SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE
WEDNESDAY 31 OCTOBER 2007

Mark Scheme:

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A: Separating out the powers of 2 and 3 we have

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}} = 2^{(r+s+2r-2s-3r)} \times 3^{(r+s+r-s-2r-4s)} = 2^{-s} \times 3^{-4s}$$

which is an integer if $s \leqslant 0$. **The answer is (b).**

B:

- $\sin(10x + 11)$ takes values between -1 and 1 as x varies;
- $\sin^2(10x + 11)$ takes values between 0 and 1 as x varies;
- $3\sin^2(10x + 11)$ takes values between 0 and 3 as x varies;
- $3\sin^2(10x + 11) - 7$ takes values between -7 and -4 as x varies;
- $\left(3\sin^2(10x + 11) - 7\right)^2$ takes values between 16 and 49 as x varies.

The answer is (c).

C: Using the identity $\sin^2 x + \cos^2 x = 1$ we see

$$\begin{aligned} 7\sin x + 2\cos^2 x &= 5 \\ \iff 2\sin^2 x - 7\sin x + 3 &= 0 \\ \iff (2\sin x - 1)(\sin x - 3) &= 0 \end{aligned}$$

Now $\sin x = 3$ has no solutions, and in the range $0 \leqslant x < 2\pi$ we note $\sin x$ takes the value $1/2$ twice (at $\pi/6$ and at $5\pi/6$). **The answer is (b).**

D: The circle with equation $(x-5)^2 + (y-4)^2 = 4$ has centre $(5, 4)$ and radius 2.

The circle with equation $(x-1)^2 + (y-1)^2 = 1$ has centre $(1, 1)$ and radius 1.

The vector from the first circle's centre to the second circle's centre is $(-4, -3)$ which has length $\sqrt{(-4)^2 + (-3)^2} = 5$. So the point on the first circle, closest to the second is

$$(5, 4) + \frac{2}{5}(-4, -3) = (5, 4) + (-1.6, -1.2) = (3.4, 2.8).$$

The answer is (a).

E: Let

$$f_n(x) = (1-x)^n (2-x)^{2n} (3-x)^{3n} (4-x)^{4n} (5-x)^{5n}.$$

- If $x = 4$ then $f_n(x) = 0$ and so (a) and (d) are false.
- If $n = 6$ then each exponent in $f_n(x)$ is even and so (c) is false.
- If $x > 5$ then each bracket is negative, and if n is odd then

$$f_n(x) = (\text{negative})(\text{positive})(\text{negative})(\text{positive})(\text{negative}) < 0.$$

The answer is (b).

F: If we set $y = 2^x$ then the equation $8^x + 4 = 4^x + 2^{x+2}$ can be rewritten as

$$\begin{aligned} y^3 + 4 &= y^2 + 4y \\ \iff y^3 - y^2 - 4y + 4 &= 0 \\ \iff (y-1)\left(y^2 - 4\right) &= 0 \end{aligned}$$

So $y = 1, 2, -2$ are the possible values for y. But as $y = 2^x > 0$ then only positive values for y will lead to real values for x. Hence $y = 1, 2$ and $x = 0, 1$ are the only possible x-values. **The answer is (c).**

G: If $y = 2^{-x}\sin^2\left(x^2\right)$ then note that $y > 0$, which discounts (b). Also $y(0) = 0$ which discounts (d). Finally the points where graph (c) meets the x-axis arise regularly – this is not the case with $y = 2^{-x}\sin^2\left(x^2\right)$ where $y = 0$ at $x = \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}, \ldots$ **The answer is (a).**

H: If we set

$$A = \int_0^1 f(x)\,\mathrm{d}x, \qquad B = \int_1^2 f(x)\,\mathrm{d}x$$

then we have the equations

$$3A + 2B = 7, \qquad (A+B) + B = 1.$$

Solving these simultaneous equations we find $A = 3$ and $B = -1$. Hence

$$\int_0^2 f(x)\,\mathrm{d}x = A + B = 3 - 1 = 2.$$

The answer is (d).

I: Note that a is largest when $\log_{10} a$ is largest. As $4(\log_{10} a)^2 + (\log_{10} b)^2 = 1$ then $(\log_{10} a)^2$ is largest when $b = 1$ and $\log_{10} b = 0$. So

$$4(\log_{10} a)^2 = 1 \implies \log_{10} a = \frac{1}{2} \implies a = \sqrt{10}.$$

The answer is (c).

J: Note that

$$(n+1) + \left(n^4 + 2\right) + \left(n^9 + 3\right) + \left(n^{16} + 4\right) + \cdots + \left(n^{10000} + 100\right)$$

increases as n increases. So the inequality will hold for all $n \geqslant 1$ if it holds for $n = 1$. So we need

$$\begin{aligned} (1+1) + (1+2) + (1+3) + (1+4) + \cdots + (1+100) &> k \\ \iff 2 + 3 + 4 + \cdots + 101 &> k \\ \iff \frac{100}{2}(2 + 101) &> k \\ \iff 5150 &> k. \end{aligned}$$

The answer is (d).

