Lecture 2 — String Diagrams
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I. INTRODUCTION

This lecture started with a review of what we did in the first lecture and then went on to show a few examples of how diagrammatic methods can help do things that would otherwise be much more difficult. There is not full overlap with these and the lectures — e.g. we covered a bit about density matrices which I did not include here. There is also more in these notes than what we talked about in class. I am sure there are still lots of typos left — If you find any, please email me.

II. DIAGRAMMATICS

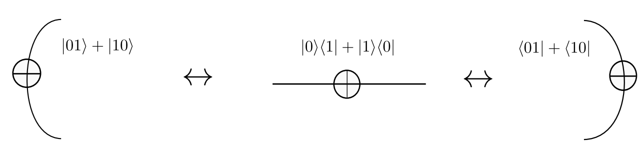
Diagrammatic methods in quantum information science have a long history. Perhaps the first to note their importance was Oxford Professor David Deutsch, who's Paul Dirac Prize in Physics citation states, "*For developing... quantum logic gates in quantum networks*". Methods to manipulate quantum circuits are taught in nearly all quantum information courses, and are presented in nearly all books on the subject. Quantum circuit methods have provided a useful tool to study algorithms, and complexity theory, as well as other things.

It turns out that Category Theory provides the exact arena of mathematics concerned with diagrammatic reasoning. It is more general than graph theory, but graphs can be manipulated as well. These diagrams capture mathematical properties of how maps, or arrows compose.

- The graphs we know of as quantum circuits are only a subclass (planar and directed acyclic) of the types of graphs constructable in a Monoidal Category.

For instance, by dropping temporality during diagrammatic manipulation, one can reason about quantum circuits in new ways, and then translate the network back into a machine readable quantum circuit, to construct implementable quantum processes, depending on the specific application.

To begin with, consider the following diagram showing the 1-1 correspondence between the NOT gate, the singlet, and it's costate:



Two concerns have been brought to my attention when presenting this diagram. The first is that we write quantum circuits as time-dependent diagrams, such that time moves from left to right on the page, depending on convention. This wire is the NOT map, and it clearly has time-direction as it transforms the state of a qubit ψ at some time into $\sigma^x\psi$ at a later time. The second concern is that a single qubit quantum circuit is a map on qubit states (e.g. a map to and from the single qubit Hilbert space $\sigma^x : \mathbb{C}^2 \to \mathbb{C}^2$), whereas a two particle singlet state resides in the Hilbert space of two qubits (the joint state $\mathbb{C}^2 \otimes \mathbb{C}^2$). People have asked me, "how can we get two particles from one?" This is not the point, but with minimal effort we can of course realise these cups and caps as Bell states/measurements. Cups are Bell states and caps are measurements in the Bell basis with simplistic correction.

Remark II.1. *(Bending quantum circuit wires) Bends in quantum circuit wires correspond to dropping temporality during diagrammatic manipulation. This allows one to reason about quantum circuits and quantum information in new ways. We can always translate the network back into a machine readable quantum circuit, to construct implementable quantum processes, depending on the specific application.*

This work attempts to be self contained. For those interested in the graphical language of string diagrams outside the area of quantum mechanics, Peter Selinger's survey offers an excellent starting place. The mathematical insight behind using pictures to represent tensor networks dates back to Oxford Professor Penrose and in quantum circuits, to Oxford Professor David Deutsch. The mathematics behind the category theory is based largely on a completeness result (originally proved by Joyal and Street) about the kinds of string diagrams we consider here.

Theorem II.2. ***Coherence for monoidal categories:*** *The geometric picture calculus in the plane faithfully represents calculations in monoidal categories.*

Samson Abramsky and Bob Coecke provided a categorical model of quantum information processing. A lot of OUCL and outside work built on different aspects of their model. In particular, Bob Coecke and Ross Duncan made an early attempt towards a categorical model of quantum theory applicable to problems in quantum information and computation.

