SHEET 1*
THE QUANTUM THEORY OF INFORMATION AND COMPUTATION — Trinity Term 2010†

* Course homepage: http://www.comlab.ox.ac.uk/activities/quantum/course/

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QUANTUM CIRCUITS, SYMMETRY AND EIGENSTATES

(i) Let $\underline{P}.\underline{\sigma} := p_0 X + p_1 Y + p_2 Z$ where $|P| = 1$ and show that $\exp(-i\frac{\theta}{2}\underline{P}.\underline{\sigma}) = \mathbf{1}\cos(\theta/2) - i(\underline{P}.\underline{\sigma})\sin(\theta/2)$ and find the values of θ, $\underline{P}$ to recover the Hadamard gate, up to a phase factor. Give a Hamiltonian and prescribed time of evolution to remove this global phase factor.

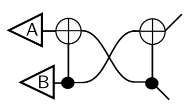
(ii) Find the time of the evolution of the Hamiltonian $|11\rangle\langle 11|$ to create a CZ-gate, then write down a quantum circuit in terms of H and CZ to create a CNOT-gate. What are the input states needed to use the CNOT-gate to prepare the singlet state $|\Psi^-\rangle = |01\rangle - |10\rangle$?

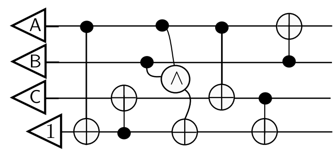
(iii) Show that the SWAP operator $\frac{1}{2}(\mathbf{1} + \underline{\sigma}_A \cdot \underline{\sigma}_B)$ permutes the values of bits A and B as

$$\text{SWAP}|i_A\rangle|i_B\rangle = |i_B\rangle|i_A\rangle \tag{1}$$

where the notation $\underline{\sigma}_A \cdot \underline{\sigma}_B$ stands for the scalar and tensor product.

(iv) Express the logic function on bits A, B, C implemented by the following circuit





where the truth table for $\mathsf{XOR}/\mathsf{AND}$ is given as

a	b	$a \oplus b$	$a \wedge b$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

(v) In the computational basis, express the general form of a two-qubit symmetric eigenstate of the SWAP operator and count the real degrees of freedom. Repeat this for anti-symmetric eigenstates (e.g. $\text{SWAP}|\psi\rangle = -|\psi\rangle$).

(vi) Using the notation from (iii) above, find a value for q to show that the two-site quantum Heisenberg model $J\underline{\sigma}_1 \cdot \underline{\sigma}_2$ can be written as $\frac{J}{2}\left((\underline{\sigma}_1 + \underline{\sigma}_2)^2 - q\mathbf{1}\right)$ and show that $|\Psi^\pm\rangle = |01\rangle \pm |10\rangle$ are energy eigenstates.

HAMILTONIAN AND STATE SYMMETRY, DUALITY AND REPRESENTATIONS

Symmetric quantum states are invariant under permutation (SWAP) of any pair of particles. Examples are the three qubit GHZ-state $|\mathsf{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and the three qubit W-state $|\mathsf{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle$. Note: a general n qubit quantum system has the 2^n orthonormal basis vectors $\{|00\ldots00\rangle, |00\ldots01\rangle, \ldots, |11\ldots11\rangle\}$. For the subspace of *symmetric* n qubit states, an orthonormal basis is given by the $n+1$ *symmetric* basis states $\{|S_0\rangle, |S_1\rangle, \ldots, |S_n\rangle\}$. They are defined as:

$$|S_k\rangle = \binom{n}{k}^{-\frac{1}{2}} \sum_{\text{perm}} \underbrace{|0\rangle|0\rangle\cdots|0\rangle}_{n-k}\underbrace{|1\rangle|1\rangle\cdots|1\rangle}_{k} \tag{2}$$

We can therefore write $|\mathsf{W}\rangle = |S_1\rangle$ and $|\mathsf{GHZ}\rangle = \frac{1}{\sqrt{2}}(|S_0\rangle + |S_3\rangle)$.