QUESTION 2: We have
$$f_n(x) = (2 + (-2)^n) x^2 + (n+3) x + n^2.$$

(i) So

$$f_3(x) = -6x^2 + 6x + 9 = -6\left(x^2 - x - \frac{3}{2}\right) = -6\left(\left(x - \frac{1}{2}\right)^2 - \frac{7}{4}\right) = -6\left(x - \frac{1}{2}\right)^2 + \frac{21}{2}.$$

So the maximum is $\frac{21}{2} = 10.5$ achieved at $x = 1/2$.

For any n, $f_n(x)$ is a quadratic in x which has a maximum when the lead coefficient is negative. If $2 + (-2)^n < 0$ then n is an odd number greater than 1.

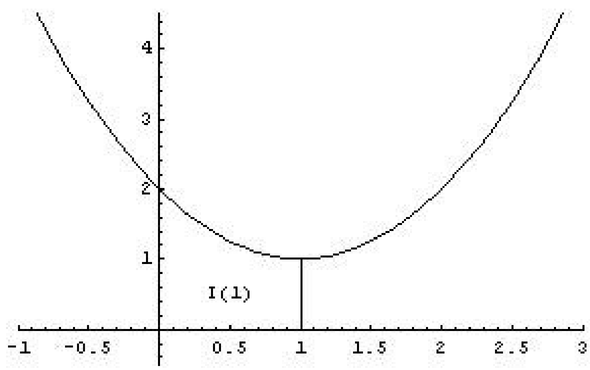
(ii) Setting $n = 1$ we have $f_1(x) = 4x + 1$. So

$$\begin{aligned} f_1(f_1(x)) &= 4(4x+1) + 1 = 16x + 5 \\ f_1(f_1(f_1(x))) &= 4(16x+5) + 1 = 64x + 21 \end{aligned}$$

More generally
$$f_1^k(x) = 4^k x + \left(1 + 4 + \cdots + 4^{k-1}\right) = 4^k x + \frac{4^k - 1}{3}.$$

(iii) Setting $n = 2$ we have $f_2(x) = 6x^2 + 5x + 4$. So $f_2^k(x)$ is a polynomial of degree 2^k.

QUESTION 3: (i)



(ii) As $\left(x - c^2\right) + c^2 \geqslant 0$ for all x then $I(c) \geqslant 0$.

(iii)
$$I(c) = \int_0^1 \left((x-c)^2 + c^2\right) \mathrm{d}x = \left[\frac{(x-c)^3}{3}\right]_0^1 + c^2 = \frac{(1-c)^3}{3} + \frac{c^3}{3} + c^2 = 2c^2 - c + \frac{1}{3}.$$

(iv) Completing the square

$$I(c) = 2\left(c^2 - \frac{c}{2} + \frac{1}{6}\right) = 2\left(\left(c - \frac{1}{4}\right)^2 + \frac{5}{48}\right) = 2\left(c - \frac{1}{4}\right)^2 + \frac{5}{24}.$$

So the minimum is 5/24.

(v) If c can only vary between ± 1 then the maximum is at $I(-1)$ as -1 is furthest from $1/4$. In this case

$$I(-1) = 2\left(\frac{5}{4}\right)^2 + \frac{5}{24} = \frac{50}{16} + \frac{5}{24} = \frac{150 + 10}{48} = \frac{160}{48} = \frac{10}{3}.$$

QUESTION 4: (i) Let $C = (1,1)$ denote the centre of the circle. then CQ makes angle θ with the vertical and is of length 1. So

$$Q = C + \overrightarrow{CQ} = (1,1) + (\sin\theta, \cos\theta) = (1+\sin\theta, 1+\cos\theta).$$

The gradient of the line is

$$\frac{-RO}{PO} = -\tan\theta$$

by looking at the triangle OPR. So using the formula $y - y_Q = m(x - x_Q)$ we have

$$y - 1 - \cos\theta = -\tan\theta\,(x - 1 - \sin\theta).$$

At P we have $y = 0$ and so we have

$$x = \cot\theta\,(1+\cos\theta) + 1 + \sin\theta = \frac{\cos\theta + \cos^2\theta + \sin\theta + \sin^2\theta}{\sin\theta} = 1 + \cot\theta + \csc\theta.$$