A. Reminder Lecture 1 Notes — Compactness

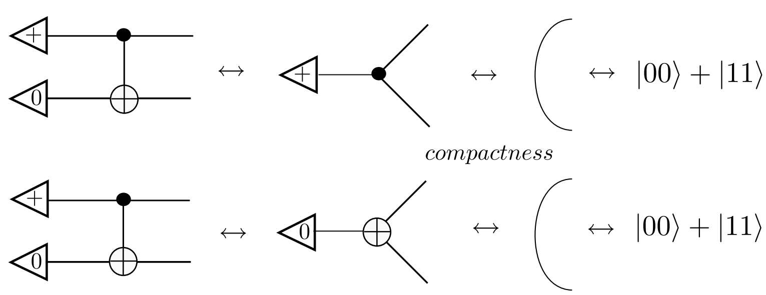
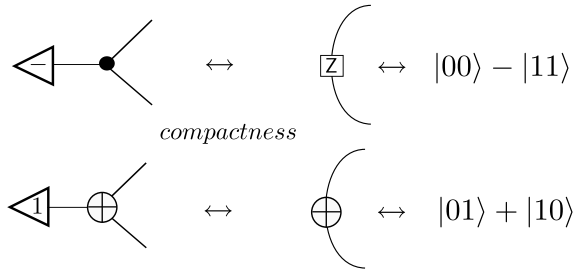


FIG. 1. Showing that the compact structure of the product $\oplus$ is the same as the compact structure of the product $\bullet$. Here the compact structure of the black-dot is used in the lower picture to recover the plus-dot. The compact structure of both dots produce the same cup as $\Phi^+ = |00\rangle + |11\rangle = |++\rangle + |--\rangle$ that is, Φ^+ is invariant under a Hadamard transform.

Remark II.3. *(Bell Basis) Using compactness and single qubit gate operations one is able to generate diagrammatic representations of the remaining three states in the Bell basis (the forth state is generated by acting on the top with a* NOT*-gate, or by acting on the bottom with a* Z*-gate).*

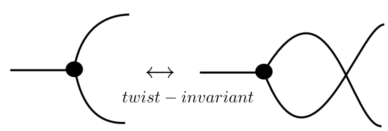


III. COPY CONSTRUCTS: 'DOTS'

A fundamental requirement of universality in computation is the ability to copy bits of data. In the case of reversible and quantum computation, the situation is no different. However, the quantum no-cloning theorem limits the types of copying possible to processes that copy any single basis. In this way, copy is defined as a linear map

$$\mathsf{copy} :: |i\rangle \rightarrow |i, i\rangle \tag{1}$$

ranging over some basis $\{|i\rangle\}$ such as the basis of a non-degenerate observable. Diagrammatically, copy is given by a dot with three legs and a specific *color*, where the color is notation meant to represent the basis (e.g. $\bullet$ for σ^z and $\oplus$ for σ^x). The copy operation is clearly invariant if one exchanges the values after performing copy, which diagrammatically is invariance under *twist* or *swap*.



We will now examine the two copy constructs that arise as sub-components of the CN-gate: the observable σ^z is given as $\bullet$ and for σ^x as $\oplus$, and these are related by a Hadamard transform. We will then generalize the copy operation to any basis in Section III D.

A. Z-copy

As $|0\rangle$ and $|1\rangle$ are eigenstates of σ^z, Z-copy is defined by considering the map that copies these eigenstates:

$$\mathsf{Z\text{-}copy} : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 :: \begin{cases} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{cases}$$

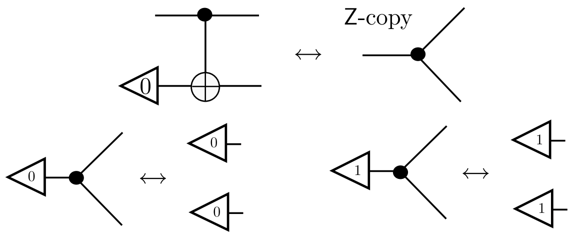
This map can be written as

$$\mathsf{Z\text{-}copy} : |00\rangle\langle 0| + |11\rangle\langle 1| \tag{2}$$

and under braket duality (on the right bra) this state becomes a GHZ-state as

$$\psi_{\mathsf{GHZ}} = |000\rangle + |111\rangle \tag{3}$$

and finally, the σ^z copy construct is given diagrammatically as



B. X-copy

Figure III B depicts X-copy. X-copy spans the truth table for XOR as follows. If we consider $f(a,b) = a \oplus b$ then $f = 0$ corresponds to $(a,b) = (0,0),(1,1)$ and $f = 1$ corresponds to $(a,b) = (1,0),(0,1)$, where the truth table is given as

