Graphs States and the necessity of Euler Decomposition

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Abstract. Coecke and Duncan recently introduced a categorical formalisation of the interaction of complementary quantum observables. In this paper we use their diagrammatic language to study graph states, a computationally interesting class of quantum states. We give a graphical proof of the fixpoint property of graph states. We then introduce a new equation, for the Euler decomposition of the Hadamard gate, and demonstrate that Van den Nest's theorem—locally equivalent graphs represent the same entanglement—is equivalent to this new axiom. Finally we prove that the Euler decomposition equation is not derivable from the existing axioms.

1 Introduction

Upon asking the question "What are the axioms of quantum mechanics?" we can expect to hear the usual story about states being vectors of some Hilbert space, evolution in time being determined by unitary transformations, etc. However, even before finishing chapter one of the textbook, we surely notice that something is amiss. Issues around normalisation, global phases, etc. point to an "impedence mismatch" between the theory of quantum mechanics and the mathematics used to formalise it. The question therefore should be "What are the axioms of quantum mechanics *without Hilbert spaces'?*"

In their seminal paper [1] Abramsky and Coecke approached this question by studying the categorical structures necessary to carry out certain quantum information processing tasks. The categorical treatment provides as an intuitive pictorial formalism where quantum states and processes are represented as certain diagrams, and equations between them are described by rewriting diagrams. A recent contribution to this programme was Coecke and Duncan's axiomatisation of the algebra of a pair complementary observables [2] in terms of the

red-green calculus. The formalism, while quite powerful, is known to be incomplete in the following sense: there exist true equations which are not derivable from the axioms.

In this paper we take one step towards its completion. We use the red-green language to study *graph states*. Graph states [3] are very important class of states used in quantum information processing, in particular with relation to the one-way model of quantum computing [4]. Using the axioms of the red-green system, we attempt to prove Van den Nest's theorem [5], which establishes the local complementation property for graph states. In so doing we show that a new equation must be added to the system, namely that expressing the Euler decomposition of the Hadamard gate. More precisely, we show that Van den Nest's theorem is equivalent to the decomposition of H, and that this equation cannot be deduced from the existing axioms of the system.

The paper procedes as follows: we introduce the graphical language and the axioms of the red-green calculus, and its basic properties; we then introduce graph states and and prove the fixpoint property of graph states within the calculus; we state Van den Nest's theorem, and prove our main result—namely that the theorem is equivalent to the Euler decomposition of H. Finally we demonstrate a model of the red-green axioms where the Euler decomposition does not hold, and conclude that this is indeed a new axiom which should be added to the system.

2 The Graphical Formalism

Definition 1. *A* diagram *is a finite undirected open graph generated by the following two families of vertices:*
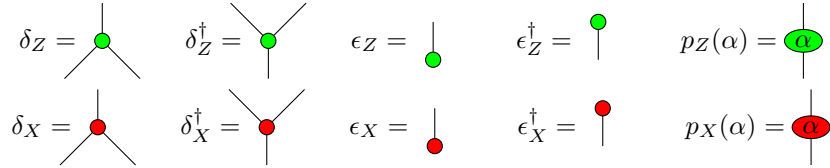
*where $\alpha \in [0, 2\pi)$, and a vertex $H = \boxed{H}$ belonging to neither family.*

Diagrams form a monoidal category $\mathcal{D}$ in the evident way: composition is connecting up the edges, while tensor is simply putting two diagrams side by side. In fact, diagrams form a †-compact category [6, 1] but we will suppress the details of this and let the pictures speak for themselves. We rely here on general results [7–9] which state that a pair diagrams are equal by the axioms of †-compact categories exactly when they may be deformed to each other.

Each family forms a *basis structure* [10] with an associated *local phase shift*. The axioms describing this structure can be subsumed by the following law. Define $\delta_0 = \epsilon^\dagger$, $\delta_1 = 1$ and $\delta_n = (\delta_{n-1} \otimes 1) \circ \delta$, and define $\delta_n^\dagger$ similarly.
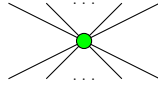
Spider Law. *Let f be a connected diagram, with n inputs and m outputs, and whose vertices are drawn entirely from one family; then*

$$f = \delta_m \circ p(\alpha) \circ \delta_n^\dagger \qquad\qquad \text{where } \alpha = \sum_{p(\alpha_i) \in f} \alpha_i \mod 2\pi$$

with the convention that $p(0) = 1$.
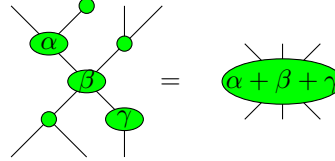
The spider law justifies the use of "spiders" in diagrams: coloured vertices of arbitrary degree labelled by some angle α. By convention, we leave the vertex empty if $\alpha = 0$.

