Fault Tolerance in the Categorical Quantum Circuit Model

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(1) The three qubit bit flip code

Suppose we send qubits through a channel which leaves the qubits untoched with a probability 1-p and flips qubits with a probability p.

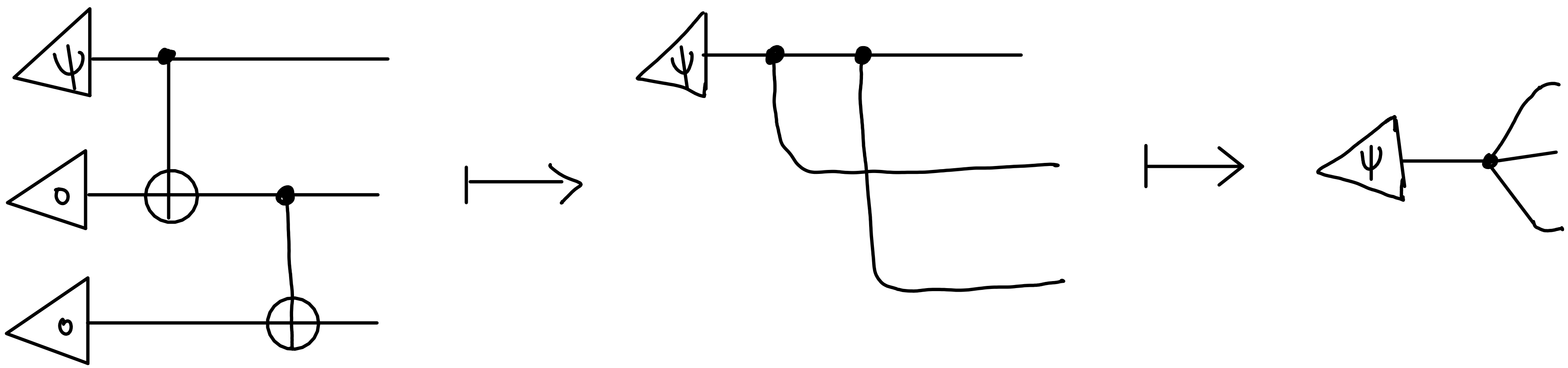
$$|0\rangle \longmapsto |0_L\rangle := |000\rangle$$

$$|1\rangle \longmapsto |1_L\rangle := |111\rangle$$

so $\alpha|0\rangle + \beta|1\rangle \longmapsto \alpha|000\rangle + \beta|111\rangle$

The "bit flip channel": $\psi \longrightarrow X\psi$ w/ prob p

Lemma 1: The state encoding is given as follows, in the graphical language



(1.1) Error detection or syndrome diagnosis

A measurement result is called the `error syndrome'

$$P_0 \equiv |000\rangle\langle 000| + |111\rangle\langle 111| \qquad \text{no error}$$

$$P_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011| \qquad \text{bit-flip on qubit 1}$$

$$P_2 \equiv |010\rangle\langle 010| + |101\rangle\langle 101| \qquad \text{bit-flip on qubit 2}$$

$$P_3 \equiv |001\rangle\langle 001| + |110\rangle\langle 110| \qquad \text{bit-flip on qubit 3}$$

Remark

$$[P_i, P_j] = 0 \text{ for } i \neq j \text{ and } = P_j \text{ for } i = j$$

(1.2) Evolving Hamiltonians to detect and correct errors

We must show a circuit acting on the encoded state space that detects each given error.

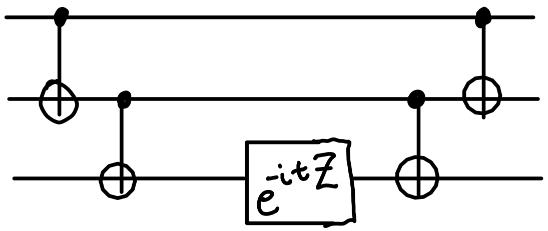
Let $\underline{x}$ be an n-bit string, a projector $\Pi_{\underline{x}}$ onto $\underline{X}$ is given as

$$\Pi_x = \frac{1}{2} \sum_{i=1}^{n} (1 + (-1)^{1-x_i} Z_i) \qquad (1)$$

following notation in [JDB, PRA 77 05231 (2008)]