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

Under braket duality, the state defined by X-copy is given as $\sum_{ab} |a\rangle|b\rangle|f(a,b)\rangle =$

$$\psi_\oplus := |000\rangle + |110\rangle + |011\rangle + |101\rangle \tag{4}$$

which is in the GHZ-class — see Section III C. The X-copy operation is defined on a basis as

$$\mathsf{X\text{-}copy} : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2 :: \begin{cases} |0\rangle \mapsto |00\rangle + |11\rangle \\ |1\rangle \mapsto |10\rangle + |01\rangle \end{cases} \qquad \text{or equivalently} \qquad \begin{cases} |+\rangle \mapsto |++\rangle \\ |-\rangle \mapsto |--\rangle \end{cases}$$

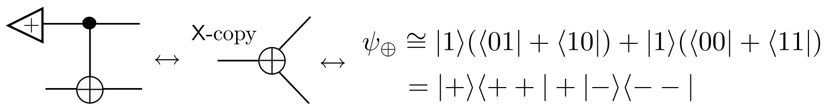
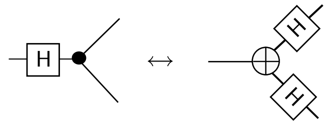


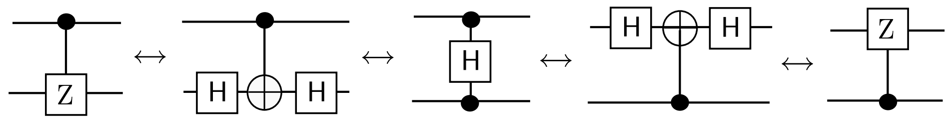
FIG. 2. The copy construct on the σ^x basis is given by a change of basis from the copy construct on the σ^z basis — see the next Section (III C).

C. Dot-duality: Hadamard transforms between Z- and X-copy

By dot-duality, the Z- and X-copy constructs are related by a Hadamard transform, applied to all of the dot's legs to transform a Z- into an X-copy and vise versa. This can be captured diagrammatically in the slightly different form



which clarifies several applications. For instance, it provides elegant Diagrammatic justification for the circuit identity allowing the control and target of a CZ-gate to be exchanged



One can consider a generating set of dots as follows. We will take $\{\bullet,\ \langle +|\}$ and its dual under the dagger which can construct any diagram with only black dots. Adding the Hadamard gate allows one to arrive at the generators

$$\{\bullet,\ \oplus,\ \langle +|,\ \langle 0|\} \tag{5}$$

which is a complete set to express any classical linear circuit. Alternatively, one can start from the CN-gate and together with the units for $\bullet$ and $\oplus$ to arrive at the set (5) see Figure 3.

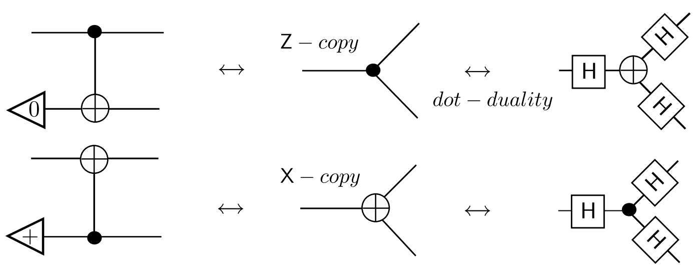
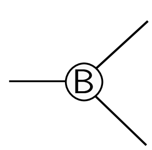


FIG. 3. Z- and X-copy constructs from the CNOT-gate.