We use the spider law as rewrite equation between graphs. It allows vertices of the same colour to be merged, or single vertices to be broken up. An important special case is when $n = m = 1$ and no angles occur in f; in this case f can be reduced to a simple line. (This implies that both families generate the same compact structure.)

Lemma 1. *A diagram without H is equal to a bipartite graph.*

Proof. If any two adjacent vertices are the same colour they may be merged by the spider law. Hence if we can do such mergings, every green vertex is adjacent only to red vertices, and vice versa.

We interpret diagrams in the category $\mathbf{FdHilb}_{\mathrm{wp}}$; this the category of complex Hilbert spaces and linear maps under the equivalence relation $f \equiv g$ iff there exists θ such that $f = e^{i\theta} g$. A diagram f with n inputs and m output defines a linear map $[\![f]\!] : \mathbb{C}^{\otimes 2n} \to \mathbb{C}^{\otimes 2m}$. Let

$$[\![\epsilon_Z^\dagger]\!] = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad\qquad\qquad [\![\epsilon_X^\dagger]\!] = |0\rangle$$

$$[\![\delta_Z^\dagger]\!] = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix} \qquad\qquad [\![\delta_X^\dagger]\!] = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1\,0\,0\,1 \\ 0\,1\,1\,0 \end{pmatrix}$$

$$[\![p_Z(\alpha)]\!] = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \qquad [\![p_X(\alpha)]\!] = e^{-\frac{i\alpha}{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$[\![H]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and set $\llbracket f^\dagger \rrbracket = \llbracket f \rrbracket^\dagger$. The map $\llbracket \cdot \rrbracket$ extends in the evident way to a monoidal functor.

The interpretation of $\mathcal{D}$ contains a universal set of quantum gates. Note that $p_Z(\alpha)$ and $p_X(\alpha)$ are the rotations around the X and Z axes, and in particular when $\alpha = \pi$ they yield the Pauli X and Z matrices. The $\wedge Z$ is defined by:

The δ_X and δ_Z maps *copy* the eigenvectors of the Pauli X and Z; the ϵ maps *erase* them. (This is why such structures are called basis structures).

Now we introduce the equations[4] which make the X and Z families into *complementary* basis structures as in [2]. Note that all of the equations are also satisfied in satisfied in the Hilbert space interpretation. We present them in one colour only; they also hold with the colours reversed.

Copying

Bialgebra

π-Commutation

A consequence of the axioms we have presented so far is the Hopf Law:

This equation, when combined with the spider law, provides a very useful property, namely that every diagram (without H) is equal to one without parallel edges.

Lemma 2. *Every diagram without H is equal to one without parallel edges.*

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[4] We have, both above and below, made some simplifications to the axioms of [2] which are specific to the case of qubits. We also supress scalar factors.

Proof. Suppose that v, u are vertices in some diagram, connected by two or more edges. If they are the same colour, they can be joined by the spider law, eliminating the edges between them. Otherwise the Hopf law allows one pair of parallel edges to be removed; the result follows by induction.

Finally, we introduce the equations for H:
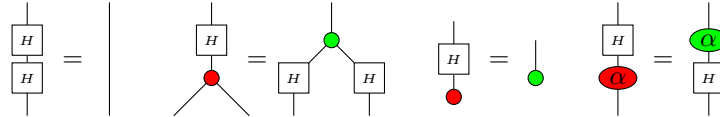
The special role of H in the system is central to our investigation in this paper.

3 Generalised Bialgebra Equations

The bialgebra law is a key equation in the graphical calculus. Notice that the left hand side of the equation is a 2-colour bipartite graph which is both a $K_{2,2}$ (i.e a complete bipartite graph) and a C_4 (i.e. a cycle composed of 4 vertices) with alternating colours. In the following we introduce two generalisations of the bialgebra equation, one for any $K_{n,m}$ and another one for any C_{2n} (even cycle).

We give graphical proofs for both generalisations; both proofs rely essentially on the primitive bialgebra equation.