By means of the so-called *Majorana Representation* every symmetric state of n qubits $|\psi_{\text{s}}\rangle$ can be unambiguously represented by n single qubit states $|\phi_i\rangle$:

$$|\psi_{\text{s}}\rangle = \frac{1}{\sqrt{K}} \sum_{\text{perm}} |\phi_{P(1)}\rangle \otimes |\phi_{P(2)}\rangle \otimes \cdots \otimes |\phi_{P(n)}\rangle \tag{3}$$

$$|\phi_i\rangle = \cos\tfrac{\theta_i}{2}\,|0\rangle + e^{-i\varphi_i}\sin\tfrac{\theta_i}{2}|1\rangle \tag{4}$$

The above sum is performed over all permutations $P : \{1,\ldots,n\} \to \{1,\ldots,n\}$, and the normalisation factor K is in general different for different $|\psi_{\text{s}}\rangle$. The $|\phi_i\rangle$ can be visualised by points on the Bloch sphere — called the *Majorana Points* (MP).

(i) Verify by direct calculation that $|\mathsf{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ is composed of the MPs: $|\phi_1\rangle = |0\rangle$, $|\phi_2\rangle = |0\rangle$ and $|\phi_3\rangle = |1\rangle$ and that $|\mathsf{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is composed of the MPs: $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|\phi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i2\pi/3}|1\rangle)$ and $|\phi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i4\pi/3}|1\rangle)$.

(ii) Find a matrix M of kets, e.g.

$$M = \begin{bmatrix} |\ \rangle & 0 \\ 0 & |\ \rangle \end{bmatrix} \tag{5}$$

such that $|\mathsf{GHZ}\rangle = \mathrm{Tr}M^3 = 2\langle +|M^3+\rangle$ and a matrix of kets Q and states $\psi := |1\rangle$ and $\phi := |0\rangle$ such that $|\mathsf{W}\rangle = 2\langle\phi|Q^3\psi\rangle$, where the *internal* matrix product is interpreted as tensor $\otimes$.

(iii) (Operator and Hamiltonian symmetry) By considering the Pauli-algebra, calculate

$$(\sigma^z \otimes \sigma^z \otimes \sigma^z \otimes \sigma^z)(\sigma^x \otimes \sigma^x \otimes \mathbf{1} \otimes \mathbf{1})(\sigma^z \otimes \sigma^z \otimes \sigma^z \otimes \sigma^z) \tag{6}$$

$$(\sigma^z \otimes \sigma^z \otimes \sigma^z \otimes \sigma^z)(\mathbf{1} \otimes \sigma^x \otimes \sigma^x \otimes \mathbf{1})(\sigma^z \otimes \sigma^z \otimes \sigma^z \otimes \sigma^z) \tag{7}$$

$$(\sigma^z \otimes \sigma^z \otimes \sigma^z \otimes \sigma^z)(\mathbf{1} \otimes \mathbf{1} \otimes \sigma^x \otimes \sigma^x)(\sigma^z \otimes \sigma^z \otimes \sigma^z \otimes \sigma^z) \tag{8}$$

and repeat the calculation by making the replacement $\sigma^x \mapsto \sigma^y$. Consider the 1D XY-model with open boundary conditions

$$H_{XY} := \sum_{j=1}^{N} \left(\frac{1+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2} \sigma_j^y \sigma_{j+1}^y \right) - \lambda \sum_{j=1}^{N} \sigma_j^z \tag{9}$$

where the real parameter λ is the intensity of the magnetic field applied in the z-direction and the parameter γ determines the degree of anisotropy of the spin-spin interaction. Hence, using the results in (iii) above or otherwise, show that the XY-model is invariant under conjugation by $\Pi_{j=1}^{N} \sigma_j^z$ as

$$\left(\Pi_{j=1}^{N} \sigma_j^z\right) H_{XY} \left(\Pi_{j=1}^{N} \sigma_j^z\right) \tag{10}$$

and hence show that $[\Pi_{j=1}^{N} \sigma_j^z, H_{XY}] = 0$.

(iv) By considering real symmetric states and the duality induced by the linear maps

$$\eta := \sum_i |i\rangle \otimes |i\rangle \tag{11}$$

$$\epsilon := \sum_i \langle i| \otimes \langle i| \tag{12}$$

show that the product induced by 3-qubit states (e.g. $\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2$) is commutative and associative. Depict these properties using string diagrams.