D. General copy constructs: a dot of arbitrary *color*

It is of course desirable to extend the definitions of copy to arbitrary basis. This can be done in general, and it turns out that the cases given above are both special cases of the more general framework we will now describe.

We will focus on the case of qubits, and then present an extension to finite dimensions in Section III F. Say you have an orthonormal basis $\{|\psi\rangle, |\psi^\perp\rangle\}$, which we will call B and you wish to have a copy construct (a dot of this *color* B) to copy this basis as $|\psi\rangle \mapsto |\psi, \psi\rangle$ and $|\psi^\perp\rangle \mapsto |\psi^\perp, \psi^\perp\rangle$. To represent such a dot diagrammatically, let us introduce the diagrammatic notation follows as:



$$|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$$
$$|\psi^\perp\rangle \mapsto |\psi^\perp\rangle \otimes |\psi^\perp\rangle$$

Building such dots using local unitary maps is shown to be possible in the next Section.

E. Transforming one dot into another

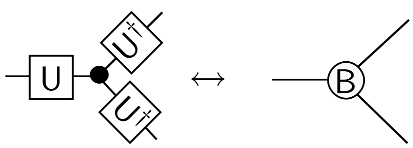
To transform the copy construct defined by the black dot into an arbitrary copy operation, define the unitary

$$U = \sum_i |i\rangle\langle\phi_i| \tag{6}$$

where $\{\phi_i\}_i$ and $\{i\}_i$ are bases for the same space such that $\langle\phi_i|\phi_j\rangle = \delta_{ij}$ and $\langle i|j\rangle = \delta_{ij}$. The map U is unitary as

$$U^\dagger U = (\sum_i |\phi_i\rangle\langle i|)(\sum_j |j\rangle\langle\phi_j|) = \sum_j |\phi_i\rangle\delta_{ij}\langle\phi_j| = \mathbf{1} \tag{7}$$

and the diagrammatic representation of the copy operation for B is given as



$$|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$$
$$|\psi^\perp\rangle \mapsto |\psi^\perp\rangle \otimes |\psi^\perp\rangle$$

Remark III.1. *(non-Hermitian Unitary) It is easy to consider a case where one wishes to copy a basis only a small angle away from the computational basis. In this case, multiple applications of* U *will continue to move the basis further and further away, whereas for any state* ϕ*, and any* $U = U^\dagger$ *then* $U^2\phi = \phi$ *ruling out Hermitian Unitaries in general for a change of basis. In general, a change of basis from* A *to* A' *is given as* $A' = UAU^\dagger$*. This means that* $U^\dagger$ *will need to be placed on the right of the black dots, as* $U \neq U^\dagger$*.*

F. Arbitrary *colors* and dimensions

The general construction for dots of any *colors* and any finite dimension is as follows. Let $\mathcal{V}$ be a Hilbert space with $\dim(\mathcal{V}) = d$ finite and let $|i\rangle$ index a basis for $\mathcal{V}$, such that $\langle i|j\rangle = \delta_{ij}$. It can be shown that $U = -2\sum_{i=1}^{d} |i\rangle\langle i| + \mathbf{1}$ is self-adjoint and unitary as

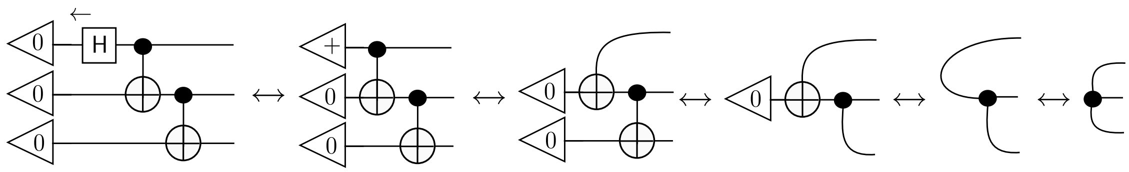
$$P^2 = (\sum_i |i\rangle\langle i|)(\sum_j |j\rangle\langle j|) = \sum_{ij} |i\rangle\delta_{ij}\langle j| = \sum_i |i\rangle\langle i| = P \tag{8}$$

The construction then follows as in Section III D given a copy construct on the computational basis in this dimension d.