**Lemma 3.** *For any n, m, "$K_{n,m} = P_2$", graphically:*
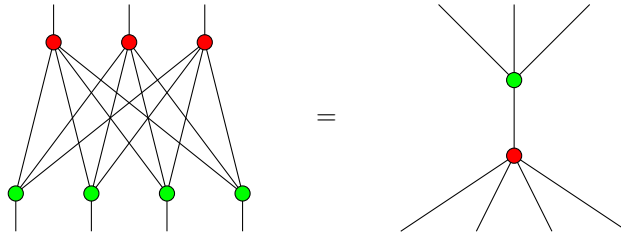
Proof. The proof is by induction on (m, n) where m (resp. n) is the number of red (resp. green) dots of the left hand side of the equation. Let $\prec$ be the lexicographical order (i.e. $(m, n) \prec (k, l)$ iff $m < n$ or $m = k \wedge m < l$). Notice that if either $m = 1$ or $n = 1$ then the resulting degree 1 vertices may simply be removed, by the spider theorem, hence the equation is trivially satisfied. Moreover, if $m = n = 2$ the equation is nothing but the bialgebra equation. Let $(m, n) \succ (2, 2)$. The following graphical proof is by induction, using the

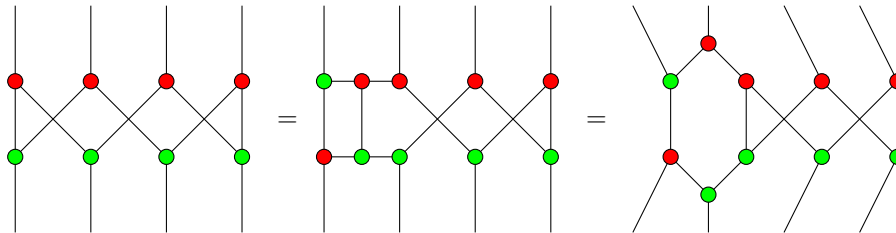hypothesis of induction twice, first with $(m, n-1)$ and then with $(m, 2)$.
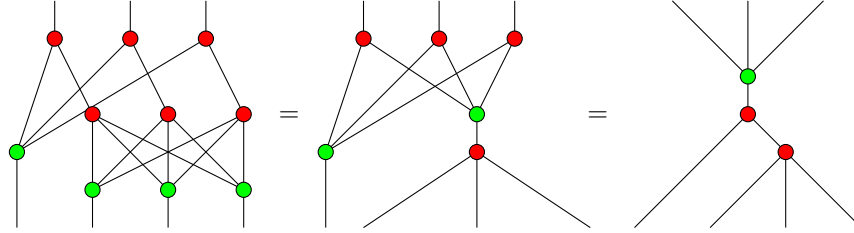
Notice in the first step we use the spider law to extract the $K_{m,n-1}$ subgraph.

Lemma 4. *For n, an even cycle of size $2n$, of alternating colours, can be rewritten into hexagons. Graphically:*
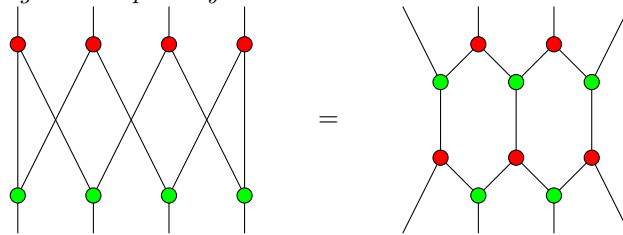
Proof. The proof is by induction, with one application of the bialgebra equation:

Note, as before, the use of the spider theorem in the first step.

4   Graph states

In order to explore the power and the limits of the axioms we have described, we now consider the example of *graph states*. Graph states provide a good testing ground for our formalism because they are relatively easy to describe, but have wide applications across quantum information, for example they form a basis for universal quantum computation, capture key properties of entanglement, are related to quantum error correction, establish links to graph theory and violate Bell inequalities.

In this section we show how graph states may be defined in the graphical language, and give a graphical proof of the fix point property, a fundamental property of graph states. The next section will expose a limitation of the theory, and we will see that proving Van den Nest's theorem requires an additional axiom.

Definition 2. *For a given simple undirected graph G, let $|G\rangle$ be the corresponding graph state*

$$|G\rangle = \left(\prod_{(u,v)\in E(G)} \wedge Z_{u,v} \right) \left(\bigotimes_{u\in V(G)} \frac{|0\rangle_u + |1\rangle_u}{\sqrt{2}} \right)$$

where $V(G)$ (resp. $E(G)$) is the set of vertices (resp. edges) of G.