(v) Find the units for the product induced by $|\mathsf{GHZ}\rangle$ and $|\mathsf{W}\rangle$, and hence show that these products form commutative monoids. Depict the iteration of these two products (e.g. the interaction of W- and GHZ-products) diagrammatically by joining two legs. This joining gives rise to a linear map $\mathbb{C}^2 \to \mathbb{C}^2$ — give it's explicit Schmidt form and determine the rank.

ENTANGLEMENT AND MAJORANA POINTS

The *Geometric Measure of Entanglement* E_G is defined as the maximal overlap of a quantum state $|\psi\rangle$ with all product states $|\lambda\rangle = |\lambda_1\rangle \otimes \cdots \otimes |\lambda_n\rangle$:

$$E_G(|\psi\rangle) = -\log_2 \left(\max_{|\lambda\rangle \in \mathcal{H}_{\mathrm{SEP}}} |\langle \lambda|\psi\rangle|^2 \right). \tag{13}$$

If $|\psi\rangle$ is a symmetric state with only positive valued coefficients, then the expression $|\langle\lambda|\psi\rangle|$ is maximized by a product state $|\lambda\rangle$ which is also symmetric and has positive coefficients: $|\lambda\rangle = |\sigma\rangle^{\otimes n}$ with $|\sigma\rangle = \cos\frac{\vartheta}{2}|0\rangle + \sin\frac{\vartheta}{2}|1\rangle$. We call the single qubit state $|\sigma\rangle$ a *Closest*

Product Point (CPP).

Note: It was proved only very recently that if $|\psi\rangle$ is symmetric, then there exists a closest product state $|\lambda\rangle$ which is symmetric itself. From this it is easy to show that if $|\psi\rangle$ is symmetric *and* positive, then $|\lambda\rangle$ is also symmetric *and* positive.

This result means that we can visualize the MPs as well as the CPPs on the Bloch sphere. Note that an n qubit symmetric state has exactly n undistinguishable MPs (which can coincide), while the number of CPPs is not fixed. By definition there is at least one CPP. See Figure 1 for examples.

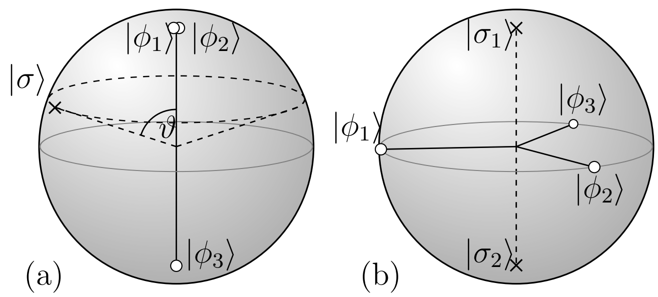


FIG. 1. The MPs (white dots) and CPPs (crosses) of the three qubit (a) W-state and (b) GHZ-state on the Bloch sphere.

(i) Determine the positive CPP of $|\mathsf{W}\rangle$. For this, use the ansatz $|\sigma\rangle = \cos\frac{\vartheta}{2}|0\rangle + \sin\frac{\vartheta}{2}|1\rangle$ and determine the maximum from Equation (13).

(ii) A given symmetric state can have more than one CPP. Show that for $|\mathsf{GHZ}\rangle$ there are two positive CPPs, namely $|\sigma_1\rangle = |0\rangle$ and $|\sigma_2\rangle = |1\rangle$.

(iii) Calculate the geometric entanglement E_G of $|\mathsf{W}\rangle$ and $|\mathsf{GHZ}\rangle$.

(iv) Find the MPs and the positive CPP of the two qubit Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and visualize them on the Bloch sphere. Determine the geometric entanglement E_G of $|\Psi^+\rangle$.