IV. APPLICATIONS

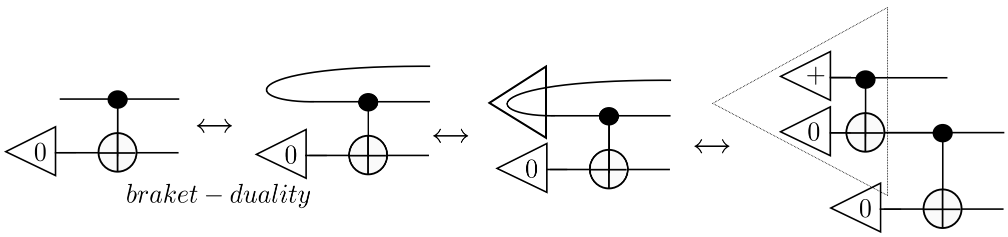
A. GHZ-state circuits

In the lectures, we showed how to build a GHZ-sate using the following circuit:



The simplication from left to right first uses the self-duality of the H-gate, and then the compact structures of the dots (allowing them to transform into bent wires).

On the other hand, one could realise this circuit by considering the braket-duality of the copy operation found from letting a CN-gate act on $|0\rangle$. Diagrammatically this provides backwards justification of the circuit as follows



B. W-state circuits

We consider

$$|001\rangle + |010\rangle + |100\rangle = \sigma_1^x(|101\rangle + |110\rangle + |000\rangle) \tag{9}$$

and so

$$\psi_{\mathsf{W}} = |101\rangle + |110\rangle + |000\rangle \tag{10}$$

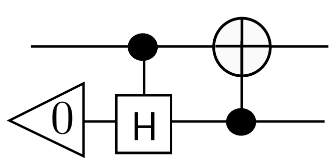
becomes our representative of the W-class. And hence, as described, the W-product forms a map as follows

$$W : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2 :: \begin{cases} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |01\rangle + |10\rangle \end{cases}$$

and so as a first step we wish to find a unitary U and a fixed state ψ such that

$$\begin{aligned} U|0\rangle \otimes |\psi\rangle &= |0\rangle \otimes |0\rangle \\ U|1\rangle \otimes |\psi\rangle &= |01\rangle + |10\rangle \end{aligned} \tag{11}$$

we let $\psi := |0\rangle$ and with a little trial and error note that the map U factors into a quantum circuit as follows



where the open control on the CNOT-gate is controlled on $|0\rangle$ and we have ignored the normalization condition. We note that the map indeed sends $|0\rangle$ to $|00\rangle$ and likewise it sends $|1\rangle$ to $|01\rangle + |10\rangle$.

One will then naturally attempt to dualise the circuit by bending the top wire. This however fails as follows. The input state to the circuit is now $(|00\rangle + |11\rangle)|0\rangle$ the controlled H-gate takes this to

$$|000\rangle + \frac{1}{\sqrt{2}}(|111\rangle + |110\rangle) \tag{12}$$

and then the CN-gate takes this to

$$|000\rangle + \frac{1}{\sqrt{2}}(|101\rangle + |110\rangle) \tag{13}$$

which does give the correct bit pattern, but the wrong normalisation as all states should have the same weights.