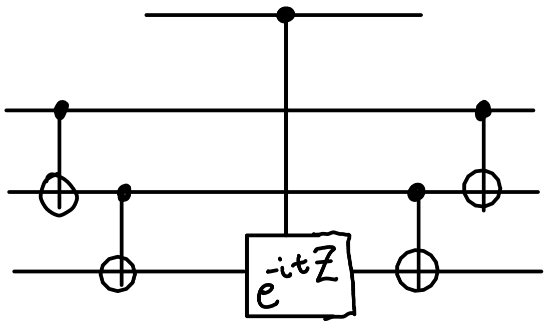
The evolution under some product of pauli operators $P \in P_n = \{1, X, Y, Z\}^n$ amounts to evolution of $\otimes$-products of just $\{1, Z\}$ as* $U_1 \otimes \cdots \otimes U_n \, P \, U_1^\dagger \otimes \cdots \otimes U_n^\dagger \in \{1, Z\}^n$. We have to show a reduction of the general form of a circuit realising the propigator generated by such a hamilton, and impliment this in the graphical language.

Lemma: The evolution of some Hamiltonian $tZZZ$ Is given in the graphical language as



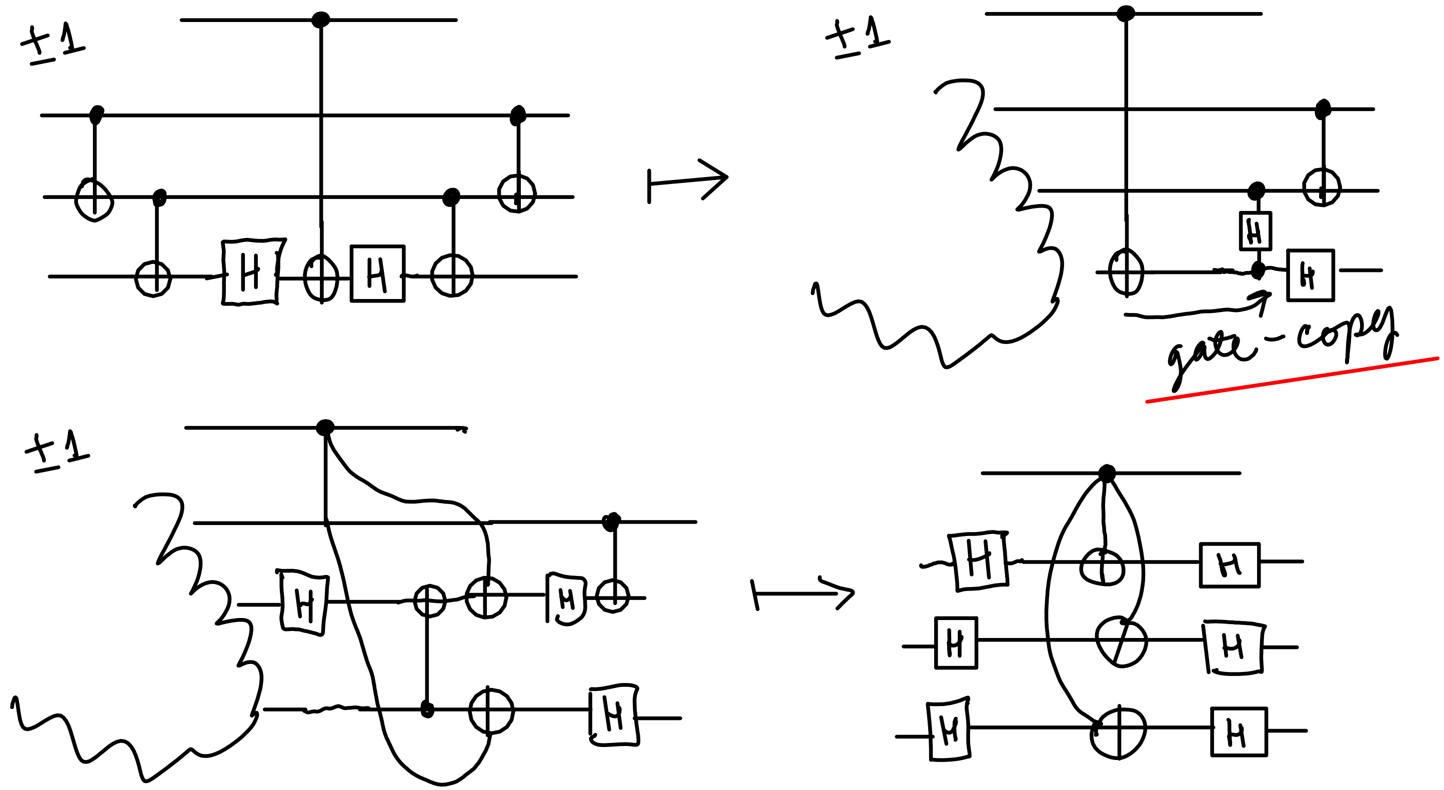
where $e^{-itZ} = 1\cos t - iZ\sin t$ as $Z^2 = 1$