Notice that for any $u, v, u', v' \in V(G)$, $\wedge Z_{u,v} = \wedge Z_{v,u}$ and $\wedge Z_{u,v} \circ \wedge Z_{u',v'} = \wedge Z_{u',v'} \circ \wedge Z_{u,v}$, which make the definition of $|G\rangle$ does not depends on the orientation or order of the edges of G.

Since both the state $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and the unitary gate $\wedge Z$ can be depicted in the graphical calculus, any graph state can be represented in the graphical language. For instance, the 3-qubit graph state associated to the triangle is represented as follows:
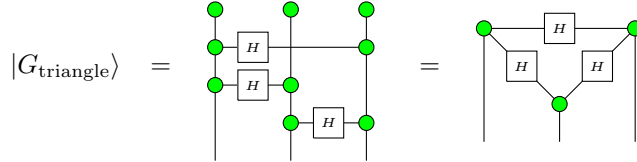
More generally, any graph G, $|G\rangle$ may be depicted by a diagram composed of $|V(G)|$ green dots. Two green dots are connected with a H gate if and only if the corresponding vertices are connected in the graph. Finally, one output wire is connected to every green dot. Note that the qubits in this picture are the output wires rather than the dots themselves; to act on a qubit with some operation we simply connect the picture for that operation to the wire.

Having introduced the graphs states we are now in position to derive one of their fundamental properties, namely the *fixpoint property*.

Property 1 (Fixpoint). Given a graph G and a vertex $u \in V(G)$,

$$R_x(\pi)^{(u)} R_z(\pi)^{(N_G(u))} |G\rangle = |G\rangle$$

The fixpoint property can shown in the graphical calculus by the following example. Consider a star-shaped graph shown below; the qubit u is shown at the top of the diagram, with its neighbours below. The fixpoint property simply

asserts that the depicted equation holds.
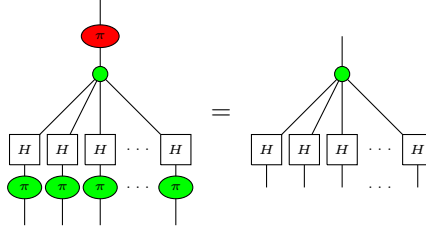
Theorem 1. *The fixpoint property is provable in the graphical language.*

Proof. First, notice it is enough to consider star graphs. Indeed, for more complicated graphs, green rotations can always be pushed through the green dots, leading to the star case.

Let S_n be the star composed of n vertices. Since the red π-rotation is a green comonoid homorphism, the fixpoint property is satisfied for S_1:
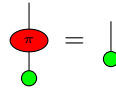
By induction, for any $n > 1$,
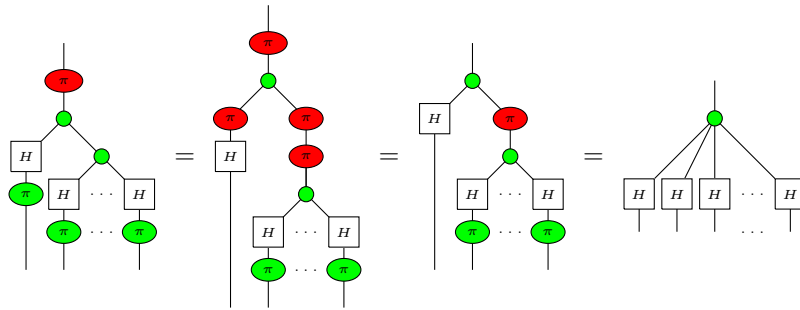
5   Local Complementation

In this section, we present the Van den Nest theorem. According to this theorem, if two graphs are locally equivalent (i.e. one graph can be transformed into the other by means of local complementations) then the corresponding quantum states are LC-equivalent, i.e. there exists a local Clifford unitary[5] which

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[5] One-qubit Clifford unitaries form a finite group generated by $\pi/2$ rotations around X and Z axis: $R_x(\pi/2), R_z(\pi/2)$. A n-qubit local Clifford is the tensor product of n one-qubit Clifford unitaries.

transforms one state into the other. We prove that the local complementation property is true if and only if H has an Euler decomposition into $\pi/2$-green and red rotations. At the end of the section, we demonstrate that the $\pi/2$ decomposition does not hold in all models of the axioms, and hence show that the axiom is truly necessary to prove Van den Nest's Theorem.

