Skeleton Process Tomography and Distance Measures on Quantum States

Jacob Biamonte*

Distance measures and process tomography are important concepts in quantum information theory. We did not have enough time to cover these topics in lectures, but the basic ideas are fairly simple and important enough to warrent study. This short note is designed to introduce the ideas needed to follow some of the derivations we have used, in particular Vlatko Vedral will assume readers would have exposure to these ideas durring his lecture on many-body entanglement theory (6/15).

* http://www.comlab.ox.ac.uk/activities/quantum/course/

I. BACKGROUND

A review of the properties of operation elements is given in Ch. 3 of the 1998 PhD Thesis by Nielsen[8]. For experimental use of these ideas see for example[3]. Chapter 8 in[2] has an introduction to state and process tomography, which differs from the presentation here.

II. QUANTUM TOMOGRAPHY

Promted by experimental needs, in the mid to late 90's a method of black box characterization known as quantum process tomography[1] was developed. A quantum process is described as a map between *input* and *output* quantum states, e.g.

$$\rho_{out} = \mathcal{E}(\rho_{in}) = \sum_j E_j \rho_{in} E_j^\dagger \tag{1}$$

where the map $\mathcal{E}$ is a *quantum operation* (we consider $\sum_j E_j E_j^\dagger = \mathbf{1}$) and the operators E_j are called operation elements. Process tomography is a procedure used to reconstruct the behavior of a *quantum network* by performing state tomography on a set of initial states ρ_i that form an operator basis for the system in question.

The input states and measurement projectors in each form a basis for the set of $n-$qubit density matrices requiring $d^2 = 2^{2n}$ elements in each set, where d is the dimension of the Hilbert space. For a two-qubit gate $d^2 = 16$, resulting in 256 different settings of input states and measurement projectors.

One of many possible input combinations (adapted from the optics experiment in[3]) forming an operator basis needed to characterize the space of two-qubit circuits is the following:[?]

$$\begin{aligned}\{&|00\rangle, |01\rangle, |10\rangle, |11\rangle, |0+\rangle, |0y_-\rangle, |1y_-\rangle, |1+\rangle,\\ &|++\rangle, |y_+y_-\rangle, |y_++\rangle, |+y_+\rangle, |+1\rangle,\\ &|y_+1\rangle, |+0\rangle, |y_+0\rangle\}.\end{aligned} \tag{2}$$

Of course there are many possible choices for such a basis. In general however, for a system of n qubits the computational basis states $|0\rangle, ..., |2^{n-1}\rangle$ and superpositions $(|q\rangle \pm |r\rangle)/\sqrt{2}$ are prepared, where $q \neq r$[4].

Given many copies of a *sample* state, state tomography is a procedure allowing one to reconstruct an arbitrary quantum state to a given accuracy. It can be done given a set of simple measurement operators that are products of Pauli matrices. The method relies on creating a set of orthogonal measurements and using the Hilbert-Schmidt inner product to expand the state of ρ based on the average outcome of each measurement. A single qubit may be reconstructed as the following density matrix:

$$\rho = \frac{tr(\rho)\sigma_i + tr(\sigma_x\rho)\sigma_x + tr(\sigma_y\rho)\sigma_y + tr(\sigma_z\rho)\sigma_z}{2}. \tag{3}$$

Expressions like $tr(\sigma_x\rho)$ in Eqn. 3 refer to an average measurement outcome where σ_x is an observable.

A similar expansion to that of Eqn. 3 applies to n–qubit systems. For example, reconstruction of any two-qubit operator requires a total of $2^{2n} = 16$ measurement observables:

$$\{\sigma_i \otimes \sigma_i, \sigma_i \otimes \sigma_x, \sigma_i \otimes \sigma_y, \sigma_i \otimes \sigma_z, \sigma_x \otimes \sigma_i, \sigma_x \otimes \sigma_x, \sigma_x \otimes \sigma_y, \sigma_x \otimes \sigma_z, \sigma_y \otimes \sigma_i, \sigma_y \otimes \sigma_x,$$
$$\sigma_y \otimes \sigma_y, \sigma_y \otimes \sigma_z, \sigma_z \otimes \sigma_i, \sigma_z \otimes \sigma_x, \sigma_z \otimes \sigma_y, \sigma_z \otimes \sigma_z\}. \tag{4}$$

A difficulty associated with quantum process tomography is that in experimental practice, the observables are not easily realized. A system with d dimensions requires $16^d - 4^d$ independent parameters to uniquely describe the process, where $d = 2^n$. The useful method of quantum process tomography was developed out of a need for black box characterization.