(ii) If we consider the diagram with $\pi/2 - \theta$ as the angle OPR rather than θ, then this is just a reflection of the θ-diagram in the $y = x$ line. Hence, comparing areas,

$$A(\theta) = B(\pi/2 - \theta).$$

So when $\theta = \pi/4$ we have, dividing up the triangle

$$A(\pi/4) + B(\pi/4) + 3\pi/4 + 1 = \frac{1}{2}P_{\pi/4}R_{\pi/4}$$

But $A(\pi/4) = B(\pi/4)$ and $P_{\pi/4} = R_{\pi/4} = 1 + 1 + \sqrt{2} = 2 + \sqrt{2}$. Hence

$$2A(\pi/4) + 3\pi/4 + 1 = \frac{1}{2}\left(2+\sqrt{2}\right)^2 = 3 + 2\sqrt{2}$$

giving

$$A(\pi/4) = 1 + \sqrt{2} - \frac{3\pi}{8}.$$

(iii) Let $D = (1,0)$. When $\theta = \pi/3$ we can calculate $A(\pi/3)$ as the area of the congruent right-angled triangles DCP and PCQ minus 1/3 of the circle. So

$$A(\pi/3) = 2\left(\frac{1}{2}\times\left(P_{\pi/3} - 1\right)\times 1\right) - \frac{\pi}{3} = \left(1 + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - 1\right) - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}.$$

QUESTION 5: (i)

$$f(5) = 2f(4) = 2(f(2))^2 = 2\left(\left(f(1)^2\right)\right)^2 = 2\left(\left(2^2\right)^2\right) = 32.$$

(ii) As we had to calculate $f(4), f(2), f(1), f(0)$ on the way then $f(5)$ has recursion depth 4.

(iii)

$$g(5) = 1 + g(4) = 1 + 1 + g(2) = 1 + 1 + 1 + g(1) = 1 + 1 + 1 + 1 + g(0) = 4.$$

(iv) For any natural number k

$$g\left(2^k\right) = 1 + g\left(2^{k-1}\right) = \cdots = k + g\left(2^0\right) = k + g(1) = k + 1.$$

(v) For natural numbers $l > k \geqslant 0$

$$g\left(2^l + 2^k\right) = k + g\left(2^{l-k} + 1\right) = k + 1 + g\left(2^{l-k}\right) = k + 1 + l - k + 1 = l + 2.$$

(vi) In the definition of $g(n)$ a further 1 is added to previously calculated values at each stage whether n is even or odd; as $g(0) = 0$ then $g(n)$ is a measure of the number of previously calculated values, i.e. $g(n)$ equals the recursion depth.

QUESTION 6:

i) Suppose that Alf says "I always tell lies" and Beth says "Yes, that's true, Alf always tells lies".

Only the random person could ever say "I always tell lies", (untruthfully), so Alf is the random person. Therefore Beth's statement is false, leaving Gemma as the truthful person.

ii) Suppose instead that Gemma says "Beth always tells the truth" and Beth says "That's wrong".

The truthful person can never say that anyone else is the truthful person. So Gemma's statement rules her out as the truthful person.

If Gemma's statement were true then Beth couldn't reply so. So the statement is false and Alf is the truthful person.

Because Beth has told the truth she is the random person and Gemma is the liar.

(iii) Suppose instead that Alf says "Beth is the one who behaves randomly" and Gemma says "Alf always lies". Then Beth says "You have heard enough to determine who always tells the truth".

This is probably best done by considering all six possibilities.

Alf's statement rules out (A,B,G) = (T,L,R) or (L,R,T)

Gemma's rules out (R,L,T).

We do *not* have enough information to determine who always tells the truth, so Beth's statement was a lie, ruling out (L,T,R) and (R,T,L).

This leaves (T,R,L); Alf was truthful and Gemma was the liar.

QUESTION 7:

(a) A match must begin with $\left[\frac{\mathsf{UU}}{\mathsf{U}}\right]$.

(b) There are two matches including four tiles.

$$\left[\frac{\mathsf{UU}}{\mathsf{U}}\right]\left[\frac{\mathsf{X}}{\mathsf{U}}\right]\left[\frac{\mathsf{Z}}{\mathsf{X}}\right]\left[\frac{\mathsf{E}}{\mathsf{ZE}}\right]$$
$$\left[\frac{\mathsf{UU}}{\mathsf{U}}\right]\left[\frac{\mathsf{Y}}{\mathsf{U}}\right]\left[\frac{\mathsf{Z}}{\mathsf{Y}}\right]\left[\frac{\mathsf{E}}{\mathsf{ZE}}\right].$$

(c) With the new tile, the shortest match consists of three copies of $\left[\frac{\mathsf{UU}}{\mathsf{U}}\right]$, followed by a free choice of three tiles from $\left\{\left[\frac{\mathsf{X}}{\mathsf{U}}\right], \left[\frac{\mathsf{Y}}{\mathsf{U}}\right]\right\}$, then three tiles from $\left\{\left[\frac{\mathsf{Z}}{\mathsf{X}}\right], \left[\frac{\mathsf{Z}}{\mathsf{Y}}\right]\right\}$ (the selection of which is determined by the previous choice), and finally the new tile $\left[\frac{\mathsf{E}}{\mathsf{ZZZE}}\right]$.

Thus the shortest match involves 10 tiles.

There are $2^3 = 8$ matches of of this length.

(d) A list of x tiles of the first type and y tiles of the second type is a match iff $7x + y = x + 10y$, i.e., $2x = 3y$.

This implies that $y = 2k$ for some $k > 0$ and $x = 3k$., giving $x + y = 5k$. So the number of tiles used has to be a multiple of 5.