Definition 3 (Local Complementation). *Given a graph G containing some vertex u, we define the* local complementation *of u in G, written $G * u$ by the complementation of the neighbourhood of u, i.e. $V(G * u) = V(G)$, $E(G * u) := E(G)\Delta(N_G(u) \times N_G(u))$, where $N_G(u)$ is the set of neighbours of u in G (u is not $N_G(u)$) and Δ is the symmetric difference, i.e. $x \in A\Delta B$ iff $x \in A$ xor $x \in B$.*

Theorem 2 (Van den Nest). *Given a graph G and a vertex $u \in V(G)$,*

$$R_x(-\pi/2)^{(u)} R_z^{(N_G(u))} \ket{G} = \ket{G * u} \ .$$

We illustrate the theorem in the case of a star graph:
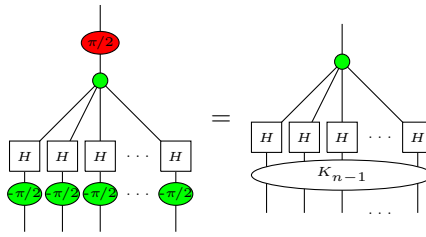
where K_{n-1} denotes the totally connected graph.

Theorem 3. *Van den Nest's theorem holds if and only if H can be decomposed into $\pi/2$ rotations as follows:*
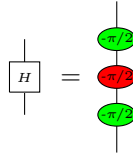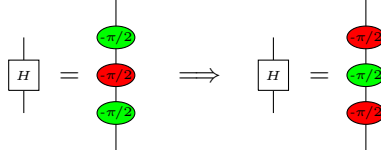

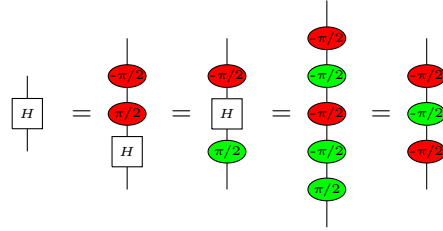Notice that this equation is nothing but the Euler decomposition of H:

$$H = R_Z(-\pi/2) \circ R_X(-\pi/2) \circ R_Z(-\pi/2)$$

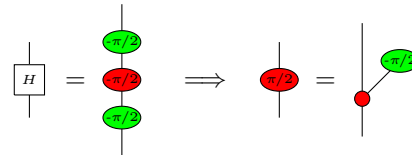Several interesting consequences follow from the decomposition. We note two:

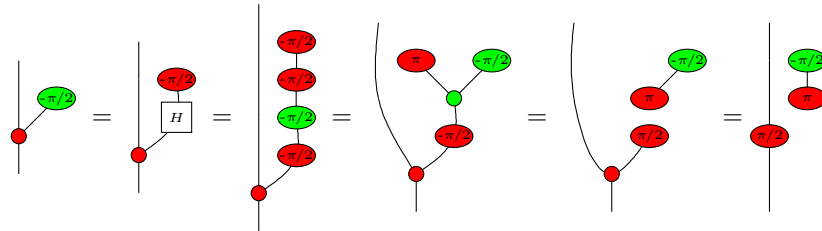**Lemma 5.** *The H-decomposition into $\pi/2$ rotations is not unique:*

Proof.

Lemma 6. *Each colour of $\pi/2$ rotation may be expressed in terms of the other colour.*
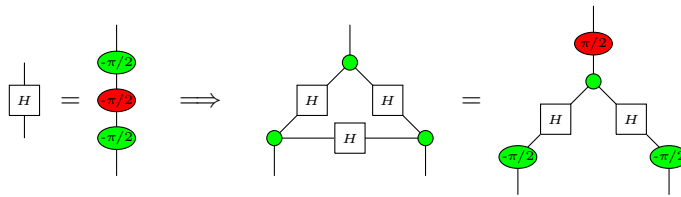

Proof.

Remark: The preceding lemmas depend only on the *existence* of a decomposition of the form $H = R_z(\alpha)\ R_x(\beta)\ R_z(\gamma)$. It is straight forward to generalise these result based on an arbitrary sequence of rotations, although in the rest of this paper we stick to the concrete case of $\pi/2$.