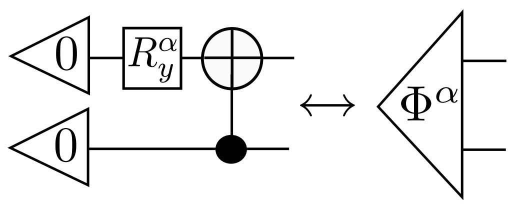
The solution is to consider a new cup. Normalisation can now be accounted for by constructing the generalised cup, which realises the following state:

$$\cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle \tag{14}$$

for $\cos(\theta) = \sqrt{2/3}$. This can be realized by the following sequence

$$(R_y(\alpha) \otimes \mathbf{1})(\mathsf{CNOT})|00\rangle = \cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle \tag{15}$$

which is given diagrammatically as



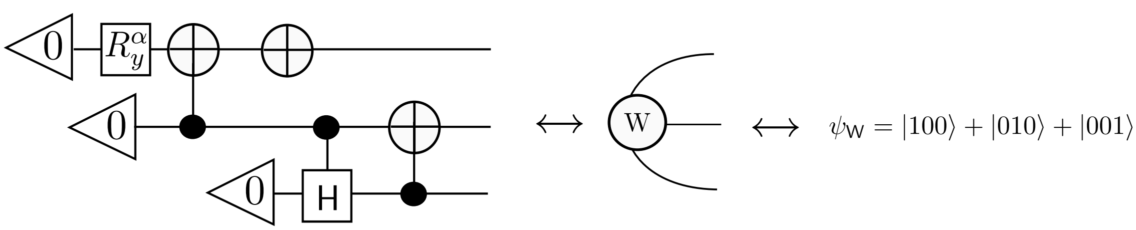


FIG. 4. Circuit realisation of a map from the state $(|00\rangle+|11\rangle)\otimes|0\rangle$ to $|001\rangle+|010\rangle+|100\rangle$, graphical circuit language depiction and W-state vector, from left to right across the page respectively.

Remark IV.1. W *is stabilised by* $-\sigma^z\sigma^z\sigma^z$ *and only has one Pauli stabiliser.*

1. Scaling of implementation

Exercise IV.2. *Construct a general circuit to realize the state* $|0....1\rangle + |0...01\rangle + ... + |1...0\rangle$ *on n-qubits.*

Exercise IV.3. *Give explicit gate counts as functions of n for the exercise above.*

V. REMARKS

That concluded the second lecture.

- Course homepage: http://www.comlab.ox.ac.uk/activities/quantum/course/
- See also the tablet PC generated notes from the lecture!
- If you see any typos please let me know.

VI. SELF-STUDY

A. A stabiliser theory for dots

Viewed in light of braket duality, *Stabiliser Theory* becomes vividly applicable to states and linear maps as both are on equal footing in the Categorical Quantum Circuit Model. We consider $P = \{1, \sigma^x, \sigma^y, \sigma^z\}$ as the Pauli group. This forms a complete operator basis for the space of operators acting on single qubits. We consider the power functor, P^n as the set of n-fold tensor products taken from the Pauli alphabet. This forms an operator basis for the space of n-qubit linear operators.

A Unitary matrix U stabilizes a quantum state ψ if $U\psi = \psi$. Stabilizers of ψ form a group. Here are some standard examples

- σ^x stabilises $|+\rangle = |0\rangle + |1\rangle$ and $-\sigma^x$ stabilises $|-\rangle = |0\rangle - |1\rangle$
- σ^y stabilises $|y_+\rangle = |0\rangle + i|1\rangle$ and $-\sigma^y$ stabilises $|y_-\rangle = |0\rangle - i|1\rangle$
- σ^z stabilises $|0\rangle$ and $-\sigma^z$ stabilises $|1\rangle$

Example VI.1. *(Heisenberg pictures and the evolution of operators) If $N\psi = \psi$ and $M\psi = \psi$ then $MN\psi = NM\psi = \psi$. Now if we consider $U\psi$ this then equals $UN\psi$ which can be written as $UNU^\dagger U\psi$ so U takes N to $UNU^\dagger$ and MN to $(UMU^\dagger)(UNU^\dagger)$.*

Definition VI.2. *(Pauli Stabilisers) If ψ can be produced from the all-$|0\rangle$ state by just* CNOT*, Hadamard, and phase gates, then ψ is stabilized by 2^n tensor products of Pauli matrices or their sign opposites (where n is the number of qubits). This means that the stabilizer group is generated by $\log(2^n) = n$ such tensor products. The state ψ is then the stabilizer state uniquely determined by these generators.*