III. DISTANCE MEASURES BETWEEN QUANTUM STATES

Distance measures between quantum states are now reviewed. First we recall the well known Fidelity measure between quantum states.

Definition III.1. The **Fidelity** between density matrices ρ and σ is defined as:

$$F(\rho,\sigma) \equiv tr\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2 \tag{5}$$

When $\rho = |\psi\rangle\langle\psi|$ is a pure state the fidelity has an easy interpretation as the overlap between ρ and σ, reducing to:

$$F(\psi,\sigma) = \langle\psi|\sigma|\psi\rangle.$$

Furthermore, the Fidelity evaluates to zero when two pure states being compared are orthogonal, it evaluates to one when two states being compared are identical, and is not a metric. Two common ways of turning the Fidelity into a metric are the *Bures metric*,

$$B(\rho,\sigma) \equiv \sqrt{2 - 2\sqrt{F(\rho,\sigma)}} \tag{6}$$

and the *angle*,

$$A(\rho,\sigma) \equiv \arccos\left(\sqrt{F(\rho,\sigma)}\right) \tag{7}$$

a very comprehensive discussion of these details can be found in Ref.[5]. The discussion can be extended to include an operational interpretation of the Fidelity for a mixed state.

A second common distance measure is the Trace Distance between quantum states.

Definition III.2. The **Trace Distance** between density matrices ρ and σ is defined as:

$$D(\rho,\sigma) \equiv \frac{1}{2} tr|\rho - \sigma| \tag{8}$$

where $|Z| = \sqrt{Z^{\dagger}Z}$ and $0 \leq D \leq 1$. The trace distance is a genuine metric on quantum states[2,5] as the following three properties hold:

***i*:** $D(\rho,\sigma) \geq 0$ with $D(\rho,\sigma) = 0$ *iff* $\sigma = \rho$

***ii*:** Symmetry: $D(\rho,\sigma) = D(\sigma,\rho)$

***iii*:** The Triangle Inequality:

$$D(\mathcal{E}(\rho),\mathcal{G}(\rho)) \leq D(\mathcal{E}(\rho),\mathcal{F}(\rho)) + D(\mathcal{F}(\rho),\mathcal{G}(\rho)) \tag{9}$$

The Trace Distance represents the statistical distribution between quantum states with respect to measurement. The Trace Distance has the property of *contractivity*, $D(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \leq D(\rho,\sigma)$ whenever $\mathcal{E}$ is a trace-preserving quantum operation. This just means that acting on arbitrary quantum states ρ and σ both with operation $\mathcal{E}$ will never increase how well one can distinguish these states with respect to measurements[2,5].

The Trace Distance and Fidelity are complementary measures and should be considered equally important when comparing two quantum states[5]. Distance measures may also be used to compare and contrast a real process $\mathcal{F}$ and an ideal process $\mathcal{E}$, such that $\Delta(\mathcal{F},\mathcal{E})$ defines an error metric on a quantum process[5].

Definition III.3. The **S-Fidelity** between real quantum process $\mathcal{F}$ and ideal quantum process $\mathcal{E}$ is defined as:

$$\Delta^{F}_{min}(\mathcal{F},\mathcal{E}) \equiv \min_{|\psi\rangle} \Delta(\mathcal{F}(\psi),\mathcal{E}(\psi)) \tag{10}$$

where the minimum is over all possible pure state inputs and Δ is a Fidelity measure on quantum states.

Definition III.4. The **S-Distance** between real quantum process $\mathcal{F}$ and ideal quantum process $\mathcal{E}$ is defined as:

$$\Delta^D_{max}(\mathcal{F}, \mathcal{E}) \equiv \max_{|\psi\rangle} \Delta(\mathcal{F}(\psi), \mathcal{E}(\psi)) \tag{11}$$

where the maximum is over all possible pure state inputs and Δ is a Distance metric on quantum states.

It is instructive to restrict our thinking to a set of inputs needed to form a complete operator basis for the system in question. In this case, experimentally determining the S-Distance and S-Fidelity amounts to performing state tomography on this complete operator basis input set while keeping track of both the worst Trace Distance (8) and the worst Fidelity (5) between the reconstructed state and that of the ideal. Ref.[5] stated that, *"...the S-Distance and S-Fidelity are the two best error measures, and should be used as the basis for comparison of real quantum information processing experiments to the theoretical ideal."*

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