigma_{M,M} = m$. Dually, an *(internal) comonoid* in a symmetric monoidal category is a triple

$$(X, \delta : X \to X \otimes X, \epsilon : \mathrm{I} \to X)$$

satisfying

$$(\delta \otimes 1_X) \circ \delta = (1_X \otimes \delta) \circ \delta \qquad \lambda_X^\dagger \circ (\epsilon \otimes 1_X) \circ \delta = \rho_X^\dagger \circ (1_X \otimes \epsilon) \circ \delta = 1_X \,,$$

and it is moreover *commutative* if we have that $\sigma_{M,M} \circ \delta = \delta$.

Now consider a set X and let $\delta : X \to X \times X$ be the function which *copies* entries, i.e. $\delta :: x \mapsto (x, x)$. Since δ is a function it is also a relation, namely

$$\delta := \{(x, (x, x)) \mid x \in X\} \subseteq (X \times X) \times X \,,$$

and as a relation it admits a *relational converse*, obtained by exchanging the two entries in the pairs which make up that relation. The relational converse to δ, i.e.

$$m := \{((x, x), x) \mid x \in X\} \subseteq (X \times X) \times X \,,$$

relates pairs $(x, x) \in X \times X$ to $x \in X$, while it does not relate pairs $(x, y) \in X \times X$ for $x \neq y$ to anything. Let $\epsilon : X \to \mathbb{I}$ be the function which *erases* entries i.e. $\epsilon : x \mapsto \star$. When conceived as a relation ϵ admits a relational converse

$$e := \{(\star, x) \mid x \in X\} \subseteq \mathbb{I} \times X \,,$$

which now relates $\star \in \mathbb{I}$ to each $x \in X$. When taking sets as objects and relations as morphisms, with the composite of relations $r \subseteq X \times Y$ and $s \subseteq Y \times Z$ to be

$$s \circ r := \{(x, z) \mid \exists y : (x, y) \in r\,, (y, z) \in s\} \subseteq X \times Z \,,$$

and when taking the cartesian product to be the tensor and the relational converse to be the adjoint, we obtain a †-SMC **Rel**. The copying/deleting pair (δ, ϵ) is a comonoid in **Rel**, and the pair (m, e) consisting of their respective converses is a monoid in **Rel**. The pair (m, δ) moreover satisfies another important remarkable property: the diagram

$$\begin{array}{ccc} X \otimes X & \xrightarrow{\quad m \quad} & X \\ {\scriptstyle \delta \otimes 1_X}\downarrow & & \downarrow{\scriptstyle \delta} \\ X \otimes X \otimes X & \xrightarrow[\ 1_X \otimes m\ ]{} & X \otimes X \end{array} \tag{5}$$

commutes. Indeed, we have

$$\delta \circ m = (1_X \otimes m) \circ (\delta \otimes 1_X) = \{(x,x),(x,x) \mid x \in X\} \subseteq (X \times X) \times (X \times X)\,.$$

This fascinating property first appeared in the literature as part of Carboni and Walters' axiomatisation of the category **Rel** in [6] where they introduced the notion of a *Frobenius algebra* in a SMC **C**, as any quintuple of morphisms

$$(X, d : X \otimes X \to X, e : \mathrm{I} \to X, \delta : X \to X \otimes X, \epsilon : X \to \mathrm{I})$$

where (X, m, e) is an internal commutative monoid and (X, δ, ϵ) is an internal commutative comonoid, which together satisfy the *Frobenius condition*, that is, they make diagram (5) commute. Such a Frobenius algebra is *special* if it moreover satisfies $m \circ \delta = 1_X$ [27].

Frobenius structure is our structural vehicle for describing and reasoning about classical contexts for quantum systems as well as classical data flows.

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