HAMILTONIANS, EVOLUTIONS, ENTANGLEMENT

The two-site Hubbard model with cyclic boundary conditions is given by

$$H = t(c^\dagger_{1\uparrow}c_{2\uparrow} + c^\dagger_{1\downarrow}c_{2\downarrow} + c^\dagger_{2\uparrow}c_{1\uparrow} + c^\dagger_{2\downarrow}c_{1\downarrow}) + U(n_{1\uparrow}n_{1\downarrow} + n_{2\uparrow}n_{2\downarrow}) \tag{14}$$

where

$$n_{j\uparrow} := c^\dagger_{j\uparrow}c_{j\uparrow} \tag{15}$$

$$n_{j\downarrow} := c^\dagger_{j\downarrow}c_{j\downarrow} \tag{16}$$

and where the Fermi-operators $c^\dagger_{j\uparrow}$, $c^\dagger_{j\downarrow}$, $c_{j\uparrow}$, $c_{j\downarrow}$ obey the anti-commutation relations

$$\{c^\dagger_{i\kappa}, c_{j\kappa'}\} = \delta_{\kappa\kappa'}\delta_{ij}\mathbf{1} \tag{17}$$

$$\{c^\dagger_{i\kappa}, c^\dagger_{j\kappa'}\} = \{c_{i\kappa}, c_{j\kappa'}\} = 0 \tag{18}$$

This Hamiltonian commutes with the total number operator $\hat{N}$ and the total spin operator $\hat{S}_z$ in the z direction

$$\hat{N} = \sum_{j=1}^{2}(c^\dagger_{j\uparrow}c_{j\uparrow} + c^\dagger_{j\downarrow}c_{j\downarrow}) \tag{19}$$

$$\hat{S}_z = \sum_{j=1}^{2}(c^\dagger_{j\uparrow}c_{j\uparrow} - c^\dagger_{j\downarrow}c_{j\downarrow}) \tag{20}$$

(i) Consider the subspace (Fock space) with $N = 2$ electrons and $S_z = 0$. Consider the two particle total spin-zero basis as $|s_1\rangle := c^\dagger_{1\uparrow}c^\dagger_{1\downarrow}|0\rangle$, $|s_2\rangle := c^\dagger_{1\uparrow}c^\dagger_{2\downarrow}|0\rangle$, $|s_3\rangle := c^\dagger_{2\uparrow}c^\dagger_{1\downarrow}|0\rangle$ and $|s_1\rangle := c^\dagger_{2\uparrow}c^\dagger_{2\downarrow}|0\rangle$. Find the matrix representation of H in this basis.

(ii) As in (i) above, express the Hamiltonian in the basis given by

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}(c^\dagger_{1\downarrow}c^\dagger_{1\uparrow}|0\rangle + c^\dagger_{2\downarrow}c^\dagger_{2\uparrow}|0\rangle) \tag{21}$$

$$|\Psi^+\rangle := \frac{1}{\sqrt{2}}(c^\dagger_{1\downarrow}c^\dagger_{2\uparrow}|0\rangle + c^\dagger_{2\downarrow}c^\dagger_{1\uparrow}|0\rangle) \tag{22}$$

$$|\Phi^-\rangle := \frac{1}{\sqrt{2}}(c^\dagger_{1\downarrow}c^\dagger_{1\uparrow}|0\rangle - c^\dagger_{2\downarrow}c^\dagger_{2\uparrow}|0\rangle) \tag{23}$$

$$|\Psi^-\rangle := \frac{1}{\sqrt{2}}(c^\dagger_{1\downarrow}c^\dagger_{2\uparrow}|0\rangle - c^\dagger_{2\downarrow}c^\dagger_{1\uparrow}|0\rangle) \tag{24}$$

and by considering the Bell-basis show that it can be written as a direct sum ($\oplus$) of 2 by 2 matrices.

(iii) Show that the two point Hubbard model admits a discrete symmetry under particle exchange $2 \mapsto 1$, $1 \mapsto 2$. Hence or otherwise, consider the irreducible representations of the finite two element group and identify two invariant subspaces.

(iv) Find the time evolution of the initial state $|p(0)\rangle = \frac{1}{\sqrt{2}}(c^\dagger_{1\downarrow}c^\dagger_{2\uparrow}|0\rangle + c^\dagger_{2\downarrow}c^\dagger_{1\uparrow}|0\rangle)$ and calculate the geometric entanglement E_G of this state as a function of time.