Example VI.3. *(Stabilisers of the black-dot) Under braket-duality, the black dot becomes a* GHZ*-state which has stabiliser generators $\sigma_1^x \otimes \sigma_2^x \otimes \sigma_3^x$ and $\sigma_i^z \otimes \sigma_j^z$ which uniquely determine $\psi_{\mathsf{GHZ}} = |000\rangle + |111\rangle$ and result in the following set of Stabilisers*

$$\{\sigma^x\sigma^x\sigma^x, -\sigma^x\sigma^y\sigma^y, -\sigma^y\sigma^x\sigma^y, -\sigma^y\sigma^y\sigma^x, \mathbf{1} \otimes \sigma^z \otimes \sigma^z, \sigma^z \otimes \mathbf{1} \otimes \sigma^z, \sigma^z \otimes \sigma^z \otimes \mathbf{1}, \mathbf{1}\} \tag{16}$$

Diagrammatically these relations are given in Figure 5.

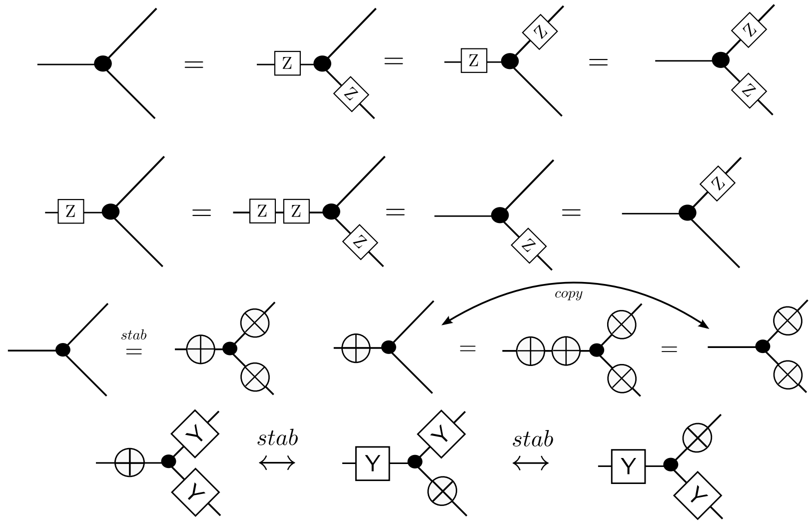
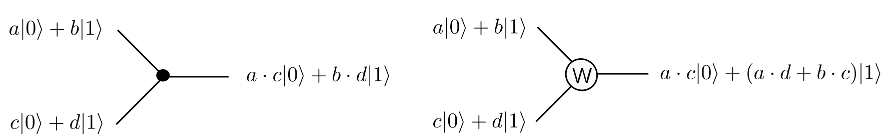


FIG. 5. (Top) Diagrammatic depiction of the stabiliser equation $\sigma_i^z\sigma_j^z(|000\rangle+|111\rangle) = |000\rangle+|111\rangle$. (Bottom) Uses the Stabiliser identity together with $\sigma_z^2 = \mathbf{1}$ to show that the σ^z commutes with the black-dot (this is a special case of the phase group property). (Middle) Diagrammatic depiction of the stabilizer equation $\sigma_1^x \otimes \sigma_2^x \otimes \sigma_3^x(|000\rangle + |111\rangle) = |000\rangle + |111\rangle$. (Bottom) Diagrammatic depiction (up to a sign) of the stabilizer equation $-\sigma_i^x \otimes \sigma_j^y \otimes \sigma_k^y(|000\rangle + |111\rangle) = |000\rangle + |111\rangle$.