Most of the rest of the paper is devoted to proving Theorem 3: the equivalence of Van den Nest's theorem and the Euler form of H. We begin by proving the easier direction: that the Euler decomposition implies the local complementation property.
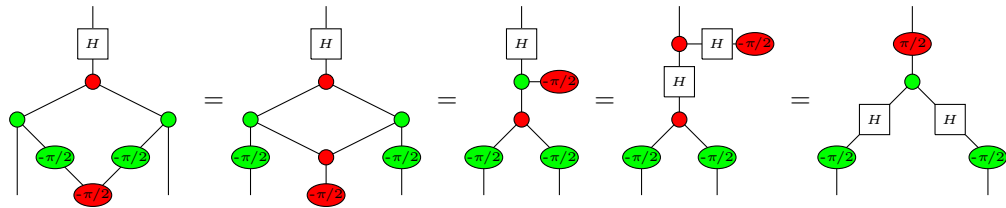
5.1   Euler Decomposition Implies Local Complementation

Triangles We begin with the simplest non trivial examples of local complementation, namely triangles. A local complementation on one vertex of the triangle removes the opposite edge.
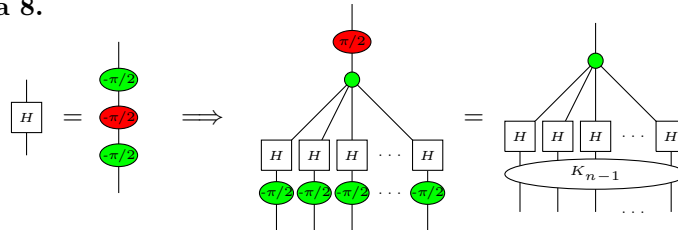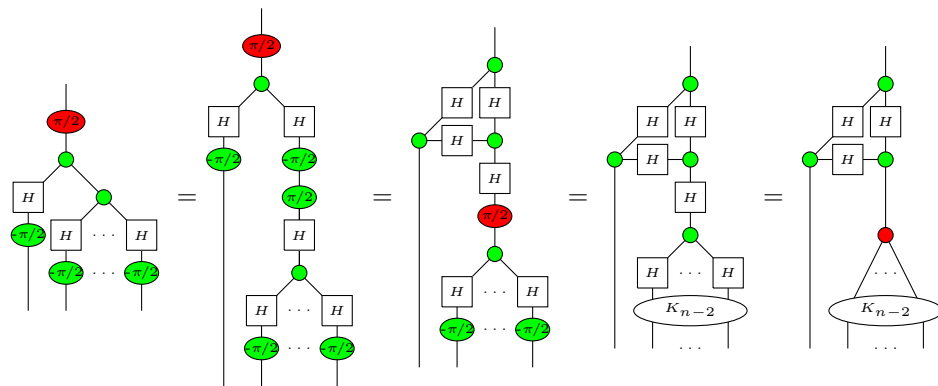
Lemma 7.

Proof.
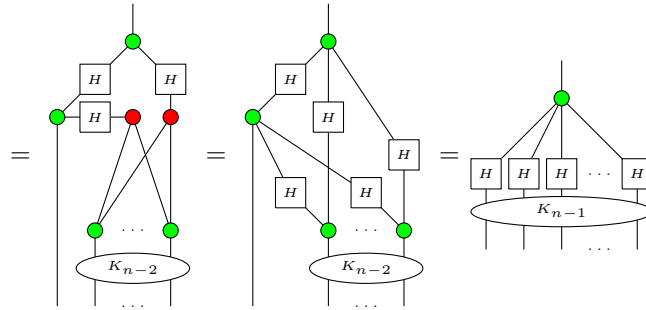

Note the use of Lemma 6 in the last equation.

Complete Graphs and Stars More generally, S_n (a star composed of n vertices) and K_n (a complete graph on n vertices) are locally equivalent for all n.

Lemma 8.

Proof.

General case The general case can be reduced to the previous case: first green rotations can always be pushed through green dots for obtaining the lhs of equation in Lemma 8. After the application of the lemma, one may have pairs of vertices having two edges (one coming from the original graph, and the other from the complete graph). The Hopf law is then used for removing these two edges.

