I. HILBERT SPACES, PROJECTORS AND SPECTRAL PROPERTIES

Let $\mathcal{V}$ be a Hilbert space with $\dim(\mathcal{V}) = d$ finite and let $|i\rangle$ index a basis for $\mathcal{V}$, such that $\langle i|j\rangle = \delta_{ij}$, where δ_{ij} is Kronecker's delta. Consider the vectors $|\phi\rangle, |\psi\rangle \in \mathcal{V}$ and some $P = \sum_{n=1}^{N} |n\rangle\langle n|$ where all $|n\rangle \in \{|i\rangle\}$, and the index n ranges over at most d vectors.

(i) Show that $P^2 = P$ and hence that P is a projector. Show that $U = -2\sum_n |n\rangle\langle n| + \mathbf{1}$ is self-adjoint and unitary.

(ii) Give values for the matrix trace and determinant of U.

(iii) Show that $(1 - \sum_n |n\rangle\langle n|)$ and $(\sum_n |n\rangle\langle n|)$ project onto eigenspaces of U, find the corresponding eigenvalues and the maximum dimension of the eigenspaces. Show that eigenvectors $|\phi\rangle$ in the $+1$ and -1 subspaces satisfy $P|\phi\rangle = 0$ and $P|\phi\rangle = |\phi\rangle$, respectively.

(iv) Hence, using these results or otherwise construct an invertible map between projectors and self-adjoint unitary maps.

II. TENSOR PRODUCTS, DENSITY OPERATORS, SINGULAR VALUES AND PURIFICATION

Let $\psi = |001\rangle + |010\rangle + |100\rangle$ be a state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\dim(\mathcal{H}_1) = 2$ and $\dim(\mathcal{H}_2) = 4$, and let

$$V^\dagger = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1}$$

(i) By writing $\phi = \sum_{ij} C_{ij}|i\rangle|j\rangle$, where $|i\rangle \in \mathcal{H}_1$, $|j\rangle \in \mathcal{H}_2$ are both orthonormal basis, state the values of $i \in \{0,1\}, j \in \{0,1,2,3\}$ that correspond to non-zero coefficients of C_{ij} and hence express ψ in the basis $|i\rangle, |j\rangle$.

(ii) Write the 2x4 matrix $\mathbf{C} = (C_{ij})_{ij}$ and show that $\mathbf{C}\mathbf{C}^\dagger$ and $\mathbf{C}^\dagger\mathbf{C}$ are (non-normalized) density operators, equivalent to ρ_1, ρ_2 found by tracing over the systems $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. (Here the adjoint $\dagger$ means matrix conjugate transpose.)

(iii) From the singular value decomposition, one can write $\mathbf{C} = U\Sigma V^\dagger$. For V given above, and U the 2x2 identity matrix, find the 2x4 matrix of singular values Σ.

(iv) Find purifications for ρ_1 and ρ_2 (other than ψ).

III. HAMILTONIANS, EVOLUTIONS, OBSERVABLES AND PROBABILITIES

Recall that the evolution of a quantum system in a static configuration can be found by solving the eigenvalue equation, $H|\psi_n\rangle = E_n|\psi_n\rangle$, where $\mathbf{H}$ is the Hamiltonian and E_n is the energy of state $|\psi_n\rangle$ written in terms of $\hbar\omega$, where ω is angular frequency. The time-evolution of some initial state $|\phi\rangle$ under $\mathbf{H}$ is then given as

$$|\phi(t)\rangle = \sum_n \exp(-itE_n/\hbar)\langle\psi_n|\phi\rangle|\psi_n\rangle \tag{2}$$

Let the Hamiltonian for a certain three-level system be given as

$$\mathbf{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and consider the observables

$$\mathbf{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\mathbf{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

for positive real constants λ and μ.

(i) Find a map P that transforms the eigenvectors of $\mathbf{A}$ (given as $|1\rangle + |2\rangle$, $|1\rangle - |2\rangle$, $|3\rangle$) into the eigenvectors of $\mathbf{B}$.

(ii) Suppose the system starts out in the generic state $|\psi(0)\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$ where $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. Find the expectation values at $t = 0$ of $\mathbf{H}$, $\mathbf{A}$ and $\mathbf{B}$.

(iii) Find the time-dependence of the static configuration and hence find $|\psi(t)\rangle$ under the Hamiltonian $\mathbf{H}$.

(iv) Determine the possible measured values of the energy of state $|\psi\rangle$ at time t. Likewise, determine the possible measured values values of $|\psi\rangle$ at time t for the observables $\mathbf{A}$ and $\mathbf{B}$.