Exercise VI.4. *(Gottesman-Knill theorem) Provide a graphical rewrite proof (by bounding the number of rewrites) of the Gottesman-Knill theorem. Show that it follows by considering the action of the black and plus dots on σ^z and σ^x. Gottesman's paper can be found on aXriv.org here http://arxiv.org/abs/quant-ph/9807006.*

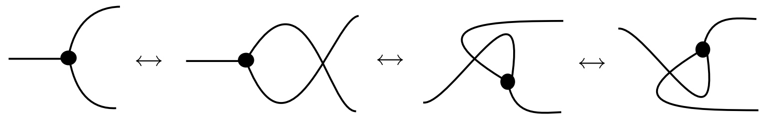
B. The GHZ- and W-products

In the Categorical Model of Quantum Circuits, every quantum state of three or more systems forms a product. By this it means that the measurement outcomes have a mathematical structure which can be viewed as a type of multiplication. For instance, if one measures the first bit of a GHZ-state in the σ^x-basis and recovers $+1$, the state of the system is left in a Bell state, which under braket duality is an identity wire. There is an elegant reason for this. We will see that the state $|+\rangle$ is a unit for the multiplication formed by the GHZ-state. Let's begin with the following depiction of the state induced GHZ- and W-products.



Here one typically considers the numbers a, b, c, d to be amplitudes of a quantum state after measurement, e.g. measuring a state in σ^y and recovering a -1 would result in $|y_-\rangle$, that is $(a, b) = (1, -i)$, etc. This will be made precise below.

The GHZ state has the nice property that it is symmetric under particle exchange — meaning the induced product is associative. Diagrammatically, the full symmetry of this state is given as



By composition of the GHZ-state with one of the cap operators, one constructs the operator

$$|00\rangle\langle 0| + |11\rangle\langle 1| : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2 :: |i\rangle \mapsto |i, i\rangle \tag{17}$$

which we already know acts as a copy operation on the basis $|0\rangle$, $|1\rangle$, for $i \in \{0, 1\}$. Now we will explain the above diagram (the induced product structure of the GHZ-state) in more detail. Consider again the map given by

$$|0\rangle\langle 00| + |1\rangle\langle 11| : \mathbb{C}^2 \to \mathbb{C} :: (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \mapsto ac|0\rangle + bd|1\rangle \tag{18}$$

which is readily verified by considering the action of the map on states with amplitude coefficients (a, b) and (c, d). Hence, the GHZ product forms a monoid $(M, ., e)$ as follows. Here M is the space $\mathbb{C}^2$, and e is given by the $|+\rangle$ now the product becomes

$$\mathsf{GHZ} : \mathbb{C}^2 \otimes \mathbb{C}^2 \mapsto \mathbb{C}^2 :: (a, b).(c, d) \mapsto (ac, bd) \tag{19}$$

with unit $(1, 1)$. Again we note that these amplitudes are measurement outcomes, and so one can think of postselection as being built into the induced product structure presented here.

Definition VI.5. *(GHZ-product) The GHZ-product for the state $|000\rangle + |111\rangle$ is given as*

$$\mathsf{GHZ} : \mathbb{C}^2 \otimes \mathbb{C}^2 \mapsto \mathbb{C}^2 :: (a, b)._G(c, d) \mapsto (ac, bd) \tag{20}$$

where (a, b), (c, d) represent the complex amplitudes of a state after measurement. The unit of the multiplication is $|+\rangle$.

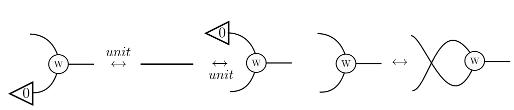
Definition VI.6. *(W-product) The W-product for the state $|000\rangle + |1\rangle(|01\rangle + |10\rangle)$ is given as*

$$\mathsf{W} : \mathbb{C}^2 \otimes \mathbb{C}^2 \mapsto \mathbb{C}^2 :: (a, b)._W(c, d) \mapsto (ac, ad + bc) \tag{21}$$

The unit of the multiplication is $|0\rangle$.

Theorem VI.7. *The GHZ-product has subgroup $\{|+\rangle, |-\rangle, |y_+\rangle, |y_-\rangle\}$ which can be verified by direct calculation.*

Theorem VI.8. *The W-product forms a commutative monoid with unit $|0\rangle$, given diagrmamatically as*



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- If you spot any typos please email me. Thanks to Shane Mansfield for doing just that!

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