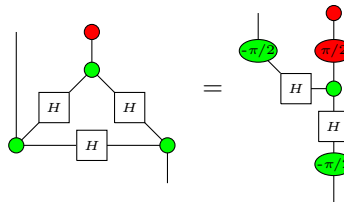### 5.2 Local Complementation Implies Euler Decomposition

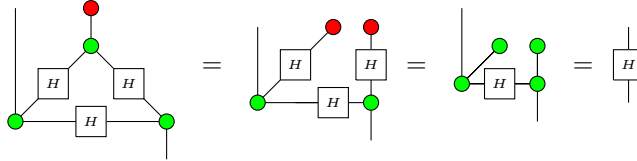**Lemma 9.** *Local complementation implies the H-decomposition:*
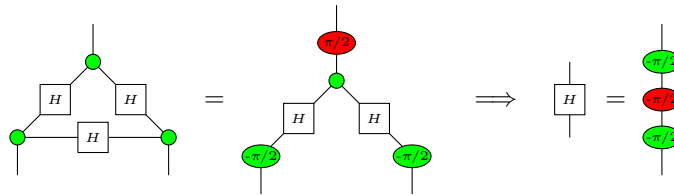
Proof. The local complementation property can be rewritten as follows:
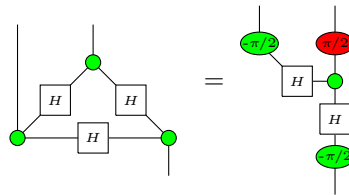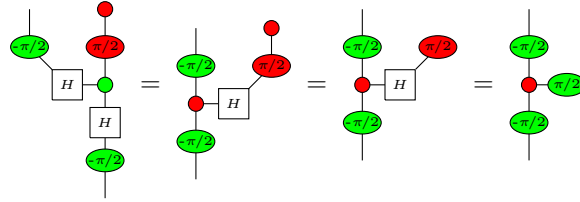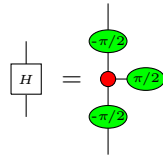

then

Since,

And

So

Complementing the above equation with H on both sides, we obtain:
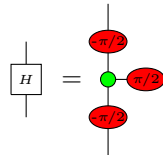
Finally,
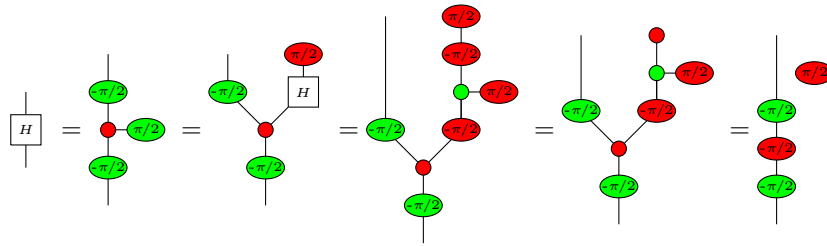
which is the desired decomposition.

This completes the proof of Theorem 3. Note that we have shown the equivalence of two equations, both of which were expressible in the graphical language. What remains to be established is that these properties—and here we focus on the decomposition of H—are not derivable from the axioms already in the system. To do so we define a new interpretation functor.

Let $n \in \mathbb{N}$ and define $[\![\cdot]\!]_n$ exactly as $[\![\cdot]\!]$ with the following change:

$$[\![p_X(\alpha)]\!]_n = [\![p_X(n\alpha)]\!] \qquad [\![p_Z(\alpha)]\!]_n = [\![p_Z(n\alpha)]\!]$$

Note that $[\![\cdot]\!] = [\![\cdot]\!]_1$. Indeed, for all n, this functor preserves all the axioms introduced in Section 2, so its image is indeed a valid model of the theory. However we have the following inequality

$$[\![H]\!]_n \neq [\![p_Z(-\pi/2)]\!]_n \circ [\![p_X(-\pi/2)]\!]_n \circ [\![p_Z(-\pi/2)]\!]_n$$

for example, in $n = 2$, hence the Euler decomposition is not derivable from the axioms of the theory.

6 Conclusions

We studied graph states in an abstract axiomatic setting and saw that we could prove Van den Nest's theorem if we added an additional axiom to the theory. Moreover, we proved that the $\pi/2$-decomposition of H is exactly the extra power which is required to prove the theorem, since we prove that the Van den Nest theorem is true if and only if H has a $\pi/2$ decomposition. It is worth noting that the system without H is already universal in the sense every unitary map is expressible, via an Euler decomposition. The original system this contained two representations of H which could not be proved equal; it's striking that removing this ugly wart on the theory turns out to necessary to prove a non-trivial theorem. In closing we note that this seemingly abstract high-level result was discovered by studying rather concrete problems of measurement-based quantum computation.

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