Each P_i term must be realised by constructing sequences of circuits as above. These in turn must be 'controlled' so that detected errors can be corrected. It is a great simplification of our approach that a controlled version of the above circuit reduces to controlling only the center most gate, as



*(note that U_i is a simple change of basis $Z \to X$, or $Z \to Y$)

From 1 a further reduction to the case where the CZ_t gate becomes $\pm CZ$ and the above circuit reduces to



This example sets the general framework to talk about error correction in the picture calculus. For instance, take the operator

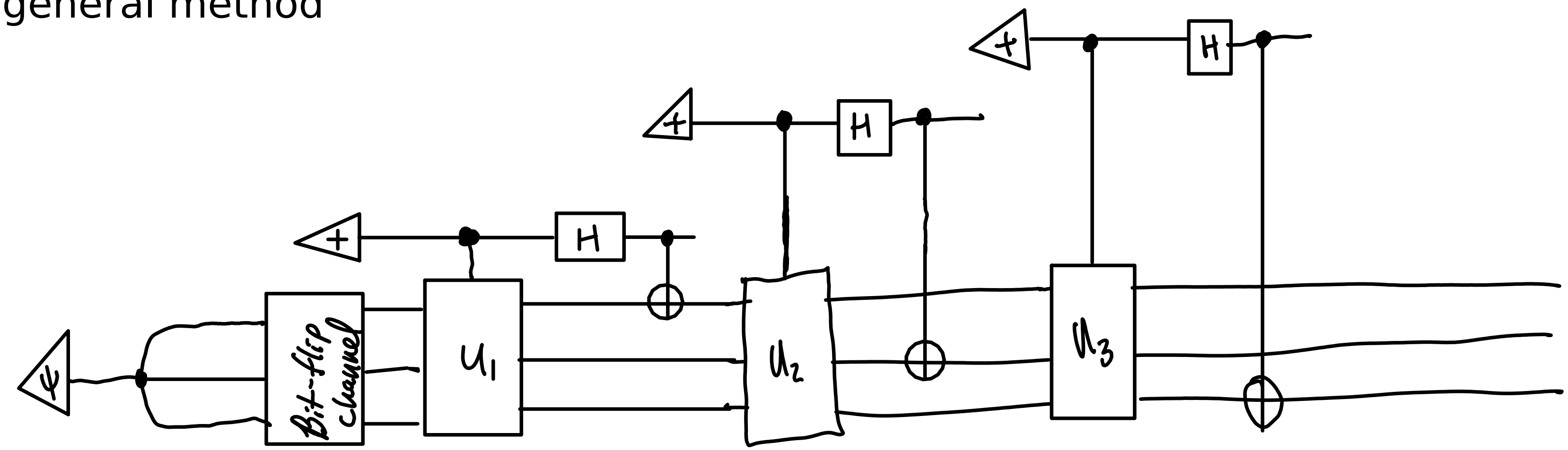
$$P_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011| \qquad \text{bit-flip on qubit } 1$$

Then $U = 1 - 2P_1$ is self adjoint, unitary with $\phi \in \text{span}\{|100\rangle, |011\rangle\}$ in the -1 subspace and with all other eigenvectors are in the $+1$ subspace.

Hence, vectors in the computational basis are eigenvectors of U.

We are then faced with constructing a controlled U, in such a way to measure (detect) and if detected correct faults. This is done as follows:

(1.3) The general method



(1.5) Conclusion

The graphical methods developed here are general as the string diagrams we presented can be quickly adjusted to detect any pauli fault. The main new trick to all of this is how to consider a controlled measurement operation in the graphical language.

References

N and C (error correction and bit flip code)

Coecke & Duncan (grapical methods)

JDB (gate copy), (projector circuits)

JDW, JDB, AAG (Hamiltonian simulations)